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Numerical methods for two-dimensional G-heat equation

Ziting Pei[1], Xingye Yue[2,3,*], Xiaotao Zheng[2]

[1]*School of Business, Suzhou University of Science and Technology, Suzhou* 215009, *China*
[2]*Center for Financial Engineering, Soochow University, Suzhou* 215006, *China*
[3]*School of Mathematical Sciences, Soochow University, Suzhou* 215006, *China*

Email: xyyue@suda.edu.cn

Abstract The G-expectation is a sublinear expectation. It is an important tool for pricing financial products and managing risk owing to its ability to deal with model uncertainty. The problem is how to efficiently quantify it since the commonly used Monte Carlo method does not work. Fortunately, the expectation of a G-normal random variable can be linked to the viscosity solution of a fully nonlinear G-heat equation. In this paper, we first identify the limits of the uncertainty in the covariance of a two-dimensional G-normal random variable and determine the corresponding G-heat equation. Then, we propose a novel numerical scheme for solving the two-dimensional G-heat equation and pay more attention to the case where there exists uncertainty on the correlation, especially in the case that the correlation ranges from negative to positive. The scheme is monotone, stable, and convergent. The numerical tests show that the scheme is highly efficient.

Keywords G-expectation, G-heat equation, model uncertainty, inner iteration, convergence, viscosity solution

2020 Mathematics Subject Classification 35K55, 60G65, 91G60

1. Introduction

In 2006, Peng [10] introduced the so-called G expectation to solve problems with model uncertainty. It has developed rapidly in response to the increasing demand for robust quantitative analysis and risk management. Moreover, according to Peng [11], G-expectation is a coherent risk measure that satisfies all the axioms proposed in Artzner et al. [1]. It is difficult to compute the G-expectation by Monte Carlo sampling since the distribution of the random variable is uncertain. However, the G-expectation is related to a fully nonlinear G-heat equation [11]. One can quantify the G-expectation by numerically solving the G-heat equation.

The work of Barles and Souganidis [3] provided a theoretical foundation for fully nonlinear second-order equations that the numerical solution of a consistent and monotonic scheme converges to the viscosity solution of the original equation. For the one-dimensional case, the G-heat equation

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*Corresponding author

appears early as an option pricing model with volatility uncertainty, Pooley et al. [12] developed numerical algorithms and discussed their convergence properties. For the multi-dimensional G-heat equations, it is non-trivial to construct a monotone scheme to ensure its convergence to the viscosity solution (Barles and Souganidis [3], Barles et al. [2]). In this paper, we take the two-dimensional G-heat equation as an example, while the three-dimensional case can be analyzed analogously.

The two-dimensional G-heat equation also appeared early as a two-factor uncertain volatility model. Pooley et al. [13] numerically solved the equation; however, the scheme was not guaranteed to be monotone. The main difficulty in constructing monotone schemes is to treat the cross-derivative term. When the sign of the correlation is determined, a compact seven-point stencil (Öksendal and Sulem [9], Clift and Forsyth [5]) that relies on the sign of the correlation is employed for the discretization of the cross-derivative. To ensure monotonicity for problems with the cross-derivative, Bonnans and Zidani [4], Debrabant and Jakobsen [6] focused on explicit wide stencil schemes, while Ma and Forsyth [8] proposed an implicit numerical scheme that combines the use of a fixed-point stencil and a wide stencil based on a local coordinate rotation.

However, so far, there has been no discussion that takes into account the situation in which the sign of the correlation is uncertain. The approximation of the second-order cross-derivative is crucial in obtaining a monotone scheme. The selection of the seven-point stencil depends on the sign of the correlation, while the correlation, in turn, depends on the selected seven-point stencil due to the nonlinearity. To break this cycle of dilemmas, we develop a novel implicit numerical scheme that is stable, consistent, and monotone. Thus, our numerical scheme guarantees the convergence to the viscosity solution.

The organization of this paper is as follows. In Section 2, we review the basic concepts of the G-expectation. We identify the limits of the uncertainty in the covariance of a two-dimensional G-normal random variable and determine the corresponding G-heat equation. In Section 3, we develop an implicit numerical scheme to solve the general two-dimensional G-heat equation for the case where the correlation varies from negative to positive. In Section 4, we show the monotonicity of the scheme, which guarantees convergence to the viscosity solution. We present an estimate of the convergence rate. In particular, we show that the nonlinear iteration at each timestep is always convergent. In Section 5, we validate the efficiency of our numerical scheme through numerical examples. Finally, we provide some conclusions in Section 6.

2. Background

In this section, we recall some basic knowledge about Peng's G-stochastic calculus [11]. We identify the ranges of the uncertainty on the covariance of a G-normal random variable and determine the corresponding G-heat equation.

Before proceeding with the definitions, we introduce the relevant mathematical spaces used throughout this paper. Let Ω be a given set and let $\mathcal{H}$ be a linear space of real valued functions defined on Ω. $\mathcal{H}$ satisfies the following two conditions: (1) $c \in \mathcal{H}$ for each constant c and (2) $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$. The space $C_{l.Lip}(\mathbb{R}^d)$ consists of local Lipschitz continuous functions φ satisfying

$$|\varphi(x) - \varphi(y)| \leqslant C(1 + |x|^m + |y|^m)|x - y|, \quad \forall x, y \in \mathbb{R}^d,$$

for some constants $C > 0$ and $m \in \mathbb{N}$ depending on φ.

Definition 2.1 *The G-expectation* $\mathbb{E}$ *is a sublinear expectation that is a functional* $\mathbb{E}\colon \mathcal{H} \mapsto \mathbb{R}$ *satisfying*

(a) *Monotonicity: If* $X \geqslant Y$, *then* $\mathbb{E}[X] \geqslant \mathbb{E}[Y]$*;*
(b) *Constant preserving:* $\mathbb{E}[c] = c, \forall c \in \mathbb{R}$*;*
(c) *Sub-additivity:* $\mathbb{E}[X+Y] \leqslant \mathbb{E}[X] + \mathbb{E}[Y]$*;*
(d) *Positive homogeneity:* $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X], \forall \lambda \geqslant 0$*.*

Definition 2.2 *Let* X_1 *and* X_2 *be two* d*-dimensional random vectors defined on the sublinear expectation spaces* $(\Omega, \mathcal{H}, \mathbb{E})$*. They are called identically distributed, denoted by* $X_1 \overset{d}{=} X_2$*, if*

$$\mathbb{E}[\varphi(X_1)] = \mathbb{E}[\varphi(X_2)], \quad \forall \varphi \in C_{l.Lip}(\mathbb{R}^d).$$

Definition 2.3 *In a sublinear expectation space* $(\Omega, \mathcal{H}, \mathbb{E})$*, a random vector* $Y \in \mathcal{H}^d$ *is said to be independent of another random vector* $X \in \mathcal{H}^d$ *under* $\mathbb{E}$ *if for each test function* $\varphi \in C_{l.Lip}(\mathbb{R}^{2d})$ *we have*

$$\mathbb{E}[\varphi(X,Y)] = \mathbb{E}\left[\mathbb{E}[\varphi(x,Y)]_{x=X}\right].$$

Definition 2.4 (G-normal distribution) *A* d*-dimensional random vector* $X = (X_1, \ldots, X_d)$ *in a sublinear expectation space* $(\Omega, \mathcal{H}, \mathbb{E})$ *is called G-normal distributed if for each* $a, b > 0$ *we have*

$$aX + b\overline{X} \overset{d}{=} \sqrt{a^2+b^2}X,$$

where $\overline{X}$ *is an independent copy of* X*.*

Given a G-normal random variable $X \in \mathbb{R}^d$, we want to quantify its G-expectation $\mathbb{E}[\varphi(X)]$ for some application φ. Due to the uncertainty in covariance, we do not know how to obtain the samples for X, so the common-used Monte Carlo simulation is inapplicable to the G-expectation. Let $u(t,x) = \mathbb{E}[\varphi(x+\sqrt{t}X)]$, Peng [11] shows that u is the viscosity solution of the following so-called G-equation

$$\begin{cases} \partial_t u - G(D^2 u) = 0, & (t,x) \in (0,\infty) \times \mathbb{R}^d, \\ u(0,x) = \varphi(x), \end{cases} \tag{2.1}$$

where $D^2 u$ is the Hessian matrix of u and

$$\begin{aligned} G(A) &:= \frac{1}{2}\mathbb{E}[\langle AX, X\rangle] \\ &= \frac{1}{2}\sup_{Q\in\Theta} \mathrm{Tr}[AQ], \end{aligned} \tag{2.2}$$

where $A \in \mathbb{S}(d)$, $\mathbb{S}(d)$ denotes the space of $d \times d$ symmetric matrices, and Θ represents the set of all possible symmetric matrices, which is a given bounded, closed, nonnegative-definite, and convex subset of $\mathbb{R}^{d\times d}$. If Θ is a singleton ($\Theta = \{Q\}$), then X is a classical zero-mean normal distributed with covariance Q. In general, Θ characterizes the covariance uncertainty in X, and

$$Q = \begin{pmatrix} \sigma_1^2 & b_{12} & \ldots & b_{1d} \\ b_{21} & \sigma_2^2 & \ldots & b_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ b_{d1} & b_{d2} & \ldots & \sigma_d^2 \end{pmatrix}, \tag{2.3}$$

is a symmetric nonnegative-definite matrix, and $b_{i,j} = b_{j,i}$.

