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On discounted infinite-time mean field games

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Abstract In this paper, we study the infinite-time mean field games with discounting, establishing an equilibrium where individual optimal strategies collectively regenerate the mean-field distribution. To solve this problem, we partition all agents into a representative player and the social equilibrium. When the optimal strategy of the representative player has the same feedback form as the strategy in the social equilibrium, we say that the system achieves a Nash equilibrium. We construct a Nash equilibrium using the stochastic maximum principle and infinite-time forward-backward stochastic differential equations (FBSDEs). By employing elliptic master equations, a class of distribution-dependent elliptic partial differential equations (PDEs), we provide a representation for the Nash equilibrium strategies. We prove the Yamada−Watanabe type theorem and show weak uniqueness for infinite-time FBSDEs. Furthermore, we prove that the solutions to a system of infinite-time FBSDEs can be employed to construct viscosity solutions for a class of distribution-dependent elliptic PDEs.

Keywords Discounted infinite-time mean field games, Infinite-time forward-backward equations, Weak uniqueness, Elliptic master equations

2020 Mathematics Subject Classification 60H30, 91A15, 91A16

1. Introduction

The study of mean field games was initiated independently by Lasry−Lions [11–13] and Huang−Malhamé−Caines [10] as an analysis of limit models for symmetric weakly interacting $(N+1)$-player differential games. It is noteworthy that current theoretical frameworks are primarily developed for finite-time problems, while infinite-time scenarios remain considerably underdeveloped. We refer the reader to [5] for a comprehensive exposition on this subject.

In this paper, we consider a generalized framework for mean field games, which extends classical finite-time settings to discounted infinite-time mean field games. In our framework, a representative player interacts with a continuum of other players (also referred to as the population or social equilibrium). Let $\mu_0 \in \mathcal{P}(\mathbb{R})$ be the initial state distribution in the society,

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and let $(\mu_t)_{t\geqslant 0}$ denote the population distribution flow starting at μ_0. For the player(x), a representative player with initial state x, the dynamic of its state is given by

$$X_t^{x,\beta} = x + \int_0^t b(X_t^{x,\beta}, \mu_s, \beta_s)\mathrm{d}s + B_t, \tag{1.1}$$

where β, an $\mathbb{R}$-valued progressively measurable stochastic process, is the strategy of the player(x). Denote by β^x the optimal strategy of the representative player(x) derived from the stochastic optimization problem:

$$\min_\beta J^\mu(\beta) \triangleq \mathbb{E}\left[\int_0^\infty e^{-rt} f(X_t^{x,\beta}, \mu_t, \beta_t)\mathrm{d}t\right]. \tag{1.2}$$

Since the population consists of a multitude of individuals, the macroscopic distribution should satisfy

$$\mu_t(\cdot) = \int \mathbb{P}(X_t^{x,\beta^x} \in \cdot)\mu_0(\mathrm{d}x). \tag{1.3}$$

This is exactly a fixed-point problem.

Note that our definition of mean field games differs from that in [1] and from those of earlier finite-time mean field games. In [1], all agents are identical and thus represented by a single representative player. For a given measure flow $(\mu_t)_{t\geqslant 0}$, the representative player seeks to minimize

$$J^\mu(\alpha) \triangleq \mathbb{E}\left[\int_0^\infty e^{-rt} f(t, X_t, \mu_t, \alpha_t)\mathrm{d}t\right] \tag{1.4}$$

under the constraint

$$\begin{cases} \mathrm{d}X_t = b\left(t, X_t, \mu_t, \alpha_t\right)\mathrm{d}t + \sigma \mathrm{d}B_t, \\ X_0 = \xi. \end{cases} \tag{1.5}$$

Then, we require that the law of X_t coincides with μ_t, which means that we need to find a fixed point.

In the conventional definition of mean field games, the primary focus on the distribution flow $(\mu_t)_{t\geqslant 0}$ stems from its role as the limit of the Nash equilibrium of an N-player game. Within this framework, the master equation also serves as a crucial tool: see [7] and [8]. In the study by Buckdahn et al. [2], the individual state process and the population state process are treated separately, allowing for an investigation into various properties of the parabolic master equation. Inspired by their approach, our definition of mean field games explicitly decouples the representative player from the population. This separation offers two key advantages: first, it provides a mathematically interpretation of the game problem even in the absence of a realistic N-player game context; second, the value function $V(x, \mu_0)$ of the representative player(x) directly embodies the relationship between the individual state x and the initial population distribution μ_0, thereby laying a solid foundation for further analysis of the value function's properties. We employ the stochastic maximum principle and infinite-time forward-backward stochastic differential equations (FBSDEs) to solve this game-theoretic problem and derive a representation of the equilibrium strategy via an elliptic master equation.

The theory of general nonlinear BSDEs was pioneered by Pardoux and Peng [15, 16] in the early 1990s, which is now a typical tool in stochastic optimization problems. We shall solve the mean field game problem using the Pontryagin's maximum principle and infinite-time FBSDEs. The foundational work in [17] proved the existence and uniqueness of solutions to infinite-time FBSDEs, and subsequent work in [18] investigated a more general setting and established connections with quasilinear elliptic partial differential equations (PDEs). Recently, [1] extended this framework to McKean−Vlasov FBSDEs, which play crucial roles in obtaining equilibrium solutions for mean field games. In this paper, we partition all agents into a representative player and the social equilibrium and characterize the equilibrium state through the following system of infinite-time FBSDE:

$$\begin{cases} \mathrm{d}X_t^\xi = b(X_t^\xi, \mathcal{L}_{X_t^\xi}, \hat{\alpha}(X_t^\xi, Y_t^\xi))\mathrm{d}t + \mathrm{d}B_t, \\ \mathrm{d}Y_t^\xi = -\partial_x \mathcal{H}(X_t^\xi, \mathcal{L}_{X_t^\xi}, \hat{\alpha}(X_t^\xi, Y_t^\xi), Y_t^\xi)\mathrm{d}t + Z_t^\xi \mathrm{d}B_t, \\ X_0^\xi = \xi. \end{cases} \tag{1.6}$$

$$\begin{cases} \mathrm{d}X_t^x = b(X_t^x, \mathcal{L}_{X_t^\xi}, \hat{\alpha}(X_t^x, Y_t^x))\mathrm{d}t + \mathrm{d}B_t, \\ \mathrm{d}Y_t^x = -\partial_x \mathcal{H}(X_t^x, \mathcal{L}_{X_t^\xi}, \hat{\alpha}(X_t^x, Y_t^x), Y_t^x)\mathrm{d}t + Z_t^x \mathrm{d}B_t, \\ X_0^x = x. \end{cases} \tag{1.7}$$

Here, $\mathcal{H}(x, \mu, a, y) \triangleq b(x, \mu, a) \cdot y + f(x, \mu, a) - rxy$ is the generalized Hamiltonian, and $\hat{\alpha}(x, y)$ is the minimizer of $\mathcal{H}(x, \mu, a, y)$ with respect to a when we assume that $\mathcal{H}$ is separable in variables μ and a. It is worth emphasizing that we set the diffusion coefficient to 1 here, which makes both equations (1.6) and (1.7) solvable. If the diffusion term in problem setup (1.1) is dependent on the state, distribution, and control, studying these two equations would be considerably more challenging. Now, we consider the solutions to these two FBSDEs. The solution X_t^ξ to Equation (1.6) represents the population's state process, whose law corresponds to the population distribution μ_t. In addition, the solution X_t^x to Equation (1.7) is the state process of the representative agent after solving the optimization problem. Notably, it exhibits the same feedback structure as the population's state process. To further elucidate the relationship in Equation (1.3), we introduce elliptic master equations.

First introduced by Lions in lectures [14], the parabolic master equation appeared in the context of the theory of mean field games. This is a time-dependent equation that bears profound connections with finite-time mean field game theory. Essentially, it describes a strategic interaction between a representative player and the collective environment. When the Nash equilibrium exists, the master equation provides a powerful tool to characterize the equilibrium cost and control pattern of this system. We refer the reader to [3, 5, 9] for a comprehensive exposition on the subject. In this paper, we propose elliptic master equations, which explicitly characterize the feedback forms of both the representative player and the social equilibrium. These equations take the following form:

$$\begin{aligned} rU(x, \mu) &= H(x, \mu, \partial_x U(x, \mu)) + \frac{1}{2}\partial_{xx} U(x, \mu) \\ &\quad + \tilde{\mathbb{E}}\left[\frac{1}{2}\partial_{\tilde{x}}\partial_\mu U(x, \mu, \tilde{\xi}) + \partial_\mu U(x, \mu, \tilde{\xi})\partial_y H(\tilde{\xi}, \mu, \partial_x U(\tilde{\xi}, \mu))\right]. \end{aligned} \tag{1.8}$$

Here, ∂_x and ∂_{xx} are standard spatial derivatives, ∂_μ and $\partial_{\tilde{x}\mu}$ are W_2-Wasserstein derivatives, $\tilde{\xi}$ is a random variable with law μ and $\tilde{\mathbb{E}}$ is the expectation with respect to its law. Under the assumption that the master equation (1.8) admits a solution with sufficient regularity, we derive the following representation for Equations (1.6) and (1.7):

$$Y_t^\xi = \partial_x U(X_t^\xi, \mathcal{L}_{X_t^\xi}), \quad Z_t^\xi = \partial_{xx} U(X_t^\xi, \mathcal{L}_{X_t^\xi}), \tag{1.9}$$

$$Y_t^x = \partial_x U(X_t^x, \mathcal{L}_{X_t^\xi}), \quad Z_t^x = \partial_{xx} U(X_t^x, \mathcal{L}_{X_t^\xi}). \tag{1.10}$$

If $\tilde{b}(x,\mu) \triangleq b(x,\mu,\hat{\alpha}(x,\partial_x U(x,\mu)))$ is Lipschitz continuous in (x,μ), we have

$$X_t^x|_{x=\xi} = X_t^{\xi}, \quad t \in [0,T] \tag{1.11}$$

for any fixed finite time T. This is precisely the relationship expressed in Equation (1.3).

Alternatively, if we define $\mathcal{V}(x,\mu) \triangleq Y_0^{x,\xi}$, we demonstrate that $\mathcal{V}$ serves as a viscosity solution to the distribution-dependent PDE below:

$$\begin{aligned} r\mathcal{U}(x,\mu) &= \partial_x H(x,\mu,\mathcal{U}(x,\mu)) + \partial_y H(x,\mu,\mathcal{U}(x,\mu)) \cdot \partial_x \mathcal{U}(x,\mu) + \frac{1}{2}\partial_{xx}\mathcal{U}(x,\mu) \\ &\quad + \tilde{\mathbb{E}}\left[\frac{1}{2}\partial_{\tilde{x}}\partial_\mu \mathcal{U}(x,\mu,\tilde{\xi}) + \partial_\mu \mathcal{U}(x,\mu,\tilde{\xi})\partial_y H(\tilde{\xi},\mu,\mathcal{U}(\tilde{\xi},\mu))\right]. \end{aligned} \tag{1.12}$$

This paper is organized as follows: in section 2, we present preliminaries for problems in this paper; in section 3, we introduce infinite-time mean field games with discounting and the definition of a Nash equilibrium; in section 4 we characterize the equilibrium states through a system of infinite-time FBSDEs; in section 5, we introduce an elliptic master equation to provide a representation for the Nash equilibrium; and in section 6, we prove the Yamada−Watanabe theorem for infinite-time FBSDEs and provide a viscosity solution for distribution-dependent elliptic PDEs by virtue of the class of FBSDEs introduced in section 4.

2. Preliminaries

We will use a filtered probability space $(\Omega,\mathcal{F},\mathbb{P},\mathbb{F})$ endowed with a Brownian motion B. Its filtration $\mathbb{F} \triangleq (\mathcal{F}_t)_{t\geqslant 0}$ is augmented by all $\mathbb{P}$-null sets and a sufficiently rich sub-σ-algebra $\mathcal{F}_0$ independent of B such that it can support any measure on $\mathbb{R}$ with finite second moment.

Let $(\Omega',\mathcal{F}',\mathbb{P}',\mathbb{F}')$ be a copy of the filtered probability space $(\Omega,\mathcal{F},\mathbb{P},\mathbb{F})$ with corresponding Brownian motion B'. We define the larger filtered probability space by

$$\tilde{\Omega} \triangleq \Omega\times\Omega', \quad \tilde{\mathcal{F}} \triangleq \mathcal{F}\otimes\mathcal{F}' \quad \tilde{\mathbb{F}} = \{\tilde{\mathcal{F}}_t\}_{t\geqslant 0} \triangleq \{\mathcal{F}_t\otimes\mathcal{F}'_t\}_{t\geqslant 0}, \quad \tilde{\mathbb{P}} \triangleq \mathbb{P}\otimes\mathbb{P}', \quad \tilde{\mathbb{E}} \triangleq \mathbb{E}^{\tilde{\mathbb{P}}}. \tag{2.1}$$

Throughout the paper, we will use the probability space $(\Omega,\mathcal{F},\mathbb{P},\mathbb{F})$. However, when we deal with the distribution-dependent master equation, independent copies of random variables or processes are needed. Then, we will tacitly use their extensions to the larger space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}},\tilde{\mathbb{F}})$.

