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Asymptotic solution for the optimal consumption, life insurance, and investment problem under the 4/2 stochastic volatility model with habit formation

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Abstract This paper explores the optimal consumption, life insurance, and investment strategies of an individual under the influence of habit formation. We assume that the individual can invest in a risk-free asset, a stock, and an index bond in the financial market, where the stock price follows the 4/2 stochastic volatility model. The primary aim of this paper is to maximize the expected utility of consumption, total bequests, and terminal wealth before retirement or death; the utility of consumption is derived from actual consumption exceeding the established habitual consumption level. By applying the dynamic programming method, we derive the Hamilton–Jacobi–Bellman (HJB) equation that the value function satisfies, obtain the asymptotic solutions for the optimal consumption, life insurance, and investment strategies using the asymptotic expansion method, and prove the corresponding verification theorem. Furthermore, we provide numerical examples to analyze the influence of consumption habit patterns and model parameters on the individual's optimal strategies.

Keywords Habit formation, 4/2 stochastic volatility model, Hamilton–Jacobi–Bellman equation, Asymptotic expansion method, Optimal strategies

2020 Mathematics Subject Classification 60H30, 91G10, 93E20, 35Q93, 35C20

1. Introduction

In recent years, the issues of investment and consumption have occupied a significant position in financial theory and practice, attracting the attention of numerous scholars. [29] made pioneering contributions to the research on optimal consumption and investment strategies. Subsequently, [10, 14], and others conducted in-depth explorations under different market assumptions, investigating optimal investment and consumption strategies. [36] investigated optimal consumption decisions and life insurance choices under uncertain lifetimes, while [32]

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explored consumption, investment, and life insurance expenditure decisions using the dynamic programming methods. [31] further discussed optimal life insurance purchase, consumption, and investment decisions under uncertain lifetimes. Further studies on consumption, investment, and multiple life insurance purchases were conducted by [1, 7, 23] and [35].

However, these studies often use the geometric Brownian motion model to simulate the price processes of risky assets. Further investigation of asset price characteristics has revealed that the geometric Brownian motion model fails to accurately describe the true trends of asset prices, thus impacting the optimal decision-making results. The emergence of stochastic volatility models has addressed the inadequacy of constant volatility in depicting the volatility trends inherent in the price processes of risky assets. Therefore, to capture the complex volatility of the stock market, [13, 24] and [26] examined optimal decision-making problems using the Heston stochastic volatility model. In order to address the limitations of the Heston stochastic volatility model in capturing extreme volatility scenarios, [20] proposed the 3/2 stochastic volatility model. Subsequently, [6, 37] and [38] conducted related research based on this model. Nevertheless, this model fails to capture the price dynamics of assets with low volatility. To jointly address the limitations of the Heston stochastic volatility model and the 3/2 stochastic volatility model, [16] proposed the 4/2 stochastic volatility model, which combined the 1/2 stochastic volatility model introduced by [19] and the 3/2 stochastic volatility model. [8] studied optimal investment strategies under the 4/2 stochastic volatility model.

Additionally, many scholars have found that individuals' consumption behavior is often influenced by their past consumption levels. Incorporating the concept of habit formation enhances the realism of the model, thereby enabling a more precise characterization of individuals' consumption behavior. For instance, [28] was the first to discuss the influence of past consumption levels on current consumption. Subsequently, [11] incorporated consumption habits into the Merton model, and [2] explored consumption and insurance strategies under habit formation. Moreover, [7, 27] and [39] incorporated habit formation into the analysis of personal life insurance demand. They examined the optimal consumption, life insurance, and investment strategies and analyzed the impact of consumption habits on these decisions.

In the practical economic environment, inflation influences the levels of market prices, the purchasing power of consumers, and the cost of living for individuals. [5] emphasized that considering inflation risk is crucial for long-term investors. [22] indicated that individuals with long-term financial plans should consider both mortality and inflation risk. Inflation risk can generally be hedged through liquid inflation-linked index bonds (hereinafter referred to as "index bonds"), as discussed in [22] and [30].

Building upon but distinct from the studies of [8] and [27], this paper makes four main contributions. First, we examine the optimal consumption, life insurance, and investment strategies under the impacts of habit formation and inflation, where the stock price process follows the 4/2 stochastic volatility model, thus extending the existing literature. Secondly, the analytical solution for the optimal strategies under the 4/2 stochastic volatility model cannot be obtained because of the complicated nonlinearity of the partial differential equation. We further employ an asymptotic expansion technique as illustrated in [18] to derive an asymptotic solution for these strategies. Thirdly, we provide and prove a verification theorem to ensure that the solutions obtained from the HJB equation are indeed optimal for the optimization problem. Finally, we provide numerical examples to analyze the impact of different habit formation models and varying model parameters on optimal strategies.

The rest of this paper is structured as follows. Section 2 proposes some necessary assumptions and describe the model. Section 3 derives the HJB equation by applying the dynamic programming principle and obtain the asymptotic solutions with a slow varying volatility component. Section 4 utilizes numerical simulations to separately demonstrate the impact of different habit formation models and changes in model parameters on optimal consumption, life insurance expenditure, and investment. Finally, Section 5 presents our conclusions. Detailed information on some proofs can be found in the Appendix.

2. Model formulation

We assume that an individual retires at time T. Under this assumption, trading occurs continuously and there are no transaction costs or taxes. The uncertainty is represented by a complete probability space $\big(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\in[0,T]}, P\big)$ that supports Brownian motions $\boldsymbol{W(t)} := (W_S(t), W_I(t))'$ and $W_v(t)$. Here, $W_I(t)$ is independent of $W_S(t)$ and $W_v(t)$ (the specific formulations of these Brownian motions are introduced in Section 2.1). Furthermore, the natural filtration $\{\mathcal{F}(t)\}_{t\in[0,T]}$, generated by Brownian motions $W_S(t)$, $W_I(t)$, and $W_v(t)$, satisfies the usual conditions and represents the information available up to time t. All stochastic processes and random variables are defined in the filtered probability space $\big(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\in[0,T]}, P\big)$. In this work, the prime denotes the transposition of a matrix.

2.1 Financial market

Assume that the individual can invest in a risk-free asset, a stock and an index bond. The risk-free asset with a constant rate of nominal return R follows the ordinary differential equation

$$\mathrm{d}B_0(t) = RB_0(t)\mathrm{d}t, \quad B_0(0) = B_0 > 0. \tag{2.1}$$

The price process of the stock $S(t)$ is given by the 4/2 stochastic volatility model as follows:

$$\begin{aligned}\mathrm{d}S(t) =& \left[R + \kappa\left(av(t) + b\right)\right] S(t)\mathrm{d}t + \left(a\sqrt{v(t)} + \frac{b}{\sqrt{v(t)}}\right) S(t)\mathrm{d}W_S(t),\\ S(0) =& S_0 > 0,\end{aligned} \tag{2.2}$$

where $\kappa \in \mathbb{R}^+$ is the control parameter of excess return, and $a, b \in \mathbb{R}^+$. The stochastic volatility $v(t)$ is defined by the CIR model:

$$\mathrm{d}v(t) = \bar{a}\left(\bar{b} - v(t)\right)\mathrm{d}t + \sigma_1\sqrt{v(t)}\mathrm{d}W_v(t), \quad v(0) = v_0 > 0,$$

where $\bar{a}, \bar{b}, \sigma_1 \in \mathbb{R}^+$ denote the mean-reversion speed, long-run mean, and volatility of volatility, respectively. Parameter $\rho_{Sv} \in (-1, 1)$ is the correlation coefficient of the Brownian motions $W_S(t)$ and $W_v(t)$. In addition, the Feller condition $2\bar{a}\bar{b} \geqslant \sigma_1^2$ is also satisfied to keep the processes $v(t)$ strictly positive.

Remark 2.1 *Note that a and b are the main control parameters of the 4/2 stochastic volatility model. When $a = 1$ and $b = 0$, the 4/2 stochastic volatility model is reduced to the Heston stochastic volatility model (see [19]). When $a = 0$ and $b = 1$, the 4/2 stochastic volatility model is reduced to the 3/2 stochastic volatility model(see [20]).*

Based on the works of [4] and [22], the commodity price level $I(t)$ is expressed as follows:

$$\frac{\mathrm{d}I(t)}{I(t)} = \mu_I(t)\mathrm{d}t + \boldsymbol{\tilde{\sigma}_I'}\mathrm{d}\boldsymbol{W(t)}, \quad I(0) = I_0 > 0, \tag{2.3}$$

where $\mu_I(t)$ represents the instantaneous expected inflation rate, and $\boldsymbol{\tilde{\sigma}_I} = (\sigma_I\rho_{IS}, \sigma_I\sqrt{1-\rho_{IS}^2})'$, with $\sigma_I > 0$ denoting the volatility of the inflation rate. ρ_{IS} is the correlation coefficient between financial risk and inflation risk. Let $r(t)$ stand for the real interest rate at time t; consequently, the index bond's behavior follows this dynamics:

$$\frac{\mathrm{d}B_1(t)}{B_1(t)} = r(t)\mathrm{d}t + \frac{\mathrm{d}I(t)}{I(t)} = (r(t) + \mu_I(t))\,\mathrm{d}t + \tilde{\boldsymbol{\sigma}}_I'\mathrm{d}\boldsymbol{W(t)}. \tag{2.4}$$

In this study, we focus on how habit formation influences optimal strategies. Following the approach of [22], we assume that $\mu_I(t) \equiv \mu_I$ and $r(t) \equiv r < R$ are constants.

2.2 Life insurance

Assume that the individual's lifetime is uncertain. Let τ denote the death time of the individual, which is a non-negative random variable defined on the probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\in[0,T]}, P\right)$. Furthermore, τ is independent of the Brownian motions $W_S(t)$, $W_v(t)$ and $W_I(t)$. Let $\lambda(t)$ (see [3]) be a hazard function at time t, which is defined as follows:

$$\lambda(t) = \lim_{\varepsilon\to 0}\frac{P\left(t \leqslant \tau < t+\varepsilon | \tau \geqslant t\right)}{\varepsilon}.$$

$S(s,t) = P\{\tau > s|\tau > t\}$ represents the probability that the individual survives at time t and also at time $s\,(s \geqslant t)$. From the definition of the force of mortality, we can see that the probability of $\tau > s$ given $\tau > t$ is

$$S(s,t) = \mathrm{e}^{-\int_t^s \lambda(u)\mathrm{d}u}.$$

Thus, $f(s,t)$ is defined as the probability that the individual dies at time s, conditional on he/she having survived to t,

$$f(s,t) = \lambda(s)\mathrm{e}^{-\int_t^s \lambda(u)\mathrm{d}u}.$$

For mortality risk, the individual can hedge it by purchasing life insurance, paying a life insurance premium at a nominal rate $p_N(t)$ continuously. Upon the individual's death at time t, the beneficiary receives an insurance benefit of $p_N(t)/\zeta(t)$, where $\zeta(t)$ represents the deterministic premium-insurance ratio. It is important to note that $p_N(t)$ could be negative; in such a scenario, [12] suggested that $p_N(t)$ resembled a term life annuity, as the wealth (amounting to $-p_N(t)/\zeta(t)$) upon death was exchanged for cash inflows while alive. Conversely, [22] considered $p_N(t)$ a pension annuity.

2.3 Habit formation

Following [21] and [34], we assume the individual's preference exhibits internal habit formation. The habit level at time t is defined as

$$h(t) = h_0\mathrm{e}^{-\xi_2 t} + \xi_1\int_0^t \mathrm{e}^{-\xi_2(t-s)}c(s)\mathrm{d}s. \tag{2.5}$$

That is,

$$\mathrm{d}h(t) = (\xi_1 c(t) - \xi_2 h(t))\,\mathrm{d}t. \tag{2.6}$$

At time t, the individual's consumption is denoted by $c(t)$, with h_0 representing the initial level of habit. Notably, the value of h_0 is influenced by the level of social development, with higher social development generally corresponding to a greater h_0. Typically, $\xi_1 > 0$ measures the individual's inclination to maintain the current level of consumption; a larger ξ_1 indicates greater desire to sustain the current consumption level in the future. $\xi_2 > 0$ measures the individual's tendency to forget past consumption; a larger ξ_2 indicates a higher degree of forgetfulness regarding past consumption.

