Probability, Uncertainty and Quantitative Risk
Vol. 11, No. 1, 2026, 1–18
https://www.aimsciences.org/PUQR
DOI:10.3934/puqr.2026006

Monotonicity and convergence for *g*-expectation of distributions

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Abstract Based on the g-expectation of distributions, we obtain the monotonicity and Jensen's inequality for the g-expectation of distributions; and for a sequence of distribution functions, we establish a monotone weak convergence theorem, Fatou's lemma, and a convergence theorem with respect to the g-expectation of distributions.

Keywords g-expectation, g-expectation of distributions, Backward stochastic differential equation, Monotonicity, Monotone weak convergence, Convergence

2020 Mathematics Subject Classification 60H10

1. Introduction

In Peng [21], the notion of g-expectation was proposed through backward stochastic differential equations (BSDEs). It preserves almost all the properties of classical mathematical expectation except linearity. Since the notion of g-expectation was introduced, reference [1] proved a converse comparison theorem for BSDEs and showed that the monotonicity of g-expectations inversely determined the comparative properties of their generators; the authors of [5] studied the time-consistency, monotonicity and regularity of nonlinear expectation and g-expectation; Jiang [17] studied the convexity, translation invariance and subadditivity of g-expectations, and obtained sufficient and necessary conditions for these properties. Jensen's inequality has also attracted much attention in this area. Jensen's inequality for g-expectation and nonlinear expectation and related problems have been studied in [3, 8, 12, 27], where many interesting results were obtained. Jiang [16] obtained a sufficient and necessary condition for Jensen' s inequality of BSDEs. More properties of g-expectation have been studied in [11, 15, 25]. By the theory of g-expectation, applications in fields such as finance, economics and insurance have been studied in [2, 10, 14, 23, 26, 30].

The expectation of a random variable is uniquely determined by its distribution in classical probability theory. However, it is often not law-invariant in nonlinear expectation theory. BSDEs have important applications in finance, particularly in characterizing wealth processes in portfolio optimization problems. Sometimes, we only need to find the optimal terminal distribution and replicate a random variable with that distribution. Therefore, people are more concerned about BSDEs with a given terminal distribution. Xu, Xu, and Zhou [29] proposed the

Received 1 July 2025; Accepted 4 February 2026; Early access 20 March 2026
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notion of g-expectation of distributions and obtained some interesting results.

Motivated by Xu, Xu, and Zhou [29], this paper aims to investigate some properties of the g-expectation of distributions. First, we obtain the monotonicity by constructing a random variable associated with a modified distribution function. Second, we derive Jensen's inequality for the g-expectation of distributions depending on this monotonicity result and the work in Jiang [16]. Based on the translation invariance and positive homogeneity of g-expectation, we demonstrate that these characteristics remain valid when considering g-expectation of distributions.

Third, based on the previously established monotonicity, we establish a monotone weak convergence theorem for the g-expectation of distributions. Limit theory in the nonlinear expectation framework is mainly conducted under sublinear expectations defined in Peng [22]. However, the convergence of a distribution function sequence under the g-expectation cannot be obtained directly through the convergence theorems studied in [13, 19, 28, 31]. For a non-decreasing sequence of distribution functions, we establish that the g-expectation of distributions is monotonically non-increasing and converges with weak convergence of distributions. However, for a non-increasing sequence of distribution functions, to obtain the result, we need some kind of local consistency condition. This condition is reasonable in the nonlinear expectation framework as additional assumptions are required when establishing monotone convergence theorems for capacities under Choquet expectations and for random variables under G-expectations.

Finally, based on the established monotone convergence theorem, we prove Fatou's lemma and a convergence theorem. Notably, unlike classical convergence conditions, our requirements are intrinsically tied to distributional properties. Specifically, we should verify that the limit of the sequence of distribution functions remains a distribution.

The structure of this paper is organized as follows. In Section 2, we introduce some assumptions and lemmas related to g-expectation. The monotonicity and Jensen's inequality for the g-expectation of distributions are proved in Section 3. In Section 4, we establish three convergence theorems with respect to the g-expectation of distributions.

2. Preliminaries

Let $(\Omega, \mathcal{F}_T, P)$ be a complete probability space with a given real number $T > 0$, and $(B_t)_{t\geqslant 0}$ be a d-dimensional standard Brownian motion on this space. Let $(\mathcal{F}_t)_{t\geqslant 0}$ be the filtration generated by Brownian motion $(B_t)_{t\geqslant 0}$ and augmented by the set of all P-null subsets. For any positive integer n and $z \in \mathbf{R}^n$, $|z|$ denotes its Euclidean norm.

We denote the set of all square-integrable $\mathcal{F}_t$-measurable random variables by $L^2(\Omega, \mathcal{F}_t, P)$, $t \in [0,T]$. We also define the following usual spaces of processes:

$$\mathbb{S}^2(0,T;\mathbf{R}) = \Big\{\psi : \psi \text{ continuous, progressively measurable and } \mathbf{E}[\sup_{t\in[0,T]} |\psi_t|^2] < +\infty\Big\},$$
$$\mathbb{H}^2(0,T;\mathbf{R}^d) = \Big\{\psi : \psi \text{ progressively measurable and } \mathbf{E}[\textstyle\int_0^T |\psi_t|^2 \mathrm{d}t] < +\infty\Big\}.$$

Throughout the paper, we consider a function $g : \Omega \times [0,T] \times \mathbf{R} \times \mathbf{R}^d \to \mathbf{R}$ such that the process $(g(t,y,z))_{t\in[0,T]}$ is progressively measurable for each pair (y,z) in $\mathbf{R} \times \mathbf{R}^d$, and furthermore, g satisfies the following assumptions (A1) and (A2):

(A1) There exists a constant $K \geqslant 0$ such that $\mathrm{d}P \times \mathrm{d}t$-*a.s.*, $\forall y_1,\ y_2 \in \mathbf{R},\ z_1,\ z_2 \in \mathbf{R}^d$, $|g(t,y_1,z_1) - g(t,y_2,z_2)| \leqslant K(|y_1 - y_2| + |z_1 - z_2|)$.

(A2) The process $(g(t,0,0))_{t\in[0,T]}$ belongs to $\mathbb{H}^2$.

(A3) $\mathrm{d}P \times \mathrm{d}t$-*a.s.*, $\forall y \in \mathbf{R}$, $g(t,y,0) \equiv 0$.

We consider the following BSDE:

$$Y_t = X + \int_t^T g(s, Y_s, Z_s)\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}B_s, \quad 0 \leqslant t \leqslant T, \tag{1}$$

where the terminal value X is an $\mathcal{F}_T$-measurable random variable.

Let g satisfy (A1) and (A2). Then for each $X \in L^2(\Omega, \mathcal{F}_T, P)$, there exists a unique pair of processes in $\mathbb{S}^2(0,T;\mathbf{R}) \times \mathbb{H}^2(0,T;\mathbf{R}^d)$, denoted by $(Y_t(g,T,X), Z_t(g,T,X))_{t\in[0,T]}$, solving the BSDE (1) (Pardoux and Peng [20]). Using the solution of BSDE, the notion of g-expectation and some properties of g-expectation were proposed in Peng [21].