Let $\mathcal{P} \triangleq \mathcal{P}(\mathbb{R})$ be the set of all probability measures on $\mathbb{R}$ and let $\mathcal{P}_p(p\geqslant 1)$ denote the set of $\mu\in\mathcal{P}$ with finite p-th moment. For any sub-σ-field $\mathcal{G}\subset\mathcal{F}$ and $\mu\in\mathcal{P}_p$, we define $\mathbb{L}^p(\mathcal{G})$ to be the set of $\mathbb{R}$-valued, $\mathcal{G}$-measurable, and p-integrable random variables ξ and $\mathbb{L}^p(\mathcal{G};\mu)$ to be the set of $\xi\in\mathbb{L}^p(\mathcal{G})$ such that the law $\mathcal{L}_\xi=\mu$. For any $\mu,\nu\in\mathcal{P}_p$, we define the $\mathcal{W}_p$-Wasserstein distance between them as follows:

$$\mathcal{W}_p(\mu,\nu) \triangleq \inf\Big\{\big(\mathbb{E}[|\xi-\eta|^q]\big)^{1/q} : \text{for all } \ \xi\in\mathbb{L}^p(\mathcal{F};\mu), \eta\in\mathbb{L}^p(\mathcal{F};\nu)\Big\}.$$

Due to our interest in discounted infinite-time mean field games, for any $K\in\mathbb{R}$, we denote by $L_K^2(t_0,\infty,\mathbb{R})$ the Hilbert space of all $\mathbb{R}-$ valued adapted stochastic processes (v_t) starting from t_0 such that

$$\mathbb{E}\left[\int_{t_0}^{\infty} e^{-Kt}|v_t|^2 \mathrm{d}t\right] < +\infty. \tag{2.2}$$

To simplify, we set $L_K^2 \triangleq L_K^2(0,\infty,\mathbb{R})$. For each $\mathcal{F}_0$-measurable square-integrable random variable ξ, we consider the following infinite-time FBSDE:

$$\begin{cases} \mathrm{d}X_t = G(t, X_t, Y_t, \mathcal{L}_{X_t})\mathrm{d}t + \mathrm{d}B_t, \\ \mathrm{d}Y_t = -F(t, X_t, Y_t, \mathcal{L}_{X_t})\mathrm{d}t + Z_t\mathrm{d}B_t, \\ X_0 = \xi. \end{cases} \tag{2.3}$$

Here, $G, F : \mathbb{R}^+ \times \mathbb{R}^2 \times \mathcal{P}_2 \to \mathbb{R}$ are two measurable functions that satisfy the following assumptions.

Assumption 2.1 (i) *There exists a positive constant* ℓ *such that for any* $x, x', y, y' \in \mathbb{R}$ *and* $\mu, \mu' \in \mathcal{P}_2$ *,*

$$\begin{aligned} &|G(t,x,y,\mu) - G(t,x',y',\mu')| + |F(t,x,y,\mu) - F(t,x',y',\mu')| \\ \leqslant\ &\ell(|x-x'| + |y-y'| + \mathcal{W}_2(\mu,\mu')), \quad \text{a.s.} \end{aligned} \tag{2.4}$$

(ii) *There exist constants* $0 < K < 2\kappa$ *such that for any* $t \geqslant 0$ *and any square-integrable random variables* X, X', Y, Y' *,*

$$\begin{aligned} &\mathbb{E}\left[-K\hat{X}\hat{Y} - \hat{X}(F(t,U) - F(t,U')) + \hat{Y}(B(t,U) - B(t,U'))\right] \\ \leqslant &- \kappa\mathbb{E}\left[\hat{X}^2 + \hat{Y}^2\right], \end{aligned} \tag{2.5}$$

where $\hat{X} = X - X'$, $\hat{Y} = Y - Y'$ *and* $U = (X, Y, \mathcal{L}_X)$, $U' = (X', Y', \mathcal{L}_{X'})$.

The following lemma establishes the existence and uniqueness of a solution to the FBSDE (2.3). For a detailed proof, we refer the reader to [1] (Theorem 2.1).

Lemma 2.2 *Under Assumption 2.1, the FBSDE (2.3) admits a unique solution in* L^2_K.

We introduce the Wasserstein space and the associated differential calculus. For a $\mathcal{W}_2$-continuous function $U : \mathcal{P}_2 \to \mathbb{R}$, its $\mathcal{W}_2$-Wasserstein derivative [5] (also called the Lions-derivative) takes the form $\partial_\mu U : (\mu, \tilde{x}) \in \mathcal{P}_2 \times \mathbb{R} \to \mathbb{R}$ and satisfies

$$U(\mathcal{L}_{\xi+\eta}) - U(\mu) = \mathbb{E}\big[\langle \partial_\mu U(\mu,\xi), \eta\rangle\big] + o(\|\eta\|_2), \ \forall\, \xi \in \mathbb{L}^2(\mathcal{F};\mu), \eta \in \mathbb{L}^2(\mathcal{F}). \tag{2.6}$$

Let $C^0(\mathcal{P}_2)$ denote the set of $\mathcal{W}_2$-continuous functions $U : \mathcal{P}_2 \to \mathbb{R}$. For $C^1(\mathcal{P}_2)$, we define the space of functions $U \in C^0(\mathcal{P}_2)$ such that $\partial_\mu U$ exists, is continuous on $\mathcal{P}_2 \times \mathbb{R}$, and is uniquely determined by (2.6). Let $C^{2,1}(\mathbb{R} \times \mathcal{P}_2)$ denote the set of continuous functions $U : \mathbb{R} \times \mathcal{P}_2 \to \mathbb{R}$ such that $\partial_x U, \partial_{xx} U$ exist and are joint continuous on $\mathbb{R} \times \mathcal{P}_2$ and $\partial_\mu U, \partial_{x\mu} U, \partial_{\tilde{x}\mu} U$ exist and are continuous on $\mathbb{R} \times \mathcal{P}_2 \times \mathbb{R}$.

Finally, we consider the space $\Theta \triangleq [0,T] \times \mathbb{R} \times \mathcal{P}_2$ for some $T > 0$; let $C^{1,2,1}(\Theta)$ denote the set of continuous functions $U : \Theta \to \mathbb{R}$ that has the following continuous derivatives: $\partial_t U$, $\partial_x U$, $\partial_{xx} U$, $\partial_\mu U$, $\partial_{x\mu} U$, $\partial_{\tilde{x}\mu} U$. One crucial property of functions $U \in C^{1,2,1}(\Theta)$ is that they satisfy the Itô's formula [2, 5]. For $i = 1, 2$, let $\mathrm{d}X^i_t \triangleq b^i_t \mathrm{d}t + \sigma^i_t \mathrm{d}B_t$, where $b^i : [0,T] \times \Omega \to \mathbb{R}$ and $\sigma^i : [0,T] \times \Omega \to \mathbb{R}$ are $\mathbb{F}$-progressively measurable and bounded (for simplicity), and let $\rho_t \triangleq \mathcal{L}_{X^2_t}$. Suppose that for every compact subset $\mathcal{K} \subset \mathbb{R} \times \mathcal{P}_2$, it holds that

$$\sup_{(t,x,\mu)\in[0,T]\times\mathcal{K}} \int_{\mathbb{R}} \left(|\partial_\mu U(t,x,\mu,\tilde{x})|^2 + |\partial_{\tilde{x}\mu} U(t,x,\mu,\tilde{x})|^2\right) \mathrm{d}\mu(\tilde{x}) < \infty. \tag{2.7}$$

We have

$$\begin{aligned}\mathrm{d}U(t,X_t^1,\rho_t) = &\Big[\partial_t U + \partial_x U \cdot b_t^1 + \frac{1}{2}\partial_{xx}U(\sigma_t^1)^2\Big](t,X_t^1,\rho_t)\mathrm{d}t\\ &+ \Big(\tilde{\mathbb{E}}_{\mathcal{F}_t}\big[\partial_\mu U(t,X_t^1,\rho_t,\tilde{X}_t^2)(\tilde{b}_t^2) + \frac{1}{2}\partial_{\tilde{x}}\partial_\mu U(t,X_t^1,\rho_t,\tilde{X}_t^2)(\tilde{\sigma}_t^2)^2\big]\Big)\mathrm{d}t\\ &+ \partial_x U(t,X_t^1,\rho_t)\sigma_t^1\mathrm{d}B_t. \end{aligned} \tag{2.8}$$

Here, $\tilde{\mathbb{E}}_{\mathcal{F}_t}$ is the conditional expectations under $\tilde{\mathbb{P}}$ given $\mathcal{F}_t$ and the process $(\tilde{X}_t^2,\tilde{b}_t^2,\tilde{\sigma}_t^2)_{0\leqslant t\leqslant T}$ is a copy of the process $(X_t^2,b_t^2,\sigma_t^2)_{0\leqslant t\leqslant T}$, defined on a copy $(\Omega',\mathcal{F}',\mathbb{P}',\mathbb{F}')$ of the probability space $(\Omega,\mathcal{F},\mathbb{P},\mathbb{F})$.

3. Infinite-time mean field games with discounting

In this section, we introduce infinite-time mean field games with discounting. Let $r>0$ represent the time discount factor and $A\subset\mathbb{R}$ be a convex control space. Define $\mathcal{A}\triangleq L_r^2(0,\infty,A)$ to be the space of all admissible controls, and $b,f:\mathbb{R}\times\mathcal{P}_2\times A\to\mathbb{R}$ are two measurable functions.

We consider a population consisting of a continuum of players, where each individual player strategically interacts with the aggregate distribution formed by all other players to minimize their own cost. Let μ_t denote the population distribution and $\xi\in\mathbb{L}^2(\mathcal{F}_0)$ denote the initial state with $\mathcal{L}_\xi=\mu_0$. The state of the representative player with initial value x is given by

$$X_t^{x,\beta} = x + \int_0^t b(X_s^{x,\beta},\mu_t,\beta_s)\mathrm{d}s + B_t, \tag{3.1}$$

where $\beta\in\mathcal{A}$ is the strategy that remains to be determined. The representative player seeks to minimize the cost

$$J^\mu(\beta) \triangleq \mathbb{E}\left[\int_0^\infty e^{-rt}f(X_t^{x,\beta},\mu_t,\beta_t)\mathrm{d}t\right]. \tag{3.2}$$

Assuming that $\beta^x\in\mathcal{A}$ minimizes the cost function, the state process of the representative player is X_t^{x,β^x}. Since the representative player can characterize the strategies of all players with the same initial state in the population, we assert that the following fundamental relationship must hold:

$$\mu_t = \mathcal{L}_{X_t^{x,\beta^x}|_{x=\xi}}. \tag{3.3}$$

To solve this mean field game problem, we work under the following assumptions.

Assumption 3.1 (i) *$b(x,\mu,a)$ is Lipschitz in (x,μ,a), and $f(x,\mu,a)$ is of at most quadratic growth in (x,μ,a). There exists a positive constant ℓ such that for any $\mu,\mu'\in\mathcal{P}_2, x\in\mathbb{R}, a\in A$,*

$$|b(x,\mu,a)-b(x,\mu',a)| \leqslant \ell\mathcal{W}_2(\mu,\mu'). \tag{3.4}$$

(ii) *There exists a constant $\lambda>\ell-r/2$ such that for any $a\in A,\mu\in\mathcal{P}_2,x,x'\in\mathbb{R}$, it holds that*

$$(x-x')\,(b(x,\mu,a)-b(x',\mu,a)) \leqslant -\lambda(x-x')^2. \tag{3.5}$$

These assumptions jointly ensure that (i) the state process remains confined within the L_r^2 space and (ii) the cost functional maintains integrability.

Then, we partition all agents into two components: (i) the *representative player*, who dynamically optimizes their strategy, and (ii) the *social equilibrium* (or *mean field*), characterizing the macroscopic state shared by the population. The state of the social equilibrium is governed by the following stochastic differential equation (SDE):

$$X_t^{\xi,\alpha} = \xi + \int_0^t b(X_s^{\xi,\alpha}, \mathcal{L}_{X_s^{\xi,\alpha}}, \alpha_s)\mathrm{d}s + B_t, \tag{3.6}$$

where $\xi \in \mathbb{L}^2(\mathcal{F}_0)$ and $\alpha \in \mathcal{A}$. We note that by Assumption 3.1, this SDE has a unique strong solution in L_r^2; see [1] (Proposition 2.2) for more details.

The state of the representative player is governed by

$$X_t^{x,\beta} = x + \int_0^t b(X_s^{x,\beta}, \mathcal{L}_{X_s^{\xi,\alpha}}, \beta_s)\mathrm{d}s + B_t. \tag{3.7}$$

Here, we also require their control $\beta \in \mathcal{A}$.