2.4 Wealth process and optimization problem

During the period $[0,T]$, the individual earns income from the nonfinancial market at a nominal rate of $Y_N(t)$. Define the real rate of income as $y(t) \equiv \frac{Y_N(t)}{I(t)}$. Let $\pi_0(t)$, $\pi_1(t)$, and $\pi_2(t)$ be the respective portfolio weights on the risk-free asset, stock, and index bond, respectively, such that $\pi_0(t)+\pi_1(t)+\pi_2(t)=1$. Furthermore, let $X_N(t)$ denote the individual's nominal wealth. The dynamics of $X_N(t)$ can be described using the following expression:

$$\begin{aligned}\mathrm{d}X_N(t) &= \pi_0(t)X_N(t)\frac{\mathrm{d}B_0(t)}{B_0(t)} + \pi_1(t)X_N(t)\frac{\mathrm{d}S(t)}{S(t)} + \pi_2(t)X_N(t)\frac{\mathrm{d}B_1(t)}{B_1(t)} - c_N(t)\mathrm{d}t \\ &\quad - p_N(t)\mathrm{d}t + y_N(t)\mathrm{d}t, \quad t \in \min\{\tau, T\}. \end{aligned} \tag{2.7}$$

Let $X(t) \equiv \frac{X_N(t)}{I(t)}$ represent the real wealth process. Applying the Itô formula, we obtain the individual's real wealth process. In other words, for all $t \in \min\{\tau,T\}$, the real wealth $X(t)$ evolves as follows:

$$\mathrm{d}X(t) = (rX(t) - c(t) - p(t) + y(t))\,\mathrm{d}t + X(t)\boldsymbol{\pi'\eta}\mathrm{d}t + X(t)\boldsymbol{\pi'\Sigma}\mathrm{d}\boldsymbol{W(t)}, \tag{2.8}$$

where $\boldsymbol{\pi(t)} := (\pi_0(t), \pi_1(t))'$,

$$\boldsymbol{\eta(v(t))} := \begin{pmatrix} R - r - \mu_I + \sigma_I^2 \\ R + \kappa\,(av(t)+b) - r - \mu_I + \sigma_I^2 - \rho_{IS}\sigma_I\left(a\sqrt{v(t)} + \dfrac{b}{\sqrt{v(t)}}\right) \end{pmatrix},$$

$$\boldsymbol{\Sigma(v(t))} := \begin{pmatrix} -\sigma_I\rho_{IS} & -\sigma_I\sqrt{1-\rho_{IS}^2} \\ \left(a\sqrt{v(t)} + \dfrac{b}{\sqrt{v(t)}}\right) - \sigma_I\rho_{IS} & -\sigma_I\sqrt{1-\rho_{IS}^2} \end{pmatrix}.$$

Definition 2.2 *(Admissible strategy) A strategy $(c,p,\boldsymbol{\pi}) := \{c(t), p(t), \boldsymbol{\pi(t)}, t \in [0,T]\}$ is said to be admissible if*

(1) $\forall t \in [0,T]$, $\{c(t)\}$, $\{p(t)\}$ and $\{\boldsymbol{\pi(t)}\}$ are adapted to $\{\mathcal{F}(t)\}_{t\in[0,T]}$;

(2) $\mathbb{E}[\int_0^T c(t)\mathrm{d}t] < \infty$, $\mathbb{E}[\int_0^T p(t)\mathrm{d}t] < \infty$, $\mathbb{E}[\int_0^T X^2(t) \parallel \boldsymbol{\pi(t)} \parallel^2 \mathrm{d}t] < \infty$;

(3) (2.8) has a pathwise unique strong solution for all $(t,x,v) \in [0,T]\times\mathbb{R}\times\mathbb{R}^+$, where $\|\cdot\|$ denotes the Euclidean norm of a vector.

Denote the set of all admissible strategies by Π.

Next, we examine the optimization problems. Define $\Pi(t)$ as the set of all admissible strategies within the time interval $[t,T]$. For any given admissible strategy $(c,p,\boldsymbol{\pi}) \in \Pi(t)$, we establish the value function at time t as follows:

$$\begin{aligned} V(t,x,v,h) = \max_{(c,p,\boldsymbol{\pi})\in\Pi(t)} \mathbb{E}_t\Bigg[&\int_t^{\tau\wedge T} \alpha_1 \mathrm{e}^{-\rho(s-t)} U_1\,(c(s)-h(s))\,\mathrm{d}s \\ &+ \alpha_2\mathrm{e}^{-\rho(\tau-t)}U_2\left(X(\tau) + \frac{p(\tau)}{\zeta(\tau)}\right)\mathbf{I}_{\{\tau\leqslant T\}} + \alpha_3\mathrm{e}^{-\rho(T-t)}U_3\,(X(T))\,\mathbf{I}_{\{\tau>T\}}\Bigg], \end{aligned} \tag{2.9}$$

where $\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot|X(t)=x, v(t)=v, h(t)=h]$; $\alpha_i > 0, i=1,2,3$ are constants; ρ is the time preference rate; $\mathbf{I}_{\{\cdot\}}$ represents the indicator function; $U_i(x), i=1,2,3$ are the utility functions (see [22]), which are strictly increasing, concave, and continuously differentiable and have the following forms:

$$U_i^{'}(0+)=\lim_{x\to 0+}U_i^{'}(x)=\infty,\quad U_i^{'}(\infty)=\lim_{x\to\infty}U_i^{'}(x)=0.$$

(2.9) illustrates that the individual's utility is derived from three aspects: $c(t)-h(t)$, the difference between real consumption $c(t)$ and real habit level $h(t)$; $X(\tau)+\frac{p(\tau)}{\zeta(\tau)}$, the total legacy in the event of premature death; and $X(T)$, the terminal wealth at retirement time. Thus, α_1, α_2, and α_3 signify the weights assigned to the three utility factors. It is important to highlight that we assume that consumption habits do not influence the utility of terminal wealth in equation (2.9). For similar settings, refer to [25].

Applying the law of iterated expectation, (2.9) can be rewritten as follows (for detailed derivation, see Appendix A). This transformation simplifies the following calculations.

$$\begin{aligned}V(t,x,v,h)=&\max_{(c,p,\boldsymbol{\pi})\in\Pi(t)}\mathbb{E}_t\Bigg[\int_t^T \mathrm{e}^{-\int_t^s(\rho+\lambda(u))\mathrm{d}u}\Bigg(\alpha_1U_1\left(c(s)-h(s)\right)\\&+\alpha_2\lambda(s)U_2\left(X(s)+\frac{p(s)}{\zeta(s)}\right)\Bigg)\mathrm{d}s+\alpha_3\mathrm{e}^{-\int_t^T(\rho+\lambda(u))\mathrm{d}u}U_3\left(X(T)\right)\Bigg].\end{aligned}\tag{2.10}$$

3. Solution for the optimization problem

3.1 HJB equation

We assume $U_i(x)$ is a power utility exhibiting constant relative risk aversion (CRRA):

$$U_i(x)=\frac{x^\delta}{\delta}:=U(x),\quad i=1,2,3,$$

where δ $(\delta<1,\delta\neq 0)$ is the relative risk aversion coefficient. Subsequently, we apply the dynamic programming method to address the individual's optimization problems related to habit formation. For convenience, we introduce $C^{1,2,2,1}:=C^{1,2,2,1}([0,T]\times\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+)=\{\varphi(t,x,v,h):\varphi(\cdot,x,v,\cdot)$ is once continuously differentiable for t on $[0,T]$ and h on $\mathbb{R}^+$, $\varphi(t,\cdot,\cdot,h)$ is twice continuously differentiable for x on $\mathbb{R}$ and v on $\mathbb{R}^+\}$.

Under the real habit level given by (2.6), we define a differential operator: for any function $\varphi(t,x,v,h)\in C^{1,2,2,1}$,

$$\begin{aligned}\mathcal{A}_1^{c,p,\boldsymbol{\pi}}\varphi(t,x,v,h)=&-\left(\rho+\lambda(t)\right)\varphi+\varphi_t+\left(rx+y(t)-c-p+\boldsymbol{\pi}'\boldsymbol{\eta}x\right)\varphi_x+\left(\xi_1c-\xi_2h\right)\varphi_h\\&+\bar{a}\left(\bar{b}-v\right)\varphi_v+\frac{1}{2}\boldsymbol{\pi}'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{\pi}x^2\varphi_{xx}+\frac{1}{2}\sigma_1^2v\varphi_{vv}+\sigma_1\sqrt{v}x\rho_{Sv}\boldsymbol{\pi}'\boldsymbol{\Phi}\varphi_{xv},\end{aligned}\tag{3.1}$$

where

$$\boldsymbol{\eta}=\left(R-r-\mu_I+\sigma_I^2,\quad R+\kappa\left(av+b\right)-r-\mu_I+\sigma_I^2-\rho_{IS}\sigma_I\left(a\sqrt{v}+\frac{b}{\sqrt{v}}\right)\right)',$$

$$\boldsymbol{\Sigma}=\begin{pmatrix}-\sigma_I\rho_{IS} & -\sigma_I\sqrt{1-\rho_{IS}^2}\\ \left(a\sqrt{v}+\frac{b}{\sqrt{v}}\right)-\sigma_I\rho_{IS} & -\sigma_I\sqrt{1-\rho_{IS}^2}\end{pmatrix},\quad \boldsymbol{\Phi}=\begin{pmatrix}-\sigma_I\rho_{IS}\\ \left(a\sqrt{v}+\frac{b}{\sqrt{v}}\right)-\sigma_I\rho_{IS}\end{pmatrix},$$

φ_x, φ_t, φ_v, φ_h, φ_{vv}, φ_{xx}, and φ_{xv} represent the partial derivatives of $\varphi(t,x,v,h)$ with respect to (w.r.t.) the corresponding variables.

According to the principle of stochastic dynamic programming, the corresponding HJB equation of the value function (2.10) can be obtained by

$$\max_{(c,p,\boldsymbol{\pi})\in\Pi(t)}\left\{\mathcal{A}_1^{c,p,\boldsymbol{\pi}}V(t,x,v,h)+\alpha_1U(c-h)+\alpha_2\lambda(t)U\left(x+\frac{p}{\zeta(t)}\right)\right\}=0,\tag{3.2}$$

with the boundary condition

$$V(T,x,v,h)=\alpha_3 U(x).$$

To solve the optimal consumption, life insurance, and investment problem outlined in (3.2), we first solve the function $V(t,x,v,h)$. Here, recall that $U(x)=\frac{x^\delta}{\delta}$, $\delta<1,\delta\neq 0$.

Let us suppose that $\tilde{V}(t,x,v,h)$ is a solution for the HJB equation (3.2) satisfying $\tilde{V}_h<0$, $\tilde{V}_x>0$, and $\tilde{V}_{xx}<0$; then, the following are the first-order optimal conditions:

$$c^*(t)=h+\left(\frac{\tilde{V}_x-\xi_1\tilde{V}_h}{\alpha_1}\right)^{\frac{1}{\delta-1}}, \tag{3.3}$$

$$p^*(t)=\zeta(t)\cdot\left(\frac{\tilde{V}_x\zeta(t)}{\alpha_2\lambda(t)}\right)^{\frac{1}{\delta-1}}-\zeta(t)x, \tag{3.4}$$

$$\boldsymbol{\pi}^*=-\left(\boldsymbol{\Sigma\Sigma'}\right)^{-1}\boldsymbol{\eta}\frac{1}{x}\frac{\tilde{V}_x}{\tilde{V}_{xx}}-\left(\boldsymbol{\Sigma\Sigma'}\right)^{-1}\boldsymbol{\Phi}\sigma_1\sqrt{v}\frac{\rho_{Sv}}{x}\frac{\tilde{V}_{xv}}{\tilde{V}_{xx}}. \tag{3.5}$$

Substituting (3.3), (3.4) and (3.5) into (3.2) yields the following:

$$\begin{aligned}&\tilde{V}_t-(\rho+\lambda(t))\tilde{V}+(\zeta(t)x+rx+y(t)-h)\tilde{V}_x+(\xi_1-\xi_2)h\tilde{V}_h+\bar{a}\left(\bar{b}-v\right)\tilde{V}_v\\&+\frac{1-\delta}{\delta}\alpha_1^{\frac{1}{1-\delta}}\left(\tilde{V}_x-\xi_1\tilde{V}_h\right)^{\frac{\delta}{\delta-1}}+\frac{1-\delta}{\delta}\left(\alpha_2\lambda(t)\right)^{\frac{1}{1-\delta}}\zeta(t)^{\frac{\delta}{\delta-1}}\tilde{V}_x^{\frac{\delta}{\delta-1}}-\frac{\boldsymbol{\eta'}\left(\boldsymbol{\Sigma\Sigma'}\right)^{-1}\boldsymbol{\eta}}{2}\frac{\tilde{V}_x^2}{\tilde{V}_{xx}}\\&+\frac{1}{2}\sigma_1^2v\tilde{V}_{vv}-\frac{1}{2}\sigma_1^2v\rho_{Sv}^2\boldsymbol{\Phi'}\left(\boldsymbol{\Sigma\Sigma'}\right)^{-1}\boldsymbol{\Phi}\frac{\tilde{V}_{xv}^2}{\tilde{V}_{xx}}-\sigma_1\sqrt{v}\rho_{Sv}\boldsymbol{\eta'}\left(\boldsymbol{\Sigma\Sigma'}\right)^{-1}\boldsymbol{\Phi}\frac{\tilde{V}_x\tilde{V}_{xv}}{\tilde{V}_{xx}}=0.\end{aligned} \tag{3.6}$$