Definition 2.1 *Let g satisfy (A1) and (A3). For each $X \in L^2(\Omega, \mathcal{F}_T, P)$, let $(Y_t, Z_t)_{t\in[0,T]}$ be the solution of BSDE. The g-expectation of X is defined as $\varepsilon_g[X] = Y_0(g,T,X)$.*

Lemma 2.1 *Let g satisfy (A1) and (A3). Then, for any $\xi,\ \eta \in L^2(\Omega, \mathcal{F}_T, P)$, the following statements hold:*

(i) *(Preservation of constants) $\varepsilon_g[c] = c$, $\forall c \in \mathbf{R}$;*
(ii) *(Monotonicity) If $\xi \geqslant \eta$, then $\varepsilon_g[\xi] \geqslant \varepsilon_g[\eta]$;*
(iii) *(Strict Monotonicity) If $\xi \geqslant \eta$ and $P(\xi > \eta) > 0$, then $\varepsilon_g[\xi] > \varepsilon_g[\eta]$.*

The probability distribution function of a real random variable is a nondecreasing right-continuous function $F: \mathbf{R} \to [0,1]$ satisfying $F(-\infty) = 0$ and $F(+\infty) = 1$. If there exists a random variable $X \in L^2(\Omega, \mathcal{F}_T, P)$ such that X is with the distribution F, we say that F is a reachable distribution.

We denote the set of all reachable distributions of random variables in $L^2(\Omega, \mathcal{F}_T, P)$ by $\mathcal{M}$, i.e., $\mathcal{M} \triangleq \{F \mid X \sim F,\ X \in L^2(\Omega, \mathcal{F}_T, P)\}$.

In Xu, Xu, and Zhou [29], the notion of g-expectation of distributions was proposed.

Definition 2.2 *Let the generator g satisfy (A1) and (A3), let F be a distribution in $\mathcal{M}$, and let $X \in L^2(\Omega, \mathcal{F}_T, P)$ satisfy $X \sim F$. The optimal solution to the problem*

$$\inf_{X\sim F} \varepsilon_g[X]$$

is denoted by X^*, and the infimum value is called the g-expectation of F, denoted by

$$\varepsilon_g[F] = \inf_{X\sim F} \varepsilon_g[X].$$

In this paper, we always assume that $(\Omega, \mathcal{F}_T, P)$ is an atomless probability space and all random variables are defined on this space. We further assume that g satisfies assumptions (A1) and (A3) and all distribution functions discussed belong to $\mathcal{M}$.

3. Monotonicity and Jensen's inequality for *g*-expectation of distributions

In this section, we study the monotonicity and Jensen's inequality for the g-expectation of distributions proposed by [29].

We first list some useful results with regard to probability distribution functions and quantile functions. For a distribution $F \in \mathcal{M}$, its (left) quantile is defined as

$$F^{-1}(u) \triangleq q(u) = \inf\{x \in \mathbf{R} \mid F(x) \geqslant u\}, \quad u \in (0,1).$$

The following Lemma 3.1 and Lemma 3.2 present some relationships between a distribution function and a random variable on $(\Omega, \mathcal{F}_T, P)$, which can be found in Föllmer and Schied [9] and Rüschendorf [24], respectively.

Lemma 3.1 *Let U be a random variable with a uniform distribution on $(0,1)$, and let q be the quantile function of F; then, $X \triangleq q(U)$ has the distribution F.*

Lemma 3.2 *Let X be a random variable with F_X, and let $\tilde{U}$ be uniformly distributed on $(0,1)$, which is independent of X. For all $x \in \mathbf{R}$ and $\lambda \in [0,1]$, the modified distribution function of X is defined as*

$$\tilde{F}_X(x,\lambda) \triangleq P(X < x) + \lambda P(X = x),$$

then, $U \triangleq \tilde{F}_X(X,\tilde{U})$ is uniformly distributed on $(0,1)$, and $X = F_X^{-1}(U)$.

The following theorem establishes monotonicity for the g-expectation of distributions.

Theorem 3.1 (Monotonicity) *Let F and G be distribution functions in $\mathcal{M}$. If $F \leqslant G$, i.e., $F(x) \leqslant G(x)$ for all $x \in \mathbf{R}$, then*

$$\varepsilon_g[F] \geqslant \varepsilon_g[G].$$

Proof Let X be a random variable with F; it is obvious $X \in L^2(\Omega, \mathcal{F}_T, P)$. Let $\tilde{U}$ be uniformly distributed on $(0,1)$ and independent of X. By Lemma 3.2, we can get

$$X = F^{-1}(U),$$

where $U \triangleq \tilde{F}_X(X,\tilde{U})$ is uniformly distributed on $(0,1)$, and $\tilde{F}_X$ is the modified distribution function of X. Using U and the distribution G, we construct a random variable

$$Z = G^{-1}(U).$$

By Lemma 3.1, we know that Z is with the distribution G.

Since $F(x) \leqslant G(x)$ holds for all $x \in \mathbf{R}$, by the definitions of F^{-1} and G^{-1}, we can immediately get

$$F^{-1}(u) \geqslant G^{-1}(u), \quad \forall u \in (0,1).$$

So, we have

$$X = F^{-1}(U) \geqslant G^{-1}(U) = Z.$$

According to Lemma 2.1 (ii), we have

$$\varepsilon_g[X] \geqslant \varepsilon_g[Z] \geqslant \inf_{Y \sim G} \varepsilon_g[Y].$$

Since the random variable X with the distribution F is arbitrary, by Definition 2.2, we can obtain

$$\varepsilon_g[F] = \inf_{X \sim F} \varepsilon_g[X] \geqslant \inf_{Y \sim G} \varepsilon_g[Y] \triangleq \varepsilon_g[G].$$

The proof of Theorem 3.1 is complete. □

Jensen's inequality is a fundamental inequality in probability theory. Some results on Jensen's inequality of g-expectation and nonlinear expectation were obtained in [1, 3, 12, 16]. By Jensen's inequality for g-expectation, some results on g-submartingales and the pricing of contingent claims for European options in an incomplete market were obtained in Chen, Kulperger, and Jiang [4]. We now study Jensen's inequality for the g-expectation of distributions. Roughly speaking, we aim to find a suitable convex function f such that the inequality

$$\varepsilon_g[f(F)] \geqslant f(\varepsilon_g[F]), \quad F \in \mathcal{M},$$

holds. We have the following Theorem 3.2.