The representative player seeks to minimize the cost

$$J(x,\xi;\alpha,\beta) = \mathbb{E}\left[\int_0^{+\infty} \mathrm{e}^{-rt} f\big(X_t^{x,\beta}, \mathcal{L}_{X_t^{\xi,\alpha}}, \beta_t\big)\mathrm{d}t\right]. \tag{3.8}$$

For any $(x,\xi) \in \mathbb{R} \times \mathbb{L}^2(\mathcal{F}_0)$ and $\alpha \in \mathcal{A}$, we consider the infimum

$$V(x,\xi;\alpha) \triangleq \inf_{\beta\in\mathcal{A}} J(x,\xi;\alpha,\beta). \tag{3.9}$$

Definition 3.2 *We say that a Lipschitz function $\alpha^*(x,\mu) : \mathbb{R} \times \mathcal{P}_2 \to \mathbb{R}$ constitutes a discounted infinite-time mean field Nash equilibrium for a given initial distribution μ_0 if for any initial state $\xi_0 \in \mathbb{L}^2(\mathcal{F}_0)$ with distribution μ_0, the closed-loop controls $\alpha_s^{\xi_0} = \alpha^*(X_s^{\xi_0,\alpha^{\xi_0}}, \mathcal{L}_{X_s^{\xi_0,\alpha^{\xi_0}}})$, $\alpha_s^x = \alpha^*(X_s^{x,\alpha^x}, \mathcal{L}_{X_s^{\xi_0,\alpha^{\xi_0}}})$ satisfy*

$$\alpha^x \in \arg\min_{\beta\in\mathcal{A}} J(x,\xi_0;\alpha^{\xi_0},\beta). \tag{3.10}$$

When the Nash equilibrium α^* exists, we have

$$\xi,\xi' \in \mathbb{L}^2(\mathcal{F}_0), \quad \mathcal{L}_{\xi'} = \mathcal{L}_\xi \quad \Longrightarrow \quad \mathcal{L}_{X_t^{\xi,\alpha^\xi}} = \mathcal{L}_{X_t^{\xi',\alpha^{\xi'}}}, \quad \textit{for a.e. } t \geqslant 0. \tag{3.11}$$

Therefore, we can define

$$V(x,\mu) \triangleq J(x,\xi_0;\alpha^{\xi_0},\alpha^x), \quad \xi_0 \in \mathbb{L}^2(\mathcal{F}_0,\mu). \tag{3.12}$$

In our framework, we separate a representative player from the population, who only needs to consider their optimization problem starting from state x. This model decouples the micro-level agent from the macro-level population distribution, enabling an interconnected analysis of their evolution. The equilibrium is characterized by two consistency conditions.

• Individual Rationality: The representative player's optimal strategy is consistent with the perceived social equilibrium.

• Macro-consistency: The aggregate distribution generated by all players adopting this strategy must coincide with the posited social equilibrium.

The representative player's state evolution depends on both their state $X^{x,\beta}$ and the overall population distribution $\mathcal{L}_{X^{\xi,\alpha}}$. Here, we would like to emphasize that since the number of players is large, any change by a representative player does not impact the measure flow $\mathcal{L}_{X^{\xi,\alpha}}$. Under the existence assumption of the Nash equilibrium α^* specified in Definition 3.2, the stochastic dynamics of both the population and representative player are characterized by the following SDE:

$$\begin{cases} X_t^{\xi,\alpha^*} = \xi + \int_0^t b(X_s^{\xi,\alpha^*}, \mathcal{L}_{X_s^{\xi,\alpha^*}}, \alpha^*(X_s^{\xi,\alpha^*}, \mathcal{L}_{X_s^{\xi,\alpha^*}}))\mathrm{d}s + B_t, \\ X_t^{x,\alpha^*} = x + \int_0^t b(X_s^{x,\alpha^*}, \mathcal{L}_{X_s^{\xi,\alpha^*}}, \alpha^*(X_s^{x,\alpha^*}, \mathcal{L}_{X_s^{\xi,\alpha^*}}))\mathrm{d}s + B_t. \end{cases} \tag{3.13}$$

We further assume that $\tilde{b}(x,\mu) \triangleq b(x,\mu,\alpha^*(x,\mu))$ is Lipschitz continuous in (x,μ). For any fixed finite time T, see [2, 4], we have

$$X_t^{x,\alpha^*}|_{x=\xi} = X_t^{\xi,\alpha^*}, \quad t \in [0,T]. \tag{3.14}$$

This implies that every sample point from the initial population follows the same evolutionary dynamics as our hypothesized representative player, which justifies the mathematical validity of using a single representative player to characterize the behavior of all individuals in the population. Moreover, when all agents in the population adopt the same strategy as the representative player, their collective behavior precisely generates the aggregate distribution $\mathcal{L}_{X_t^{\xi,\alpha^*}}$ derived from our solution. This justifies why we refer to X_t^{ξ,α^*} as the social equilibrium.

4. Connection with infinite-time McKean−Vlasov FBSDEs

In this section, we employ the maximum principle to solve the optimization problem for the representative player and then use infinite-time McKean−Vlasov FBSDEs to construct the optimal strategy for the representative player such that the controls of the representative player and the social equilibrium share the same feedback form. Our derivation is based on the following key assumptions on b, f:

Assumption 4.1 (i) *$b(x,\mu,a) = b_0(x,\mu) + b_1(x,a)$ and $f(x,\mu,a) = f_0(x,\mu) + f_1(x,a)$ where b_0, f_0 are measurable functions on $\mathbb{R} \times \mathcal{P}_2$ and b_1, f_1 are measurable functions on $\mathbb{R} \times A$.*

(ii) *b, f are differentiable with respect to (x,a), and $\partial_a b, \partial_a f$ are Lipschitz continuous in (x,a).*

(iii) *$H_0(x,\mu,a,y) \triangleq b(x,\mu,a)\cdot y + f(x,\mu,a)$ is convex with respect to (x,a). $\min_{a\in A} H_0(x,\mu,a,y)$ has a unique minimizer $\hat{\alpha}(x,y)$ that is Lipschitz continuous in (x,y).*

4.1 Pontryagin's maximum principle

Assuming that the state of the social equilibrium X_t^{ξ} is given, we consider the optimization problem for the representative player, whose state is given by

$$X_t^{x,\beta} = x + \int_0^t b(X_s^{x,\beta}, \mathcal{L}_{X_s^{\xi}}, \beta_s)\mathrm{d}s + B_t. \tag{4.1}$$

The cost functional takes the form

$$J(\beta) \triangleq \mathbb{E}\left[\int_0^\infty e^{-rt} f(X_t^{x,\beta}, \mathcal{L}_{X_t^{\xi}}, \beta_t)\mathrm{d}t\right], \tag{4.2}$$

and the representative player wants to solve the minimization problem

$$\inf_{\beta\in\mathcal{A}} J(\beta). \tag{4.3}$$

Let us formally derive the maximum principle for the infinite-time control problem. Suppose β is an optimal control, choose another admissible control γ, and denote by $X^{x,\beta+\epsilon\gamma}$ the state trajectory corresponding to the control $\beta+\epsilon\gamma$. Let

$$R_t = \lim_{\epsilon\to 0} \frac{X_t^{x,\beta+\epsilon\gamma} - X_t^{x,\beta}}{\epsilon} \tag{4.4}$$

be the variation process. Then, it can be shown that R satisfies

$$\begin{cases} \mathrm{d}R_t = \Big(\partial_x b(X_t^{x,\beta}, \mathcal{L}_{X_t^\xi}, \beta_t) \cdot R_t + \partial_a b(X_t^{x,\beta}, \mathcal{L}_{X_t^\xi}, \beta_t) \cdot \gamma_t\Big)\,\mathrm{d}t, \\ R_0 = 0. \end{cases} \tag{4.5}$$

The function $\beta \to J(\beta)$ is Gâteaux differentiable in the direction β and its derivative is given by

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} J(\beta + \epsilon\gamma)\bigg|_{\epsilon=0} = \mathbb{E}\left[\int_0^\infty e^{-rt}\left(\partial_x f(X_t^{x,\beta}, \mathcal{L}_{X_t^\xi}, \beta_t) \cdot R_t + \partial_a f(X_t^{x,\beta}, \mathcal{L}_{X_t^\xi}, \beta_t) \cdot \gamma_t\right)\mathrm{d}t\right]. \tag{4.6}$$

Define the generalized Hamiltonian

$$\mathcal{H}(x, \mu, a, y) \triangleq b(x, \mu, a) \cdot y + f(x, \mu, a) - rxy, \tag{4.7}$$

and introduce the adjoint process, which is determined by an infinite-time BSDE:

$$\mathrm{d}Y_t^{x,\beta} = -\left(\partial_x \mathcal{H}(X_t^{x,\beta}, \mathcal{L}_{X_t^\xi}, \beta_t, Y_t^{x,\beta})\right)\mathrm{d}t + Z_t^{x,\beta}\mathrm{d}B_t. \tag{4.8}$$

Applying Itô's formula to the process $(e^{-rt}R_tY_t^{x,\beta})$ and by simple computation, we can deduce that

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} J(\beta + \epsilon\gamma)\bigg|_{\epsilon=0} = \mathbb{E}\left[\int_0^\infty e^{-rt}\partial_a \mathcal{H}(X_t^{x,\beta}, \mathcal{L}_{X_t^\xi}, \beta_t, Y_t^{x,\beta}) \cdot \gamma_t \mathrm{d}t\right]. \tag{4.9}$$

Thus when β is an optimal admissible control with the associated stochastic processes $(X_t^{x,\beta}, Y_t^{x,\beta}, Z_t^{x,\beta})$, it holds that

$$\mathcal{H}(X_t^{x,\beta}, \mathcal{L}_{X_t^\xi}, \beta_t, Y_t^{x,\beta}) = \min_{a \in A} \mathcal{H}(X_t^{x,\beta}, \mathcal{L}_{X_t^\xi}, a, Y_t^{x,\beta}). \tag{4.10}$$

Recalling our convexity assumptions on b, f in Assumption 4.1, we know that the representative player's optimal control β takes a feedback form, that is,

$$\beta_t = \hat{\alpha}(X_t^{x,\beta}, Y_t^{x,\beta}). \tag{4.11}$$

Now, we consider the following two McKean−Vlasov FBSDEs:

$$\begin{cases} \mathrm{d}X_t^\xi = b(X_t^\xi, \mathcal{L}_{X_t^\xi}, \hat{\alpha}(X_t^\xi, Y_t^\xi))\mathrm{d}t + \mathrm{d}B_t, \\ \mathrm{d}Y_t^\xi = -\partial_x \mathcal{H}(X_t^\xi, \mathcal{L}_{X_t^\xi}, \hat{\alpha}(X_t^\xi, Y_t^\xi), Y_t^\xi)\mathrm{d}t + Z_t^\xi \mathrm{d}B_t, \\ X_0^\xi = \xi. \end{cases} \tag{4.12}$$

$$\begin{cases} \mathrm{d}X_t^x = b(X_t^x, \mathcal{L}_{X_t^\xi}, \hat{\alpha}(X_t^x, Y_t^x))\mathrm{d}t + \mathrm{d}B_t, \\ \mathrm{d}Y_t^x = -\partial_x \mathcal{H}(X_t^x, \mathcal{L}_{X_t^\xi}, \hat{\alpha}(X_t^x, Y_t^x), Y_t^x)\mathrm{d}t + Z_t^x \mathrm{d}B_t, \\ X_0^x = x. \end{cases} \tag{4.13}$$

Here, (4.12) and (4.13) denote the state processes of the social equilibrium and the representative player, respectively. Observe that their admissible controls both take the identical feedback form $\hat{\alpha}(x, y)$. We shall prove that when the social equilibrium employs this feedback control, the representative player's loss function is minimized if they use the same feedback form, thereby constituting a Nash equilibrium.