We assume that a solution for (3.6) has the form

$$\tilde{V}(t,x,v,h)=m(t,v)\frac{Y(t,x,h)^\delta}{\delta}, \tag{3.7}$$

with the boundary conditions $m(T,v)=\alpha_3$ and $Y(T,x,h)=x$. Then,

$$\begin{aligned}&\tilde{V}_t=m_t\frac{Y^\delta}{\delta}+mY^{\delta-1}Y_t,\quad \tilde{V}_h=mY^{\delta-1}Y_h,\\&\tilde{V}_v=m_v\frac{Y^\delta}{\delta},\quad \tilde{V}_{vv}=m_{vv}\frac{Y^\delta}{\delta},\quad \tilde{V}_{xv}=m_vY^{\delta-1}Y_x,\\&\tilde{V}_x=mY^{\delta-1}Y_x,\quad \tilde{V}_{xx}=mY^{\delta-1}Y_{xx}+m(\delta-1)Y^{\delta-2}Y_x^2,\end{aligned} \tag{3.8}$$

where $\tilde{V}=\tilde{V}(t,x,v,h)$, $m=m(t,v)$ and $Y=Y(t,x,h)$. By substituting (3.8) and (3.7) into (3.6), we obtain the following:

$$\begin{aligned}&m_t\frac{Y^\delta}{\delta}-(\rho+\lambda(t))m\frac{Y^\delta}{\delta}+(\zeta(t)x+rx+y(t)-h)mY^{\delta-1}Y_x+(\xi_1-\xi_2)hmY^{\delta-1}Y_h\\&+\frac{1-\delta}{\delta}\alpha_1^{\frac{1}{1-\delta}}\left(mY^{\delta-1}Y_x-\xi_1mY^{\delta-1}Y_h\right)^{\frac{\delta}{\delta-1}}+\frac{1-\delta}{\delta}\left(\alpha_2\lambda(t)\right)^{\frac{1}{1-\delta}}\zeta(t)^{\frac{\delta}{\delta-1}}m^{\frac{\delta}{\delta-1}}Y^\delta Y_x^{\frac{\delta}{\delta-1}}\\&+mY^{\delta-1}Y_t+\bar{a}\left(\bar{b}-v\right)m_v\frac{Y^\delta}{\delta}+\frac{1}{2}\sigma_1^2v\frac{m_{vv}Y^\delta}{\delta}-\frac{\sigma_1\sqrt{v}\rho_{Sv}\boldsymbol{\eta'}\left(\boldsymbol{\Sigma\Sigma'}\right)^{-1}\boldsymbol{\Phi}m_vY^\delta Y_x^2}{YY_{xx}+(\delta-1)Y_x^2}\\&-\frac{\boldsymbol{\eta'}\left(\boldsymbol{\Sigma\Sigma'}\right)^{-1}\boldsymbol{\eta}}{2}\frac{mY^\delta Y_x^2}{YY_{xx}+(\delta-1)Y_x^2}-\frac{1}{2}\boldsymbol{\Phi'}\left(\boldsymbol{\Sigma\Sigma'}\right)^{-1}\boldsymbol{\Phi}\frac{\sigma_1^2v\rho_{Sv}^2m_v^2Y^\delta Y_x^2}{mYY_{xx}+m(\delta-1)Y_x^2}=0.\end{aligned} \tag{3.9}$$

Let us assume that $m(t,v)$ is given by

$$m(t,v)=f(t,v)^{1-\delta}, \tag{3.10}$$

with boundary condition $f(T,v)=\alpha_3^{\frac{1}{1-\delta}}$. Then, it follows that

$$\begin{aligned}&m_t=(1-\delta)\,f^{-\delta}f_t,\quad m_v=(1-\delta)\,f^{-\delta}f_v,\\&m_{vv}=-\delta\,(1-\delta)\,f^{-\delta-1}f_v^2+(1-\delta)\,f^{-\delta}f_{vv}.\end{aligned}\tag{3.11}$$

By inputting (3.11) into (3.9) and simplifying, we derive the following:

$$\begin{aligned}&f_t+\frac{\delta}{1-\delta}\frac{Y_t}{Y}f-\frac{\rho+\lambda(t)}{1-\delta}f+\frac{\delta\left(\zeta(t)x+rx+y(t)-h\right)}{1-\delta}\frac{Y_x}{Y}f+\alpha_1^{\frac{1}{1-\delta}}\left(Y_x-\xi_1Y_h\right)^{\frac{\delta}{\delta-1}}\\&+\left(\alpha_2\lambda(t)\right)^{\frac{1}{1-\delta}}\zeta(t)^{\frac{\delta}{\delta-1}}Y_x^{\frac{\delta}{\delta-1}}+\frac{\delta h\left(\xi_1-\xi_2\right)}{1-\delta}\frac{Y_h}{Y}f-\frac{\delta\boldsymbol{\eta}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\eta}}{2\,(1-\delta)}\frac{Y_x^2f}{YY_{xx}+(\delta-1)\,Y_x^2}\\&+\bar a\left(\bar b-v\right)f_v+\frac{1}{2}\sigma_1^2v\left(f_{vv}-\delta\frac{f_v^2}{f}\right)-\boldsymbol{\eta}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\Phi}\frac{\sigma_1\sqrt{v}\rho_{Sv}\delta Y_x^2}{Y_{xx}Y+(\delta-1)\,Y_x^2}f_v\\&-\frac{1}{2}\sigma_1^2v\rho_{Sv}^2\boldsymbol{\Phi}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\Phi}\frac{\delta\,(1-\delta)\,Y_x^2}{Y_{xx}Y+(\delta-1)\,Y_x^2}\frac{f_v^2}{f}=0.\end{aligned}\tag{3.12}$$

Let us assume that $Y(t,x,h)$ is given by

$$\begin{aligned}&Y(t,x,h)=x+A(t)+B(t)h,\\&Y(T,x,h)=x,\quad A(T)=0,\quad B(T)=0,\end{aligned}\tag{3.13}$$

then, the first- and second-order partial derivatives of $Y(t,x,h)$ are obtained as follows:

$$\begin{aligned}&Y_t=A'(t)+B'(t)h,\quad Y_h=B(t),\\&Y_x=1,\quad Y_{xx}=0.\end{aligned}\tag{3.14}$$

Substituting (3.14) into (3.12), we obtain the following:

$$\begin{aligned}&f_t+\frac{\delta}{1-\delta}\cdot\frac{A'(t)+B'(t)h}{x+A(t)+B(t)h}f-\frac{\rho+\lambda(t)}{1-\delta}f+\frac{\delta}{1-\delta}\cdot\frac{\zeta(t)x+rx+y(t)-h}{x+A(t)+B(t)h}f\\&+\frac{\delta}{1-\delta}\frac{B(t)\left(\xi_1-\xi_2\right)h}{x+A(t)+B(t)h}f+\frac{\delta\boldsymbol{\eta}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\eta}}{2\,(1-\delta)^2}f+\frac{1}{2}\sigma_1^2v\left(f_{vv}-\delta\frac{f_v^2}{f}\right)\\&-\sigma_1\sqrt{v}\rho_{Sv}\boldsymbol{\eta}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\Phi}\frac{\delta}{(\delta-1)}f_v+\frac{1}{2}\sigma_1^2v\rho_{Sv}^2\boldsymbol{\Phi}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\Phi}\delta\frac{f_v^2}{f}\\&+\alpha_1^{\frac{1}{1-\delta}}\left(1-\xi_1B(t)\right)^{\frac{\delta}{\delta-1}}+\bar a\left(\bar b-v\right)f_v+\left(\alpha_2\lambda(t)\right)^{\frac{1}{1-\delta}}\zeta(t)^{\frac{\delta}{\delta-1}}=0,\end{aligned}\tag{3.15}$$

which means

$$\begin{aligned}&x\left[\frac{\delta}{1-\delta}\left(\zeta(t)+r\right)f+M(t,v)\right]+\left[\frac{\delta}{1-\delta}A'(t)f+\frac{\delta}{1-\delta}y(t)f+A(t)M(t,v)\right]\\&+h\left[\frac{\delta}{1-\delta}B'(t)f-\frac{\delta}{1-\delta}f+\frac{\delta}{1-\delta}\left(\xi_1-\xi_2\right)B(t)f+B(t)M(t,v)\right]=0,\end{aligned}\tag{3.16}$$

where

$$\begin{aligned}M(t,v)=&f_t-\frac{\rho+\lambda(t)}{1-\delta}f+\alpha_1^{\frac{1}{1-\delta}}\left(1-\xi_1B(t)\right)^{\frac{\delta}{\delta-1}}+\left(\alpha_2\lambda(t)\right)^{\frac{1}{1-\delta}}\zeta(t)^{\frac{\delta}{\delta-1}}\\&+\frac{1}{2}\boldsymbol{\eta}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\eta}\frac{\delta}{(1-\delta)^2}f+\bar a\left(\bar b-v\right)f_v+\frac{1}{2}\sigma_1^2v\left(f_{vv}-\delta\frac{f_v^2}{f}\right)\\&-\sigma_1\sqrt{v}\rho_{Sv}\boldsymbol{\eta}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\Phi}\frac{\delta}{(\delta-1)}f_v+\frac{1}{2}\sigma_1^2v\rho_{Sv}^2\boldsymbol{\Phi}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\Phi}\delta\frac{f_v^2}{f}.\end{aligned}\tag{3.17}$$

Drawing inspiration from (3.16) and (3.17), we present (3.18), (3.19) and (3.20) as follows:

$$\frac{\delta}{1-\delta}\left(\zeta(t)+r\right)f+M(t,v)=0, \tag{3.18}$$

$$\frac{\delta}{1-\delta}A'(t)f+\frac{\delta}{1-\delta}y(t)f+A(t)M(t,v)=0, \tag{3.19}$$

$$\frac{\delta}{1-\delta}B'(t)f-\frac{\delta}{1-\delta}f+\frac{\delta}{1-\delta}\left(\xi_1-\xi_2\right)B(t)f+B(t)M(t,v)=0. \tag{3.20}$$

Furthermore, we have

$$\begin{aligned} &A'(t)-\left(\zeta(t)+r\right)A(t)+y(t)=0, \quad \textit{with} \quad A(T)=0,\\ &B'(t)+\left(\xi_1-\xi_2-\left(\zeta(t)+r\right)\right)B(t)-1=0, \quad \textit{with} \quad B(T)=0. \end{aligned}$$

We can obtain the following information:

$$A(t)=\int_t^T y(s)\mathrm{e}^{-\int_t^s r+\zeta(u)\mathrm{d}u}\mathrm{d}s, \tag{3.21}$$

$$B(t)=-\int_t^T \mathrm{e}^{-\int_t^s[\xi_2-\xi_1+\zeta(u)+r]\mathrm{d}u}\mathrm{d}s. \tag{3.22}$$

According to (3.17) and (3.18), we have

$$\begin{aligned} &f_t+\left[\frac{\delta\left(\zeta(t)+r\right)-\rho-\lambda(t)}{1-\delta}+\frac{\delta\boldsymbol{\eta}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\eta}}{2\left(1-\delta\right)^2}\right]f+\alpha_1^{\frac{1}{1-\delta}}\left(1-\xi_1B(t)\right)^{\frac{\delta}{\delta-1}}\\ &+\left[\bar{a}\left(\bar{b}-v\right)+\boldsymbol{\eta}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\Phi}\frac{\rho_{Sv}\sigma_1\sqrt{v}\delta}{(1-\delta)}\right]f_v+\frac{\delta\sigma_1^2v\left(\rho_{Sv}^2\boldsymbol{\Phi}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\Phi}-1\right)f_v^2}{2f}\\ &+\frac{1}{2}\sigma_1^2vf_{vv}+\left(\alpha_2\lambda(t)\right)^{\frac{1}{1-\delta}}\zeta(t)^{\frac{\delta}{\delta-1}}=0. \end{aligned} \tag{3.23}$$

The above formula can be written as follows:

$$\left(L_0+L_1+L_2\right)f+\frac{\delta\sigma_1^2v\left(\rho_{Sv}^2\boldsymbol{\Phi}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\Phi}-1\right)f_v^2}{2f}=0, \tag{3.24}$$

where

$$L_0f=\bar{a}\left(\bar{b}-v\right)f_v+\frac{1}{2}\sigma_1^2vf_{vv}, \tag{3.25}$$

$$\begin{aligned} L_1f=f_t&+\left[\frac{\delta\left(\zeta(t)+r\right)-\rho-\lambda(t)}{1-\delta}+\frac{\delta\boldsymbol{\eta}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\eta}}{2\left(1-\delta\right)^2}\right]f+\alpha_1^{\frac{1}{1-\delta}}\left(1-\xi_1B(t)\right)^{\frac{\delta}{\delta-1}}\\ &+\left(\alpha_2\lambda(t)\right)^{\frac{1}{1-\delta}}\zeta(t)^{\frac{\delta}{\delta-1}}, \end{aligned} \tag{3.26}$$

$$L_2f=\frac{\delta\sigma_1\sqrt{v}\rho_{Sv}\boldsymbol{\eta}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\Phi}}{1-\delta}f_v. \tag{3.27}$$

We opted to utilize an asymptotic method due to the complexities involved in obtaining an exact solution for the nonlinear partial differential equation (3.24). In fact, [15] has already proposed this method to solve the nonlinear partial differential equations with multi-scale stochastic volatility models, examining a two-factor stochastic volatility model driven by a fast mean-reverting diffusion and a slowly varying diffusion. Building on [15], we explore the optimal consumption, life insurance, and investment strategies in the next subsection.