Theorem 3.2 (i) *(Jensen's Inequality) Let* $f: \mathbf{R} \to \mathbf{R}$ *be convex and continuous on the closed interval* $[0,1]$*; let* $F \in \mathcal{M}$*. If* $f(F) \in \mathcal{M}$ *and* $\varepsilon_g[F] \in [0,1]$*, then*

$$\varepsilon_g[f(F)] \geqslant f(\varepsilon_g[F]).$$

(ii) *Let* g *be independent of* y *and superhomogeneous with respect to* z*; let* ϕ *be a strictly increasing and convex function on* $\mathbf{R}$*. For any* $F \in \mathcal{M}$*, if* $F^\phi \in \mathcal{M}$*, then*

$$\varepsilon_g[F^\phi] \geqslant \phi(\varepsilon_g[F]),$$

where $F^\phi(x) \triangleq F(\phi^{-1}(x))$ *for all* $x \in \mathbf{R}$*.*

Proof (i) Since F and $f(F)$ are distribution functions in $\mathcal{M}$, we have

$$0 = \lim_{x \to -\infty} F(x) = \lim_{x \to -\infty} [f(F)](x), \quad 1 = \lim_{x \to +\infty} F(x) = \lim_{x \to +\infty} [f(F)](x).$$

Since f is continuous on $[0,1]$, we have

$$\begin{aligned} f(0) &= \lim_{x \to -\infty} f(F(x)) = \lim_{x \to -\infty} [f(F)](x) = 0, \\ f(1) &= \lim_{x \to +\infty} f(F(x)) = \lim_{x \to +\infty} [f(F)](x) = 1. \end{aligned}$$

Using the convexity of f on $[0,1]$, we obtain

$$f(t) = f((1-t) \times 0 + t \times 1) \leqslant (1-t) f(0) + t f(1) = t, \quad \forall t \in [0,1].$$

Thus, we have

$$f(t) \leqslant t, \quad \forall t \in [0,1]. \tag{2}$$

It follows that

$$f(F(x)) \leqslant F(x), \quad \forall x \in \mathbf{R}.$$

Note that $f(F)$ is a distribution function and $f(F) \in \mathcal{M}$; by Theorem 3.1, we have

$$\varepsilon_g[f(F)] \geqslant \varepsilon_g[F]. \tag{3}$$

Due to $0 \leqslant \varepsilon_g[F] \leqslant 1$, it follows from (2) that

$$f(\varepsilon_g[F]) \leqslant \varepsilon_g[F].$$

Together with (3), the above inequality yields

$$\varepsilon_g[f(F)] \geqslant f(\varepsilon_g[F]).$$

(ii) Since F, $F^\phi \in \mathcal{M}$, then for each random variable X, we can prove that

$$X \sim F \quad \text{if and only if} \quad \phi(X) \sim F^\phi.$$

Indeed, if $X \sim F$, since ϕ is strictly increasing, we have

$$\begin{aligned} P(\phi(X) \leqslant y) &= P(X \leqslant \phi^{-1}(y)) \\ &= F(\phi^{-1}(y)) = F^\phi(y), \quad \forall y \in \mathbf{R}. \end{aligned}$$

So $\phi(X) \sim F^\phi$. If $Y \sim F^\phi$, then

$$
\begin{aligned}
P(\phi^{-1}(Y) \leqslant x) &= P(Y \leqslant \phi(x)) \\
&= F^{\phi}(\phi(x)) = F(\phi^{-1}(\phi(x)) = F(x), \quad \forall x \in \mathbf{R}.
\end{aligned}
$$

Hence, $\phi^{-1}(Y) \sim F$.

Since g is independent of y and superhomogeneous with respect to z, by Jensen's inequality of g-expectation in [16], it follows that

$$
\varepsilon_g[\phi(X)] \geqslant \phi(\varepsilon_g[X]), \quad \forall X \in L^2(\Omega, \mathcal{F}_T, P). \tag{4}
$$

From (4), Definition 2.2, and the fact that ϕ is strictly increasing, we obtain

$$
\begin{aligned}
\phi(\varepsilon_g[F]) &= \phi(\inf_{X \sim F} \varepsilon_g[X]) \leqslant \inf_{X \sim F} \phi(\varepsilon_g[X]) \\
&\leqslant \inf_{X \sim F} \varepsilon_g[\phi(X)] = \inf_{\phi(X) \sim F^{\phi}} \varepsilon_g[\phi(X)] = \inf_{Y \sim F^{\phi}} \varepsilon_g[Y] = \varepsilon_g[F^{\phi}].
\end{aligned}
$$

This completes the proof. □

Example 3.1 *Let $f(x) = x^2$ for all $x \in \mathbf{R}$, let $a \in \mathbf{R}$, and let F_a denote the degenerate distribution function at a, i.e.,*

$$
F_a(x) = \begin{cases} 0, & x < a, \\ 1, & x \geqslant a. \end{cases}
$$

It follows that $f(F_a) = F_a$. With the help of Lemma 2.1 (i) and Definition 2.2, we obtain that $\varepsilon_g[F_a] = a$. Consequently,

$$
f(\varepsilon_g[F_a]) = (\varepsilon_g[F_a])^2 = a^2, \quad \varepsilon_g[f(F_a)] = \varepsilon_g[F_a] = a.
$$

When $\varepsilon_g[F_a] \notin [0,1]$, i.e., $a \notin [0,1]$, then

$$
f(\varepsilon_g[F_a]) > \varepsilon_g[f(F_a)],
$$

which means that Jensen's inequality for the g-expectation of distributions does not hold in this case. When $\varepsilon_g[F_a] \in [0,1]$, we have

$$
f(\varepsilon_g[F_a]) \leqslant \varepsilon_g[f(F_a)],
$$

which is exactly the statement of Theorem 3.2 (i).

In the following proposition, we establish the translation invariance and positive homogeneity for the g-expectation of distributions, which can be obtained from the translation invariance and positive homogeneity of g-expectation (see Peng [21] and Jiang [17]) immediately, and we omit the proof.

Proposition 3.1 *Let $F \in \mathcal{M}$. The g-expectation $\varepsilon_g[\cdot]$ satisfies the following properties.*

(i) *(Translation Invariance) If g is independent of y, then for all $c \in \mathbf{R}$,*

$$
\varepsilon_g[F] = \varepsilon_g[F^c] + c,
$$

where $F^c(x) \triangleq F(x+c)$ for all $x \in \mathbf{R}$.

(ii) *(Positive Homogeneity) If g is positive homogeneous with respect to (y,z), i.e., $g(t, \lambda y, \lambda z) = \lambda g(t,y,z)$ for all $\lambda > 0$ and $(y,z) \in \mathbf{R} \times \mathbf{R}^d$, then*

$$
\lambda \varepsilon_g[F] = \varepsilon_g[F^{\lambda}],
$$

where $F^{\lambda}(x) \triangleq F(\frac{1}{\lambda}x)$ *for all* $x \in \mathbf{R}$*.*

4. Convergence theorems for *g*-expectation of distributions

It is well known that the monotone convergence theorem, Fatou's lemma, and the dominated convergence theorem hold for classical linear mathematical expectations. This raises a natural inquiry: do these results still hold for the g-expectation of distributions? In this section, we will discuss these convergence theorems for the g-expectation of distributions.