Theorem 4.2 *Let (b, f) be differentiable in (x, a) and $\mathcal{H}$ be convex in (x, a) . Suppose that $\hat{\alpha}$ is Lipschitz continuous and that both (4.12) and (4.13) admit unique strong solutions in L_r^2. If we denote $\hat{\alpha}(X_t^x, Y_t^x)$ as α_t^* , which is an admissible control in $\mathcal{A}$, then we have*

$$J(\alpha^*) = \min_{\beta\in\mathcal{A}} J(\beta). \tag{4.14}$$

Proof For an arbitrary admissible control $\beta \in \mathcal{A}$ and its associated process

$$X_t^{x,\beta} = x + \int_0^t b(X_s^{x,\beta}, \mathcal{L}_{X_s^\xi}, \beta_s)\mathrm{d}s + B_t, \tag{4.15}$$

we have

$$\begin{aligned} J(\alpha^*) - J(\beta) = &\mathbb{E}\left[\int_0^\infty e^{-rt}\left(\mathcal{H}(X_t^x, \mathcal{L}_{X_t^\xi}, \alpha_t^*, Y_t^x) - \mathcal{H}(X_t^{x,\beta}, \mathcal{L}_{X_t^\xi}, \beta_t, Y_t^x)\right)\mathrm{d}t\right] \\ &- \mathbb{E}\left[\int_0^\infty e^{-rt}\left(b(X_t^x, \mathcal{L}_{X_t^\xi}, \alpha_t^*) - b(X_t^{x,\beta}, \mathcal{L}_{X_t^\xi}, \beta_t)\right)\cdot Y_t^x\mathrm{d}t\right] \\ &+ r\mathbb{E}\left[\int_0^\infty e^{-rt}(X_t^x - X_t^{x,\beta})\cdot Y_t^x\mathrm{d}t\right]. \end{aligned} \tag{4.16}$$

Since $X_t^x, X_t^{x,\beta}, Y_t^x$ all belong to L_r^2, we can find a sequence of $T_i \to \infty$ such that

$$\mathbb{E}\left[e^{-rT_i}(X_{T_i}^x - X_{T_i}^{x,\beta})\cdot Y_{T_i}^x\right] \to 0. \tag{4.17}$$

Applying Itô's formula to $e^{-rT_i}(X_{T_i}^x - X_{T_i}^{x,\beta})\cdot Y_{T_i}^x$ and letting $T_i \to \infty$, we obtain that

$$\begin{aligned} &\mathbb{E}\left[\int_0^\infty e^{-rt}(X_t^x - X_t^{x,\beta})\left(\partial_x\mathcal{H}(X_t^x, \mathcal{L}_{X_t^\xi}, \alpha_t^*, Y_t^x)\right)\mathrm{d}t\right] \\ = &\mathbb{E}\left[\int_0^\infty e^{-rt}\left(-r(X_t^x - X_t^{x,\beta}) + b(X_t^x, \mathcal{L}_{X_t^\xi}, \alpha_t^*) - b(X_t^{x,\beta}, \mathcal{L}_{X_t^\xi}, \beta_t)\right)\cdot Y_t^x\mathrm{d}t\right]. \end{aligned} \tag{4.18}$$

According to the convexity and differentiability of $\mathcal{H}$, we have

$$\begin{aligned} &\mathcal{H}(X_t^{x,\beta}, \mathcal{L}_{X_t^\xi}, \beta_t, Y_t^x) - \mathcal{H}(X_t^x, \mathcal{L}_{X_t^\xi}, \alpha_t^*, Y_t^x) \\ \geqslant &(X_t^{x,\beta} - X_t^x)\cdot\partial_x\mathcal{H}(X_t^x, \mathcal{L}_{X_t^\xi}, \alpha_t^*, Y_t^x) + (\beta_t - \alpha_t^*)\cdot\partial_a\mathcal{H}(X_t^x, \mathcal{L}_{X_t^\xi}, \alpha_t^*, Y_t^x) \\ = &(X_t^{x,\beta} - X_t^x)\cdot\partial_x\mathcal{H}(X_t^x, \mathcal{L}_{X_t^\xi}, \alpha_t^*, Y_t^x). \end{aligned} \tag{4.19}$$

The last equality follows from the fact that $\alpha_t^* = \hat{a}(X_t^x, Y_t^x)$ is the minimizer of $\min_{a\in A}\mathcal{H}(X_t^x, \mathcal{L}_{X_t^\xi}, a, Y_t^x)$. Combining equations (4.16), (4.18), and (4.19), we obtain

$$J(\alpha^*) - J(\beta) \leqslant 0 \tag{4.20}$$

for all admissible controls β. Thus, we complete the proof. □

4.2 Solvability of mean field game FBSDEs

In this subsection, we will find sufficient conditions for the existence and uniqueness of solutions to (4.12) and (4.13). Considering the linear case, we assume that $b(x, \mu, a) = b_1x + b_2\bar{\mu} + b_3a$ and $f(x, \mu, a) = f_0(x, \mu) + f_1(x, a)$, where $\bar{\mu}$ is the mean value of the probability measure μ and b_1, b_2, b_3, b_4 are constants.

We require a technical lemma about the Lipschitz and convex properties of the minimizer $\hat{\alpha}$, the detailed proof of which can be found in [1] (Lemma 3.1).

Lemma 4.3 *Suppose f_1 is once continuously differentiable in (x, a), $\partial_a f_1$ is ℓ-Lipschitz in x and f_1 is η-convex in a, which means that*

$$f_1(x, a') - f_1(x, a) - (a' - a)\cdot\partial_a f_1(x, a) \geqslant \eta|a' - a|^2, \quad \textit{for all } x \in \mathbb{R}. \tag{4.21}$$

Then, it holds that

$$|\hat{\alpha}(x,y)-\hat{\alpha}(x',y')| \leqslant \frac{\ell}{2\eta}|x'-x|+\frac{|b_3|}{2\eta}|y'-y|, \tag{4.22}$$

and for some $a_0 \in A$,

$$|\hat{\alpha}(x,y)| \leqslant \eta^{-1}(|\partial_a f_1(x,a_0)|+|b_2 y|)+|a_0|. \tag{4.23}$$

Furthermore, if $A=\mathbb{R}$ *and* $\partial_a f$ *is* ζ*-Lipschitz in* a *, it follows that*

$$b_3(y'-y)\cdot(\hat{\alpha}(x,y')-\hat{\alpha}(x,y)) \leqslant -\frac{2b_3^2\eta}{\zeta^2}(y'-y)^2. \tag{4.24}$$

Theorem 4.4 *Let* $b(x,\mu,a)=b_1x+b_2\bar{\mu}+b_3a$ *and* $f(x,\mu,a)=f_0(x,\mu)+f_1(x,a)$. *Under the following conditions, Assumptions 3.1 and 4.1 are satisfied and both (4.12) and (4.13) admit unique strong solutions in* L_r^2.

(i) *There exists a positive constant* k *such that* $|b_2| \leqslant k$ *and* $-b_1 \geqslant k-\frac{r}{2}$. $f_0(x,\mu)$ *is once continuously differentiable in* x *and of at most quadratic growth in* (x,μ). $f_1(x,a)$ *is once continuously differentiable and of at most quadratic growth in* (x,a).

(ii) $f_0(x,\mu)$ *is convex in* x *, and there exists a positive constant* $b_4>0$ *such that*

$$|\partial_x f_0(x,\mu)-\partial_x f_0(x',\mu')| < b_4\left(|x-x'|+\mathcal{W}_2(\mu,\mu')\right). \tag{4.25}$$

(iii) *There exist some positive constants* η,ι *such that the following convexity condition holds:*

$$\begin{aligned} &f_1(x',a')-f_1(x,a)-\partial_{(x,a)}f_1(x,a)\cdot(x'-x,a'-a)\\ &\geqslant \iota(x'-x)^2+\eta(a'-a)^2. \end{aligned} \tag{4.26}$$

(iv) *There exist some positive constants* ζ,ℓ *such that* $\partial_a f_1$ *is* ℓ*-Lipschitz in* x *and* ζ*-Lipschitz in* a. $\partial_x f_1$ *is Lipschitz continuous in* (x,a).

(v) $A=\mathbb{R}$ *, and it holds that*

$$\min\left\{2\iota-\frac{\ell^2}{2\eta}-\frac{b_3\ell}{2\eta}-\frac{|b_2|}{2}-2b_4,\ \frac{2b_3^2\eta}{\zeta^2}-\frac{b_3\ell}{2\eta}-\frac{|b_2|}{2}\right\} > \frac{r}{2}. \tag{4.27}$$

Remark 4.5 *If we set* $f_0(x,\mu)=b_4x\bar{\mu}, f_1(x,a)=Ax^2+Ca^2, A>0, C>0$ *, we have* $\iota=A$, $\eta=C,\zeta=2C,\ell=0$. *Then, the requirement in (4.27) becomes*

$$\min\left\{2A-\frac{|b_2|}{2}-|b_4|,\ \frac{b_3^2}{2C}-\frac{|b_2|}{2}\right\} > \frac{r}{2}. \tag{4.28}$$

Fixing C,b_2,b_4, *we take a large* b_3 *such that* $\frac{b_3^2}{2C}-\frac{|b_2|}{2}$ *is greater than* $r/2$. *Then we choose a sufficiently large* A *such that* $2A-\frac{|b_2|}{2}-|b_4|$ *exceeds* $r/2$. *This construction satisfies all the required conditions.*

Proof It is clear that Assumption 3.1 is satisfied. And by Lemma 4.3, $\hat{\alpha}$ is Lipschitz. In addition, $\mathcal{H}(x,\mu,a,y)=(b_1x+b_2\bar{\mu}+b_3a)\cdot y+f_0(x,\mu)+f_1(x,a)$ is convex in (x,a) according to the assumption on f_0,f_1; thus, Assumption 4.1 is satisfied. To prove that the FBSDEs (4.12) and (4.13) admit unique strong solutions in L_r^2, the only condition that remains to be verified is (2.5).

We consider the FBSDE (4.12) and set

$$\begin{aligned} B(t,x,y,\mu) &= b_1 x + b_2\overline{\mu} + b_3\hat{\alpha}(x,y), \\ F(t,x,y,\mu) &= b_1 y + \partial_x f_0(x,\mu) + \partial_x f_1(x,\hat{\alpha}(x,y)) - ry. \end{aligned} \tag{4.29}$$

Take four arbitrary square-integrable random variables X, Y, X', Y'. Define $\hat{X} = X - X', \hat{Y} = Y - Y'$ and $U = (X, Y, \mathcal{L}_X), U' = (X', Y', \mathcal{L}_{X'})$. We have

$$\begin{aligned} & - r\hat{X}\hat{Y} - \hat{X}[F(t,U) - F(t,U')] + \hat{Y}[B(t,U) - B(t,U')] \\ =& - r\hat{X}\hat{Y} - \hat{X}\Big((b_1 - r)\hat{Y} + \partial_x f_0(X,\mathcal{L}_X) - \partial_x f_0(X',\mathcal{L}_{X'}) + \partial_x f_1(X,\hat{\alpha}(X,Y)) - \partial_x f_1(X',\hat{\alpha}(X',Y'))\Big) \\ & + \hat{Y}\left(b_1\hat{X} + b_2\mathbb{E}[\hat{X}] + b_3(\hat{\alpha}(X,Y) - \hat{\alpha}(X',Y'))\right) \\ \leqslant& - \hat{X}\left(\partial_x f_1(X,\hat{\alpha}(X,Y)) - \partial_x f_1(X',\hat{\alpha}(X',Y'))\right) + b_3\hat{Y}\left(\hat{\alpha}(X,Y) - \hat{\alpha}(X',Y')\right) \\ & + b_2\hat{Y}\mathbb{E}[\hat{X}] + b_4|\hat{X}| \cdot \left(|\hat{X}| + (\mathbb{E}[\hat{X}^2])^{\frac{1}{2}}\right). \end{aligned} \tag{4.30}$$

Since f_1 is ι-convex in x, we have

$$\left[\partial_x f_1(x',a) - \partial_x f_1(x,a)\right](x' - x) \geqslant 2\iota(x' - x)^2. \tag{4.31}$$

Moreover, $\partial_x f_1$ is l-Lipschitz in a and $\hat{\alpha}$ satisfies (4.22), we have that

$$\begin{aligned} & - \hat{X}\left(\partial_x f_1(X,\hat{\alpha}(X,Y)) - \partial_x f_1(X',\hat{\alpha}(X,Y))\right) \\ =& - \hat{X}\left(\partial_x f_1(X,\hat{\alpha}(X,Y)) - \partial_x f_1(X',\hat{\alpha}(X,Y))\right) \\ & - \hat{X}\left(\partial_x f_1(X',\hat{\alpha}(X,Y)) - \partial_x f_1(X',\hat{\alpha}(X',Y'))\right) \\ \leqslant& - 2\iota\hat{X}^2 + l|\hat{X}|(\frac{l}{2\eta}|\hat{X}| + \frac{|b_3|}{2\eta}|\hat{Y}|). \end{aligned} \tag{4.32}$$

From Lemma 4.3, $\hat{\alpha}$ satisfies (4.24), it follows that

$$\begin{aligned} & b_3\hat{Y}\left(\hat{\alpha}(X,Y) - \hat{\alpha}(X',Y')\right) \\ =& b_3\hat{Y}\left(\hat{\alpha}(X,Y) - \hat{\alpha}(X,Y')\right) + b_3\hat{Y}\left(\hat{\alpha}(X,Y') - \hat{\alpha}(X',Y')\right) \\ \leqslant& - \frac{2b_3^2\eta}{\zeta^2}\hat{Y}^2 + |b_3\hat{Y}|\left(\frac{l}{2\eta}|\hat{X}|\right). \end{aligned} \tag{4.33}$$

By applying elementary estimates, we derive

$$\begin{aligned} & \mathbb{E}\left[-r\hat{X}\hat{Y} - \hat{X}(F(t,U) - F(t,U')) + \hat{Y}(B(t,U) - B(t,U'))\right] \\ \leqslant& (-2\iota + \frac{l^2}{2\eta} + \frac{b_3 l}{2\eta} + \frac{|b_2|}{2} + 2b_4)\mathbb{E}[\hat{X}^2] + (-\frac{2b_3^2\eta}{\zeta^2} + \frac{b_3 l}{2\eta} + \frac{|b_2|}{2})\mathbb{E}[\hat{Y}^2] \\ <& - \frac{r}{2}\mathbb{E}[\hat{X}^2 + \hat{Y}^2]. \end{aligned} \tag{4.34}$$

Now we know (4.12) admits a unique solution $X_t^\xi \in L_r^2$ and set $\bar{\mu}_t = \mathbb{E}[X_t^\xi]$ For the FBSDE (4.13), we set

$$\begin{aligned} B'(t,x,y) &= b_1 x + b_2\overline{\mu}_t + b_3\hat{\alpha}(x,y), \\ F'(t,x,y) &= b_1 y + \partial_x f_0(x,\mu_t) + \partial_x f_1(x,\hat{\alpha}(x,y)) - ry. \end{aligned} \tag{4.35}$$

Following the identical analytical procedure, we have

$$\begin{aligned}&\mathbb{E}\left[-r\hat{X}\hat{Y}-\hat{X}(F'(t,X,Y)-F'(t,X',Y'))+\hat{Y}(B'(t,X,Y)-B'(t,X',Y'))\right]\\ \leqslant&(-2\iota+\frac{l^2}{2\eta}+\frac{b_3 l}{2\eta}+b_4)\mathbb{E}[\hat{X}^2]+(-\frac{2b_3^2\eta}{\zeta^2}+\frac{b_3 l}{2\eta})\mathbb{E}[\hat{Y}^2]\\ <&-\frac{r}{2}\mathbb{E}[\hat{X}^2+\hat{Y}^2].\end{aligned}\tag{4.36}$$

By Lemma 2.2, we know both FBSDEs (4.12) and (4.13) admit unique strong solutions in L_r^2. □

5. Master equation representation

While we have derived a Nash equilibrium solution through FBSDEs that yields identical feedback forms for both the representative player and social equilibrium, this feedback structure differs from our previously defined formulation in Definition 3.2. In this section, we shall establish an alternative representation of the Nash equilibrium using classical solutions to the elliptic master equation (1.8).