Hence, if we can solve the G-equation (2.1), we get $\mathbb{E}[\varphi(X)] = u(1,0)$.

In this paper, we will only consider two-dimensional problems, that is, $X = (X_1, X_2)$. Equation (2.1) can be rewritten as

$$\begin{cases} u_t - \sup_{Q\in\Theta}\left(\frac{\sigma_1^2}{2}u_{xx}+\frac{\sigma_2^2}{2}u_{yy}+b_{12}u_{xy}\right)=0, \quad (t,x,y)\in(0,\infty)\times\mathbb{R}\times\mathbb{R}, \\ u(0,x,y)=\varphi(x,y), \end{cases} \tag{2.4}$$

where the uncertainty in Q can be identified by testing the proper symmetric matrices A in equation (2.2). Choosing, respectively, in equation (2.2),

$$A_1=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2=\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.5}$$

we can derive the uncertainty in the variance of X_1:

$$\begin{cases} \sup_{\sigma_1\in\Gamma_1}\sigma_1^2=\mathbb{E}\left[X_1^2\right], \\ \inf_{\sigma_1\in\Gamma_1}\sigma_1^2=-\mathbb{E}\left[-X_1^2\right]. \end{cases} \tag{2.6}$$

It is evident that $\sigma_1^2\in\Gamma_1\triangleq[-\mathbb{E}[-X_1^2],\mathbb{E}[X_1^2]]$.

Similarly, by choosing matrices A in equation (2.2) as

$$A_3=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_4=\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.7}$$

we get $\sigma_2^2\in\Gamma_2\triangleq[-\mathbb{E}[-X_2^2],\mathbb{E}[X_2^2]]$. Meanwhile, by choosing specific matrices A as

$$A_5=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_6=\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \tag{2.8}$$

we get the uncertainty in the covariance between X_1 and X_2:

$$\begin{cases} \sup_{b_{12}\in\Gamma_{12}} b_{12}=\mathbb{E}[X_1X_2], \\ \inf_{b_{12}\in\Gamma_{12}} b_{12}=-\mathbb{E}[-X_1X_2]. \end{cases} \tag{2.9}$$

Thus, we obtain $b_{12}\in\Gamma_{12}\triangleq[-\mathbb{E}[-X_1X_2],\mathbb{E}[X_1X_2]]$. In summary, equation (2.1) can be written as follows:

$$\begin{cases} u_t - \sup_{\substack{\sigma_1^2\in\Gamma_1,\sigma_2^2\in\Gamma_2,\\ b_{12}\in\Gamma_{12}}}\left(\frac{\sigma_1^2}{2}u_{xx}+\frac{\sigma_2^2}{2}u_{yy}+b_{12}u_{xy}\right)=0, (t,x,y)\in(0,\infty)\times\mathbb{R}\times\mathbb{R}, \\ u(0,x,y)=\varphi(x,y). \end{cases} \tag{2.10}$$

Specifically, if X_1 is independent of X_2 (or if X_2 is independent of X_1), we have $\mathbb{E}\left[X_1X_2\right]=0$, and $b_{12}=0$. The G-heat equation (2.1) is as follows:

$$\begin{cases} u_t - \sup_{\sigma_1^2\in\Gamma_1,\sigma_2^2\in\Gamma_2}\left(\frac{\sigma_1^2}{2}u_{xx}+\frac{\sigma_2^2}{2}u_{yy}\right)=0, (t,x,y)\in(0,\infty)\times\mathbb{R}\times\mathbb{R}, \\ u(0,x,y)=\varphi\left(x,y\right). \end{cases} \tag{2.11}$$

Remark 2.5 *If there is no uncertainty in the volatility σ_1^2 and σ_2^2, that is, $-\mathbb{E}[-X_1^2]=\mathbb{E}[X_1^2]=\sigma_1^2$ and $-\mathbb{E}[-X_2^2]=\mathbb{E}[X_2^2]=\sigma_2^2$, then we can define the uncertainty in the 'correlation coefficient' as*

$$\rho\in\left[\frac{-\mathbb{E}[-X_1X_2]}{\sigma_1\sigma_2},\frac{\mathbb{E}[X_1X_2]}{\sigma_1\sigma_2}\right].$$

Otherwise, we do not have the concept of the correlation coefficient.

Equations (2.10) and (2.11) are HJB equations of some stochastic control problems, and numerical methods have been extensively investigated when the sign of b_{12} is definite [5, 7–9]. If $\mathbb{E}[-X_1X_2]\cdot\mathbb{E}[X_1X_2]] > 0$, then the sign of b_{12} varies. There does not exist any reliable numerical scheme for this case.

3. Finite difference scheme

In this section, we consider the case where the sign of covariance b_{12} in the G-heat equation (2.10) is uncertain.

$$\begin{cases} u_t - \sup\limits_{\substack{\sigma_1^2\in\Gamma_1,\sigma_2^2\in\Gamma_2,\\ b_{12}\in\Gamma_{12}}} \left(\dfrac{\sigma_1^2}{2}u_{xx} + \dfrac{\sigma_2^2}{2}u_{yy} + b_{12}u_{xy}\right) = 0, (t,x,y)\in(0,\infty)\times\mathbb{R}\times\mathbb{R}, \\ u(0,x,y) = \varphi(x,y), \end{cases} \tag{3.1}$$

where $\Gamma_1 = [\underline{\sigma_1}^2, \overline{\sigma_1}^2]$, $\Gamma_2 = [\underline{\sigma_2}^2, \overline{\sigma_2}^2]$, and $\Gamma_{12} = [\underline{b_{12}}, \overline{b_{12}}]$ and, specially,

$$\underline{b_{12}} < 0 < \overline{b_{12}}. \tag{3.2}$$

Here, we used the notations $\overline{\sigma_i}^2 = \mathbb{E}\left[X_i^2\right]$, $\underline{\sigma_i}^2 = -\mathbb{E}\left[-X_i^2\right]$, and $\overline{b_{12}} = \mathbb{E}[X_1X_2]$, $\underline{b_{12}} = -\mathbb{E}[-X_1X_2]$.

The optimal variance and covariance are achieved at the endpoints of the intervals, depending only on the sign of the second-order partial derivatives, that is,

$$\sigma_1^2(u_{xx}) = \begin{cases} \overline{\sigma_1}^2 & u_{xx}\geqslant 0, \\ \underline{\sigma_1}^2 & u_{xx} < 0, \end{cases} \tag{3.3}$$

$$\sigma_2^2(u_{yy}) = \begin{cases} \overline{\sigma_2}^2 & u_{yy}\geqslant 0, \\ \underline{\sigma_2}^2 & u_{yy} < 0, \end{cases} \tag{3.4}$$

and

$$b_{12}(u_{xy}) = \begin{cases} \overline{b_{12}} & u_{xy}\geqslant 0, \\ \underline{b_{12}} & u_{xy} < 0. \end{cases} \tag{3.5}$$

For computational purpose, we confine the problem (3.1) within a truncated domain,

$$0\leqslant t\leqslant T \text{ and } (x,y)\in\Omega, \quad \Omega = \{(x,y)\,||x|<L,\ |y|<L\},$$

with the Dirichlet boundary condition. Subsequently, the problem is reformulated as

$$\begin{cases} u_t - \dfrac{\sigma_1^2(u_{xx})}{2}u_{xx} - \dfrac{\sigma_2^2(u_{yy})}{2}u_{yy} - b_{12}(u_{xy})u_{xy} = 0,\ (t,x,y)\in(0,T)\times\Omega, \\ u|_{t=0} = \varphi(x,y), \\ u|_{(x,y)\in\partial\Omega} = \phi(t,x,y). \end{cases} \tag{3.6}$$

Remark 3.1 *The Dirichlet boundary condition is imposed. We can expect the errors incurred by imposing approximate boundary conditions to be small in areas of interest if the truncated domain is sufficiently large [2].*

Taking an equi-distance partition with a spatial step size $h = 2L/M$, $\Delta t = T/N$, we have grid points $x_i = -L + i*h$, $y_j = -L + j*h$, $t^n = n\Delta t$, for $i,j = 0,\dots,M$ and $n = 0,\dots,N$.