We recall the definition of weak convergence for distribution functions. The following definition can be found in Durrett [6].

Definition 4.1 *Let* $\{F_n\}_{n\geqslant 1}$ *and* F *be distribution functions on* $\mathbf{R}$*. We say* $\{F_n\}_{n\geqslant 1}$ *converges weakly to* F*, denoted by* $F_n \xrightarrow{w} F$*, if* $\lim_{n\to+\infty} F_n(x) = F(x)$ *holds for all* $x \in C_F$*, where* $C_F \subseteq \mathbf{R}$ *is the set of continuous points of* F*.*

Theorem 4.1 (Monotone Weak Convergence Theorem) *Let* $\{F_n\}_{n\geqslant 1}$ *and* F *be distribution functions in* $\mathcal{M}$*, and* $F_n \xrightarrow{w} F$*.*

(i) *If* $\{F_n\}_{n\geqslant 1}$ *is a nondecreasing sequence, then* $\varepsilon_g[F_n] \downarrow \varepsilon_g[F]$*;*

(ii) *If* $\{F_n\}_{n\geqslant 1}$ *is a nonincreasing sequence, and* $\{F_n^{-1}\}_{n\geqslant 1}$ *converges uniformly to* F^{-1} *on* $C_{F^{-1}}$*, then* $\varepsilon_g[F_n] \uparrow \varepsilon_g[F]$*.*

Proof (i) For a nondecreasing sequence $\{F_n\}_{n\geqslant 1}$, we first illustrate that

$$F_n(x) \leqslant F(x), \quad \forall x \in \mathbf{R}.$$

For any $x \in \mathbf{R}$, there exists a sequence of continuity points of $F(x)$, denoted as $\{x_m\}_{m\in\mathbb{N}^+}$, such that $x_m \downarrow x$. It is obvious that

$$F(x) \leqslant F(x_m), \quad F_n(x) \leqslant F_n(x_m), \quad n = 1, 2, \ldots.$$

Note that $F_n \xrightarrow{w} F$ and we have

$$\lim_{n\to+\infty} F_n(x_m) = F(x_m), \quad m = 1, 2, \ldots.$$

Considering that $\{F_n\}_{n\geqslant 1}$ is a nondecreasing sequence, we have

$$F_n(x) \leqslant F_n(x_m) \leqslant F(x_m), \quad n = 1, 2, \ldots.$$

Since $x_m \downarrow x$, by the right-continuity of F and F_n, we have

$$F_n(x) = \lim_{m\to+\infty} F_n(x_m) \leqslant \lim_{m\to+\infty} F(x_m) = F(x), \quad \forall x \in \mathbf{R}.$$

By Theorem 3.1, we have

$$\varepsilon_g[F_n] \geqslant \varepsilon_g[F_{n+1}] \geqslant \varepsilon_g[F].$$

It follows that

$$\lim_{n\to+\infty} \varepsilon_g[F_n] \geqslant \varepsilon_g[F]. \tag{5}$$

Having proven one direction, we now demonstrate that the opposing inequality holds. Remember that the quantile functions of F and F_n are defined as

$$F^{-1}(u) = \inf\{x \in \mathbf{R} \mid F(x) \geqslant u\}, \quad F_n^{-1}(u) = \inf\{x \in \mathbf{R} \mid F_n(x) \geqslant u\}, \quad u \in (0,1).$$

Note that $F^{-1}(u)$ is a nondecreasing function and we know that $(0,1)\backslash C_{F^{-1}}$ is a countable set. Since $F_n \xrightarrow{w} F$, by Skorokhod's Representation Theorem in [6], we obtain that

$$\lim_{n\to+\infty} F_n^{-1}(u) = F^{-1}(u) \tag{6}$$

holds for all but a countable number of u.

Let X be a random variable on $(\Omega, \mathcal{F}_T, P)$ with the distribution F; it is obvious that $X \in L^2(\Omega, \mathcal{F}_T, P)$. Let $\tilde{U}$ be independent of X and uniformly distributed on (0,1). By Lemma 3.2, there exists a modified distribution function $\tilde{F}(X, \tilde{U})$ such that

$$X = F^{-1}(U),$$

where $U \triangleq \tilde{F}(X, \tilde{U})$ is uniformly distributed on $(0,1)$. For the distribution functions sequence $\{F_n\}_{n\geqslant 1}$, we define

$$X_n \triangleq F_n^{-1}(U), \quad n = 1, 2, \ldots.$$

By Lemma 3.1, we know that the random variable X_n is with distribution F_n for each $n \in \mathbb{N}^+$. Clearly, $X_n \in L^2(\Omega, \mathcal{F}_T, P)$. Since $(\Omega, \mathcal{F}_T, P)$ is atomless, it follows from (6) that

$$\lim_{n\to+\infty} X_n = X, \quad \text{a.s.}$$

Using $(X_n - X)^2 \leqslant (X_1 - X)^2 \leqslant 2(X_1^2 + X^2)$, we have $\lim_{n\to+\infty} E[X_n - X]^2 = 0$. By the prior estimate in [7], we know that

$$|\varepsilon_g[X_n] - \varepsilon_g[X]|^2 \leqslant KE[X_n - X]^2, \tag{7}$$

where K is a positive constant only depending on g and T. We can deduce that

$$\lim_{n\to+\infty} \varepsilon_g[X_n] = \varepsilon_g[X]. \tag{8}$$

By Definition 2.2, we know that

$$\varepsilon_g[F_n] \leqslant \varepsilon_g[X_n]. \tag{9}$$

From (8) and (9), we have

$$\lim_{n\to+\infty} \varepsilon_g[F_n] \leqslant \lim_{n\to+\infty} \varepsilon_g[X_n] = \varepsilon_g[X].$$

Note that X with the distribution F is arbitrary; then, we have

$$\lim_{n\to+\infty} \varepsilon_g[F_n] \leqslant \inf_{X\sim F} \varepsilon_g[X] = \varepsilon_g[F].$$

Together with (5), the above inequality yields

$$\lim_{n\to+\infty} \varepsilon_g[F_n] = \varepsilon_g[F].$$

Since $\{F_n\}_{n\geqslant 1}$ is a nondecreasing sequence, we can deduce that $\varepsilon_g[F_n] \downarrow \varepsilon_g[F]$.