3.2 Asymptotic solution for the slow varying volatility component

In this subsection, we assume that $v^\epsilon(t)$ obeys the following equation:

$$\mathrm{d}v^\epsilon(t) = \epsilon\bar{a}\left(\bar{b} - v^\epsilon(t)\right)\mathrm{d}t + \sqrt{\epsilon}\sigma_1\sqrt{v^\epsilon(t)}\mathrm{d}W_v(t), \tag{3.28}$$

in which ϵ is a small parameter.

Applying (3.28) to (3.24) and replacing $\bar{a}\left(\bar{b} - v\right)$ and $\sigma_1\sqrt{v}$ with $\epsilon\bar{a}\left(\bar{b} - v\right)$ and $\sqrt{\epsilon}\sigma_1\sqrt{v}$, respectively, (3.24) can be written as follows:

$$\left(\epsilon L_0 + L_1 + \sqrt{\epsilon}L_2\right) f^\epsilon + \frac{\epsilon\delta\sigma_1^2 v\left(\rho_{Sv}^2\boldsymbol{\Phi}'\left(\boldsymbol{\Sigma\Sigma}'\right)^{-1}\boldsymbol{\Phi} - 1\right)\left(f_v^\epsilon\right)^2}{2f^\epsilon} = 0, \tag{3.29}$$

where $L_0,\ L_1,\ L_2$ are given by (3.25)–(3.27). It is assumed that a solution to (3.29) can be expressed in the following form:

$$f^\epsilon(t,v) = f^{(0)}(t,v) + \sqrt{\epsilon}f^{(1)}(t,v) + \epsilon f^{(2)}(t,v). \tag{3.30}$$

Additionally, substituting (3.30) into (3.29), we obtain

$$\begin{aligned} L_1 f^{(0)} + \sqrt{\epsilon}\left(L_1 f^{(1)} + L_2 f^{(0)}\right) + \epsilon\Bigg(L_0 f^{(0)} + L_1 f^{(2)} + L_2 f^{(1)} \\ + \frac{\delta\sigma_1^2 v\left(\rho_{Sv}^2\boldsymbol{\Phi}'\left(\boldsymbol{\Sigma\Sigma}'\right)^{-1}\boldsymbol{\Phi} - 1\right)\left(f_v^{(0)}\right)^2}{2f^{(0)}}\Bigg) = 0. \end{aligned} \tag{3.31}$$

Collecting the same order of the terms in (3.31), three equations can be presented as follows. For the term of ϵ^0, we have

$$L_1 f^{(0)} = 0, \quad \text{with} \quad f^{(0)}(T,v) = \alpha_3^{\frac{1}{1-\delta}}. \tag{3.32}$$

For the term of $\sqrt{\epsilon}$,

$$L_1 f^{(1)} + L_2 f^{(0)} = 0, \quad \text{with} \quad f^{(1)}(T,v) = \alpha_3^{\frac{1}{1-\delta}}. \tag{3.33}$$

For the term of ϵ, the equation satisfies

$$\begin{aligned} &L_0 f^{(0)} + L_1 f^{(2)} + L_2 f^{(1)} + \frac{\delta\sigma_1^2 v\left(\rho_{Sv}^2\boldsymbol{\Phi}'\left(\boldsymbol{\Sigma\Sigma}'\right)^{-1}\boldsymbol{\Phi} - 1\right)\left(f_v^{(0)}\right)^2}{2f^{(0)}} = 0, \\ &\text{with} \quad f^{(2)}(T,v) = \alpha_3^{\frac{1}{1-\delta}}. \end{aligned} \tag{3.34}$$

In the subsequent lemmas, we aim to solve the three equations presented in (3.32), (3.33) and (3.34).

Lemma 3.1 *The solution to* (3.32) *can be obtained as follows:*

$$f^{(0)}(t,v) = \alpha_3^{\frac{1}{1-\delta}}\mathrm{e}^{\int_t^T G_1(s,v)\mathrm{d}s} + \int_t^T G_0(s)\mathrm{e}^{\int_t^s G_1(y,v)\mathrm{d}y}\mathrm{d}s, \tag{3.35}$$

where $G_0(t)$ *and* $G_1(t,v)$ *can be found in* (3.37) *and* (3.38), *respectively.*

Proof Utilizing (3.26), (3.32) can be rewritten as follows:

$$\begin{aligned} &f_t^{(0)} + \left[\frac{\delta\left(\zeta(t) + r\right) - \rho - \lambda(t)}{1-\delta} + \frac{\delta\boldsymbol{\eta}'\left(\boldsymbol{\Sigma\Sigma}'\right)^{-1}\boldsymbol{\eta}}{2\left(1-\delta\right)^2}\right] f^{(0)} + \alpha_1^{\frac{1}{1-\delta}}\left(1 - \xi_1 B(t)\right)^{\frac{\delta}{\delta-1}} \\ &+ \left(\alpha_2\lambda(t)\right)^{\frac{1}{1-\delta}}\zeta(t)^{\frac{\delta}{\delta-1}} = 0, \end{aligned} \tag{3.36}$$

with the boundary condition $f^{(0)}(T,v)=\alpha_3^{\frac{1}{1-\delta}}$. To simplify the notation, we make the following definitions,

$$G_0(t)=\alpha_1^{\frac{1}{1-\delta}}\left(1-\xi_1 B(t)\right)^{\frac{\delta}{\delta-1}}+\left(\alpha_2\lambda(t)\right)^{\frac{1}{1-\delta}}\zeta(t)^{\frac{\delta}{\delta-1}}, \tag{3.37}$$

$$G_1(t,v)=\frac{\delta\left(\zeta(t)+r\right)-\rho-\lambda(t)}{1-\delta}+\frac{\delta\boldsymbol{\eta}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\eta}}{2\left(1-\delta\right)^2}, \tag{3.38}$$

then, (3.36) becomes

$$f_t^{(0)}+G_1(t,v)f^{(0)}+G_0(t)=0. \tag{3.39}$$

We can think of the formula (3.39) as a non-homogeneous ordinary differential equation with respect to t and boundary condition $f^{(0)}(T,v)=\alpha_3^{\frac{1}{1-\delta}}$; (3.35) can be easily obtained, thus the lemma is verified. □

Lemma 3.2 *The solution to* (3.33) *can be derived as follows:*

$$f^{(1)}(t,v)=\alpha_3^{\frac{1}{1-\delta}}\mathrm{e}^{\int_t^T G_1(s,v)\mathrm{d}s}+\int_t^T G_2(s,v)\mathrm{e}^{\int_t^s G_1(y,v)\mathrm{d}y}\mathrm{d}s, \tag{3.40}$$

where $G_1(t,v)$ *and* $G_2(t,v)$ *are given by* (3.38) *and* (3.52).

Proof From (3.26) and (3.27), (3.33) can be rewritten as follows:

$$f_t^{(1)}+G_1(t,v)f^{(1)}+G_0(t)+\frac{\delta\sigma_1\sqrt{v}\rho_{Sv}\boldsymbol{\eta}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\Phi}}{1-\delta}f_v^{(0)}=0 \tag{3.41}$$

with the boundary condition $f^{(1)}(T,v)=\alpha_3^{\frac{1}{1-\delta}}$. Differentiating $\boldsymbol{\eta}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\Phi}$ with respect to v, we obtain

$$\begin{aligned}\frac{\partial\left(\boldsymbol{\eta}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\Phi}\right)}{\partial v}&=\left(R-r-\mu_I+\sigma_I^2\right)\left(g_2(v)g_5(v)-\sigma_I\rho_{IS}g_1(v)\right)\\&\quad+\frac{v\left(R-r-\mu_I+\sigma_I^2\right)\left(\rho_{IS}\left(av^{\frac12}+bv^{-\frac12}\right)-\sigma_I\right)g_{5v}(v)}{\sigma_I\left(1-\rho_{IS}^2\right)\left(av+b\right)^2}\\&\quad-\frac{\rho_{IS}v\left(\rho_{IS}\left(av^{\frac12}+bv^{-\frac12}\right)-\sigma_I\right)g_{3v}(v)}{\left(1-\rho_{IS}^2\right)\left(av+b\right)^2}-\sigma_I\rho_{IS}g_2(v)g_3(v)\\&\quad+g_3(v)g_4(v)g_5(v)+\frac{v\left(g_{3v}(v)g_5(v)+g_3(v)g_{5v}(v)\right)}{\left(1-\rho_{IS}^2\right)\left(av+b\right)^2}\\&\triangleq g_6(v),\end{aligned} \tag{3.42}$$

where

$$g_1(v)=\frac{(av-b)\left(\rho_{IS}\left(av^{\frac12}+bv^{-\frac12}\right)-\sigma_I\right)}{\sigma_I\left(1-\rho_{IS}^2\right)\left(av+b\right)^3}, \tag{3.43}$$

$$g_2(v)=\frac{(av-b)\left(\sigma_I-\frac12\rho_{IS}\left(av^{\frac12}+bv^{-\frac12}\right)\right)}{\sigma_I\left(1-\rho_{IS}^2\right)\left(av+b\right)^3}, \tag{3.44}$$

$$g_3(v)=R+\kappa\left(av+b\right)-r-\mu_I+\sigma_I^2-\rho_{IS}\sigma_I\left(a\sqrt{v}+\frac{b}{\sqrt{v}}\right), \tag{3.45}$$

$$g_{3v}(v) = a\kappa - \frac{1}{2}\rho_{IS}\sigma_I\left(av^{-\frac{1}{2}} - bv^{-\frac{3}{2}}\right), \tag{3.46}$$

$$g_4(v) = \frac{b - av}{\left(1 - \rho_{IS}^2\right)(av + b)^3}, \tag{3.47}$$

$$g_5(v) = av^{\frac{1}{2}} + bv^{-\frac{1}{2}} - \sigma_I\rho_{IS}, \tag{3.48}$$

$$g_{5v}(v) = \frac{1}{2}\left(av^{-\frac{1}{2}} - bv^{-\frac{3}{2}}\right). \tag{3.49}$$

Secondly, based on (3.38), we obtain

$$G_{1v}(t,v) = \frac{\partial G_1(t,v)}{\partial v} = \frac{\delta}{2(1-\delta)^2}\frac{\partial\left(\boldsymbol{\eta}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\Phi}\right)}{\partial v} = \frac{\delta}{2(1-\delta)^2}g_6(v). \tag{3.50}$$

Meanwhile,

$$\begin{aligned}
f_v^{(0)} &= f_v^{(0)}(t,v) \\
&= \alpha_3^{\frac{1}{1-\delta}}\mathrm{e}^{\int_t^T G_1(s,v)\mathrm{d}s}\cdot\int_t^T G_{1v}(s,v)\mathrm{d}s + \int_t^T G_0(s)\mathrm{e}^{\int_t^s G_1(y,v)\mathrm{d}y}\cdot\int_t^s G_{1v}(y,v)\mathrm{d}y\mathrm{d}s \\
&= \alpha_3^{\frac{1}{1-\delta}}\frac{\delta(T-t)g_6(v)}{2(1-\delta)^2}\mathrm{e}^{\int_t^T G_1(s,v)\mathrm{d}s} + \int_t^T\frac{G_0(s)\delta g_6(v)(s-t)}{2(1-\delta)^2}\mathrm{e}^{\int_t^s G_1(y,v)\mathrm{d}y}\mathrm{d}s \\
&\triangleq g_7(t,v).
\end{aligned} \tag{3.51}$$

Based on (3.41) and (3.51), define

$$G_2(t,v) = G_0(t) + \frac{\delta\sigma_1\sqrt{v}\rho_{Sv}\boldsymbol{\eta}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\Phi}}{1-\delta}f_v^{(0)}. \tag{3.52}$$

The formula (3.41) can be seen as an ordinary differential equation with respect to t, with the boundary condition $f^{(1)}(T,v) = \alpha_3^{\frac{1}{1-\delta}}$. Therefore, (3.40) follows easily, confirming the lemma. □

Lemma 3.3 *The solution for* (3.34) *is given by*

$$f^{(2)}(t,v) = \alpha_3^{\frac{1}{1-\delta}}\mathrm{e}^{\int_t^T G_1(s,v)\mathrm{d}s} + \int_t^T G_3(s,v)\mathrm{e}^{\int_t^s G_1(y,v)\mathrm{d}y}\mathrm{d}s, \tag{3.53}$$

where the definitions of $G_1(t,v)$ *and* $G_3(t,v)$ *can be found in* (3.38) *and* (3.59), *respectively.*

Proof From (3.25)–(3.27) and (3.37)–(3.38), (3.34) can be rewritten as follows:

$$\begin{aligned}
&f_t^{(2)} + G_1(t,v)f^{(2)} + G_0(t) + \bar{a}\left(\bar{b} - v\right)f_v^{(0)} + \frac{\sigma_1^2 v}{2}f_{vv}^{(0)} + \frac{\delta\sigma_1\sqrt{v}\rho_{Sv}\boldsymbol{\eta}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\Phi}}{1-\delta}f_v^{(1)} \\
&+ \frac{\delta\sigma_1^2 v\left(\rho_{Sv}^2\boldsymbol{\Phi}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\Phi} - 1\right)\left(f_v^{(0)}\right)^2}{2f^{(0)}} = 0,
\end{aligned} \tag{3.54}$$

with the boundary condition $f^{(2)}(T,v) = \alpha_3^{\frac{1}{1-\delta}}$.