Let $U_{i,j}^n$ denote the approximate solution at (t^n, x_i, y_j). An implicit scheme for solving equation (3.6) is expressed as, for $n = 0,\dots,N-1$,

$$\begin{cases} \delta_t U_{i,j}^{n+1} - \dfrac{\sigma_1^2\left(\delta_x^2 U_{i,j}^{n+1}\right)}{2}\delta_x^2 U_{i,j}^{n+1} - \dfrac{\sigma_2^2\left(\delta_y^2 U_{i,j}^{n+1}\right)}{2}\delta_y^2 U_{i,j}^{n+1} - (b_{12}\delta_{xy}U)_{i,j}^{n+1} = 0, \quad 0 < i,j < M, \\ U_{i,j}^0 = \varphi(x_i, y_j), \quad i,j = 0,\dots,M, \\ U_{i,j}^{n+1}|_{(x_i,y_j)\in\partial\Omega} = \phi\left(t^{n+1}, x_i, y_j\right), \end{cases} \tag{3.7}$$

where

$$\delta_t U_{i,j}^{n+1} = \frac{U_{i,j}^{n+1} - U_{i,j}^n}{\Delta t},$$

$$\delta_x^2 U_{i,j}^{n+1} = \frac{U_{i+1,j}^{n+1} - 2U_{i,j}^{n+1} + U_{i-1,j}^{n+1}}{h^2}, \ \delta_y^2 U_{i,j}^{n+1} = \frac{U_{i,j+1}^{n+1} - 2U_{i,j}^{n+1} + U_{i,j-1}^{n+1}}{h^2},$$

$$(b_{12}\delta_{xy}U)_{i,j}^{n+1} = \max\left(\overline{b_{12}}\delta_{xy}^+ U_{i,j}^{n+1}, \underline{b_{12}}\delta_{xy}^- U_{i,j}^{n+1}\right), \tag{3.8a}$$

$$\delta_{xy}^+ U_{i,j}^{n+1} = \frac{U_{i+1,j+1}^{n+1} + 2U_{i,j}^{n+1} + U_{i-1,j-1}^{n+1} - \left(U_{i+1,j}^{n+1} + U_{i-1,j}^{n+1} + U_{i,j+1}^{n+1} + U_{i,j-1}^{n+1}\right)}{2h^2}, \tag{3.8b}$$

$$\delta_{xy}^- U_{i,j}^{n+1} = \frac{U_{i+1,j}^{n+1} + U_{i-1,j}^{n+1} + U_{i,j+1}^{n+1} + U_{i,j-1}^{n+1} - \left(U_{i+1,j-1}^{n+1} + 2U_{i,j}^{n+1} + U_{i-1,j+1}^{n+1}\right)}{2h^2}, \tag{3.8c}$$

and

$$\sigma_1^2(s) = \begin{cases} \overline{\sigma_1}^2 & \text{if } s \geqslant 0, \\ \underline{\sigma_1}^2 & \text{if } s < 0, \end{cases} \qquad \sigma_2^2(s) = \begin{cases} \overline{\sigma_2}^2 & \text{if } s \geqslant 0, \\ \underline{\sigma_2}^2 & \text{if } s < 0. \end{cases} \tag{3.9}$$

Remark 3.2 *The approximation of the second-order cross-derivative in* (3.8a) *plays a key role in obtaining a monotone scheme. As is known, whether* $\delta_{xy}^+ U_{i,j}^{n+1}$ *in* (3.8b) *or* $\delta_{xy}^- U_{i,j}^{n+1}$ *in* (3.8c) *should be applied to approximate the second-order cross-derivative depends on the sign of* b_{12}*, but the sign of* b_{12} *depends on the sign of* $\delta_{xy}U_{i,j}^{n+1}$ *in equations* (3.2) *and* (3.5)*. The choice in* (3.8a) *breaks this cycle of dilemmas. When* $\delta_{xy}^+ U_{i,j}^{n+1} < 0$ *and* $\delta_{xy}^- U_{i,j}^{n+1} > 0$*, the choice in* (3.8a) *yields* $b_{12} * \delta_{xy}U_{i,j}^{n+1} < 0$*, which breaks the constraint* $b_{12} * u_{xy} \geqslant 0$ *in* (3.5)*. However, with this choice, the scheme* (3.7) *is monotone and works well. In fact, when* $\delta_{xy}^+ U_{i,j}^{n+1} < 0$ *and* $\delta_{xy}^- U_{i,j}^{n+1} > 0$*, we have* $u_{xy} \approx 0$*, which means that the second-order cross-derivative is ignorable.*

Since equation (3.7) is a nonlinear system, an inner iteration is needed to obtain the solution $U_{i,j}^n$ at each time step. Let $U_{i,j}^{n+1,k}$ be the k^{th} estimate of $U_{i,j}^{n+1}$. $U_{i,j}^{n+1,k+1}$ is given by the following Picard's iteration:

$$\begin{cases} \delta_t U_{i,j}^{n+1,k+1} - \dfrac{\sigma_1^2\left(\delta_x^2 U_{i,j}^{n+1,k}\right)}{2}\delta_x^2 U_{i,j}^{n+1,k+1} - \dfrac{\sigma_2^2\left(\delta_y^2 U_{i,j}^{n+1,k}\right)}{2}\delta_y^2 U_{i,j}^{n+1,k+1} - (b_{12})_{i,j}^k\,\delta_{xy}^{\alpha_k} U_{i,j}^{n+1,k+1} = 0, \\ U_{i,j}^0 = \varphi(x_i, y_j), \\ U_{i,j}^{n+1,k+1}|_{(x_i,y_j)\in\partial\Omega} = \phi\left(t^{n+1}, x_i, y_j\right), \end{cases} \tag{3.10}$$

with $U_{i,j}^{n+1,0} = U_{i,j}^n$, where

$$(b_{12})_{i,j}^k = \begin{cases} \overline{b_{12}}, & \text{if } \overline{b_{12}}\delta_{xy}^+ U_{i,j}^{n+1,k} \geqslant \underline{b_{12}}\delta_{xy}^- U_{i,j}^{n+1,k}, \\ \underline{b_{12}}, & \text{if } \overline{b_{12}}\delta_{xy}^+ U_{i,j}^{n+1,k} < \underline{b_{12}}\delta_{xy}^- U_{i,j}^{n+1,k}, \end{cases} \tag{3.11}$$

and

$$\alpha_k = \begin{cases} + & \text{if } (b_{12})^k = \overline{b_{12}}, \\ - & \text{if } (b_{12})^k = \underline{b_{12}}. \end{cases} \tag{3.12}$$

In the next section, we discuss the convergence of the iterative scheme (3.10) and some theoretical convergence issues of the discrete scheme (3.7).

4. Numerical analysis

The implicit scheme (3.10) leads to a nonlinear algebraic system which must be solved by an inner iteration at each time step. In this section, we first prove the convergence of the inner iteration and then check the properties of consistency, stability, and monotonicity.

4.1 Convergence of inner iteration

We first present an assumption, then prove the convergence of the inner iteration.

Assumption 4.1 *The covariance matrix* $\begin{pmatrix} \sigma_1^2 & b_{12} \\ b_{12} & \sigma_2^2 \end{pmatrix}$ *is a diagonally dominated, where* $\sigma_1^2 \in \left\{\underline{\sigma_1}^2, \overline{\sigma_1}^2\right\}$, $\sigma_2^2 \in \left\{\underline{\sigma_2}^2, \overline{\sigma_2}^2\right\}$ *and* $b_{12} \in \left\{\underline{b_{12}}, \overline{b_{12}}\right\}$.

Proposition 4.2 (Maximum principle) *If the covariance matrix* $\begin{pmatrix} \sigma_1^2 & b_{12} \\ b_{12} & \sigma_2^2 \end{pmatrix}$ *is diagonally dominated, and* $\{U_{i,j}^n\}$ *satisfies*

$$\mathbb{L}_h U_{i,j}^n \equiv \delta_t U_{i,j}^n - \frac{\sigma_1^2}{2}\delta_x^2 U_{i,j}^n - \frac{\sigma_2^2}{2}\delta_y^2 U_{i,j}^n - b_{12}\delta_{xy}^{\alpha} U_{i,j}^n \geqslant 0 (\leqslant 0),\ n = 1, \ldots, N,\ 0 < i, j < M, \tag{4.1}$$

where

$$\alpha = \begin{cases} + & if\ b_{12} \geqslant 0, \\ - & if\ b_{12} < 0, \end{cases} \tag{4.2}$$

then the minimum (maximum) of $\{U_{i,j}^n\}$ *can only be achieved at the initial or boundary points, unless* $\{U_{i,j}^n\}$ *is constant.*

Reformulate the right-hand side of the system (4.1) into an operator form as

$$\mathbb{L}_h U^n = \left(\frac{1}{\Delta t} I - \frac{\sigma_1^2}{2}\delta_x^2 - \frac{\sigma_2^2}{2}\delta_y^2 - b_{12}\delta_{xy}^{\alpha}\right) U^n - \frac{1}{\Delta t} U^{n-1} \equiv A U^n - \frac{1}{\Delta t} U^{n-1}.$$

It is easy to check that, owing to the appropriate choice of the approximation of the second-order cross-derivative, $A = (a_{ij})$ is an M-matrix, that is, $a_{ii} > 0$, $a_{ij} \leqslant 0$, for $i \neq j$, and $\sum_{j \neq i} |a_{ij}| < a_{ii}$. Then, the above proposition follows.

Based on the maximum principle established above, we now prove that the nonlinear inner iteration defined in equation (3.10) converges monotonically to the unique solution of the fully implicit scheme (3.7).

Theorem 4.3 (Convergence of inner iteration) *If Assumption 4.1 holds, then for any initial guess* $U^{n+1,0}$, *the iterative sequence* $\{U^{n+1,k}\}_{k>0}$ *in equation* (3.10) *is bounded and monotonically increasing, and hence converges to the unique solution of* (3.7).