Following Assumption 4.1, we define

$$H(x,\mu,y)=H_0(x,\mu,\hat{\alpha}(x,y),y).\tag{5.1}$$

Through the assumptions on b,f, we can easily deduce the relationship

$$\partial_y H(x,\mu,y)=b(x,\mu,\hat{\alpha}(x,y)).\tag{5.2}$$

Assume that the master equation (1.8) has a classical solution $U(x,\mu)\in C^{2,1}(\mathbb{R}\times\mathcal{P}_2)$ with H defined above such that $\partial_x U(x,\mu)\in C^{2,1}(\mathbb{R}\times\mathcal{P}_2)$ satisfying PDE (1.12) and $\hat{b}(x,\mu)=b(x,\mu,\hat{\alpha}(x,\partial_x U(x,\mu)))$ is Lipschitz continuous in (x,μ). For any $\xi\in\mathbb{L}^2(\mathcal{F}_0)$, suppose the following SDE admits a unique solution in L_r^2:

$$\mathcal{X}_t^\xi=\xi+\int_0^t b\left(\mathcal{X}_s^\xi,\mathcal{L}_{\mathcal{X}_s^\xi},\hat{\alpha}\left(\mathcal{X}_s^\xi,\partial_x U\left(\mathcal{X}_s^\xi,\mathcal{L}_{\mathcal{X}_s^\xi}\right)\right)\right)\mathrm{d}s+B_t.\tag{5.3}$$

Then, we take $\mathcal{Y}_t^\xi=\partial_x U(\mathcal{X}_t^\xi,\mathcal{L}_{\mathcal{X}_t^\xi})$. By differentiating both sides of the master equation (1.8) with respect to x and applying Itô's formula, we obtain

$$\begin{aligned}\mathrm{d}\mathcal{Y}_t^\xi=&\Bigg[\partial_{xx}U\left(\mathcal{X}_t^\xi,\mathcal{L}_{\mathcal{X}_t^\xi}\right)\cdot b\left(\mathcal{X}_t^\xi,\mathcal{L}_{\mathcal{X}_t^\xi},\hat{\alpha}\left(\mathcal{X}_t^\xi,\partial_x U\left(\mathcal{X}_t^\xi,\mathcal{L}_{\mathcal{X}_t^\xi}\right)\right)\right)\\ &+\frac{1}{2}\partial_{xx}\partial_x U\left(\mathcal{X}_t^\xi,\mathcal{L}_{\mathcal{X}_t^\xi}\right)+\tilde{\mathbb{E}}_{\mathcal{F}_t}\Bigg[\frac{1}{2}\partial_{\tilde{x}}\partial_\mu\partial_x U\left(\mathcal{X}_t^\xi,\mathcal{L}_{\mathcal{X}_t^\xi},\tilde{\mathcal{X}}_t^\xi\right)\\ &+\partial_y H\left(\tilde{\mathcal{X}}_t^\xi,\mathcal{L}_{\mathcal{X}_t^\xi},\partial_x U\left(\tilde{\mathcal{X}}_t^\xi,\mathcal{L}_{\mathcal{X}_t^\xi}\right)\right)\cdot\partial_\mu\partial_x U\left(\mathcal{X}_t^\xi,\mathcal{L}_{\mathcal{X}_t^\xi},\tilde{\mathcal{X}}_t^\xi\right)\Bigg]\Bigg]\mathrm{d}t\\ &+\partial_{xx}U\left(\mathcal{X}_t^\xi,\mathcal{L}_{\mathcal{X}_t^\xi}\right)\mathrm{d}B_t\\ =&\left(r\partial_x U\left(\mathcal{X}_t^\xi,\mathcal{L}_{\mathcal{X}_t^\xi}\right)-\partial_x H\left(\mathcal{X}_t^\xi,\mathcal{L}_{\mathcal{X}_t^\xi},\partial_x U\left(\mathcal{X}_t^\xi,\mathcal{L}_{\mathcal{X}_t^\xi}\right)\right)\right)\mathrm{d}t+\partial_{xx}U\left(\mathcal{X}_t^\xi,\mathcal{L}_{\mathcal{X}_t^\xi}\right)\mathrm{d}B_t.\end{aligned}\tag{5.4}$$

By comparing it with (4.12), we derive the following relationship for social equilibrium:

$$Y_t^\xi=\partial_x U(X_t^\xi,\mathcal{L}_{X_t^\xi}),\quad Z_t^\xi=\partial_{xx}U(X_t^\xi,\mathcal{L}_{X_t^\xi}).\tag{5.5}$$

Applying the same argument to (4.13), we obtain:

$$Y_t^x = \partial_x U(X_t^x, \mathcal{L}_{X_t^\xi}), \quad Z_t^x = \partial_{xx} U(X_t^x, \mathcal{L}_{X_t^\xi}). \tag{5.6}$$

This demonstrates that both the representative player and social equilibrium employ the same closed-loop control

$$\alpha^*(x,\mu) = \hat{\alpha}(x, \partial_x U(x,\mu)). \tag{5.7}$$

We now revisit the mean field games through the master equation. Let $\xi \in \mathbb{L}^2(\mathcal{F}_0)$ be the initial state with distribution μ. We prove that under the assumption that the master equation admits a classical solution, the feedback control defined by (5.7) constitutes a Nash equilibrium. Moreover, the solution to the master equation is precisely the value function of the representative player.

Theorem 5.1 *Assume that the master equation (1.8) admits a classical solution $U(x,\mu) \in C^{2,1}(\mathbb{R} \times \mathcal{P}_2)$, which is of at most quadratic growth, and the following SDEs admit unique solutions in L_r^2 for each $x \in \mathbb{R}$ and $\xi \in \mathbb{L}^2(\mathcal{F}_0;\mu)$*

$$\begin{cases} X_t^\xi = \xi + \int_0^t b(X_s^\xi, \mathcal{L}_{X_s^\xi}, \hat{\alpha}(X_s^\xi, \partial_x U(X_s^\xi, \mathcal{L}_{X_s^\xi}))) \mathrm{d}t + B_t, \\ X_t^x = x + \int_0^t b(X_s^x, \mathcal{L}_{X_s^x}, \hat{\alpha}(X_s^x, \partial_x U(X_s^x, \mathcal{L}_{X_s^x}))) \mathrm{d}t + B_t. \end{cases} \tag{5.8}$$

Then, for every initial distribution μ_0, $\alpha^*(x,\mu) = \hat{\alpha}(x, \partial_x U(x,\mu))$ is a Nash equilibrium, and $U(x,\mu_0)$ is exactly the value function given α^*.

Proof Given α^* and initial $\xi \in \mathbb{L}^2(\mathcal{F}_0;\mu_0)$, the state of social equilibrium is governed by the following SDE:

$$X_t^\xi = \xi + \int_0^t \partial_y H(X_s^\xi, \rho_s, \partial_x U(X_s^\xi, \rho_s)) \mathrm{d}s + B_t, \quad \rho_s \triangleq \mathcal{L}_{X_s^\xi}. \tag{5.9}$$

We note that this SDE is the same as (5.3). Meanwhile, the state of the representative player is governed by

$$X_t^x = x + \int_0^t b(X_s^x, \rho_s, \beta_s) \mathrm{d}s + B_t, \tag{5.10}$$

where $\beta \in \mathcal{A}$ remains to be determined. Applying Itô's formula to $\mathrm{e}^{-rt} U(X_t^x, \rho_t)$ yields

$$\begin{aligned} \mathbb{E}\mathrm{e}^{-rT} U(X_T^x, \rho_T) &= U(x,\mu_0) + \mathbb{E} \int_0^T \big[-r\mathrm{e}^{-rt} U(X_t^x, \rho_t) + \mathrm{e}^{-rt} \partial_x U(X_t^x, \rho_t) \cdot b(X_t^x, \rho_t, \beta_t) \\ &\quad + \frac{1}{2} \mathrm{e}^{-rt} \partial_{xx} U(X_t^x, \rho_t) + \mathrm{e}^{-rt} \tilde{\mathbb{E}}_{\mathcal{F}_t} \big[\frac{1}{2} \partial_{\tilde{x}} \partial_\mu U(X_t^x, \rho_t, \tilde{X}_t^\xi) \\ &\quad + \partial_y H(\tilde{X}_t^\xi, \rho_t, \partial_x U(\tilde{X}_t^\xi, \rho_t)) \cdot \partial_\mu U(X_t^x, \rho_t, \tilde{X}_t^\xi) \big] \big] \mathrm{d}t \\ &= U(x,\mu_0) + \mathbb{E} \int_0^T \mathrm{e}^{-rt} \left[\partial_x U(X_t^x, \rho_t) \cdot b(X_t^x, \rho_t, \beta_t) - H(X_t^x, \rho_t, \partial_x U(X_t^x, \rho_t)) \right] \mathrm{d}t \\ &\geqslant U(x,\mu_0) - \mathbb{E} \int_0^T \mathrm{e}^{-rt} f(X_t^x, \rho_t, \beta_t) \mathrm{d}t. \end{aligned} \tag{5.11}$$

Since U is of at most quadratic growth, taking the limit as $T \to +\infty$ yields

$$U(x,\mu_0) \leqslant \mathbb{E} \int_0^{+\infty} \mathrm{e}^{-rt} f(X_t^x, \rho_t, \beta_t) \mathrm{d}t \tag{5.12}$$

for every feasible control β, and the equality holds when $\beta_t = \hat{\alpha}(X_t^x, \partial_x U(X_t^x, \rho_t))$. This shows that $\alpha^*(x,\mu) = \hat{\alpha}(x, \partial_x U(x,\mu))$ is a Nash equilibrium. In addition, we have

$$U(x,\mu_0)=\mathbb{E}\int_0^{+\infty}\mathrm{e}^{-rt}f(X_t^x,\rho_t,\hat{\alpha}(X_t^x,\partial_x U(X_t^x,\rho_t)))\mathrm{d}t. \tag{5.13}$$

This completes the proof of our desired results. □

At the end of this section, we provide an example of a solvable elliptic master equation and demonstrate that its solutions are generally nonunique. Set $b(x,\mu,a)=b_1x+b_2\bar{\mu}+b_3a,\ f(x,\mu,a)=b_4x\bar{\mu}+Ax^2+Ca^2$, where b_1,b_2,b_3,b_4,A,C are constants satisfying all conditions in Theorem 4.2 and Remark 4.5. Then, we have $\hat{\alpha}(x,y)=-\frac{b_3y}{2C}$ and

$$H(x,\mu,y)=(b_1x+b_2\bar{\mu})\cdot y+b_4x\bar{\mu}+Ax^2-\frac{b_3^2}{4C}y^2. \tag{5.14}$$

The master equation (1.8) now becomes

$$\begin{aligned}rU(x,\mu)=&(b_1x+b_2\bar{\mu})\cdot\partial_xU(x,\mu)+b_4x\bar{\mu}+Ax^2\\&-\frac{b_3^2}{4C}(\partial_xU(x,\mu))^2+\frac{1}{2}\partial_{xx}U(x,\mu)\\&+\tilde{\mathbb{E}}\left[\frac{1}{2}\partial_{\tilde{x}}\partial_\mu U(x,\mu,\tilde{\xi})+\partial_\mu U(x,\mu,\tilde{\xi})(b_1\tilde{\xi}+b_2\bar{\mu}-\frac{b_3^2}{2C}\partial_xU(\tilde{\xi},\mu))\right].\end{aligned} \tag{5.15}$$

We assume that the solution takes the form

$$U(x,\mu)=a_1x^2+a_2x\bar{\mu}+a_3(\bar{\mu})^2+a_4, \tag{5.16}$$

where a_1,a_2,a_3,a_4 are constants that remain to be determined. Then we have

$$\begin{aligned}&\partial_xU(x,\mu)=2a_1x+a_2\bar{\mu}, &&\partial_{xx}U(x,\mu)=2a_1,\\&\partial_\mu U(x,\mu,\tilde{x})=a_2x+2a_3\bar{\mu}, &&\partial_{\tilde{x}}\partial_\mu U(x,\mu,\tilde{x})=0.\end{aligned} \tag{5.17}$$