First, we define

$$\begin{aligned}
G_3(t,v) = {}& G_0(t) + \bar{a}\left(\bar{b} - v\right)f_v^{(0)} + \frac{1}{2}\sigma_1^2 v f_{vv}^{(0)} + \frac{\delta\sigma_1\sqrt{v}\rho_{Sv}\boldsymbol{\eta}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\Phi}}{1-\delta}f_v^{(1)} \\
&+ \frac{\delta\sigma_1^2 v\left(\rho_{Sv}^2\boldsymbol{\Phi}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\Phi} - 1\right)\left(f_v^{(0)}\right)^2}{2f^{(0)}},
\end{aligned} \tag{3.55}$$

in which $G_0(t)$, $f_v^{(0)}$ can be found in (3.37) and (3.51).

Then, based on (3.51), we further differentiate to obtain

$$\begin{aligned}
f_{vv}^{(0)} &= f_{vv}^{(0)}(t,v)\\
&= \alpha_3^{\frac{1}{1-\delta}} \frac{\delta\left(T-t\right)}{2\left(1-\delta\right)^2} \mathrm{e}^{\int_t^T G_1(s,v)\mathrm{d}s} \left(g_{6v}(v) + g_6(v) \int_t^T G_{1v}(s,v)\mathrm{d}s \right)\\
&\quad + \frac{\delta}{2\left(1-\delta\right)^2} \int_t^T G_0(s)\left(s-t\right) \mathrm{e}^{\int_t^s G_1(y,v)\mathrm{d}y} \left(g_{6v}(v) + g_6(v) \int_t^s G_{1v}(y,v)\mathrm{d}y \right) \mathrm{d}s\\
&= \alpha_3^{\frac{1}{1-\delta}} \frac{\delta\left(T-t\right)}{2\left(1-\delta\right)^2} \mathrm{e}^{\int_t^T G_1(s,v)\mathrm{d}s} \left(g_{6v}(v) + \frac{\delta(T-t)}{2(1-\delta)^2} g_6^2(v) \right)\\
&\quad + \frac{\delta}{2\left(1-\delta\right)^2} \int_t^T G_0(s)\left(s-t\right) \mathrm{e}^{\int_t^s G_1(y,v)\mathrm{d}y} \left(g_{6v}(v) + \frac{\delta\left(s-t\right)}{2\left(1-\delta\right)^2} g_6^2(v) \right) \mathrm{d}s\\
&\triangleq g_8(t,v),
\end{aligned} \tag{3.56}$$

where according to (3.42)–(3.49),

$$\begin{aligned}
g_{6v}(v) = {} & \left(R-r-\mu_I+\sigma_I^2\right)\left(g_{2v}(v)g_5(v)+g_2(v)g_{5v}(v)-\sigma_I\rho_{IS}g_{1v}(v)\right)\\
&+ \frac{\left(\left(R-r-\mu_I+\sigma_I^2\right)g_{5v}(v)-\sigma_I\rho_{IS}g_{3v}(v)\right)\left(\rho_{IS}\left(3av^{\frac{1}{2}}+bv^{-\frac{1}{2}}\right)-2\sigma_I\right)}{2\sigma_I\left(1-\rho_{IS}^2\right)\left(av+b\right)^2}\\
&+ \frac{\left(\left(R-r-\mu_I+\sigma_I^2\right)g_{5vv}(v)-\sigma_I\rho_{IS}g_{3vv}(v)\right)\left(\rho_{IS}\left(av^{\frac{3}{2}}+bv^{\frac{1}{2}}\right)-\sigma_I v\right)}{\sigma_I\left(1-\rho_{IS}^2\right)\left(av+b\right)^2}\\
&+ \frac{2a\left(\sigma_I\rho_{IS}g_{3v}(v)-\left(R-r-\mu_I+\sigma_I^2\right)g_{5v}(v)\right)\left(\rho_{IS}\left(av^{\frac{3}{2}}+bv^{\frac{1}{2}}\right)-\sigma_I v\right)}{\sigma_I\left(1-\rho_{IS}^2\right)\left(av+b\right)^3}\\
&- \frac{g_{3v}\sigma_I\rho_{IS}\left(\rho_{IS}\left(3av^{\frac{1}{2}}+bv^{-\frac{1}{2}}\right)-2\sigma_I\right)}{2\sigma_I\left(1-\rho_{IS}^2\right)\left(av+b\right)^2} - \sigma_I\rho_{IS}g_{2v}(v)g_3(v)\\
&- \sigma_I\rho_{IS}g_{3v}(v)g_2(v)+g_{3v}(v)g_4(v)g_5(v)+g_3(v)\left(g_{4v}(v)g_5(v)+g_4(v)g_{5v}(v)\right)\\
&+ \frac{v\left(g_{3vv}(v)g_5(v)+g_{3v}(v)g_{5v}(v)\right)+g_{3v}(v)g_5(v)}{\left(1-\rho_{IS}^2\right)\left(av+b\right)^2} - \frac{2avg_{3v}(v)g_5(v)}{\left(1-\rho_{IS}^2\right)\left(av+b\right)^3}\\
&+ \frac{v\left(g_{5vv}(v)g_3(v)+g_{5v}(v)g_{3v}(v)\right)+g_{5v}(v)g_3(v)}{\left(1-\rho_{IS}^2\right)\left(av+b\right)^2} - \frac{2avg_{5v}(v)g_3(v)}{\left(1-\rho_{IS}^2\right)\left(av+b\right)^3},
\end{aligned}$$

and

$$\begin{aligned}
g_{1v}(v) = {} & \frac{a\left(\rho_{IS}\left(av^{\frac{1}{2}}+bv^{-\frac{1}{2}}\right)-\sigma_I\right)+\frac{1}{2}\rho_{IS}\left(av-b\right)\left(av^{-\frac{1}{2}}-bv^{-\frac{3}{2}}\right)}{\sigma_I\left(1-\rho_{IS}^2\right)\left(av+b\right))^3}\\
&- \frac{3a\left(av-b\right)\left(\rho_{IS}\left(av^{\frac{1}{2}}+bv^{-\frac{1}{2}}\right)-\sigma_I\right)}{\sigma_I\left(1-\rho_{IS}^2\right)\left(av+b\right)^4},\\
g_{2v}(v) = {} & \frac{a\left(\sigma_I-\frac{1}{2}\rho_{IS}\left(av^{\frac{1}{2}}+bv^{-\frac{1}{2}}\right)\right)-\frac{1}{4}\rho_{IS}\left(av-b\right)\left(av^{-\frac{1}{2}}-bv^{-\frac{3}{2}}\right)}{\sigma_I\left(1-\rho_{IS}^2\right)\left(av+b\right)^3}\\
&- \frac{3a\left(av-b\right)\left(\sigma_I-\frac{1}{2}\rho_{IS}\left(av^{\frac{1}{2}}+bv^{-\frac{1}{2}}\right)\right)}{\sigma_I\left(1-\rho_{IS}^2\right)\left(av+b\right)^4},
\end{aligned}$$

$$g_{3vv}(v) = \frac{1}{4}\rho_{IS}\sigma_I\left(av^{-\frac{3}{2}} - 3bv^{-\frac{5}{2}}\right),$$
$$g_{4v}(v) = \frac{3a(av-b) - a(av+b)}{(1-\rho_{IS}^2)(av+b)^4},$$
$$g_{5vv}(v) = \frac{1}{4}\left(3bv^{-\frac{5}{2}} - av^{-\frac{3}{2}}\right).$$

Third, from (3.40) and (3.50), we obtain

$$\begin{aligned}
f_v^{(1)} &= f_v^{(1)}(t,v) \\
&= \alpha_3^{\frac{1}{1-\delta}} e^{\int_t^T G_1(s,v)\mathrm{d}s} \cdot \int_t^T G_{1v}(s,v)\mathrm{d}s + \int_t^T G_{2v}(s,v) e^{\int_t^s G_1(y,v)\mathrm{d}y}\mathrm{d}s \\
&\quad + \int_t^T G_2(s,v) e^{\int_t^s G_1(y,v)\mathrm{d}y} \cdot \int_t^s G_{1v}(y,v)\mathrm{d}y\mathrm{d}s \\
&= \alpha_3^{\frac{1}{1-\delta}} \frac{\delta(T-t)g_6(v)}{2(1-\delta)^2} e^{\int_t^T G_1(s,v)\mathrm{d}s} + \int_t^T G_{2v}(s,v) e^{\int_t^s G_1(y,v)\mathrm{d}y}\mathrm{d}s \\
&\quad + \int_t^T \frac{G_2(s,v)\delta g_6(v)(s-t)}{2(1-\delta)^2} e^{\int_t^s G_1(y,v)\mathrm{d}y}\mathrm{d}s \\
&\triangleq g_9(t,v),
\end{aligned} \tag{3.57}$$

where

$$\begin{aligned}
G_{2v}(t,v) &= \frac{\delta\sigma_1\rho_{Sv}\boldsymbol{\eta}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\Phi}}{2(1-\delta)\sqrt{v}} f_v^{(0)} + \frac{\delta\sigma_1\rho_{Sv}\sqrt{v}}{1-\delta} \frac{\partial\left(\boldsymbol{\eta}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\Phi}\right)}{\partial v} f_v^{(0)} \\
&\quad + \frac{\delta\sigma_1\rho_{Sv}\sqrt{v}\boldsymbol{\eta}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\Phi}}{1-\delta} f_{vv}^{(0)} \\
&= \frac{\delta\sigma_1\rho_{Sv}\boldsymbol{\eta}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\Phi}}{2(1-\delta)\sqrt{v}} g_7(t,v) + \frac{\delta\sigma_1\rho_{Sv}\sqrt{v}}{1-\delta} g_6(v) g_7(t,v) \\
&\quad + \frac{\delta\sigma_1\rho_{Sv}\sqrt{v}\boldsymbol{\eta}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\Phi}}{1-\delta} g_8(t,v).
\end{aligned} \tag{3.58}$$

Ultimately, using (3.54), we define

$$\begin{aligned}
G_3(t,v) &= G_0(t) + \bar{a}\left(\bar{b}-v\right) f_v^{(0)} + \frac{1}{2}\sigma_1^2 v f_{vv}^{(0)} + \frac{\delta\sigma_1\sqrt{v}\rho_{Sv}\boldsymbol{\eta}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\Phi}}{1-\delta} f_v^{(1)} \\
&\quad + \frac{\delta\sigma_1^2 v\left(\rho_{Sv}^2\boldsymbol{\Phi}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\Phi} - 1\right)\left(f_v^{(0)}\right)^2}{2f^{(0)}} \\
&= G_0(t) + \bar{a}\left(\bar{b}-v\right) g_7(t,v) + \frac{\sigma_1^2 v g_8(t,v)}{2} + \frac{\delta\sigma_1\sqrt{v}\rho_{Sv}\boldsymbol{\eta}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\Phi}}{1-\delta} g_9(t,v) \\
&\quad + \frac{\delta\sigma_1^2 v\left(\rho_{Sv}^2\boldsymbol{\Phi}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\Phi} - 1\right) g_7^2(t,v)}{2f^{(0)}}.
\end{aligned} \tag{3.59}$$

The formula (3.54) becomes

$$f_t^{(2)} + G_1(t,v) f^{(2)} + G_3(t,v) = 0, \tag{3.60}$$

with the boundary condition $f^{(2)}(T,v) = \alpha_3^{\frac{1}{1-\delta}}$. Using the method for solving (3.39), (3.53) can similarly easily obtained, and the proof is complete. □

Theorem 3.4 *Let us assume that for any* $t \in [0,T]$, $X(t)+A(t)+B(t)h(t)>0$. *Under the real habit level, a solution to HJB equation* (3.2) *with the boundary condition* $\tilde{V}(T,x,v,h) = \alpha_3 \cdot \frac{x^\delta}{\delta} (\alpha_3 > 0, \delta < 1, \delta \neq 0)$ *is given by*

$$\tilde{V}(t,x,v,h) = (f^\epsilon(t,v))^{1-\delta} \frac{(x+A(t)+B(t)h)^\delta}{\delta}. \tag{3.61}$$