Proof Without loss of generality (WLOG), we assume that there is no uncertainty in σ_1^2 and σ_2^2, and pay more attentions to the second-order cross-derivative. Denote by $\overline{U}^k = U^{n+1,k}$ and $W^k = \overline{U}^{k+1} - \overline{U}^k$, where $k \geqslant 1$. Then, $W_{i,j}^k$ satisfies the following difference equation:

$$\begin{cases} \dfrac{W_{i,j}^k}{\Delta t} - \dfrac{\sigma_1^2}{2}\delta_x^2 W_{i,j}^k - \dfrac{\sigma_2^2}{2}\delta_y^2 W_{i,j}^k - \left((b_{12})_{i,j}^k \delta_{xy}^{\alpha_k} \overline{U}_{i,j}^{k+1} - (b_{12})_{i,j}^{k-1} \delta_{xy}^{\alpha_{k-1}} \overline{U}_{i,j}^k\right) = 0, \\ W_{i,j}^k|_{(x_i,y_j)\in\partial\Omega} = 0. \end{cases} \tag{4.3}$$

If $(b_{12})_{i,j}^k = (b_{12})_{i,j}^{k-1}$, for example, it equals to $\overline{b_{12}}$, we have

$$\frac{W_{i,j}^k}{\Delta t} - \frac{\sigma_1^2}{2}\delta_x^2 W_{i,j}^k - \frac{\sigma_2^2}{2}\delta_y^2 W_{i,j}^k - \overline{b_{12}}\delta_{xy}^+ W_{i,j}^k = 0. \tag{4.4}$$

If $(b_{12})_{i,j}^k \neq (b_{12})_{i,j}^{k-1}$, for example,

$$(b_{12})_{i,j}^k = \underline{b_{12}}, \quad (b_{12})_{i,j}^{k-1} = \overline{b_{12}},$$

which means that, from (3.11), $\underline{b_{12}}\delta_{xy}^- \overline{U}_{ij}^k \geqslant \overline{b_{12}}\delta_{xy}^+ \overline{U}_{ij}^k$, then we have

$$\frac{W_{i,j}^k}{\Delta t} - \frac{\sigma_1^2}{2}\delta_x^2 W_{i,j}^k - \frac{\sigma_2^2}{2}\delta_y^2 W_{i,j}^k - \underline{b_{12}}\delta_{xy}^- W_{i,j}^k = \underline{b_{12}}\delta_{xy}^- \overline{U}_{ij}^k - \overline{b_{12}}\delta_{xy}^+ \overline{U}_{ij}^k \geqslant 0. \tag{4.5}$$

Otherwise, if

$$(b_{12})_{i,j}^k = \overline{b_{12}}, \quad (b_{12})_{i,j}^{k-1} = \underline{b_{12}},$$

which means that, from (3.11), $\overline{b_{12}}\delta_{xy}^+ \overline{U}_{ij}^k \geqslant \underline{b_{12}}\delta_{xy}^- \overline{U}_{ij}^k$, then we have

$$\frac{W_{i,j}^k}{\Delta t} - \frac{\sigma_1^2}{2}\delta_x^2 W_{i,j}^k - \frac{\sigma_2^2}{2}\delta_y^2 W_{i,j}^k - \overline{b_{12}}\delta_{xy}^+ W_{i,j}^k = \overline{b_{12}}\delta_{xy}^+ \overline{U}_{ij}^k - \underline{b_{12}}\delta_{xy}^- \overline{U}_{ij}^k \geqslant 0. \tag{4.6}$$

Combining (4.4)–(4.6), we have, for any $0 < i, j < M$,

$$\frac{W_{i,j}^k}{\Delta t} - \frac{\sigma_1^2}{2}\delta_x^2 W_{i,j}^k - \frac{\sigma_2^2}{2}\delta_y^2 W_{i,j}^k - (b_{12})_{i,j}^k \delta_{xy}^{\alpha_k} W_{i,j}^k \geqslant 0,$$

subject to the boundary condition $W_{i,j}^k = 0$, for $(x_i, y_j) \in \partial\Omega$. From the maximum principle, we have $W_{i,j}^k \geqslant 0$, for $0 < i, j < M$, that is, $\overline{U}_{i,j}^{k+1} \geqslant \overline{U}_{i,j}^k$, for $k \geqslant 1$.

Therefore, $\{\overline{U}^k\}_{k>0}$ is a monotonic increasing sequence. Next, we check the boundedness of the sequence. From Proposition 4.2, the maximum principle is valid for the system (3.10). It follows that

$$\left|\left|\overline{U}^{k+1}\right|\right|_\infty \leqslant \max\left(||\varphi||_\infty, \max_n ||\phi^n||_{\infty,\partial\Omega}\right). \tag{4.7}$$

Consequently, as a monotonic and bounded sequence, $\overline{U}^k$ converges. □

4.2 Monotonicity and convergence of the implicit scheme (3.7)

From the work of Barles and Souganidis [3], we know that numerical solution of (3.7) converges to the viscosity solution of the fully nonlinear PDE (3.6) if the scheme (3.7) is consistent, stable (in the l_∞ norm) and monotone.

To prove convergence to the viscosity solution, we first verify the consistency of the proposed scheme.

Lemma 4.4 (Consistency) *The implicit scheme* (3.7) *is consistent.*

Proof It is easy to check that the difference equation in (3.7) tends to the G-equation (3.1) as $h, \Delta t \to 0$, since the 'sup' operation is continuous. □

We now give the definition of monotonicity. Denote it by

$$g_{i,j} = g\left(U_{i,j}^{n+1}, U_{i,j}^{n}, \{U_{k,l}^{n+1}\}_{(k,l)\in N_{i,j}}\right),$$

the left-hand side of the difference equation (3.7), where $N_{i,j} = \{(k,l) \neq (i,j) : |k-i| \leqslant 1, |l-j| \leqslant 1\}$ represents the set of all nearest-neighbor indexes of (i,j).

Definition 4.5 (Monotonicity) *The scheme* (3.7) *is monotone if for all* (i,j)*,*

$$\begin{aligned} &g_{i,j}\left(U_{i,j}^{n+1} + \epsilon_{i,j}^{n+1}, U_{i,j}^{n}, \{U_{k,l}^{n+1}\}_{(k,l)\in N_{i,j}}\right) \\ &\geqslant g_{i,j}\left(U_{i,j}^{n+1}, U_{i,j}^{n}, \{U_{k,l}^{n+1}\}_{(k,l)\in N_{i,j}}\right), \quad \forall \epsilon_{i,j}^{n+1} \geqslant 0, \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} &g_{i,j}\left(U_{i,j}^{n+1}, U_{i,j}^{n} + \epsilon_{i,j}^{n}, \{U_{k,l}^{n+1} + \epsilon_{k,l}^{n+1}\}_{(k,l)\in N_{i,j}}\right) \\ &\leqslant g_{i,j}\left(U_{i,j}^{n+1}, U_{i,j}^{n}, \{U_{k,l}^{n+1}\}_{(k,l)\in N_{i,j}}\right), \quad \forall \epsilon_{i,j}^{n}, \epsilon_{k,l}^{n+1} \geqslant 0. \end{aligned} \tag{4.9}$$

When the sign of the correlation in the equation is uncertain, proving the monotonicity of the scheme becomes challenging. The following lemma shows that, under Assumption 4.1, the proposed implicit scheme preserves monotonicity.

Lemma 4.6 (Monotonicity) *If Assumption 4.1 holds, then the implicit scheme* (3.7) *is monotone.*

Proof Considering that the proof process is quite technical and lengthy, we have included the detailed proof in Appendix A. □

We next establish an l^{∞}-stability result, which follows naturally from the discrete maximum principle.

Lemma 4.7 (Stability) *If Assumption 4.1 holds, then the fully implicit scheme* (3.7) *is stable, in the sense that*

$$\max_n ||U^n||_\infty \leqslant \max\left(||\varphi||_\infty, \max_n ||\phi^n||_{\infty,\partial\Omega}\right). \tag{4.10}$$

Proof Under the diagonal-dominance assumption, the maximum principle is valid for the system (3.7). The estimate on l_∞ (4.10) follows directly. □

We also have the following comparison principle.

Lemma 4.8 (Comparison Principle) *If Assumption 4.1 holds,* $\{U_{i,j}^{n+1}\}$ *satisfies equation* (3.7)*,* $\{V_{i,j}^{n+1}\}$ *satisfies the same difference equation*

$$\delta_t V_{i,j}^{n+1} - \frac{\sigma_1^2\left(\delta_x^2 V_{i,j}^{n+1}\right)}{2}\delta_x^2 V_{i,j}^{n+1} - \frac{\sigma_2^2\left(\delta_y^2 V_{i,j}^{n+1}\right)}{2}\delta_y^2 V_{i,j}^{n+1} - (b_{12}\delta_{xy}V)_{i,j}^{n+1} = 0,$$

but subject to different initial and boundary values such that

$$\begin{cases} U_{i,j}^{0} \geqslant V_{i,j}^{0}, \\ U_{i,j}^{n+1}|_{(x_i,y_j)\in\partial\Omega} \geqslant V_{i,j}^{n+1}|_{(x_i,y_j)\in\partial\Omega}, \end{cases}$$

then $U_{i,j}^{n+1} \geqslant V_{i,j}^{n+1}$ *for all* $i,j = 1,\ldots,M-1$ *and* $n < N$*.*

Proof This lemma can be proved using the techniques in the proof of Theorem 4.3 and Lemma 4.6 by applying the maximum principle to the governing system for $W_{i,j}^n := U_{i,j}^n - V_{i,j}^n$. □

Having established consistency, monotonicity, and stability, we can now prove the convergence of the numerical scheme to the viscosity solution of the G-heat equation.

Theorem 4.9 (Convergence to the viscosity solution) *Let Assumption 4.1 hold, then the implicit scheme* (3.7) *converges to the viscosity solution of the nonlinear PDE* (3.6).

Proof Since the scheme (3.7) is consistent, l_∞-stable, and monotone, the convergence follows from the results of Barles and Souganidis [3] directly. □

Remark 4.10 *If Assumption 4.1 does not hold, we can also use a wide stencil (see Ma and Forsyth [8]) to construct a numerical scheme with relaxed conditions, and we leave it to future research.*

In addition to convergence, we derive an explicit rate of convergence for the numerical scheme under additional regularity assumptions on the viscosity solution.