(ii) For a nonincreasing sequence $\{F_n\}_{n\geqslant 1}$, we similarly obtain

$$\lim_{n\to+\infty} F_n^{-1}(u) = F^{-1}(u), \tag{10}$$

which holds for all but a countable number of u. We can also obtain that

$$F_n(x) \geqslant F_{n+1}(x) \geqslant F(x)$$

for all $x \in \mathbf{R}$. By Theorem 3.1, we have

$$\varepsilon_g[F_n] \leqslant \varepsilon_g[F_{n+1}] \leqslant \varepsilon_g[F],$$

which implies that

$$\lim_{n\to+\infty} \varepsilon_g[F_n] \leqslant \varepsilon_g[F]. \tag{11}$$

We now turn to the reverse inequality. According to Definition 2.2, for each distribution function F_n, there exists $X^{(n)} \in L^2(\Omega, \mathcal{F}_T, P)$, $X^{(n)} \sim F_n$, and

$$\varepsilon_g[F_n] \leqslant \varepsilon_g[X^{(n)}] \leqslant \varepsilon_g[F_n] + \frac{1}{2^n}, \quad n = 1, 2, \ldots.$$

Thus, we have

$$\lim_{n\to+\infty} \varepsilon_g[X^{(n)}] = \lim_{n\to+\infty} \varepsilon_g[F_n]. \tag{12}$$

For each $X^{(n)}$, by Lemma 3.2, there exists a random variable $U^{(n)}$ uniformly distributed on $(0,1)$ such that

$$X^{(n)} = F_n^{-1}(U^{(n)}), \quad n = 1, 2, \ldots.$$

For each $U^{(n)}$, we can define a sequence of random variables as

$$Y_{n,m} \triangleq F_m^{-1}(U^{(n)}), \quad Y_{n,+\infty} \triangleq F^{-1}(U^{(n)}), \quad m = 1, 2, \ldots,$$

it is obvious that $Y_{n,n} = F_n^{-1}(U^{(n)}) = X^{(n)}$, and by Lemma 3.1, we know that

$$Y_{n,m} \sim F_m, \quad Y_{n,+\infty} \sim F.$$

By (10) and the fact that $(\Omega, \mathcal{F}_T, P)$ is atomless, we deduce that

$$\lim_{m\to+\infty} Y_{n,m} = Y_{n,+\infty}, \quad \text{a.s.}$$

It follows from the prior estimate in [7] that

$$\lim_{m\to+\infty} \varepsilon_g[Y_{n,m}] = \varepsilon_g[Y_{n,+\infty}].$$

Since $\{F_n^{-1}\}_{n\geqslant 1}$ converges uniformly to F^{-1} on $C_{F^{-1}}$, we know that for any $\epsilon > 0$, there exists a positive integer N such that for any $n \geqslant N$, we have

$$|F_n^{-1}(u) - F^{-1}(u)| < \epsilon, \quad \forall u \in C_{F^{-1}}.$$

Noticing that $(0,1)\backslash C_{F^{-1}}$ has at most a countable number of u and that $(\Omega, \mathcal{F}_T, P)$ is atomless, we have, for any $n \geqslant N$,

$$|F_n^{-1}(U^{(n)}) - F^{-1}(U^{(n)})| < \epsilon, \quad a.s.$$

Then, by the prior estimate in [7], see (7), we have

$$|\varepsilon_g[F_n^{-1}(U^{(n)})] - \varepsilon_g[F^{-1}(U^{(n)})]| \leqslant K\epsilon,$$

where K is a positive constant only depending on g and T. Hence, we have

$$\lim_{n\to+\infty} \varepsilon_g[F_n^{-1}(U^{(n)})] = \lim_{n\to+\infty} \varepsilon_g[F^{-1}(U^{(n)})],$$

which implies that

$$\lim_{n\to+\infty} \varepsilon_g[Y_{n,n}] = \lim_{n\to+\infty} \varepsilon_g[Y_{n,+\infty}]. \tag{13}$$

From (12) and (13), we have

$$\begin{aligned}\lim_{n\to+\infty} \varepsilon_g[F_n] &= \lim_{n\to+\infty} \varepsilon_g[X^{(n)}] = \lim_{n\to+\infty} \varepsilon_g[Y_{n,n}] = \lim_{n\to+\infty} \varepsilon_g[Y_{n,+\infty}] \\ &\geqslant \inf_{Y\sim F} \varepsilon_g[Y] = \varepsilon_g[F].\end{aligned}$$

Together with (11), the above inequality yields

$$\lim_{n\to+\infty} \varepsilon_g[F_n] = \varepsilon_g[F].$$

Note that $\{\varepsilon_g[F_n]\}_{n\geqslant 1}$ is a nondecreasing sequence; we can deduce that $\varepsilon_g[F_n] \uparrow \varepsilon_g[F]$.

The proof of Theorem 4.1 is finished. □

The following Example 4.1 gives an application of Theorem 4.1.

Example 4.1 *Define a nondecreasing sequence of distribution functions* $\{F_n\}_{n\geqslant 1}$ *and a nonincreasing sequence of distribution functions* $\{G_n\}_{n\geqslant 1}$ *as follows:*

$$F_n(x) = \begin{cases} 0, & x < 1, \\ \frac{1}{2}, & 1 \leqslant x < 2 + \frac{1}{n}, \\ 1, & x \geqslant 2 + \frac{1}{n}, \end{cases} \quad G_n(x) = \begin{cases} 0, & x < -\frac{1}{n}, \\ \frac{1}{2}, & -\frac{1}{n} \leqslant x < 1, \\ 1, & x \geqslant 1, \end{cases}$$

and define two distribution functions:

$$F(x) = \begin{cases} 0, & x < 1, \\ \frac{1}{2}, & 1 \leqslant x < 2, \\ 1, & x \geqslant 2, \end{cases} \qquad G(x) = \begin{cases} 0, & x < 0, \\ \frac{1}{2}, & 0 \leqslant x < 1, \\ 1, & x \geqslant 1. \end{cases}$$

We can verify that $F_n \xrightarrow{w} F$ *and* $G_n \xrightarrow{w} G$.

By Theorem 4.1 (i), we know that

$$\varepsilon_g[F_n] \downarrow \varepsilon_g[F].$$

The quantile functions of G_n *and* G *are respectively*

$$G_n^{-1}(p) = \begin{cases} -\frac{1}{n}, & 0 < p \leqslant \frac{1}{2}, \\ 1, & \frac{1}{2} < p < 1, \end{cases} \quad G^{-1}(p) = \begin{cases} 0, & 0 < p \leqslant \frac{1}{2}, \\ 1, & \frac{1}{2} < p < 1. \end{cases}$$

Hence,

$$\sup_{p\in(0,1)} |G_n^{-1}(p) - G^{-1}(p)| = \frac{1}{n},$$

which implies that $\{G_n^{-1}\}_{n\geqslant 1}$ *converges uniformly to* G^{-1} *on* $C_{G^{-1}}$*. From Theorem 4.1 (ii), we obtain*

$$\varepsilon_g[G_n]\uparrow\varepsilon_g[G].$$

Remark 4.1 *The uniform convergence condition of quantile functions in Theorem 4.1 (ii) is likely just a technical condition. The following example demonstrates that this condition is not necessary for some special distribution sequences.*

Define a nonincreasing sequence of distribution functions $\{Q_n\}_{n\geqslant 1}$ *and a distribution function* Q *as follows:*

$$Q_n(x)=\begin{cases}0, & x<\frac{1}{2},\\ \frac{1}{n}, & \frac{1}{2}\leqslant x<1,\\ 1, & x\geqslant 1,\end{cases}\qquad Q(x)=\begin{cases}0, & x<1,\\ 1, & x\geqslant 1.\end{cases}$$