The definitions of $A(t)$ *and* $B(t)$ *can be found in* (3.21) *and* (3.22). $f^\epsilon(t,v) = f^{(0)}(t,v) + \sqrt{\epsilon} f^{(1)}(t,v) + \epsilon f^{(2)}(t,v)$, $f^{(0)}(t,v)$, $f^{(1)}(t,v)$ *and* $f^{(2)}(t,v)$ *can be found in* (3.35), (3.40) *and* (3.53). *Candidate optimal consumption and life insurance premium rates are given by*

$$c^*(t) = h + \frac{X(t)+A(t)+B(t)h(t)}{f^\epsilon(t,v)} \left(\frac{\alpha_1}{1-\xi_1 B(t)} \right)^{\frac{1}{1-\delta}}, \tag{3.62}$$

$$p^*(t) = \frac{X(t)+A(t)+B(t)h(t)}{f^\epsilon(t,v)} (\alpha_2 \lambda(t))^{\frac{1}{1-\delta}} \zeta(t)^{\frac{\delta}{\delta-1}} - \zeta(t) X(t). \tag{3.63}$$

Candidate optimal portfolio weights invested on the risk-free asset, stock, and index bond can be written as follows:

$$\begin{aligned} \pi_0^*(t) = \frac{X(t)+A(t)+B(t)h(t)}{(1-\delta)X(t)} \Bigg[& \left(g_3(v) g_{10}(v) + \left(R - r - \mu_I + \sigma_I^2 \right) g_{11}(v) \right) \\ & + \sigma_1 \rho_{Sv} \sqrt{v} \left(\left(a\sqrt{v} + \frac{b}{\sqrt{v}} - \sigma_I \rho_{IS} \right) g_{10}(v) - \sigma_I \rho_{IS} g_{11}(v) \right) \Bigg], \end{aligned} \tag{3.64}$$

$$\begin{aligned} \pi_1^*(t) = \frac{X(t)+A(t)+B(t)h(t)}{(1-\delta)X(t)} \Bigg[& \left(R - r - \mu_I + \sigma_I^2 \right) g_{10}(v) + \frac{g_3(v)}{(1-\rho_{IS}^2)\left(a\sqrt{v} + \frac{b}{\sqrt{v}} \right)^2} \\ & + \frac{\sigma_1 \rho_{Sv} \sqrt{v} \left(a\sqrt{v} + \frac{b}{\sqrt{v}} - \sigma_I \rho_{IS} \right)}{(1-\rho_{IS}^2)\left(a\sqrt{v} + \frac{b}{\sqrt{v}} \right)^2} - \sigma_1 \rho_{Sv} \sqrt{v} \sigma_I \rho_{IS} g_{10}(v) \Bigg], \end{aligned} \tag{3.65}$$

and

$$\begin{aligned} \pi_2^*(t) = 1 - \frac{X(t)+A(t)+B(t)h(t)}{(1-\delta)X(t)} \Bigg[& g_3(v) g_{10}(v) + \frac{g_3(v)}{(1-\rho_{IS}^2)\left(a\sqrt{v} + \frac{b}{\sqrt{v}} \right)^2} \\ & + \frac{\sigma_1 \rho_{Sv} \sqrt{v} \left(a\sqrt{v} + \frac{b}{\sqrt{v}} - \sigma_I \rho_{IS} \right)}{(1-\rho_{IS}^2)\left(a\sqrt{v} + \frac{b}{\sqrt{v}} \right)^2} - \sigma_1 \rho_{Sv} \sqrt{v} \sigma_I \rho_{IS} g_{10}(v) \\ & + \sigma_1 \rho_{Sv} \sqrt{v} \left(\left(a\sqrt{v} + \frac{b}{\sqrt{v}} - \sigma_I \rho_{IS} \right) g_{10}(v) - \sigma_I \rho_{IS} g_{11}(v) \right) \\ & + \left(R - r - \mu_I + \sigma_I^2 \right) \left(g_{10}(v) + g_{11}(v) \right) \Bigg], \end{aligned} \tag{3.66}$$

where

$$g_{10}(v) = \frac{\rho_{IS} \left(a\sqrt{v} + \frac{b}{\sqrt{v}} \right) - \sigma_I}{\sigma_I (1-\rho_{IS}^2) \left(a\sqrt{v} + \frac{b}{\sqrt{v}} \right)^2}, \tag{3.67}$$

$$g_{11}(v)=\frac{\left(a\sqrt{v}+\frac{b}{\sqrt{v}}\right)^2-2\left(a\sqrt{v}+\frac{b}{\sqrt{v}}\right)\sigma_I\rho_{IS}+\sigma_I^2}{\sigma_I^2\left(1-\rho_{IS}^2\right)\left(a\sqrt{v}+\frac{b}{\sqrt{v}}\right)^2}. \tag{3.68}$$

Proof Combining (3.3), (3.8), (3.10), (3.13), and (3.30) and Lemmas 3.1–3.3, we obtain

$$\begin{aligned}
c^*(t) &= h(t)+\left(\frac{\tilde{V}_x-\xi_1\tilde{V}_h}{\alpha_1}\right)^{\frac{1}{\delta-1}}\\
&= h(t)+\left(\frac{mY^{\delta-1}Y_x-\xi_1 mY^{\delta-1}Y_h}{\alpha_1}\right)^{\frac{1}{\delta-1}}\\
&= h(t)+\left(\frac{f^{1-\delta}Y^{\delta-1}-\xi_1 B(t)f^{1-\delta}Y^{\delta-1}}{\alpha_1}\right)^{\frac{1}{\delta-1}}\\
&= h(t)+\alpha_1^{\frac{1}{1-\delta}}\left(X(t)+A(t)+B(t)h(t)\right)\left(f^{\epsilon}(t,v)\right)^{-1}\left(1-\xi_1 B(t)\right)^{\frac{1}{\delta-1}}\\
&= h(t)+\frac{X(t)+A(t)+B(t)h(t)}{f^{\epsilon}(t,v)}\left(\frac{\alpha_1}{1-\xi_1 B(t)}\right)^{\frac{1}{1-\delta}},
\end{aligned} \tag{3.69}$$

where the definitions of $A(t)$ and $B(t)$ can be found in (3.21) and (3.22). Combining (3.4), (3.8), (3.10), (3.13), (3.14), and (3.30) and Lemmas 3.1–3.3, we obtain the following results:

$$\begin{aligned}
p^*(t) &= \left(\frac{\tilde{V}_x\zeta(t)}{\alpha_2\lambda(t)}\right)^{\frac{1}{\delta-1}}\zeta(t)-\zeta(t)X(t)\\
&= \left(\frac{mY^{\delta-1}Y_x\zeta(t)}{\alpha_2\lambda(t)}\right)^{\frac{1}{\delta-1}}\zeta(t)-\zeta(t)X(t)\\
&= \left(\frac{f^{1-\delta}Y^{\delta-1}\zeta(t)}{\alpha_2\lambda(t)}\right)^{\frac{1}{\delta-1}}\zeta(t)-\zeta(t)X(t)\\
&= \frac{X(t)+A(t)+B(t)h(t)}{f^{\epsilon}(t,v)}\left(\alpha_2\lambda(t)\right)^{\frac{1}{1-\delta}}\zeta(t)^{\frac{\delta}{\delta-1}}-\zeta(t)X(t).
\end{aligned} \tag{3.70}$$

By applying (3.5), (3.8), (3.10), (3.13), and (3.14), we obtain

$$\begin{aligned}
\boldsymbol{\pi}^*(t) &= \left(\pi_0(t),\pi_1(t)\right)'\\
&= -\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\eta}\frac{1}{X(t)}\frac{\tilde{V}_x}{\tilde{V}_{xx}}-\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\Phi}\sigma_1\sqrt{v}\frac{\rho_{Sv}}{X(t)}\frac{\tilde{V}_{xv}}{\tilde{V}_{xx}}\\
&= -\left[\frac{\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\eta}}{X(t)}+\frac{\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\Phi}\sigma_1\sqrt{v}\rho_{Sv}}{X(t)}\right]\frac{mY^{\delta-1}Y_x}{mY^{\delta-1}Y_{xx}+m\left(\delta-1\right)Y^{\delta-2}Y_x^2}\\
&= -\left[\frac{\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\eta}}{X(t)}+\frac{\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\Phi}\sigma_1\sqrt{v}\rho_{Sv}}{X(t)}\right]\frac{YY_x}{YY_{xx}+\left(\delta-1\right)Y_x^2}\\
&= \left[\frac{\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\eta}}{X(t)}+\frac{\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\Phi}\sigma_1\sqrt{v}\rho_{Sv}}{X(t)}\right]\frac{X(t)+A(t)+B(t)h(t)}{1-\delta}\\
&= \left[\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\eta}+\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\Phi}\sigma_1\sqrt{v}\rho_{Sv}\right]\frac{X(t)+A(t)+B(t)h(t)}{\left(1-\delta\right)X(t)}.
\end{aligned}$$

Thus, we obtain (3.64), (3.65), (3.67), and (3.68). By incorporating $\pi_0(t)+\pi_1(t)+\pi_2(t)=1$ along with (3.64) and (3.65), we obtain (3.66). Thus, the theorem is verified. □

Theorem 3.6 verifies that a solution for the HJB equation (3.2) is indeed the solution for the problem (2.10). We first introduce a lemma (see [9] and [17]).

Lemma 3.5 *Assume that* $\tilde{V}(t,x,v,h)$ *is given by* (3.61)*, then for any* $t\in[0,T]$*,*

$$\mathbb{E}[\tilde{V}^2(t,X(t),v(t),h(t))]<\infty, \tag{3.71}$$

$$\mathbb{E}[\tilde{V}_x^2(t,X(t),v(t),h(t))]<\infty, \tag{3.72}$$

$$\mathbb{E}[\tilde{V}_v^2(t,X(t),v(t),h(t))]<\infty. \tag{3.73}$$

Proof First, by deriving the expression for $\tilde{V}(t,x,v,h)$ from (3.61) and following the proof process outlined in [33], we substitute (3.62), (3.63), and (3.64)–(3.66) into (2.8), yielding the following result:

$$\begin{aligned}&\frac{\mathrm{d}\left(X(t)+A(t)+B(t)h(t)\right)}{X(t)+A(t)+B(t)h(t)}\\ =&\left(\zeta(t)+r-\frac{g_{12}(t)}{f^{\epsilon}(t,v)}+\frac{1}{1-\delta}\left(\boldsymbol{\eta}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\eta}+\sigma_1\sqrt{v}\rho_{Sv}\boldsymbol{\Phi}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\eta}\right)\right)\mathrm{d}t\\ &+\frac{\boldsymbol{\eta}'\left(\boldsymbol{\Sigma}'\right)^{-1}+\sigma_1\sqrt{v}\rho_{Sv}\boldsymbol{\Phi}'\left(\boldsymbol{\Sigma}'\right)^{-1}}{1-\delta}\mathrm{d}\boldsymbol{W(t)},\end{aligned}$$

where $g_{12}(t)=\alpha_1^{\frac{1}{1-\delta}}\left(1-\xi_1B(t)\right)^{\frac{1}{\delta-1}}+\left(\alpha_2\lambda(t)\right)^{\frac{1}{1-\delta}}\left(\zeta(t)\right)^{\frac{\delta}{\delta-1}}$.

Then, we can obtain

$$\begin{aligned}X(t)+A(t)+B(t)h(t)=\ &\left(X(0)+A(0)+B(0)h(0)\right)\cdot\exp\left(\int_0^t\tilde{Q}(u)\mathrm{d}u+\right.\\ &\left.\frac{\boldsymbol{\eta}'\left(\boldsymbol{\Sigma}'\right)^{-1}+\sigma_1\sqrt{v}\rho_{Sv}\boldsymbol{\Phi}'\left(\boldsymbol{\Sigma}'\right)^{-1}}{1-\delta}\mathrm{d}\boldsymbol{W(t)}\right),\end{aligned}\tag{3.74}$$

where

$$\begin{aligned}\tilde{Q}(t)=&\,\zeta(t)+r-\frac{g_{11}(t)}{f^{\epsilon}(t,v)}+\frac{1}{1-\delta}\left[\boldsymbol{\eta}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\eta}+\sigma_1\sqrt{v}\rho_{Sv}\boldsymbol{\Phi}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\eta}\right]\\ &-\frac{\boldsymbol{\eta}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\eta}+2\sigma_1\sqrt{v}\rho_{Sv}\boldsymbol{\Phi}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\eta}+\sigma_1v\rho_{Sv}^2\boldsymbol{\Phi}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\Phi}}{2\left(1-\delta\right)^2}\end{aligned}\tag{3.75}$$

is a deterministic and bounded function of $[0,T]$. Then, by substituting (3.74) into (3.61), since

$$\begin{aligned}&\mathbb{E}\left[\exp\left(2\delta\frac{\boldsymbol{\eta}'\left(\boldsymbol{\Sigma}'\right)^{-1}+\sigma_1\sqrt{v}\rho_{Sv}\boldsymbol{\Phi}'\left(\boldsymbol{\Sigma}'\right)^{-1}}{1-\delta}\boldsymbol{W(t)}\right)\right]\\ =&\exp\left(2\delta^2\frac{\boldsymbol{\eta}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\eta}+2\sigma_1\sqrt{v}\rho_{Sv}\boldsymbol{\Phi}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\eta}+\sigma_1v\rho_{Sv}^2\boldsymbol{\Phi}'\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\right)^{-1}\boldsymbol{\Phi}}{\left(1-\delta\right)^2}t\right),\end{aligned}$$

we can obtain $\mathbb{E}[\tilde{V}^2(t,X(t),v(t),h(t))]<\infty$ under the strategies (3.62), (3.63), and (3.64)–(3.66). Since $A(t)$ and $B(t)$ are bounded functions on $[0,T]$, together with (3.74), we can similarly prove that the conclusions (3.72) and (3.73) are true. □

Theorem 3.6 *Assume that for any* $t\in[0,T]$*,* $X(t)+A(t)+B(t)h(t)>0$*. Under the habit level, if* $\tilde{V}(t,x,v,h)$ *is a solution for the HJB equation* (3.2) *with the boundary condition* $\tilde{V}(T,x,v,h)=\alpha_3\cdot\frac{x^{\delta}}{\delta}(\alpha_3>0,\delta<1,\delta\neq0)$*, then the value function* $V(t,x,v,h)=\tilde{V}(t,x,v,h)$*, and the optimal strategies are given by* (3.62)*,* (3.63)*, and* (3.64)–(3.66).