Substituting these into the equation (5.15), we obtain

$$\begin{aligned}&r\left(a_1x^2+a_2x\bar{\mu}+a_3(\bar{\mu})^2+a_4\right)\\&=(b_1x+b_2\bar{\mu})\cdot(2a_1x+a_2\bar{\mu})+b_4x\bar{\mu}+Ax^2-\frac{b_3^2}{4C}(2a_1x+a_2\bar{\mu})^2+a_1\\&+(a_2x+2a_3\bar{\mu})\cdot\left(b_1\bar{\mu}+b_2\bar{\mu}-\frac{b_3^2}{2C}(2a_1\bar{\mu}+a_2\bar{\mu})\right).\end{aligned} \tag{5.18}$$

Comparing all coefficients, we get the following system of linear and quadratic equations:

$$\begin{cases}ra_1=2b_1a_1+A-\frac{b_3^2}{C}a_1^2,\\ra_2=2b_2a_1+b_1a_2+b_4-\frac{b_3^2}{C}a_1a_2+a_2(b_1+b_2-\frac{b_3^2}{C}a_1-\frac{b_3^2}{2C}a_2),\\ra_3=b_2a_2-\frac{b_3^2}{4C}a_2^2+2a_3(b_1+b_2-\frac{b_3^2}{C}a_1-\frac{b_3^2}{2C}a_2),\\ra_4=a_1.\end{cases} \tag{5.19}$$

Note that a_1,a_2,a_3, and a_4 can be solved sequentially, and the equations for a_1 and a_2 are quadratic. Since $A>0$ and $-\frac{b_3^2}{C}<0$, the equation for a_1 must have one positive and one negative root. If the coefficients are sufficiently well-behaved, for instance, $2b_2a_1+b_4>0$, then the equation for a_2 will also have two distinct real roots. Therefore, the system of equations (5.19) can have at most four solutions. It must be emphasized that there exists at most one set of solutions for which the solution of SDE (5.3) lies in L_r^2, as the L_r^2 solution of FBSDE (4.12) is unique. In a companion article, we establish the uniqueness of solutions to the master equation under certain monotonicity and growth assumptions.

Example 5.2 *We consider a case that* $r=2,\ b_1=b_2=b_4=0,\ b_3=2,\ A=2,\ C=1$ *, then equation (5.15) becomes*

$$\begin{aligned}2U(x,\mu)=&\,2x^2-(\partial_x U(x,\mu))^2+\frac{1}{2}\partial_{xx}U(x,\mu)\\&+\tilde{\mathbb{E}}\left[\frac{1}{2}\partial_{\tilde{x}}\partial_\mu U(x,\mu,\tilde{\xi})-2\partial_\mu U(x,\mu,\tilde{\xi})\cdot\partial_x U(\tilde{\xi},\mu)\right].\end{aligned}\tag{5.20}$$

Solving the corresponding Equation (5.19) yields the following solutions:

$$\begin{aligned}U_1(x,\mu)&=\frac{1}{2}x^2+\frac{1}{4},\\U_2(x,\mu)&=\frac{1}{2}x^2-3x\bar{\mu}+\frac{3}{2}(\bar{\mu})^2+\frac{1}{4},\\U_3(x,\mu)&=-x^2-\frac{1}{2},\\U_4(x,\mu)&=-x^2+3x\bar{\mu}-\frac{3}{2}(\bar{\mu})^2-\frac{1}{2}.\end{aligned}\tag{5.21}$$

However, to ensure that the solution of Equation (5.3), now written as

$$\mathrm{d}X_t=-4\partial_x U(X_t,\mathcal{L}_{X_t})\mathrm{d}t+\mathrm{d}B_t,\quad X_0=\xi,\tag{5.22}$$

belong to L_2^2 *, only the solution* $U_1(x,\mu)=\frac{1}{2}x^2+\frac{1}{4}$ *satisfies this condition.*

6. Viscosity solution to distribution-dependent elliptic PDE

We have proven that the classical solutions of the master equations can be employed to resolve the infinite-time FBSDEs. In this section, the solution of the infinite-time FBSDEs will be utilized to characterize the viscosity solutions of the distribution-dependent elliptic PDEs. Specifically, we consider the following FBSDEs with initial state $\xi\in\mathbb{L}^2(\mathcal{F}_0)$ and $x\in\mathbb{R}$:

$$\begin{cases}\mathrm{d}X_t^\xi=\partial_y H(X_t^\xi,\mathcal{L}_{X_t^\xi},Y_t^\xi)\mathrm{d}t+\mathrm{d}B_t,\\\mathrm{d}Y_t^\xi=-\left[\partial_x H(X_t^\xi,\mathcal{L}_{X_t^\xi},Y_t^\xi)-rY_t^\xi\right]\mathrm{d}t+Z_t^\xi\mathrm{d}B_t,\\X_0^\xi=\xi.\end{cases}\tag{6.1}$$

$$\begin{cases}\mathrm{d}X_t^{x,\xi}=\partial_y H(X_t^{x,\xi},\mathcal{L}_{X_t^\xi},Y_t^{x,\xi})\mathrm{d}t+\mathrm{d}B_t,\\\mathrm{d}Y_t^{x,\xi}=-\left[\partial_x H(X_t^{x,\xi},\mathcal{L}_{X_t^\xi},Y_t^{x,\xi})-rY_t^{x,\xi}\right]\mathrm{d}t+Z_t^x\mathrm{d}B_t,\\X_0^x=x.\end{cases}\tag{6.2}$$

Here, $H(x,\mu,y)$ is defined in (5.1), and the above equations are the same as (1.6) and (1.7).

We define $\mathcal{V}(x,\mu)\triangleq Y_0^{x,\xi}$ and attempt to prove that it satisfies the following elliptic PDE in the viscosity sense:

$$\begin{aligned}r\mathcal{U}(x,\mu)=&\,\partial_x H(x,\mu,\mathcal{U}(x,\mu))+\partial_y H(x,\mu,\mathcal{U}(x,\mu))\cdot\partial_x\mathcal{U}(x,\mu)+\frac{1}{2}\partial_{xx}\mathcal{U}(x,\mu)\\&+\tilde{\mathbb{E}}\left[\frac{1}{2}\partial_{\tilde{x}}\partial_\mu\mathcal{U}(x,\mu,\tilde{\xi})+\partial_\mu\mathcal{U}(x,\mu,\tilde{\xi})\partial_y H(\tilde{\xi},\mu,\mathcal{U}(\tilde{\xi},\mu))\right].\end{aligned}\tag{6.3}$$

This equation is derived by taking the partial derivative of both sides of the master equation (1.8) with respect to x. We have the relationship $\mathcal{U}(x,\mu)=\partial_x U(x,\mu)$.

Now, let us give the definition of a viscosity solution for PDE (6.3). We rewrite the PDE as follows:

$$(\mathcal{L}\mathcal{U})[\mathcal{U}](x,\mu)+F(x,\mu,\mathcal{U}(x,\mu))=0,\tag{6.4}$$

where

$$(\mathcal{L}\Phi)[\Psi](x,\mu) \triangleq \partial_y H(x,\mu,\Psi(x,\mu)) \cdot \partial_x \Phi(x,\mu) + \frac{1}{2}\partial_{xx}\Phi(x,\mu) + \tilde{\mathbb{E}}\left[\frac{1}{2}\partial_{\tilde{x}}\partial_\mu \Phi(x,\mu,\tilde{\xi}) + \partial_\mu \Phi(x,\mu,\tilde{\xi})\partial_y H(\tilde{\xi},\mu,\Psi(\tilde{\xi},\mu))\right], \tag{6.5}$$

and

$$F(x,\mu,\Psi(x,\mu)) \triangleq \partial_x H(x,\mu,\Psi(x,\mu)) - r\Psi(x,\mu). \tag{6.6}$$

Definition 6.1 *A function $\Psi \in C^{2,1}(\mathbb{R}\times\mathcal{P}_2)$ is said to be a test function if the quantities:*

$$\int_{\mathbb{R}} |\partial_\mu \Psi(x,\mu,\tilde{x})|^2 \,\mathrm{d}\mu(\tilde{x}) \tag{6.7}$$

and

$$\sup_{\tilde{x}\in\mathbb{R}} |\partial_{\tilde{x}}\partial_\mu \Psi(x,\mu,\tilde{x})| \tag{6.8}$$

are finite, uniformly in (x,μ) in any compact subset of $\mathbb{R}\times\mathcal{P}_2$.

Definition 6.2 *Let $\mathcal{U} \in C(\mathbb{R}\times\mathcal{P}_2)$. Then $\mathcal{U}$ is called a viscosity subsolution (resp. supersolution) of PDE (6.4) if, whenever Ψ is a test function, and $(x^0,\mu^0)\in\mathbb{R}\times\mathcal{P}_2$ is a local maximum (resp. minimum) of $\mathcal{U}-\Psi$, we have*

$$(\mathcal{L}\Psi)[\mathcal{U}](x^0,\mu^0) + F(x^0,\mu^0,\mathcal{U}(x^0,\mu^0)) \geqslant 0, \tag{6.9}$$

(respectively,

$$(\mathcal{L}\Psi)[\mathcal{U}](x^0,\mu^0) + F(x^0,\mu^0,\mathcal{U}(x^0,\mu^0)) \leqslant 0 \tag{6.10}$$

).

The function $\mathcal{U}$ is called a viscosity solution of PDE (6.4) if it is both a viscosity subsolution and a viscosity supersolution.

The proof of $\mathcal{V}$ being a viscosity solution to equation (6.3) rests on two fundamental results:

- The value of $Y_0^{x,\xi}$ depends solely on the distribution of ξ, but not on the specific realization of ξ.
- $\mathcal{V}(x,\mu) \triangleq Y_0^{x,\xi}$ is jointly continuous on $\mathbb{R}\times\mathcal{P}_2$.

We will next establish them sequentially in this section and ultimately prove that $\mathcal{V}$ is a viscosity solution to equation (6.3)

6.1 Yamada–Watanabe theorem for infinite-time FBSDEs

We shall prove the weak uniqueness for the infinite-time FBSDE:

$$\begin{cases} \mathrm{d}X_t = G(t,X_t,Y_t,\mathcal{L}_{X_t})dt + \mathrm{d}B_t, \\ \mathrm{d}Y_t = -F(t,X_t,Y_t,\mathcal{L}_{X_t})dt + Z_t\mathrm{d}B_t. \end{cases} \tag{6.11}$$

We know that under Assumption 2.1, the above FBSDE has a unique solution (X_t,Y_t,Z_t) in L_r^2 with initial $X_0\in\mathbb{L}^2(\mathcal{F}_0)$. We then give the definitions of strong and weak uniqueness.

Definition 6.3 (Strong uniqueness) *We say that the strong uniqueness holds for FBSDE (6.11) if on any admissible setup $(\Omega,\mathcal{F},\mathbb{P},\mathbb{F})$ with input (X_0,B) , for any two $\mathbb{F}$-progressively measurable L_r^2 three-tuples*

$$(X_t^1, Y_t^1, Z_t^1)_{t\geqslant 0}, \quad (X_t^2, Y_t^2, Z_t^2)_{t\geqslant 0} \tag{6.12}$$

satisfying the FBSDE (6.11) with the same initial condition X_0 (up to an exceptional event), it holds that

$$\mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt}\left(|X_t^1 - X_t^2|^2 + |Y_t^1 - Y_t^2|^2 + |Z_t^1 - Z_t^2|^2\right)\mathrm{d}t\right] = 0 \tag{6.13}$$

Definition 6.4 (Weak uniqueness) *For any two set-ups $(\Omega^1, \mathcal{F}^1, \mathbb{P}^1, \mathbb{F}^1)$ and $(\Omega^2, \mathcal{F}^2, \mathbb{P}^2, \mathbb{F}^2)$ with inputs (X_0^1, B^1) and (X_0^2, B^2), X_0^1, X_0^2 have the same law on $\mathbb{R}$. We say the weak uniqueness holds for FBSDE (6.11) if for the L_r^2 solutions $(X_t^1, Y_t^1, Z_t^1)_{t\geqslant 0}$ and $(X_t^2, Y_t^2, Z_t^2)_{t\geqslant 0}$ on corresponding set-ups, the processes $(X_t^1, Y_t^1, \int_0^t Z_s^1 \mathrm{d}s)_{t\geqslant 0}$ and $(X_t^2, Y_t^2, \int_0^t Z_s^2 \mathrm{d}s)_{t\geqslant 0}$ have the same distribution.*

We use the same scheme as the one developed by Yamada and Watanabe to prove that pathwise uniqueness of solutions of FBSDE's implies uniqueness in the sense of probability law.