Theorem 4.11 (Rate of convergence) *Let u be the viscosity solution of equation* (3.6), *U be the numerical solution of equation* (3.7). *If Assumption 4.1 holds and there exists some $\beta \in (0,1)$, such that $u \in C^{1+\beta/2,2+\beta}([0,T]\times\Omega)$, then*

$$\max_n ||u^n - U^n||_\infty \leqslant C\left(\Delta t^{\frac{\beta}{2}} + h^\beta\right), \tag{4.11}$$

where C is a positive constant independent of Δt and h.

Proof Approximating the derivatives by corresponding difference quotients in (3.6), we obtain

$$\begin{aligned} 0 = \delta_t u_{i,j}^n + R_t^n - \sup_{\sigma_1^2\in\Gamma_1}\left(\frac{\sigma_1^2}{2}\left(\delta_x^2 u_{i,j}^n + R_x^n\right)\right) - \sup_{\sigma_2^2\in\Gamma_2}\left(\frac{\sigma_2^2}{2}\left(\delta_y^2 u_{i,j}^n + R_y^n\right)\right) \\ - \max\{\overline{b_{12}}\left(\delta_{xy}^+ u_{i,j}^n + R_{xy}^+\right), \underline{b_{12}}\left(\delta_{xy}^- u_{i,j}^n + R_{xy}^-\right)\}, \end{aligned} \tag{4.12}$$

since $\sup_{b_{12}\in\Gamma_{12}}(b_{12}u_{xy}) = \max\{\overline{b_{12}}u_{xy},\ \underline{b_{12}}u_{xy}\}$. Here, we have the following truncation error terms:

$$R_t^n = u_t^n - \delta_t u^n = O((\Delta t)^{\beta/2}), \quad R_x^n = u_{xx}^n - \delta_x^2 u^n = O(h^\beta), \quad R_y^n = u_{yy}^n - \delta_y^2 u^n = O(h^\beta),$$

and

$$R_{xy}^+ = u_{xy} - \delta_{xy}^+ u = O(h^\beta), \quad R_{xy}^- = u_{xy} - \delta_{xy}^- u = O(h^\beta).$$

Owing to $\sup(f+g) \leqslant \sup f + \sup g$, we have

$$\delta_t u_{i,j}^n - \sup_{\sigma_1^2\in\Gamma_1}\left(\frac{\sigma_1^2}{2}\delta_x^2 u_{i,j}^n\right) - \sup_{\sigma_2^2\in\Gamma_2}\left(\frac{\sigma_2^2}{2}\delta_y^2 u_{i,j}^n\right) - \max\{\overline{b_{12}}\delta_{xy}^+ u_{i,j}^n, \underline{b_{12}}\delta_{xy}^- u_{i,j}^n\} \leqslant R_{up}^n, \tag{4.13}$$

where

$$R_{up}^n = -R_t^n + \sup_{\sigma_1^2\in\Gamma_1}\left(\frac{\sigma_1^2}{2}R_x^n\right) + \sup_{\sigma_2^2\in\Gamma_2}\left(\frac{\sigma_2^2}{2}R_y^n\right) + \max\{\overline{b_{12}}R_{xy}^+, \underline{b_{12}}R_{xy}^-\}. \tag{4.14}$$

Given $\sup(f+g) \geqslant \sup f + \inf g$, we have

$$\delta_t u_{i,j}^n - \sup_{\sigma_1^2\in\Gamma_1}\left(\frac{\sigma_1^2}{2}\delta_x^2 u_{i,j}^n\right) - \sup_{\sigma_2^2\in\Gamma_2}\left(\frac{\sigma_2^2}{2}\delta_y^2 u_{i,j}^n\right) - \max\{\overline{b_{12}}\delta_{xy}^+ u_{i,j}^n, \underline{b_{12}}\delta_{xy}^- u_{i,j}^n\} \geqslant R_{low}^n, \tag{4.15}$$

where

$$R_{low}^n = -R_t^n + \inf_{\sigma_1^2\in\Gamma_1}\left(\frac{\sigma_1^2}{2}R_x^n\right) + \inf_{\sigma_2^2\in\Gamma_2}\left(\frac{\sigma_2^2}{2}R_y^n\right) + \min\{\overline{b_{12}}R_{xy}^+, \underline{b_{12}}R_{xy}^-\}. \tag{4.16}$$

Let $V_{i,j}^n = u_{i,j}^n - U_{i,j}^n - t^n * \max_n \|R_{up}^n\|_\infty$. Then by the fact $\sup(f-g) \geqslant \sup f - \sup g$, we have the following for $0 < i, j < M$:

$$
\begin{aligned}
&\delta_t V_{i,j}^n - \sup_{\sigma_1^2\in\Gamma_1}\left(\frac{\sigma_1^2}{2}\delta_x^2 V_{i,j}^n\right) - \sup_{\sigma_2^2\in\Gamma_2}\left(\frac{\sigma_2^2}{2}\delta_y^2 V_{i,j}^n\right) - \max\left\{\overline{b_{12}}\delta_{xy}^+ V_{i,j}^n,\ \underline{b_{12}}\delta_{xy}^- V_{i,j}^n\right\}\\
&= \delta_t(u_{i,j}^n - U_{i,j}^n) - \sup_{\sigma_1^2\in\Gamma_1}\left(\frac{\sigma_1^2}{2}\delta_x^2(u_{i,j}^n - U_{i,j}^n)\right) - \sup_{\sigma_2^2\in\Gamma_2}\left(\frac{\sigma_2^2}{2}\delta_y^2(u_{i,j}^n - U_{i,j}^n)\right)\\
&\quad - \max\left\{\overline{b_{12}}\delta_{xy}^+(u_{i,j}^n - U_{i,j}^n),\ \underline{b_{12}}\delta_{xy}^-(u_{i,j}^n - U_{i,j}^n)\right\} - \max_n \|R_{up}^n\|_\infty\\
&\leqslant \delta_t u_{i,j}^n - \sup_{\sigma_1^2\in\Gamma_1}\left(\frac{\sigma_1^2}{2}\delta_x^2 u_{i,j}^n\right) - \sup_{\sigma_2^2\in\Gamma_2}\left(\frac{\sigma_2^2}{2}\delta_y^2 u_{i,j}^n\right) - \max\left\{\overline{b_{12}}\delta_{xy}^+ u_{i,j}^n,\ \underline{b_{12}}\delta_{xy}^- u_{i,j}^n\right\}\\
&\quad - \max_n \|R_{up}^n\|_\infty - \Bigg(\delta_t U_{i,j}^n - \sup_{\sigma_1^2\in\Gamma_1}\left(\frac{\sigma_1^2}{2}\delta_x^2 U_{i,j}^n\right) - \sup_{\sigma_2^2\in\Gamma_2}\left(\frac{\sigma_2^2}{2}\delta_y^2 U_{i,j}^n\right)\\
&\quad - \max\left\{\overline{b_{12}}\delta_{xy}^+ U_{i,j}^n,\ \underline{b_{12}}\delta_{xy}^- U_{i,j}^n\right\}\Bigg)\\
&\leqslant R_{up}^n - \max_n \|R_{up}^n\|_\infty \leqslant 0,
\end{aligned}
\tag{4.17}
$$

where we have used (3.7) and (4.13). Owing to the initial condition $V_{i,j}^0 = 0$ and the boundary condition $V_{i,j}^n|_{(x_i,y_j)\in\partial\Omega} = -t^n * \max_n \|R_{up}^n\|_\infty$, we have, by the maximum principle Proposition (4.2),

$$V_{i,j}^n \leqslant 0, \quad \text{for } 0 \leqslant i, j \leqslant M, \quad 0 \leqslant n \leqslant N.$$

That is,

$$u_{i,j}^n - U_{i,j}^n \leqslant t^n * \max_n \|R_{up}^n\|_\infty. \tag{4.18}$$

Similarly, let $W_{i,j}^n = U_{i,j}^n - u_{i,j}^n - t^n * \max_n \|R_{low}^n\|_\infty$. Then, from (4.15), we have

$$U_{i,j}^n - u_{i,j}^n \leqslant t^n * \max_n \|R_{low}^n\|_\infty. \tag{4.19}$$

Hence, we finally have

$$\max_n \|u^n - U^n\|_\infty \leqslant T * \max_n(\|R_{up}^n\|_\infty + \|R_{low}^n\|_\infty) \leqslant C((\Delta t)^{\beta/2} + h^\beta).$$

□

Remark 4.12 *In Theorem 4.11, we assume that the solution is $u \in C^{1+\beta/2,\,2+\beta}([0,T]\times\Omega)$. It is well known that for the nonlinear uniformly G-heat equation, the viscosity solution exhibits the interior regularity $u \in C^{1+\beta/2,\,2+\beta}([\varepsilon,T]\times\mathbb{R}^2)$, where $0 < \epsilon < T$ and $\beta \in (0,1)$, see, for example, Krylov [15] and Peng [11]. The possible loss of regularity at $t = 0$ is a standard phenomenon in parabolic problems and is related to the incompatibility between the initial and boundary data at the parabolic boundary. If the initial and Dirichlet boundary conditions are compatible at $t = 0$ and the data are sufficiently smooth, then no initial layer is generated near $t = 0$. In this case, classical parabolic regularity theory implies that the solution remains smooth up to $t = 0$, and hence $u \in C^{1+\beta/2,\,2+\beta}([0,T]\times\Omega)$. Such compatibility conditions and the resulting boundary regularity for fully nonlinear parabolic equations can be found, for instance, in Krylov [14, 15].*

5. Numerical examples

In this section, we present some numerical examples to show the efficiency of our numerical scheme.