Proof First, for any admissible strategy $\boldsymbol{\pi}$,

$$\int_0^t \tilde{V}_x(s, X(s), v(s), h(s)) X(s) \boldsymbol{\pi'(s)} \boldsymbol{\Sigma} \mathrm{d}\boldsymbol{W(t)},$$

is a martingale (see [40]) by (3.72). By (3.73),

$$\int_0^t \tilde{V}_v(s, X(s), v(s), h(s)) \sigma_1 \sqrt{v(s)} \mathrm{d}W_v(s),$$

is also a martingale. Therefore, applying the Dynkin formula, we obtain the following results:

$$\begin{aligned} &\mathbb{E}_t \left[\mathrm{e}^{-\int_t^T (\rho+\lambda(u))\mathrm{d}u} \tilde{V}(T, X(T), v(T), h(T)) \right] - \tilde{V}(t, x, v, h) \\ &= \mathbb{E}_t \left[\int_t^T \mathrm{e}^{-\int_t^s (\rho+\lambda(u))\mathrm{d}u} \mathcal{A}_1 \tilde{V}(s, X(s), v(s), h(s)) \mathrm{d}s \right] \\ &\leqslant -\mathbb{E}_t \left[\int_t^T \mathrm{e}^{\int_t^s (\rho+\lambda(u))\mathrm{d}u} \left(\alpha_1 U\left(c(s) - h(s)\right) + \alpha_2 \lambda(s) U\left(X(s) + \frac{p(s)}{\zeta(s)} \right) \right) \mathrm{d}s \right], \end{aligned} \tag{3.76}$$

where $U(x) = \frac{x^\delta}{\delta}$, $\delta < 1, \delta \neq 0$. Consequently, we derive

$$\begin{aligned} &\tilde{V}(t, x, v, h) \\ &\geqslant \mathbb{E}_t \Bigg[\int_t^T \mathrm{e}^{\int_t^s (\rho+\lambda(u))\mathrm{d}u} \left(\alpha_1 U\left(c(s) - h(s)\right) + \alpha_2 \lambda(s) U\left(X(s) + \frac{k(s)}{\zeta(s)} \right) \right) \mathrm{d}s \\ &\quad + \mathrm{e}^{-\int_t^T (\rho+\lambda(u))\mathrm{d}u} \tilde{V}(T, X(T), Y(T), h(T)) \Bigg] \\ &= \mathbb{E}_t \Bigg[\int_t^T \mathrm{e}^{\int_t^s (\rho+\lambda(u))\mathrm{d}u} \left(\alpha_1 U\left(c(s) - h(s)\right) + \alpha_2 \lambda(s) U\left(X(s) + \frac{k(s)}{\zeta(s)} \right) \right) \mathrm{d}s \\ &\quad + \mathrm{e}^{-\int_t^T (\rho+\lambda(u))\mathrm{d}u} \alpha_3 U(x) \Bigg]. \end{aligned} \tag{3.77}$$

By taking the supremum w.r.t. $(c, k, \boldsymbol{\pi})$ in (3.77) and combining with (2.10), we have

$$\tilde{V}(t, x, v, h) \geqslant V(t, x, v, h). \tag{3.78}$$

Moreover, under the strategy $(c, k, \boldsymbol{\pi})$, the inequalities in (3.76) and (3.77) become equalities. We have

$$\begin{aligned} &\tilde{V}(t, x, v, h) \\ &= \mathbb{E}_t \Bigg[\int_t^T \mathrm{e}^{\int_t^s (\rho+\lambda(u))\mathrm{d}u} \left(\alpha_1 U\left(c(s) - h(s)\right) + \alpha_2 \lambda(s) U\left(X(s) + \frac{k(s)}{\zeta(s)} \right) \right) \mathrm{d}s \\ &\quad + \mathrm{e}^{-\int_t^T (\rho+\lambda(u))\mathrm{d}u} \tilde{V}_1(T, X(T), v(T), h(T)) \Bigg] \\ &\leqslant V(t, x, v, h). \end{aligned} \tag{3.79}$$

Finally, we derive that $\tilde{V}(t, x, v, h) = V(t, x, v, h)$ by combining (3.78) and (3.79). □

4. Numerical analysis

Based on the previous analysis, we have derived asymptotic solutions for optimal consumption, life insurance, and investment that maximize the individual's objective utility function. This section systematically analyzes the impact of habit formation and model parameters on these optimal decisions through numerical simulations. The analysis is divided into two parts: Section 4.1 investigates the influences of consumption habit parameters, and Section 4.2 analyzes the effects of model parameters.

The basic values of the parameters cited in [8, 27] and [33], are listed in Table 1. Based on [27], we can define the following expressions: $\lambda(t) = 0.000959 - 0.00012t + 0.000008t^2$, $\zeta(t) = \lambda(t)$, $y(t) = 67569\mathrm{e}^{0.075t}$, $h_0 = 0.11X(0)$.

Table 1 Values of parameters

t_0	T	σ_1	ξ_1	ξ_2	κ	a	b
30	60	0.6612	0.1	0.134	2.9428	0.9051	0.0023
v_0	α_1	α_2	α_3	δ	ρ	μ_I	σ_I
0.0328	1	300	300	−1.2	0.87	0.023	0.05
ρ_{Sv}	$X(0)$	r	ϵ	ρ_{IS}	$\bar{a}$	$\bar{b}$	R
−0.7689	270270	0.02	0.00001	−0.07	7.3479	0.17	0.04

4.1 Effects of habit formation on the optimal strategies

Inspired by [27], we explore the optimal strategies under four scenarios: nohabit formation, low habit formation, medium habit formation, and high habit formation. Subsequently, we conduct a sensitivity analysis of the variations in optimal consumption, life insurance, and investment decisions among individuals with different degrees of habit formation. The settings of the consumption habit parameters are based on [27], and the specific details are presented in Table 2.

Table 2 Assumptions of consumption habit parameters

No habit formation	Low habit formation	Medium habit formation	High habit formation
$\xi_1 = \xi_2 = 0$	$\xi_1 = \xi_2 = 0.1$	$\xi_1 = \xi_2 = 0.3$	$\xi_1 = \xi_2 = 0.5$

Figure 1 illustrates the optimal consumption paths under the four scenarios in Table 2 by varying the parameters ξ_1 and ξ_2 while keeping other parameters constant. It can be observed that the steepness of the optimal consumption path is influenced by the habit parameters. In general, larger habit parameters correspond to steeper consumption paths in the latter part of the graph. When ξ_1 and ξ_2 are both zero, meaning that habit formation is not considered, the optimal consumption path is relatively smooth.

Figure 2 illustrates the optimal life insurance premium $p^*(t)$ for individuals with different habit formation. We observe that the optimal life insurance premium follows a peak-shaped pattern,

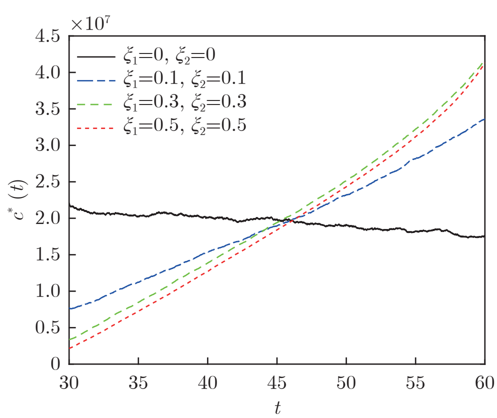

Figure 1 $c^*(t)$ under different levels of consumption habit

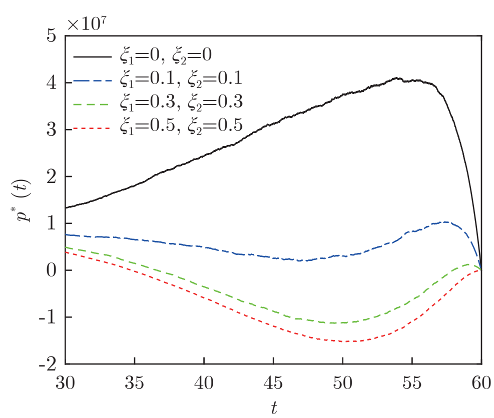

Figure 2 $p^*(t)$ under different levels of consumption habit

initially increasing and then decreasing when ξ_1 and ξ_2 are both zero. This conclusion is also validated by [27]. This could be attributed to the fact that individuals typically have less wealth in the early stages and are more inclined to leave a positive inheritance. Consequently, they tend to increase their life insurance premium to avoid leaving any unpaid debts when they pass away. In contrast, during the later stages of life, as people accumulate wealth and their debt levels decrease, their life insurance premium payments also tend to decline. Figure 2 also shows that habit formation can decrease the amount of life insurance purchased under the parameter settings in the text.

Figures 3 and 4 show that the investment amounts in stock and index bonds increase in the later stages, which is likely driven by the accumulation of personal wealth. Notably, individuals with habit formation allocate less to risky investments than those without habit formation in this late period.

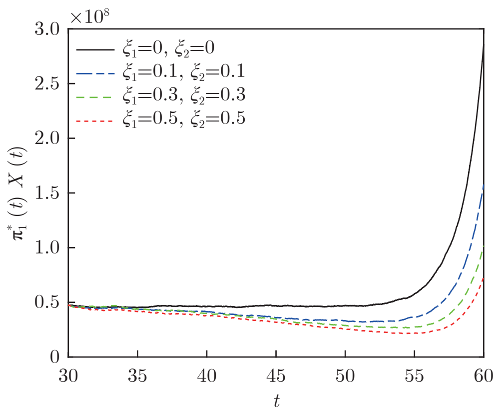

Figure 3 $\pi_1^*(t)X(t)$ under different levels of consumption habit

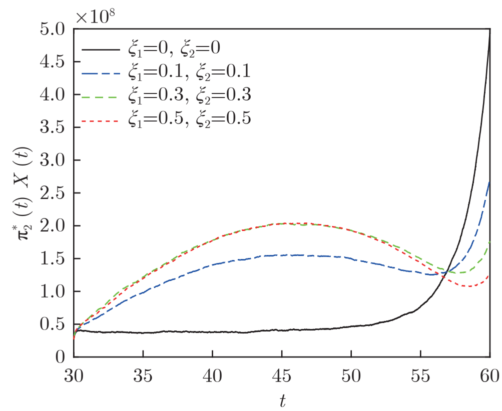

Figure 4 $\pi_2^*(t)X(t)$ under different levels of consumption habit

4.2 Effects of parameters on the optimal strategies

In Figures 5–16, we use the control variable method to examine the impact of parameter variations on individual optimal decision-making. Specifically, we adjust the value of a single target parameter while keeping all other parameters fixed.