Theorem 6.5 *Assume that on $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ with input $(X_0, B_t)_{t\geqslant 0}$, the FBSDE has a unique strong solution (X_t, Y_t, Z_t). Then, the law of $(X_0, B_t, X_t, Y_t, \int_0^t Z_s \mathrm{d}s)$ only depends on $\mathcal{L}(X_0)$.*

Proof First, we denote by $C([0,\infty);\mathbb{R})$ the space of continuous $\mathbb{R}$-valued functions on $[0,\infty)$ equipped with the metric of uniform convergence on compacts:

$$\mathrm{d}(\omega^1, \omega^2) = \sum_{n\geqslant 0} 2^{-n} \sup_{t\in[0,n]} \max(|\omega_t^1 - \omega_t^2|, 1). \tag{6.14}$$

And define

$$\begin{aligned}
&\Omega_{\text{input}} \triangleq \mathbb{R} \times C([0,\infty);\mathbb{R}),\\
&\Omega_{\text{output}} \triangleq C([0,\infty);\mathbb{R}) \times C([0,\infty);\mathbb{R}) \times C([0,\infty);\mathbb{R}),\\
&\Omega_{\text{canon}} \triangleq \Omega_{\text{input}} \times \Omega_{\text{output}}.
\end{aligned} \tag{6.15}$$

Let us consider two filtered probability spaces $(\Omega^i, \mathcal{F}^i, \mathbb{P}^i, \mathbb{F}^i)$ with identically distributed inputs (X_0^i, B^i), $i = 1, 2$, on each of which a solution $(X_t^i, Y_t^i, \int_0^t Z_s^i \mathrm{d}s)_{t\geqslant 0}$ to the FBSDE (6.11) is defined. Denote by Q^1 and Q^2 the distribution of $(X_0^1, B_t^1, X_t^1, Y_t^1, \int_0^t Z_s^1 \mathrm{d}s)_{t\geqslant 0}$ and $(X_0^2, B_t^2, X_t^2, Y_t^2, \int_0^t Z_s^2 \mathrm{d}s)_{t\geqslant 0}$ on $\Omega_{\text{canon}} = \Omega_{\text{input}} \times \Omega_{\text{output}}$, by Q_{input} the common distribution of the processes $(X_0^1, B_t^1), (X_0^2, B_t^2)$ on Ω_{input}.

Let us now define for $i \in \{1, 2\}$,

$$q^i(\omega_{\text{input}}; F) : \Omega_{\text{input}} \times \mathcal{B}(\Omega_{\text{output}}) \to [0,1] \tag{6.16}$$

as the regular conditional probability for $\mathcal{B}(\Omega_{\text{output}})$ given $\omega_{\text{input}} \in \Omega_{\text{input}}$ (under Q^i). It satisfies:

- $\forall \omega_{\text{input}} \in \Omega_{\text{input}}$, $q^i(\omega_{\text{input}}; \cdot)$ is a probability measure on $(\Omega_{\text{output}}, \mathcal{B}(\Omega_{\text{output}}))$.
- $\forall F \in \mathcal{B}(\Omega_{\text{output}})$, the mapping $(\xi, w) \to q^i(\xi, w; F)$ is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(C([0,\infty), \mathbb{R}))$-measurable.
- $\forall F \in \mathcal{B}(\Omega_{\text{output}}), \forall G \in \mathcal{B}(\Omega_{\text{input}})$:

$$Q^i(G \times F) = \int_G q^i(\omega_{\text{input}}; F) Q_{\text{input}}(\mathrm{d}\omega_{\text{input}}). \tag{6.17}$$

Next, we need an enlarged space $(\Omega_{\text{total}}, \mathcal{G}, Q)$ to support all processes. Define

$$\Omega_{\text{total}} \triangleq \Omega_{\text{input}} \times \Omega_{\text{output}} \times \Omega_{\text{output}}, \tag{6.18}$$

and $\mathcal{G}$ is the completion of the σ-field $\mathcal{B}(\Omega_{\text{canon}}) \otimes \mathcal{B}(\Omega_{\text{output}})$ by the collection $\mathcal{N}$ of all null sets under the probability measure

$$Q(G \times F_1 \times F_2) = \int_G q^1(\omega_{\text{input}}; F_1) q^2(\omega_{\text{input}}; F_2) Q_{\text{input}}(\mathrm{d}\omega_{\text{input}}), \tag{6.19}$$

where $F_1, F_2 \in \mathcal{B}(\Omega_{\text{output}}), G \in \mathcal{B}(\Omega_{\text{input}})$.

We observe that $Q(G \times F_1 \times \Omega_{\text{output}}) = Q^1(G \times F_1)$, $Q(G \times \Omega_{\text{output}} \times F_2) = Q^2(G \times F_2)$. Inparticular, we denote by $(\xi, w, x^1, y^1, \zeta^1, x^2, y^2, \zeta^2)$ the canonical process on Ω_{total}, then $(\xi, w, x^1, y^1, \zeta^1)$ has distribution Q^1 and $(\xi, w, x^2, y^2, \zeta^2)$ has distribution Q^2.

We define $(z_t^i)_{t \geqslant 0}$, $i \in \{1, 2\}$ by

$$z_t^i = \begin{cases} \lim\limits_{n \to \infty} n \left(\zeta_t^i - \zeta_{(t - \frac{1}{n})+}^i \right) & \text{if the limit exists,} \\ 0 \quad \text{otherwise.} \end{cases} \tag{6.20}$$

Since Q^i-a.s., $\int_0^t Z_s^i \mathrm{d}s$ is absolutely continuous, we know that Q-a.s., ζ_t^i is absolutely continuous. So

$$\zeta_t^i = \int_0^t z_s^i \mathrm{d}s, \quad t \geqslant 0. \tag{6.21}$$

Moreover, we have

$$\begin{aligned} \mathbb{E}^Q \int_0^\infty \mathrm{e}^{-rt} |z_t^i| \mathrm{d}t &\leqslant \varliminf_{n \to \infty} \mathbb{E}^Q \int_0^\infty \mathrm{e}^{-rt} |n(\zeta_t^i - \zeta_{(t - \frac{1}{n})+}^i)|^2 \mathrm{d}t \\ &= \varliminf_{n \to \infty} n^2 \mathbb{E}^{\mathbb{P}^i} \int_0^\infty \mathrm{e}^{-rt} |\int_{(t - \frac{1}{n})+}^t Z_s^i \mathrm{d}s|^2 \mathrm{d}t \\ &\leqslant \varliminf_{n \to \infty} n \mathbb{E}^{\mathbb{P}^i} \int_0^\infty \mathrm{e}^{-rt} \int_{(t - \frac{1}{n})+}^t |Z_s^i|^2 \mathrm{d}s \mathrm{d}t \\ &\leqslant \mathbb{E}^{Q^i} \int_0^\infty \mathrm{e}^{-rs} |Z_s^i|^2 \mathrm{d}s, \end{aligned} \tag{6.22}$$

the last inequality following from Fubini's theorem.

Let us now endow $(\Omega_{\text{total}}, \mathcal{G}, Q)$ with the filtration $\mathbb{G}$, where $\mathbb{G} = \{\mathcal{G}_t\}_{t \geqslant 0}$ is the complete and right-continuous augmentation under Q of the canonical filtration

$$\mathcal{H}_t = \{(\xi, w_s, x_s^1, y_s^1, \zeta_s^1, x_s^2, y_s^2, \zeta_s^2); 0 \leqslant s \leqslant t\} \tag{6.23}$$

on Ω_{total}. It is easy to see that ξ is $\mathcal{G}_0$-measurable and that $(w_s, x_s^1, y_s^1, z_s^1, x_s^2, y_s^2, z_s^2)_{t \geqslant 0}$ is $\{\mathcal{G}_t\}_{t \geqslant 0}$-progressively measurable. Moreover, for $i \in \{1, 2\}$,

$$Q \left\{ \omega \in \Omega_{\text{total}}; (x, w, x^i, y^i, \zeta^i) \in A \right\} = Q^i \left\{ (X_0^i, B^i, X^i, Y^i, \int_0^\cdot Z_s^i \mathrm{d}s) \in A \right\}; \quad A \in \mathcal{B}(\Omega_{\text{canon}}). \tag{6.24}$$

Actually, we just have to prove that $(w_t)_{t \geqslant 0}$ is a $\{\mathcal{G}_t\}_{t \geqslant 0}$ Brownian motion: we follow the proof given in [6] (Remark 1.6). Let us firstly define

$$\pi_t : C([0, \infty); \mathbb{R}) \to C([0, t]; \mathbb{R}), \quad h \to h|_{[0,t]}, \tag{6.25}$$

and

$$\pi_t' : \gamma \to \gamma_t, \quad h \to h|_{[0,t]}, \tag{6.26}$$

where $\gamma = \Omega_{\text{output}}$ and

$$\gamma_t = C([0,t];\mathbb{R}) \times C([0,t];\mathbb{R}) \times C([0,t];\mathbb{R}). \tag{6.27}$$

Endowing $C([0,t];\mathbb{R})$ and γ_t with their borelian σ-fields, we define

$$\mathcal{K}_t \triangleq \sigma\{\pi_t\}, \quad \mathcal{K}'_t \triangleq \sigma\{\pi'_t\}. \tag{6.28}$$

Using the separability of the spaces $C([0,t];\mathbb{R})$ and γ_t, we see that

$$\mathcal{K}_t = \sigma\{w_s; 0 \leqslant s \leqslant t\}, \tag{6.29}$$

and that $\forall i \in \{1,2\}, \forall A \in \mathcal{K}'_t$, the set $\{(X^i, Y^i, \int_0^\cdot Z_s^i \mathrm{d}s) \in A\}$ belongs to $\mathcal{F}_t^i$.

Now, considering $A \in \mathcal{K}'_t$, we want to show that, for $i \in \{1,2\}$, the map

$$\Omega_{\text{input}} \to [0,1], \quad (\xi, w) \to q^i(\xi, w; A) \tag{6.30}$$

is measurable with respect to the completion of the σ-field $\mathcal{B}(\mathbb{R}) \otimes \mathcal{K}_t$ under the probability measure Q_{input}, denoted $\overline{\mathcal{B}(\mathbb{R}) \otimes \mathcal{K}_t}$. Indeed, let us consider $F \in \mathcal{B}(\mathbb{R})$, $G_1 \in \mathcal{K}_t$ and $G_2 \in \sigma\{w_s - w_t; s \geqslant t\}$. Then, $\forall i \in \{1,2\}$

$$\begin{aligned}
&\int I_F(\xi) I_{G_1}(w) I_{G_2}(w) q^i(\xi, w; A) Q_{\text{input}}(\mathrm{d}\xi \mathrm{d}w) \\
&= \mathbb{E}^{\mathbb{P}^i}\left[I_F(X_0) I_{G_1}(B^i) I_{G_2}(B^i) I_A(X^i, Y^i, \int_0^\cdot Z_s^i \mathrm{d}s) \right] \\
&= \mathbb{E}^{\mathbb{P}^i}\left[I_F(X_0) I_{G_1}(B^i) I_A(X^i, Y^i, \int_0^\cdot Z_s^i \mathrm{d}s) \right] \mathbb{E}^{\mathbb{P}^i}\left[I_{G_2}(B^i) \right] \\
&= \int I_F(\xi) I_{G_1}(w) q^i(\xi, w; A) Q_{\text{input}}(\mathrm{d}\xi \mathrm{d}w) \int I_{G_2}(w) Q_{\text{input}}(\mathrm{d}\xi \mathrm{d}w).
\end{aligned} \tag{6.31}$$

Hence, the map $(\xi, w) \to q^i(\xi, w; A)$ is measurable with respect to $\overline{\mathcal{B}(\mathbb{R}) \otimes \mathcal{K}_t}$.

Now we prove that $(w_t)_{t \geqslant 0}$ is a $\{\mathcal{G}_t\}_{t \geqslant 0}$ Brownian motion. Let us consider $(A, A') \in (\mathcal{K}'_t)^2, F \in \mathcal{B}(\mathbb{R}), G_1 \in \mathcal{K}_t$ and $G_2 \in \sigma\{w_s - w_t; s \geqslant t\}$. Then,

$$\begin{aligned}
&\mathbb{E}^Q\left[I_F(\xi) I_{G_1}(w) I_{G_2}(w) I_A(x^1, y^1, \zeta^1) I_{A'}(x^2, y^2, \zeta^2) \right] \\
&= \int I_F(\xi) I_{G_1}(w) I_{G_2}(w) q^1(\xi, w; A) q^2(\xi, w; A') Q_{\text{input}}(\mathrm{d}\xi \mathrm{d}w) \\
&= \int I_F(\xi) I_{G_1}(w) q^1(\xi, w; A) q^2(\xi, w; A') Q_{\text{input}}(\mathrm{d}\xi \mathrm{d}w) \int I_{G_2}(w) Q_{\text{input}}(\mathrm{d}\xi \mathrm{d}w) \\
&= \mathbb{E}^Q\left[I_F(\xi) I_{G_1}(w) I_A(x^1, y^1, \zeta^1) I_{A'}(x^2, y^2, \zeta^2) \right] \mathbb{E}^Q\left[I_{G_2}(w) \right].
\end{aligned} \tag{6.32}$$

Noting that $\mathcal{H}^t = \mathcal{B}(\mathbb{R}) \otimes \mathcal{K}_t \otimes \mathcal{K}'_t \otimes \mathcal{K}'_t$, we conclude that $(w_t)_{t \geqslant 0}$ is a $\{\mathcal{G}_t\}_{t \geqslant 0}$ Brownian motion.