Example 5.1 *The following problem has an exact solution* $u = \sin(5(x+y+t))$.

$$\begin{cases} u_t - \max\limits_{\substack{\sigma_1^2\in\Gamma_1,\sigma_2^2\in\Gamma_2,\\ b_{12}\in\Gamma_{12}}} \left(\dfrac{\sigma_1^2}{2}u_{xx} + \dfrac{\sigma_2^2}{2}u_{yy} + b_{12}u_{xy}\right) = f, \\ u|_{t=0} = \sin(5(x+y)), \\ u|_{(x,y)\in\partial\Omega} = \sin(5(x+y+t))|_{(x,y)\in\partial\Omega}, \end{cases}$$

where $t \in (0,1)$, $\Omega = (-1,1)\times(-1,1)$, $\sigma_1 \in [0.2, 0.3]$, $\sigma_2 \in [0.25, 0.35]$, $b_{12} \in [-0.04, 0.03]$, *and*

$$f = 5\cos(w) + 25 \min_{\substack{\sigma_1^2\in\Gamma_1,\sigma_2^2\in\Gamma_2,\\ b_{12}\in\Gamma_{12}}} \left(\frac{\sigma_1^2}{2}\sin(w) + \frac{\sigma_2^2}{2}\sin(w) + b_{12}\sin(w)\right),$$

with $w = 5(x+y+t)$.

Due to the high regularity of the solutions of this equation, it is theoretically not difficult to derive the following error estimation between the numerical and exact solutions:

$$||u^n - U^n||_\infty \leqslant O\left(\Delta t + h^2\right).$$

For sufficiently smooth solutions, we assess the convergence behavior by refining the mesh in the standard parabolic scaling, namely by doubling the number of spatial grid points while refining the time step by a factor of four. Under this refinement strategy, from Table 1, the numerical results clearly demonstrate first-order accuracy in time and second-order accuracy in space in terms of the norm of $\mathcal{L}^\infty(0,1;\Omega)$ and $\mathcal{L}^2(0,1;\Omega)$.

Table 1 Error accuracy

Time steps	Nodes	$\mathcal{L}^\infty(0,1;\Omega)$		$\mathcal{L}^2(0,1;\Omega)$	
		Error	Error order	Error	Error order
50	11 × 11	1.9013e-01		1.6062e-01	
200	21 × 21	5.1659e-02	1.8799	4.3689e-02	1.8783
800	41 × 41	1.3075e-02	1.9822	1.1067e-02	1.9810
3200	81 × 81	3.2597e-03	2.0040	2.7594e-03	2.0038

In Figure 1, we present the number of inner iterations at each time step. The number of iterations per time step varies only from 3 to 5, demonstrating the high efficiency of the implicit numerical algorithm.

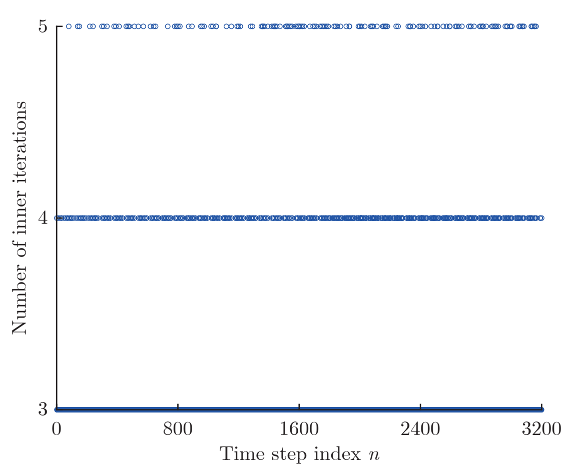

Figure 1 Number of inner iterations per time step in Example 5.1

Example 5.2 *Consider a problem as follows:*

$$\begin{cases} u_t - \max\limits_{\substack{\sigma_1^2\in\Gamma_1,\sigma_2^2\in\Gamma_2,\\ b_{12}\in\Gamma_{12}}} \left(\dfrac{\sigma_1^2}{2}u_{xx} + \dfrac{\sigma_2^2}{2}u_{yy} + b_{12}u_{xy}\right) = 0, \\ u|_{t=0} = \sin(5(x+y)), \\ u|_{(x,y)\in\partial\Omega} = \sin(5(x+y+t))|_{(x,y)\in\partial\Omega}, \end{cases}$$

where the parameter settings are identical to those in Example 5.1. A reference "exact" solution is taken as the numerical solution on a sufficiently fine grid (time step $\Delta t = 1/5000$, space step $h = 1/180$).

Table 2 shows that the error order between the numerical solution and the exact solution is approximately first-order in time and second-order in space in terms of the norms of $\mathcal{L}^\infty(0,1;\Omega)$ and $\mathcal{L}^2(0,1;\Omega)$. It should be noted that the error order is higher than our theoretical results, which is a very interesting phenomenon.

Table 2 Error accuracy

Time steps	Nodes	$\mathcal{L}^\infty(0,1;\Omega)$		$\mathcal{L}^2(0,1;\Omega)$	
		Error	Error order	Error	Error order
50	11 × 11	2.3641e-01		1.4388e-01	
200	21 × 21	9.0651e-02	1.3829	4.9501e-02	1.5393
800	41 × 41	2.8399e-02	1.6475	1.3452e-02	1.8796
3200	81 × 81	7.3077e-03	1.9584	3.3629e-03	2.000

We also present the number of inner iterations in each time step. Figure 2 shows that the number of iterations per time step typically ranges from 3 to 4, which also indicates the high efficiency of our implicit numerical algorithm. The changes in volatilities σ_1, σ_2 and covariance b_{12} over time are illustrated in Figures 3, 4 and 5, respectively, on the mesh with 3200 time steps and 81×81 spatial grid points. From the perspective of stochastic control, the parameters $\sigma_1 \in [\underline{\sigma_1}, \overline{\sigma_1}]$, $\sigma_2 \in [\underline{\sigma_2}, \overline{\sigma_2}]$ and $b_{12} \in [\underline{b_{12}}, \overline{b_{12}}]$ act as control variables in the associated HJB equation. The numerical results in Figures 3–5 show that the optimal controls take values almost exclusively at the extreme points of the admissible intervals. This indicates that the optimal control strategy is of the *bang–bang* type.