We can verify that $Q_n\xrightarrow{w}Q$*, and the quantile functions of* Q_n *and* Q *are respectively*

$$Q_n^{-1}(p)=\begin{cases}\frac{1}{2}, & 0<p\leqslant\frac{1}{n},\\ 1, & \frac{1}{n}<p<1,\end{cases}\qquad Q^{-1}(p)=1,\ \ 0<p<1.$$

By direct calculation, one gets

$$\sup_{p\in(0,1)}|Q_n^{-1}(p)-Q^{-1}(p)|=\frac{1}{2},$$

which means that $\{Q_n^{-1}\}_{n\geqslant 1}$ *does not converge uniformly to* Q^{-1}*.*

With the help of Lemma 2.1 (i) and Definition 2.2, we have

$$\varepsilon_g[Q_n]=\inf_{X_n\sim Q_n}\varepsilon_g[X_n],\quad \varepsilon_g[Q]=1.$$

For any random variable $X_n\sim Q_n$ *and* $X\sim Q$*, the prior estimate in [7] yields*

$$|\varepsilon_g[X_n]-\varepsilon_g[X]|^2\leqslant KE[|X_n-X|^2]=\frac{K}{4n},$$

where K *is a positive constant only depending on* g *and* T*. Hence,* $\lim_{n\to+\infty}\varepsilon_g[X_n]=\varepsilon_g[X]$*, and consequently,*

$$\varepsilon_g[Q_n]\uparrow\varepsilon_g[Q].$$

As this example illustrates, the uniform convergence condition for quantile functions is unnecessary in the above context. However, due to the difficulties arising from the high nonlinearity of the g-expectation of distributions, it remains challenging to ascertain whether this condition is necessary for general conclusions.

Theorem 4.2 (Fatou's Lemma) *Let* $\{F_n\}_{n\geqslant 1}$ *be distribution functions in* $\mathcal{M}$*.*

(i) *If* $\liminf_{n\to+\infty}F_n\in\mathcal{M}$*, then* $\varepsilon_g[\liminf_{n\to+\infty}F_n]\geqslant\limsup_{n\to+\infty}\varepsilon_g[F_n]$*;*

(ii) *If* $F\triangleq\limsup_{n\to+\infty}F_n\in\mathcal{M}$ *and* $\{H_n^{-1}\}_{n\geqslant 1}$ *converges uniformly to* F^{-1} *on* $C_{F^{-1}}$*, where* $H_n(x)\triangleq\inf_{y>x}\sup_{k\geqslant n}F_k(y)$*,* $\forall x\in\mathbf{R}$*,* $n\in\mathbb{N}^+$*, then* $\varepsilon_g[\limsup_{n\to+\infty}F_n]\leqslant\liminf_{n\to+\infty}\varepsilon_g[F_n]$*.*

Proof (i) For each $n\in\mathbb{N}^+$, we define a function

$$G_n(x)\triangleq\inf_{k\geqslant n}F_k(x),\quad \forall x\in\mathbf{R}.\tag{14}$$

Since $\{F_n\}_{n\geqslant 1}$ is a sequence of distribution functions, it is obvious that for each n, $G_n(x)$ is nondecreasing with respect to x, and for each x, $G_n(x)$ is nondecreasing with respect to n. Note that $G_n(x) \leqslant F_n(x)$ for all $x \in \mathbf{R}$ and $F_n(-\infty) = 0$, we have

$$G_n(-\infty) \triangleq \lim_{x\to-\infty} G_n(x) = 0.$$

Now for each n, we want to prove that G_n is a distribution function. We only need to prove that $\lim_{x\to+\infty} G_n(x) = 1$, and $G_n(x)$ is right-continuous.

Since $\liminf_{n\to+\infty} F_n \in \mathcal{M}$, we know that

$$\lim_{x\to+\infty}\lim_{n\to+\infty} G_n(x) = \lim_{x\to+\infty}\liminf_{n\to+\infty} F_n(x) = 1.$$

Therefore, for any $\epsilon > 0$, there exists a constant $M > 0$ such that

$$\lim_{n\to+\infty} G_n(x) > 1-\epsilon, \quad \forall x \geqslant M.$$

Hence, for every $x \geqslant M$, there exists a positive integer N_M such that $G_n(x) > 1-\epsilon$ holds for all $n \geqslant N_M$. Remember that $\{G_n\}_{n\geqslant 1}$ is a nondecreasing sequence, and $G_n(x)$ is nondecreasing with x for each n, for all $x \geqslant M$, we have

$$G_n(x) > 1-\epsilon, \quad n = N_M, N_M+1, \ldots. \tag{15}$$

For $n \leqslant N_M - 1$, we know that $F_1(x)$, $F_2(x), \ldots, F_{N_M-1}(x)$ are distribution functions. Since $\lim_{x\to+\infty} F_1(x) = \lim_{x\to+\infty} F_2(x) = \cdots = \lim_{x\to+\infty} F_{N_M-1}(x) = 1$, there exists $M' > 0$ such that

$$F_1(x) > 1-\epsilon,\ F_2(x) > 1-\epsilon, \ldots,\ F_{N_M-1}(x) > 1-\epsilon, \quad \forall x \geqslant M'. \tag{16}$$

Thus, for any $\epsilon > 0$, combining (15) and (16), we know that there exists $M^* = \max\{M, M'\}$ such that

$$G_n(x) = \inf_{k\geqslant n} F_k(x) = \begin{cases} \inf\{F_n(x), \ldots, F_{N_M-1}(x), G_{N_M}(x)\} > 1-\epsilon, & \forall x \geqslant M^*, \forall n \leqslant N_M - 1, \\ G_n(x) > 1-\epsilon, & \forall x \geqslant M^*, \forall n \geqslant N_M. \end{cases}$$

Since ϵ is arbitrary, it implies that

$$\lim_{x\to+\infty} G_n(x) = 1, \quad n = 1, 2, \ldots.$$

Next we prove that G_n is right-continuous. For any $a > 0$, note that

$$G_n(x+a) = \inf_{k\geqslant n} F_k(x+a) \leqslant F_k(x+a).$$

When $a \downarrow 0$, by the right continuity of F_k, we can obtain that

$$\limsup_{a\downarrow 0} G_n(x+a) \leqslant \limsup_{a\downarrow 0} F_k(x+a) = F_k(x), \quad k = n, n+1, \ldots.$$

It implies that

$$\lim_{a\downarrow 0} G_n(x+a) \leqslant \inf_{k\geqslant n} F_k(x) = G_n(x), \quad n = 1, 2, \ldots. \tag{17}$$