At last, applying the same procedure to $(z_t^i)_{t \geqslant 0}, i \in \{1,2\}$ in [5] (Volume II, Lemma 1.27), we obtain that Q-*a.s.*: for all $0 \leqslant t \leqslant T, i \in \{1,2\}$,

$$\begin{cases}
x_t^i = \xi + \int_0^t G(s, x_s^i, y_s^i, \mathcal{L}_{x_s^i}) \mathrm{d}s + w_t, \\
y_t^i = y_T^i + \int_t^T F(s, x_s^i, y_s^i, \mathcal{L}_{x_s^i}) \mathrm{d}s - \int_t^T z_s^i \mathrm{d}w_s, \\
\mathbb{E}^Q\left[\int_0^\infty \mathrm{e}^{-rt} \left(|x_t^i|^2 + |y_t^i|^2 + |z_t^i|^2 \right) \mathrm{d}t \right] < \infty.
\end{cases} \tag{6.33}$$

Through the strong uniqueness, we know that under Q, the processes $(x_t^1, y_t^1, \zeta_t^1)_{t \geqslant 0}$ and $(x_t^2, y_t^2, \zeta_t^2)_{t \geqslant 0}$ have the same law, then we get the desired result.

□

Remark 6.6 *For any two initial states $\xi_1, \xi_2 \in \mathbb{L}^2(\mathcal{F}_0)$ with the same distribution and $x \in \mathbb{R}$, the solutions Y^{x,ξ_1}, Y^{x,ξ_2} to (6.2) are the same. We can say that $Y_0^{x,\xi}$ depends solely on the distribution of ξ.*

6.2 Connection with distribution-dependent elliptic PDE

For FBSDEs (6.1) and (6.2), we define $\mathcal{V}(x,\mu) \triangleq Y_0^{x,\xi}$ for some initial state ξ with distribution μ. We first present the continuity result for $\mathcal{V}$.

Lemma 6.7 *Assuming all assumptions in Theorem 4.4 are satisfied, we have, for any $x, x' \in \mathbb{R}$ and $\mu, \mu' \in \mathcal{P}_2$, there exists a constant $C > 0$ such that*

$$|\mathcal{V}(x,\mu) - \mathcal{V}(x',\mu')| \leqslant C(|x - x'| + \mathcal{W}_2(\mu,\mu')) \tag{6.34}$$

Proof Let $(X^{\xi_1}, Y^{\xi_1}, X^{x_1,\xi_1}, Y^{x_1,\xi_1})$ and $(X^{\xi_2}, Y^{\xi_2}, X^{x_2,\xi_2}, Y^{x_2,\xi_2})$ be the solutions of Equations (6.1) and (6.2) with $\mathcal{L}_{\xi_1} = \mu_1,\ \mathcal{L}_{\xi_2} = \mu_2$. Set

$$\begin{aligned}
&\hat{X}^{\xi} = X^{\xi_1} - X^{\xi_2}, \quad \hat{Y}^{\xi} = Y^{\xi_1} - Y^{\xi_2}, \\
&\hat{X}^{x,\xi} = X^{x_1,\xi_1} - X^{x_2 \xi_2}, \quad \hat{Y}^{x,\xi} = Y^{x_1,\xi_1} - Y^{x_2,\xi_2}.
\end{aligned} \tag{6.35}$$

C_1, C_2, C_3, C_4 appeared in the following proof are positive constants.

Applying Itô's formula to $\mathrm{e}^{-rt}|Y_t^{x_1,\xi_1} - Y_t^{x_2,\xi_2}|^2$, we get

$$\begin{aligned}
|Y_0^{x_1,\xi_1} - Y_0^{x_2,\xi_2}|^2 &\leqslant C_1 \mathbb{E} \int_0^\infty \mathrm{e}^{-rt} \left[(\hat{X}_t^{x,\xi})^2 + (\hat{Y}_t^{x,\xi})^2 + W_2^2(X_t^{\xi_1}, X_t^{\xi_2}) \right] \mathrm{d}t \\
&\leqslant C_1 \mathbb{E} \int_0^\infty \mathrm{e}^{-rt} \left[(\hat{X}_t^{x,\xi})^2 + (\hat{Y}_t^{x,\xi})^2 + (\hat{X}_t^{\xi})^2 \right] \mathrm{d}t.
\end{aligned} \tag{6.36}$$

Applying Itô's formula to $\mathrm{e}^{-rt}\hat{X}^{x,\xi}\hat{Y}^{x,\xi}$, we get

$$\mathbb{E} \int_0^\infty \mathrm{e}^{-rt} \left[(\hat{X}_t^{x,\xi})^2 + (\hat{Y}_t^{x,\xi})^2 \right] \mathrm{d}t \leqslant C_2 \left(\hat{X}_0^{x,\xi} \hat{Y}_0^{x,\xi} + \mathbb{E} \int_0^\infty \mathrm{e}^{-rt} (\hat{X}_t^{\xi})^2 \mathrm{d}t \right). \tag{6.37}$$

It turns out that

$$\begin{aligned}
(\hat{Y}_0^{x,\xi})^2 &\leqslant C_1 C_2 \hat{X}_0^{x,\xi} \hat{Y}_0^{x,\xi} + (C_1 C_2 + C_1) \mathbb{E} \int_0^\infty \mathrm{e}^{-rt} (\hat{X}_t^{\xi})^2 \mathrm{d}t \\
&\leqslant \frac{1}{2} (\hat{Y}_0^{x,\xi})^2 + \frac{C_1^2 C_2^2}{2} (\hat{X}_0^{x,\xi})^2 + (C_1 C_2 + C_1) \mathbb{E} \int_0^\infty \mathrm{e}^{-rt} (\hat{X}_t^{\xi})^2 \mathrm{d}t.
\end{aligned} \tag{6.38}$$

Thus,

$$(\hat{Y}_0^{x,\xi})^2 \leqslant C_1^2 C_2^2 (\hat{X}_0^{x,\xi})^2 + 2(C_1 C_2 + C_1) \mathbb{E} \int_0^\infty \mathrm{e}^{-rt} (\hat{X}_t^{\xi})^2 \mathrm{d}t. \tag{6.39}$$

Applying the same arguments to $\mathrm{e}^{-rt}|\hat{Y}_t^{\xi}|^2$ and $\mathrm{e}^{-rt}\hat{X}^{\xi}\hat{Y}^{\xi}$, we can get that

$$\mathbb{E} \int_0^\infty \mathrm{e}^{-rt} (\hat{X}_t^{\xi})^2 \mathrm{d}t \leqslant C_3 \mathbb{E}|\xi_1 - \xi_2|^2. \tag{6.40}$$

So, we can deduce that

$$|Y_0^{x_1,\xi_1} - Y_0^{x_2,\xi_2}|^2 \leqslant C_4 \left(|x_1 - x_2|^2 + \mathbb{E}|\xi_1 - \xi_2|^2 \right). \tag{6.41}$$

Since $Y_0^{x,\xi}$ depends solely on the distribution of ξ, we have

$$|\mathcal{V}(x_1,\mu_1) - \mathcal{V}(x_2,\mu_2)|^2 \leqslant C_4 \left(|x_1 - x_2|^2 + \mathcal{W}_2^2(\mu_1,\mu_2) \right). \tag{6.42}$$

Then, we get the desired result. □

We now assert the following theorem.

Theorem 6.8 *Considering all assumptions in Theorem 4.4 are satisfied, the function $\mathcal{V}(x,\mu)$ is a viscosity solution of PDE (6.4).*

Proof We only show that $\mathcal{V}$ is a viscosity subsolution of PDE (6.4). A similar argument will show that it is also a viscosity supersolution of PDE (6.4).

Due to the uniqueness of solutions of FBSDEs, it is not hard to see that for any $t \geqslant 0$, $\mathcal{V}(X_t^{x,\xi}, \mathcal{L}_{X_t^\xi}) = Y_t^{x,\xi}$. Let Ψ be a test function and $(x^0,\mu^0) \in \mathbb{R}\times\mathcal{P}_2$ be a local maximum of $\mathcal{U}-\Psi$. We assume without loss of generality that $\mathcal{V}(x^0,\mu^0)=\Psi(x^0,\mu^0)$. We suppose that

$$(\mathcal{L}\Psi)[\mathcal{V}](x^0,\mu^0) + F(x^0,\mu^0,\mathcal{V}(x^0,\mu^0)) < 0, \tag{6.43}$$

It follows from the above that there exists an open subset $O \subset \mathbb{R}\times\mathcal{P}_2$ that contains (x^0,μ^0) such that for all $(x,\mu)\in O$,

$$\begin{cases} \mathcal{V}(x,\mu) \leqslant \Psi(x,\mu), \\ (\mathcal{L}\Psi)[\mathcal{V}](x,\mu) + F(x,\mu,\mathcal{V}(x,\mu)) < 0. \end{cases} \tag{6.44}$$

Taking an initial state $\xi^0 \in \mathbb{L}^2(\mathcal{F}_0;\mu^0)$, we consider the processes $(X_t^{\xi^0}, Y_t^{\xi^0}, Z_t^{\xi^0})$ and $(X_t^{x^0,\xi^0}, Y_t^{x^0,\xi^0}, Z_t^{x^0,\xi^0})$, which are solutions to FBSDEs (6.1) and (6.2), respectively. For some $T>0$, let τ denote the stopping time

$$\tau \triangleq \inf\{t>0 \,|\, (X_t^{x^0,\xi^0}, \mathcal{L}_{X_t^{\xi^0}}) \notin O\} \wedge T. \tag{6.45}$$

Let $\rho_t \triangleq \mathcal{L}_{X_t^{\xi^0}}$, we first note that the pair of processes

$$(\bar{Y}(t), \bar{Z}(t)) \triangleq (Y_{t\wedge\tau}^{x^0,\xi^0}, I_{[0,\tau]}(t) Z_t^{x^0,\xi^0}), \quad 0 \leqslant t \leqslant T, \tag{6.46}$$

is the solution of the BSDE

$$\begin{aligned} \bar{Y}_t = {} & \mathcal{V}(X_\tau^{x^0,\xi^0}, \rho_\tau) + \int_t^T I_{[0,\tau]}(s) F(X_s^{x^0,\xi^0}, \rho_s, \mathcal{V}(X_s^{x^0,\xi^0}, \rho_s)) \mathrm{d}s \\ & - \int_t^T \bar{Z}_s \mathrm{d}B_s, \quad 0 \leqslant t \leqslant T. \end{aligned} \tag{6.47}$$

Next, it follows from Itô's formula that the pair of processes

$$(\hat{Y}_t, \hat{Z}_t) \triangleq (\Psi(X_{t\wedge\tau}^{x^0,\xi^0}, \rho_{t\wedge\tau}), I_{[0,\tau]}(t)\partial_x\Psi(X_{t\wedge\tau}^{x^0,\xi^0}, \rho_{t\wedge\tau})) \tag{6.48}$$

is the solution of the BSDE

$$\begin{aligned} \hat{Y}_t = {} & \Psi(X_\tau^{x^0,\xi^0}, \rho_\tau) - \int_t^T I_{[0,\tau]}(s) \mathcal{L}\Psi[\mathcal{V}](X_s^{x^0,\xi^0}, \rho_s) \mathrm{d}s \\ & - \int_t^T \hat{Z}_s \mathrm{d}B_s, \quad 0 \leqslant t \leqslant T. \end{aligned} \tag{6.49}$$

Define

$$\beta_s = -\mathcal{L}\Psi[\mathcal{V}](X_s^{x^0,\xi^0}, \rho_s) - F(X_s^{x^0,\xi^0}, \rho_s, \mathcal{V}(X_s^{x^0,\xi^0}, \rho_s)) \tag{6.50}$$

and

$$\left(\tilde{Y}(t), \tilde{Z}(t)\right) = \left(\hat{Y}(t) - \bar{Y}(t), \hat{Z}(t) - \bar{Z}(t)\right). \tag{6.51}$$

We have

$$\tilde{Y}_t = \Psi(X_\tau^{x^0,\xi^0}, \rho_\tau) - \mathcal{V}(X_\tau^{x^0,\xi^0}, \rho_\tau) + \int_t^T I_{[0,\tau]}(s)\beta_s \mathrm{d}s - \int_t^T \tilde{Z}_s \mathrm{d}B_s, \quad 0 \leqslant t \leqslant T. \tag{6.52}$$

Therefore,

$$\tilde{Y}_0 = \mathbb{E}\left[\tilde{Y}_\tau + \int_0^\tau \beta_s \mathrm{d}s\right] \tag{6.53}$$

Now, from the choice of O and τ, a.s.

$$\tilde{Y}_\tau = \Psi(X_\tau^{x^0,\xi^0}, \rho_\tau) - \mathcal{V}(X_\tau^{x^0,\xi^0}, \rho_\tau) \geqslant 0, \quad \beta_s > 0, \quad s \in [0, \tau]. \tag{6.54}$$

Consequently, $\tilde{Y}_0 = \Psi(x^0, \mu^0) - \mathcal{V}(x^0, \mu^0) > 0$, which contradicts the earlier assumption. □

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