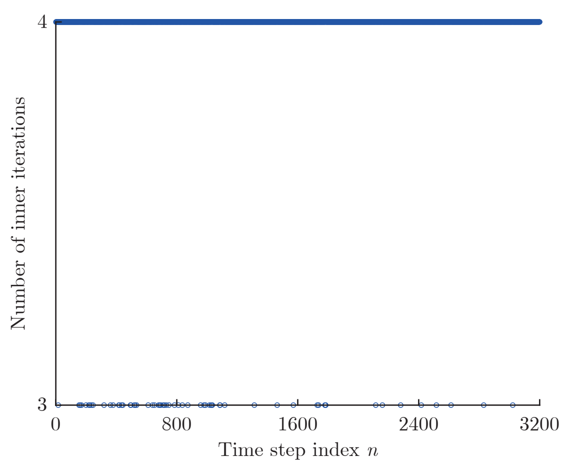

Figure 2 Number of inner iterations per time step in Example 5.2

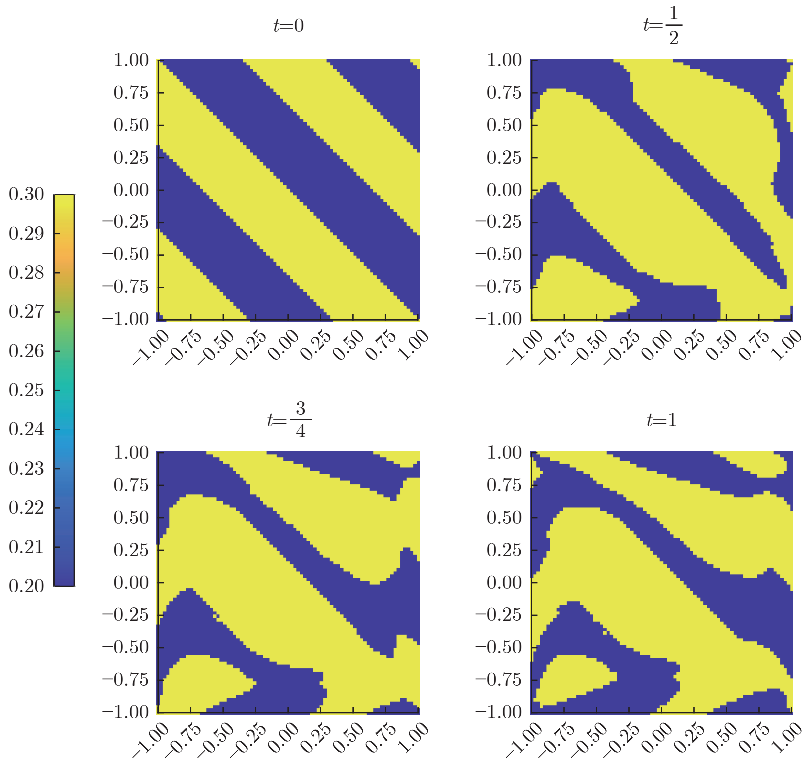

Figure 3 Graph of the variation of σ_1 over time

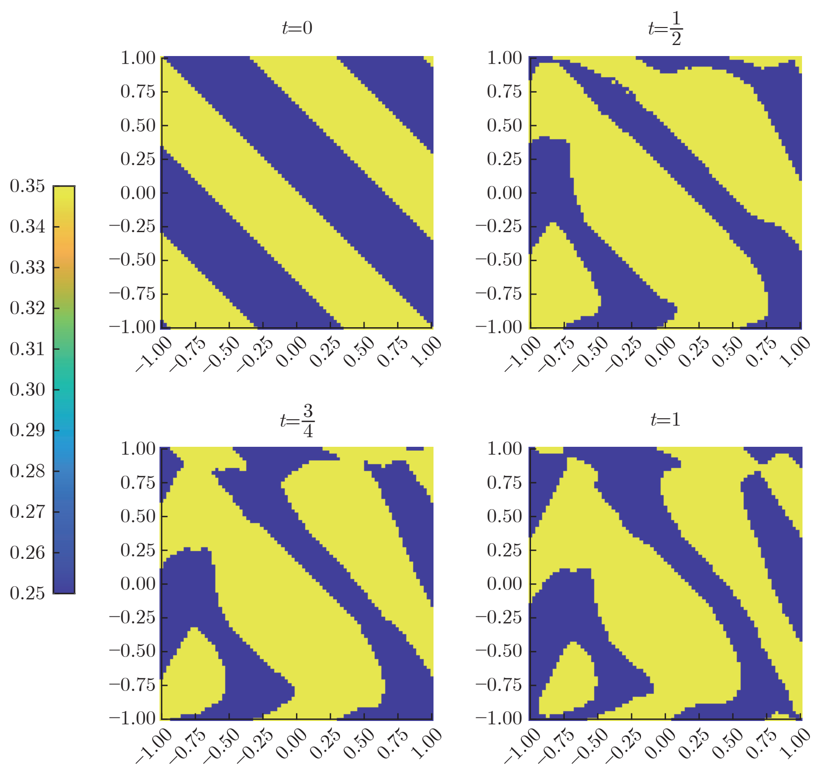

Figure 4 Graph of the variation of σ_2 over time

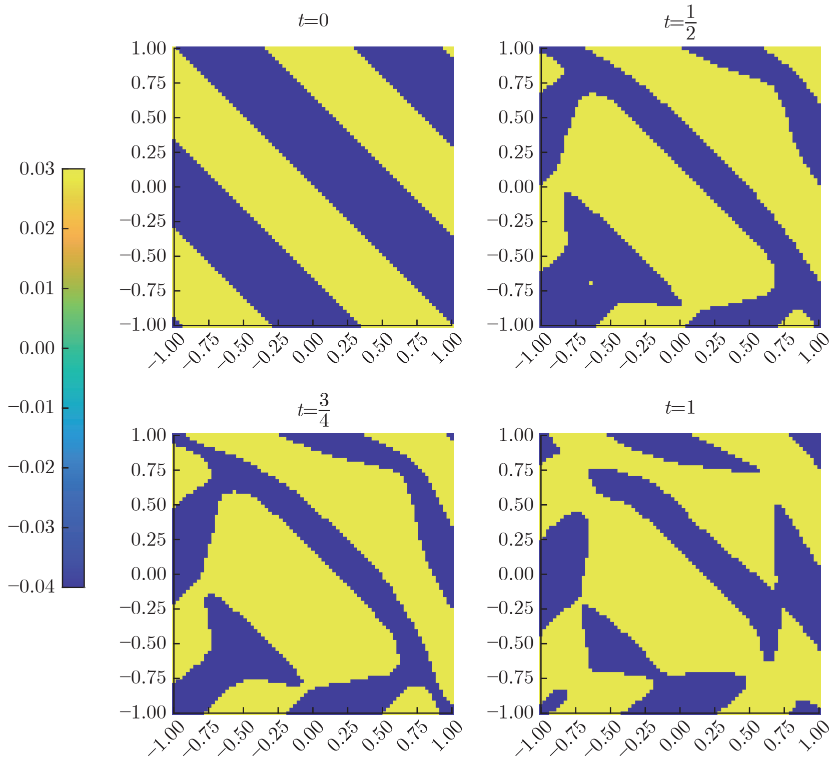

Figure 5 The graph of the variation of b_{12} over time

Example 5.3 *Influence of boundary conditions on the computation of G-expectation. Given a 2-dimensional G-normal random variable* X *and the test function* $\varphi(x,y) = sin(5(x+y))$*, the G-expectation* $\mathbb{E}[\varphi(X)]$ *can be obtained by solving the G-heat equation, with* $\mathbb{E}[\varphi(X)] = u(1,0,0)$*. We investigate the influence of boundary conditions on the numerical evaluation of the G-expectation for the standard two-dimensional G-equation. The problem is solved in a sufficiently large truncated domain* $\Omega = [-8,8] \times [-8,8]$ *with* $t \in [0,1]$ *using the fully implicit scheme* (3.7)*. We compute* $u(1,0,0)$ *under several boundary conditions, including different Dirichlet conditions and a homogeneous Neumann condition. The numerical results are reported in Table 3. Although the implicit discretization leads to a globally coupled linear system and boundary conditions affect the solution globally, their influence on the localized quantity* $u(1,0,0)$ *is expected to be small when the computational domain is large (*$N = 160,\ M = 400$*). This justifies the use of simple Dirichlet boundary conditions in* (3.6).

Table 3 Numerical values of $u(1,0,0)$ under different boundary conditions

Boundary condition	$u(1,0,0)$
$\phi = \sin(5(x+y+t))$	0.480511319599
$\phi = \cos(5(x+y+t))$	0.480511319597
$\phi = 0$	0.480511319605
$\phi = 1$	0.480511319582
Neumann: $\partial u/\partial n = 0$	0.480511318208

6. Conclusions

We have developed an implicit discretization scheme to solve the general two-dimensional

G-heat equation, which particularly addresses cases where the sign of the correlation is uncertain. First, we identified the uncertainties ranges of the covariance of a G-normal random variable and determined the corresponding G-heat equation. Then, we proved the monotonic convergence of the nonlinear inner iterations at each time step and demonstrated the consistency, stability, and monotonicity of the numerical scheme. Furthermore, we provided an estimate of the convergence order. Finally, we presented some numerical examples and showed that although the implicit numerical algorithm involves inner iterations at each time step, it remains highly efficient with a computational complexity about 3–4 times that of solving linear expectations.

A. Proof of Monotonicity

Proof of Lemma 4.6 We first consider a perturbation in $U_{i,j}^{n+1}$, and denote it by $\widetilde{U}_{i,j}^{n+1} = U_{i,j}^{n+1} + \epsilon_{i,j}^{n+1}$, for $\epsilon_{i,j}^{n+1} \geqslant 0$. We also denote it by $\widetilde{U}_{k,l}^{n+1} = U_{k,l}^{n+1}$, for $(k,l) \in N_{i,j}$. Then, the difference between the two sides of the inequality (4.8) is

$$\begin{aligned}
T &= g_{i,j}\left(\widetilde{U}_{i,j}^{n+1}, U_{i,j}^{n}, \{\widetilde{U}_{k,l}^{n+1}\}_{(k,l)\in N_{i,j}}\right) - g_{i,j}\left(U_{i,j}^{n+1}, U_{i,j}^{n}, \{U_{k,l}^{n+1}\}_{(k,l)\in N_{i,j}}\right) \\
&= \frac{\epsilon_{i,j}^{n+1}}{\Delta t} - \left(\frac{\widetilde{\sigma}_1^2}{2}\delta_x^2 \widetilde{U}_{i,j}^{n+1} - \frac{\widehat{\sigma}_1^2}{2}\delta_x^2 U_{i,j}^{n+1}\right) - \left(\frac{\widetilde{\sigma}_2^2}{2}\delta_y^2 \widetilde{U}_{i,j}^{n+1} - \frac{\widehat{\sigma}_2^2}{2}\delta_y^2 U_{i,j}^{n+1}\right) \\
&\quad - \left(\widetilde{b_{12}}\delta_{xy}^{\tilde{\alpha}} \widetilde{U}_{i,j}^{n+1} - \widehat{b}_{12}\delta_{xy}^{\alpha} U_{i,j}^{n+1}\right) \\
&= \frac{\epsilon_{i,j}^{n+1}}{\Delta t} + T_1 + T_2 + T_3,
\end{aligned} \tag{A.1}$$

where $\widetilde{\sigma}_1 = \sigma_1(\delta_x^2 \widetilde{U}_{i,j}^{n+1}), \widehat{\sigma}_1 = \sigma_1(\delta_x^2 U_{i,j}^{n+1})$, $\widetilde{\sigma}_2 = \sigma_2(\delta_y^2 \widetilde{U}_{i,j}^{n+1})$, and $\widehat{\sigma}_2 = \sigma_1(\delta_y^2 U_{i,j}^{n+1})$ are defined by (3.9), $\widetilde{b_{12}}, \widehat{b}_{12}$ and the indexes $\tilde{\alpha}$ and α are determined by the rule (3.8a) and (4.2).