Meanwhile, for any $a > 0$, it follows from $G_n(x+a) \geqslant G_n(x)$ that

$$\lim_{a\downarrow 0} G_n(x+a) \geqslant G_n(x), \quad n = 1, 2, \ldots.$$

By (17), the above inequality yields

$$\lim_{a\downarrow 0} G_n(x+a) = G_n(x), \quad n=1,2,\ldots.$$

Hence, for each n, $G_n(x)$ is right-continuous. Therefore, G_n is a distribution function. It follows from (14) that $\{G_n\}_{n\geqslant 1}$ is a nondecreasing sequence of distribution functions, and

$$\liminf_{n\to+\infty} F_n(x) = \lim_{n\to+\infty} G_n(x), \quad \forall x \in \mathbf{R}.$$

By Theorem 4.1(i), we know that

$$\varepsilon_g[\liminf_{n\to+\infty} F_n] = \varepsilon_g[\lim_{n\to+\infty} G_n] = \lim_{n\to+\infty} \varepsilon_g[G_n]. \tag{18}$$

Since $G_n(x) \leqslant F_n(x)$ for all $x \in \mathbf{R}$, by Theorem 3.1, we have

$$\varepsilon_g[G_n] \geqslant \varepsilon_g[F_n], \quad n=1,2,\ldots.$$

It implies that

$$\limsup_{n\to+\infty} \varepsilon_g[G_n] \geqslant \limsup_{n\to+\infty} \varepsilon_g[F_n].$$

By (18), we can obtain that

$$\limsup_{n\to+\infty} \varepsilon_g[F_n] \leqslant \lim_{n\to+\infty} \varepsilon_g[G_n] = \varepsilon_g[\liminf_{n\to+\infty} F_n].$$

(ii) Since $H_n(x)$ is defined as

$$H_n(x) \triangleq \inf_{y>x} \sup_{k\geqslant n} F_k(y), \quad \forall x \in \mathbf{R}, n \in \mathbb{N}^+, \tag{19}$$

the definition implies that $H_n(x)$ is a right-continuous function.

Since $\{F_n\}_{n\geqslant 1}$ is a distribution function sequence, it is obvious that for each n, $H_n(x)$ is nondecreasing with respect to x, and $\{H_n\}_{n\geqslant 1}$ is a nonincreasing sequence. Note that $H_n(x) \geqslant F_n(x)$ and $F_n(+\infty)=1$, we have

$$H_n(+\infty) \triangleq \lim_{x\to+\infty} H_n(x) = 1, \quad \forall n \in \mathbb{N}^+.$$

We now prove that

$$\lim_{x\to-\infty} \sup_{n\geqslant 1} F_n(x) = 0. \tag{20}$$

As $F \in \mathcal{M}$, we know that $\lim_{x\to-\infty} F(x) = 0$. Hence, for any $\varepsilon > 0$, there exists $M_1 > 0$ such that

$$F(x) < \varepsilon, \quad \forall x \leqslant -M_1.$$

Remember that $\lim_{n\to+\infty} \sup_{k\geqslant n} F_k(x) = F(x)$ holds for all $x \in \mathbf{R}$. Taking $x = -M_1$, we have

$$\lim_{n\to+\infty} \sup_{k\geqslant n} F_k(-M_1) = F(-M_1),$$

which implies that there exists a positive integer N such that for all $n \geqslant N$, we have

$$|\sup_{k\geqslant n} F_k(-M_1) - F(-M_1)| < \varepsilon.$$

It follows that $\sup_{k\geqslant n} F_k(-M_1) < 2\varepsilon$ holds for all $n \geqslant N$, and we can deduce that

$$F_n(x) \leqslant F_n(-M_1) < 2\varepsilon, \quad \forall x \leqslant -M_1, n \geqslant N. \tag{21}$$

For $n \leqslant N-1$, since $\lim_{x\to-\infty} F_n(x) = 0$, we know that there exists $M_2 > 0$ such that for all $x \leqslant -M_2$, we have

$$F_n(x) < \varepsilon, \quad n = 1, 2, \ldots, N-1. \tag{22}$$

Thus, for any $\varepsilon > 0$, combining (21) and (22), we conclude that there exists $M = \max\{M_1, M_2\}$ such that

$$\sup_{n\geqslant 1} F_n(x) = \sup\{F_1(x), \ldots, F_{N-1}(x), \sup_{k\geqslant N} F_k(x)\} \leqslant 2\varepsilon, \quad \forall x \leqslant -M.$$

Since ε is arbitrary, we know that (20) holds.

It follows from (19) and (20) that

$$\lim_{x\to-\infty} H_1(x) = \lim_{x\to-\infty} \sup_{n\geqslant 1} F_n(x) = 0.$$

Thus, we can deduce that

$$\lim_{x\to-\infty} H_n(x) = 0, \quad \forall n \in \mathbb{N}^+.$$

Therefore, $H_n(x)$ is a distribution function and $\{H_n\}_{n\geqslant 1}$ is a nonincreasing distribution function sequence.

We next want to prove that

$$\limsup_{n\to+\infty} F_n(x) = \lim_{n\to+\infty} H_n(x), \quad \forall x \in \mathbf{R}. \tag{23}$$

Note that $\limsup_{n\to+\infty} F_n \in \mathcal{M}$, so $\limsup_{n\to+\infty} F_n$ is right-continuous, and we have

$$\inf_{y>x} \limsup_{n\to+\infty} F_n(y) = \limsup_{n\to+\infty} F_n(x), \quad \forall x \in \mathbf{R}. \tag{24}$$

For any fixed $y > x$, we have

$$\inf_{y'>x} \sup_{k\geqslant n} F_k(y') \leqslant \sup_{k\geqslant n} F_k(y).$$

It follows that

$$\lim_{n\to+\infty} \inf_{y'>x} \sup_{k\geqslant n} F_k(y') \leqslant \lim_{n\to+\infty} \sup_{k\geqslant n} F_k(y) = \limsup_{n\to+\infty} F_n(y), \quad \forall y > x.$$

Since $y > x$ is arbitrary, combining with (24), we can deduce that

$$\lim_{n\to+\infty} H_n(x) = \lim_{n\to+\infty} \inf_{y>x} \sup_{k\geqslant n} F_k(y) \leqslant \inf_{y>x} \limsup_{n\to+\infty} F_n(y) = \limsup_{n\to+\infty} F_n(x). \tag{25}$$

Meanwhile, for any $y > x$, we know that

$$\sup_{k\geqslant n} F_k(y) \geqslant \sup_{k\geqslant n} F_k(x),$$

which implies that

$$\inf_{y>x} \sup_{k\geqslant n} F_k(y) \geqslant \sup_{k\geqslant n} F_k(x).$$

It follows that

$$\lim_{n\to+\infty} H_n(x) = \lim_{n\to+\infty} \inf_{y>x} \sup_{k\geqslant n} F_k(y) \geqslant \limsup_{n\to+\infty} F_n(x). \tag{26}$$

From (25) and (26), we have

$$\lim_{n\to+\infty} H_n(x) = \lim_{n\to+\infty} \inf_{y>x} \sup_{k\geqslant n} F_k(y) = \limsup_{n\to+\infty} F_n(x), \quad \forall x \in \mathbf{R}.$$

Since $\{H_n\}_{n\geqslant 1}$ is a nonincreasing sequence of distribution functions, and $\{H_n^{-1}\}_{n\geqslant 1}$ converges uniformly to F^{-1} on $C_{F^{-1}}$, by (23) and Theorem 4.1 (ii), we have

$$\varepsilon_g[\limsup_{n\to+\infty} F_n] = \varepsilon_g[\lim_{n\to+\infty} H_n] = \lim_{n\to+\infty} \varepsilon_g[H_n]. \tag{27}$$

Note that $H_n(x) \geqslant F_n(x)$ for all $x \in \mathbf{R}$. By Theorem 3.1, we can obtain

$$\lim_{n\to+\infty} \varepsilon_g[H_n] \leqslant \liminf_{n\to+\infty} \varepsilon_g[F_n].$$

It follows from (27) and the above inequality that

$$\varepsilon_g[\limsup_{n\to+\infty} F_n] \leqslant \liminf_{n\to+\infty} \varepsilon_g[F_n].$$

This completes the proof. □

Theorem 4.3 (Convergence Theorem) *Let $\{F_n\}_{n\geqslant 1}$ and F be distribution functions in $\mathcal{M}$. If $\{F_n\}_{n\geqslant 1}$ converges to F on $\mathbf{R}$ and $\{H_n^{-1}\}_{n\geqslant 1}$ converges uniformly to F^{-1} on $C_{F^{-1}}$, where $H_n(x) \triangleq \inf_{y>x} \sup_{k\geqslant n} F_k(y)$, $\forall x \in \mathbf{R}$, $n \in \mathbb{N}^+$, then*

$$\lim_{n\to+\infty} \varepsilon_g[F_n] = \varepsilon_g[F].$$

Proof Since $\lim_{n\to+\infty} F_n(x) = F(x)$ holds for all $x \in \mathbf{R}$, we know that

$$\liminf_{n\to+\infty} F_n(x) = \limsup_{n\to+\infty} F_n(x) = F(x),$$

which implies that $\liminf_{n\to+\infty} F_n \in \mathcal{M}$ and $\limsup_{n\to+\infty} F_n \in \mathcal{M}$. By Theorem 4.2 (i), we have

$$\varepsilon_g[F] = \varepsilon_g[\liminf_{n\to+\infty} F_n] \geqslant \limsup_{n\to+\infty} \varepsilon_g[F_n]. \tag{28}$$

As $\limsup_{n\to+\infty} F_n \in \mathcal{M}$ and $\{H_n^{-1}\}_{n\geqslant 1}$ converges uniformly to F^{-1} on $C_{F^{-1}}$, by Theorem 4.2 (ii), we have

$$\varepsilon_g[F] = \varepsilon_g[\limsup_{n\to+\infty} F_n] \leqslant \liminf_{n\to+\infty} \varepsilon_g[F_n].$$

In the view of (28) and the above inequality, we can obtain

$$\lim_{n\to+\infty} \varepsilon_g[F_n] = \varepsilon_g[F].$$

□

Now, we present an example of convergence for the g-expectation of distributions.

Example 4.2 *For each $n \in \mathbb{N}^+$, a distribution function F_n is defined as*

$$F_n(x) = \Phi(x + \frac{(-1)^n}{n}), \quad \forall x \in \mathbf{R},$$

where Φ is the standard Gaussian distribution function.

It is obvious that $\{F_n\}_{n\geqslant 1}$ is not a monotone sequence of distribution functions, and F_n converges to Φ. We set

$$H_n(x) \triangleq \inf_{y>x} \sup_{k\geqslant n} F_k(y), \quad \forall x \in \mathbf{R}, n \in \mathbb{N}^+.$$

For all $x \in \mathbf{R}$, $m \in \mathbb{N}^+$, we deduce that

$$H_n(x) = \begin{cases} F_n(x), & n = 2m, \\ F_{n+1}(x), & n = 2m-1. \end{cases}$$

Note that Φ *is strictly increasing and continuous; we know that*

$$H_n(H_n^{-1}(u)) = u, \quad \Phi(\Phi^{-1}(u)) = u, \quad \forall u \in (0,1).$$

When $n = 2m$*, we know* $H_n(x) = \Phi(x + \frac{1}{n})$*, hence,*

$$H_n(H_n^{-1}(u)) = \Phi(H_n^{-1}(u) + \frac{1}{n}) = u,$$

which implies that

$$H_n^{-1}(u) + \frac{1}{n} = \Phi^{-1}(u), \quad \forall u \in (0,1).$$

When $n = 2m-1$*, we know* $H_n(x) = F_{n+1}(x) = \Phi(x + \frac{1}{n+1})$*, similarly, we have*

$$H_n^{-1}(u) + \frac{1}{n+1} = \Phi^{-1}(u), \quad \forall u \in (0,1).$$

So, we can deduce that

$$\sup_{u\in(0,1)} \left| H_n^{-1}(u) - \Phi^{-1}(u) \right| \leqslant \frac{1}{n}, \quad \forall n \in \mathbb{N}^+.$$

Hence, $\{H_n^{-1}\}_{n\geqslant 1}$ *converges uniformly to* Φ^{-1} *on* $C_{\Phi^{-1}}$*. By Theorem 4.3, we have*

$$\lim_{n\to+\infty} \varepsilon_g[F_n] = \varepsilon_g[\Phi].$$

Remark 4.2 *Since our results mainly depend on the existence and uniqueness and comparison theorem of BSDEs, our conditions on the generator* g *can be further weakened.*

For example, we can consider the generator g *of BSDEs, defined on* $\Omega \times [0,T] \times \mathbf{R}^d \to \mathbf{R}$ *such that the process* $(g(t,z))_{0\leqslant t\leqslant T}$ *is progressively measurable for each* $z \in \mathbf{R}^d$*. Let* g *and* $F \in \mathcal{M}$ *satisfy the following conditions.*

(A4) *There exists a constant* $C > 0$ *such that* $\mathrm{d}P \times \mathrm{d}t$*-a.s., for any* $z, z_1, z_2 \in \mathbf{R}^d$*,*

$$|g(t,z)| \leqslant C(1 + \|z\|^2); |g(t,z_1) - g(t,z_2)| \leqslant C(1 + \|z_1\| + \|z_2\|)\|z_1 - z_2\|.$$

(A5) $\exists\, c,\, d \in \mathbf{R}$*, s.t.* $F(c) = 0$ *and* $F(d) = 1$*.*

Under the condition $(A5)$*, the random variables with distribution* F *are bounded. Under the conditions* $(A4)$ *and* $(A5)$*, the results of Kobylanski [18] indicate that the g-expectation of* F*,* $\varepsilon_g[F] = \inf_{X\sim F} \varepsilon_g[X]$*, is well defined. In this situation, the results in this paper can be similarly investigated under some milder conditions.*

Acknowledgements

The authors acknowledge financial support from the Fundamental Research Funds for the Central Universities (Grant No. 2024KYJD2008). The authors would like to thank the editor and the referee for their very careful reading of this paper and valuable comments and suggestions.

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