The term T_1 can be treated as

$$\begin{aligned}
T_1 = &-\frac{\widetilde{\sigma}_1^2}{2}\left(\delta_x^2 \widetilde{U}_{i,j}^{n+1} - \delta_x^2 U_{i,j}^{n+1}\right) + \left(\frac{\widehat{\sigma}_1^2}{2}\delta_x^2 U_{i,j}^{n+1} - \frac{\widetilde{\sigma}_1^2}{2}\delta_x^2 U_{i,j}^{n+1}\right) \\
\geqslant &\frac{\widetilde{\sigma}_1^2}{h^2}\epsilon_{i,j}^{n+1},
\end{aligned} \tag{A.2}$$

since $\frac{\widehat{\sigma}_1^2}{2}\delta_x^2 U_{i,j}^{n+1} = \sup_{\sigma_1} \frac{\sigma_1^2}{2}\delta_x^2 U_{i,j}^{n+1} \geqslant \frac{\widetilde{\sigma}_1^2}{2}\delta_x^2 U_{i,j}^{n+1}$. Similarly, we have

$$T_2 \geqslant \frac{\widetilde{\sigma}_2^2}{h^2}\epsilon_{i,j}^{n+1}. \tag{A.3}$$

We now turn to the term T_3.

$$\begin{aligned}
T_3 = &-\widetilde{b_{12}}(\delta_{xy}^{\tilde{\alpha}} \widetilde{U}_{i,j}^{n+1} - \delta_{xy}^{\tilde{\alpha}} U_{i,j}^{n+1}) + (\widehat{b}_{12}\delta_{xy}^{\alpha} U_{i,j}^{n+1} - \widetilde{b_{12}}\delta_{xy}^{\tilde{\alpha}} U_{i,j}^{n+1}) \\
\geqslant &-\frac{|\widetilde{b_{12}}|}{h^2}\epsilon_{i,j}^{n+1},
\end{aligned} \tag{A.4}$$

where we have used $\widehat{b}_{12}\delta_{xy}^{\alpha} U_{i,j}^{n+1} = \max\{\overline{b_{12}}\delta_{xy}^{+} U_{i,j}^{n+1}, \underline{b_{12}}\delta_{xy}^{-} U_{i,j}^{n+1}\} \geqslant \widetilde{b_{12}}\delta_{xy}^{\tilde{\alpha}} U_{i,j}^{n+1}$ and the definitions (3.8b) and (3.8c).

Substituting the above three terms into (A.1), we get, by Assumption (4.1),

$$T \geqslant \epsilon_{i,j}^{n+1}\left(\frac{1}{\Delta t} + \frac{\widetilde{\sigma}_1^2 + \widetilde{\sigma}_2^2 - |\widetilde{b_{12}}|}{h^2}\right) \geqslant 0.$$

Therefore, (4.8) is valid.

We now turn to a perturbation in $U_{k,l}^{n+1}$ and $U_{i,j}^{n}$ and denote it by $\widetilde{U}_{k,l}^{n+1} = U_{k,l}^{n+1} + \epsilon_{k,l}^{n+1}$ for $\epsilon_{k,l}^{n+1} \geqslant 0$ and $(k,l) \in N_{i,j}$. We also denote it by $\widetilde{U}_{i,j}^{n} = U_{i,j}^{n} + \epsilon_{i,j}^{n}$, for $\epsilon_{i,j}^{n} \geqslant 0$ and $\widetilde{U}_{i,j}^{n+1} = U_{i,j}^{n+1}$.

Then, the difference between the two sides of the inequality (4.9) is

$$\begin{aligned}
T' &= g_{i,j}\left(\widetilde{U}_{i,j}^{n+1}, \widetilde{U}_{i,j}^{n}, \{\widetilde{U}_{k,l}^{n+1}\}_{(k,l)\in N_{i,j}}\right) - g_{i,j}\left(U_{i,j}^{n+1}, U_{i,j}^{n}, \{U_{k,l}^{n+1}\}_{(k,l)\in N_{i,j}}\right) \\
&= -\frac{\epsilon_{i,j}^{n}}{\Delta t} - \left(\frac{\widetilde{\sigma}_1^2}{2}\delta_x^2\widetilde{U}_{i,j}^{n+1} - \frac{\widehat{\sigma}_1^2}{2}\delta_x^2 U_{i,j}^{n+1}\right) - \left(\frac{\widetilde{\sigma}_2^2}{2}\delta_y^2\widetilde{U}_{i,j}^{n+1} - \frac{\widehat{\sigma}_2^2}{2}\delta_y^2 U_{i,j}^{n+1}\right) \\
&\quad - \left(\widetilde{b_{12}}\delta_{xy}^{\tilde{\alpha}}\widetilde{U}_{i,j}^{n+1} - \widehat{b}_{12}\delta_{xy}^{\alpha}U_{i,j}^{n+1}\right) \\
&= -\frac{\epsilon_{i,j}^{n}}{\Delta t} + T_4 + T_5 + T_6, \qquad (A.5)
\end{aligned}$$

where $\widetilde{\sigma}_1 = \sigma_1(\delta_x^2\widetilde{U}_{i,j}^{n+1}), \widehat{\sigma}_1 = \sigma_1(\delta_x^2 U_{i,j}^{n+1})$, $\widetilde{\sigma}_2 = \sigma_2(\delta_y^2\widetilde{U}_{i,j}^{n+1}), \widehat{\sigma}_2 = \sigma_1(\delta_y^2 U_{i,j}^{n+1})$ are defined by (3.9), $\widetilde{b_{12}}$, $\widehat{b}_{12}$ and the indexes $\tilde{\alpha}$ and α are determined by the rule (3.8a) and (4.2).

The term T_4 can be treated as

$$\begin{aligned}
T_4 &= \left(-\frac{\widetilde{\sigma}_1^2}{2}\delta_x^2\widetilde{U}_{i,j}^{n+1} + \frac{\widehat{\sigma}_1^2}{2}\delta_x^2\widetilde{U}_{i,j}^{n+1}\right) + \frac{\widehat{\sigma}_1^2}{2}\left(\delta_x^2 U_{i,j}^{n+1} - \delta_x^2\widetilde{U}_{i,j}^{n+1}\right) \\
&\leqslant -\frac{\widehat{\sigma}_1^2}{2h^2}\sum_{|k-i|=1}\epsilon_{k,j}^{n+1}, \qquad (A.6)
\end{aligned}$$

since $\frac{\widetilde{\sigma}_1^2}{2}\delta_x^2\widetilde{U}_{i,j}^{n+1} = \sup_{\sigma_1}\frac{\sigma_1^2}{2}\delta_x^2\widetilde{U}_{i,j}^{n+1} \geqslant \frac{\widehat{\sigma}_1^2}{2}\delta_x^2\widetilde{U}_{i,j}^{n+1}$. Similarly, we have

$$T_5 \leqslant -\frac{\widehat{\sigma}_2^2}{2h^2}\sum_{|l-j|=1}\epsilon_{i,l}^{n+1}. \qquad (A.7)$$

The term T_6 can be treated as

$$T_6 = \left(-\widetilde{b_{12}}\delta_{xy}^{\tilde{\alpha}}\widetilde{U}_{i,j}^{n+1} + \widehat{b}_{12}\delta_{xy}^{\alpha}\widetilde{U}_{i,j}^{n+1}\right) + \widehat{b}_{12}\left(\delta_{xy}^{\alpha}U_{i,j}^{n+1} - \delta_{xy}^{\alpha}\widetilde{U}_{i,j}^{n+1}\right) \qquad (A.8)$$

$$\begin{aligned}
&= -\frac{|\widehat{b}_{12}|}{2h^2} * \left((\epsilon_{i+1,j+1}^{n+1} + \epsilon_{i-1,j-1}^{n+1}) * H(\widehat{b}_{12}) + (\epsilon_{i+1,j-1}^{n+1} + \epsilon_{i-1,j+1}^{n+1}) * H(-\widehat{b}_{12})\right) \\
&\quad + \frac{|\widehat{b}_{12}|}{2h^2}\left(\sum_{|k-i|=1}\epsilon_{k,j}^{n+1} + \sum_{|l-j|=1}\epsilon_{i,l}^{n+1}\right) \\
&\leqslant \widehat{b}_{12}\left(\delta_{xy}^{\alpha}U_{i,j}^{n+1} - \delta_{xy}^{\alpha}\widetilde{U}_{i,j}^{n+1}\right), \qquad (A.9)
\end{aligned}$$

where we have used $\widetilde{b_{12}}\delta_{xy}^{\tilde{\alpha}}\widetilde{U}_{i,j}^{n+1} = \max\{\overline{b_{12}}\delta_{xy}^{+}\widetilde{U}_{i,j}^{n+1}, \underline{b_{12}}\delta_{xy}^{-}\widetilde{U}_{i,j}^{n+1}\} \geqslant \widehat{b}_{12}\delta_{xy}^{\alpha}\widetilde{U}_{i,j}^{n+1}$ and the definitions (3.8b) and (3.8c). $H(\cdot)$ is the Heaviside function.

Substituting the above three terms into (A.5), we get, by Assumption (4.1), that

$$T' \leqslant -\frac{\widehat{\sigma}_1^2 - |\widehat{b}_{12}|}{2h^2}\sum_{|k-i|=1}\epsilon_{k,j}^{n+1} - \frac{\widehat{\sigma}_2^2 - |\widehat{b}_{12}|}{2h^2}\sum_{|l-j|=1}\epsilon_{i,l}^{n+1} \leqslant 0.$$

Therefore, (4.9) is true, and the scheme (3.7) is monotone. □

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