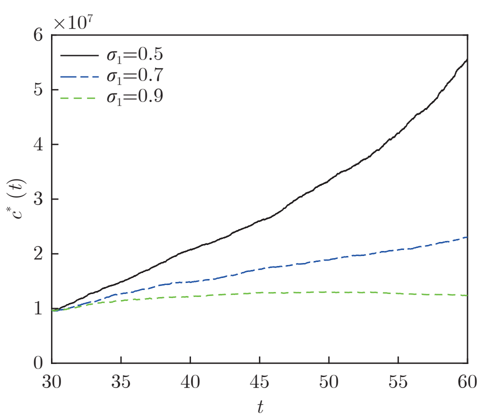

Figure 5 Effect of σ_1 on $c^*(t)$

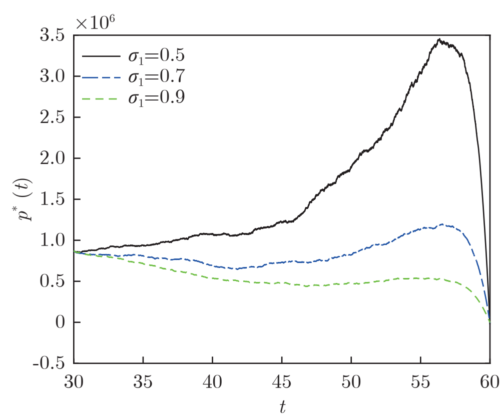

Figure 6 Effect of σ_1 on $p^*(t)$

4.2.1 Effects of σ_1 on the optimal strategies

As shown in Figure 5, we find that the individual's optimal consumption level $c^*(t)$ shows a relatively increasing trend in the latter stages of the graph as σ_1 decreases. Here σ_1 represents the volatility of volatility. At the same time, Figure 6 shows that the individual's optimal life insurance premium $p^*(t)$ increases, as does the amount invested in the stock $\pi_1^*(t)X(t)$ and index bond $\pi_2^*(t)X(t)$, as shown in Figures 7 and 8. This suggests that a decrease in the diffusion coefficient σ_1 encourages the individual to increase their investments and expenditures.

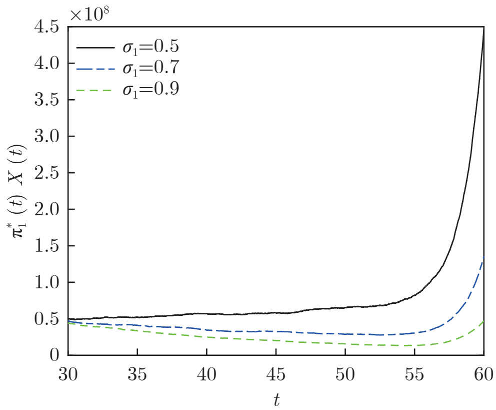

Figure 7 Effect of σ_1 on $\pi_1^*(t)X(t)$

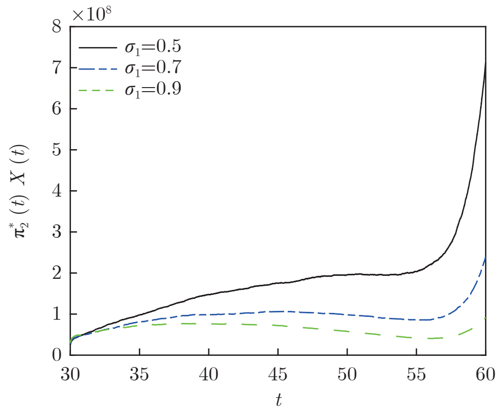

Figure 8 Effect of σ_1 on $\pi_2^*(t)X(t)$

4.2.2 Effects of κ on the optimal strategies

Based on the analysis of Figures 9–12, we find that as the control parameter of excess return κ increases, Figure 9 shows that the individual has a higher $c^*(t)$, while Figure 10 demonstrates that $p^*(t)$ is larger for greater κ before $t = 38$, but becomes smaller for greater κ after $t = 43$. Figures 11 and 12 indicate that $\pi_1^*(t)X(t)$ and $\pi_2^*(t)X(t)$ also increase as κ increases, which means that the individual allocates more assets in the stock and the index bond. This suggests that the increase in κ leads the individual to adopt more proactive investing and consumption behaviors.

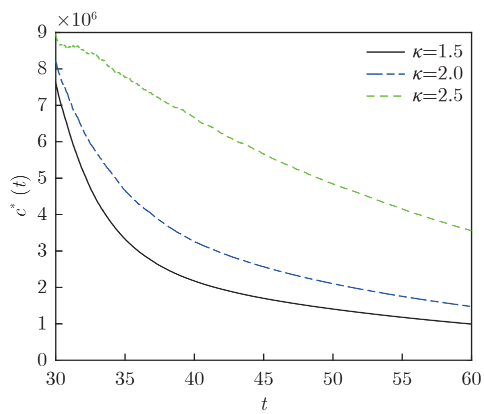

Figure 9 Effect of κ on $c^*(t)$

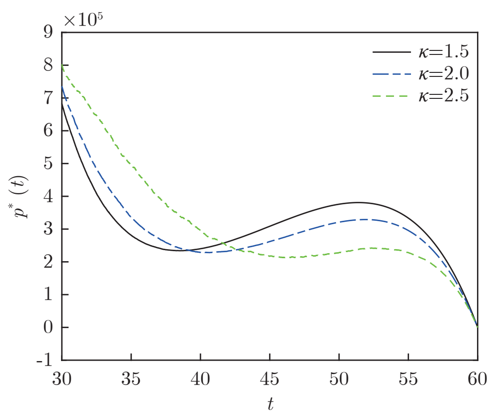

Figure 10 Effect of κ on $p^*(t)$

4.2.3 Effects of ρ_{Sv} on the optimal strategies

Based on the analysis of Figures 13–16, it is evident that the correlation coefficient ρ_{Sv}

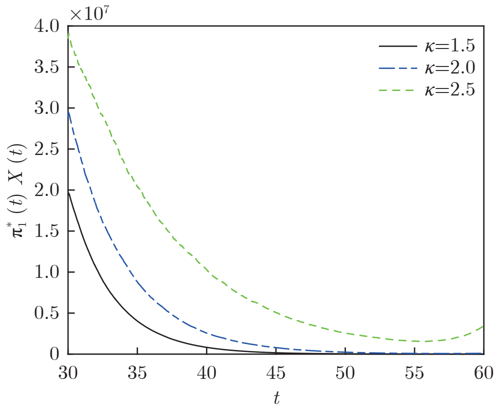

Figure 11 Effect of κ on $\pi_1^*(t)X(t)$

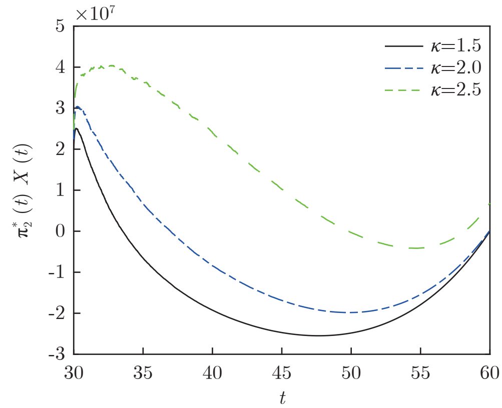

Figure 12 Effect of κ on $\pi_2^*(t)X(t)$

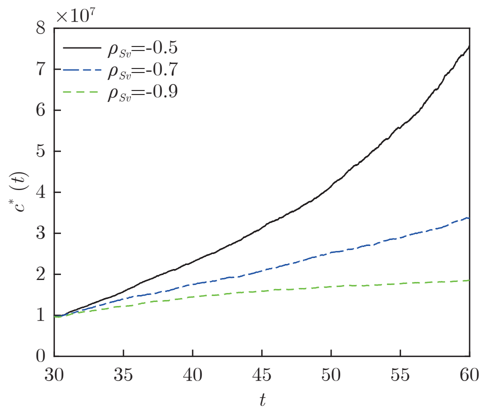

Figure 13 Effect of ρ_{Sv} on $c^*(t)$

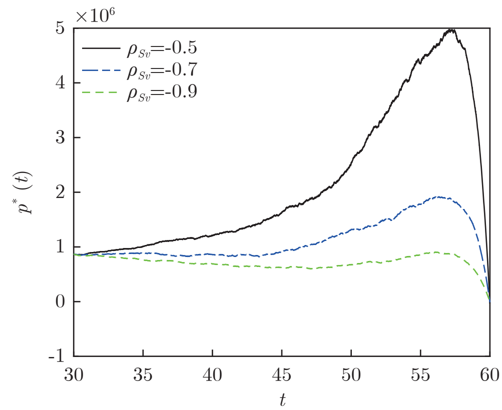

Figure 14 Effect of ρ_{Sv} on $p^*(t)$

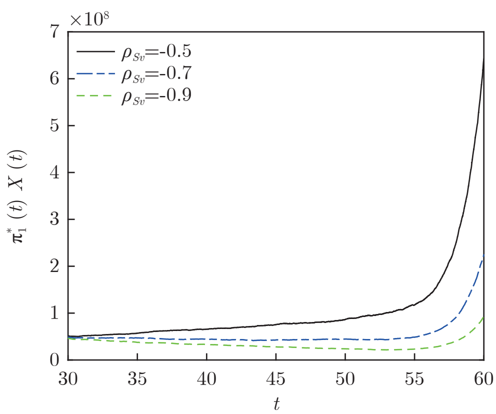

Figure 15 Effect of ρ_{Sv} on $\pi_1^*(t)X(t)$

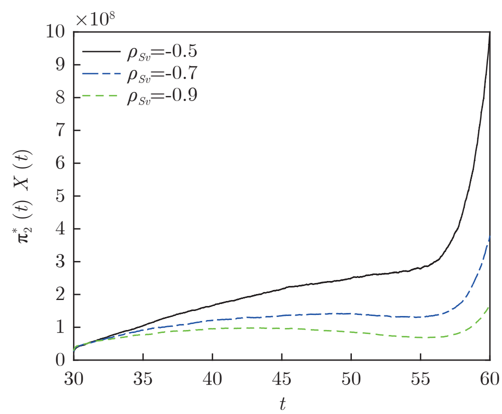

Figure 16 Effect of ρ_{Sv} on $\pi_2^*(t)X(t)$

between W_S and W_v significantly influences $c^*(t)$, $p^*(t)$, $\pi_1^*(t)X(t)$ and $\pi_2^*(t)X(t)$. As illustrated in Figure 13, when the negative correlation between W_S and W_v weakens, $c^*(t)$ increases. Similarly, Figure 14 indicates an upward trend in $p^*(t)$, while Figure 15 demonstrates that $\pi_1^*(t)X(t)$ rises and Figure 16 shows that the optimal investment amount in the index bond also grows. This suggests that the weakening of the negative correlation coefficient ρ_{Sv} leads to more aggressive investment and active consumption behavior by the individual.

5. Conclusion

This study examines individuals' optimal decisions regarding consumption, life insurance, and investment with habit formation under inflation. The financial market includes a risk-free asset, a stock, and an index bond, with the price of the stock following the 4/2 stochastic volatility model. It is challenging to obtain precise solutions for optimal decisions in the CRRA case. Therefore, asymptotic expansion techniques are employed to derive approximate solutions. Numerical results show that different types of consumption habits lead to clear differences in optimal consumption, insurance, and investment strategies, and that key model parameters significantly affect these optimal decisions. These findings provide important implications for personal financial planning, portfolio choice, and life insurance demand. However, this study assumes deterministic income, which limits its realism. Future research could therefore extend the model to incorporate stochastic income, which would require more complex derivations.

Appendix

A The derivation process of (2.10)

Proof of (2.10) Using the law of iterated expectation, (2.9) can be rewritten as follows:

$$
\begin{aligned}
V(t,x,v,h) = & \max_{(c,p,\boldsymbol{\pi})\in\Pi(t)} \mathbb{E}_t\Bigg[\int_t^{\tau\wedge T} \alpha_1 \mathrm{e}^{-\rho(s-t)} U_1\left(c(s)-h(s)\right)\mathrm{d}s \\
& + \alpha_2 \mathrm{e}^{-\rho(\tau-t)} U_2\left(X(\tau)+\frac{p(\tau)}{\zeta(\tau)}\right)\mathbf{I}_{\{\tau\leqslant T\}} + \alpha_3 \mathrm{e}^{-\rho(T-t)} U_3\left(X(T)\right)\mathbf{I}_{\{\tau>T\}}\Bigg] \\
= & \max_{(c,p,\boldsymbol{\pi})\in\Pi(t)} \mathbb{E}_t\Bigg[\int_t^{T} \mathbf{I}_{\{s\leqslant\tau\}}\alpha_1 \mathrm{e}^{-\rho(s-t)} U_1\left(c(s)-h(s)\right)\mathrm{d}s \\
& + \alpha_2 \mathrm{e}^{-\rho(\tau-t)} U_2\left(X(\tau)+\frac{p(\tau)}{\zeta(\tau)}\right)\mathbf{I}_{\{\tau\leqslant T\}} + \alpha_3 \mathrm{e}^{-\rho(T-t)} U_3\left(X(T)\right)\mathbf{I}_{\{\tau>T\}}\Bigg] \\
= & \max_{(c,p,\boldsymbol{\pi})\in\Pi(t)} \mathbb{E}_t\Bigg[\mathbb{E}_t\Bigg[\int_t^{T} \mathbf{I}_{\{s\leqslant\tau\}}\alpha_1 \mathrm{e}^{-\rho(s-t)} U_1\left(c(s)-h(s)\right)\mathrm{d}s \\
& + \alpha_2 \mathrm{e}^{-\rho(\tau-t)} U_2\left(X(\tau)+\frac{p(\tau)}{\zeta(\tau)}\right)\mathbf{I}_{\{\tau\leqslant T\}} + \alpha_3 \mathrm{e}^{-\rho(T-t)} U_3\left(X(T)\right)\mathbf{I}_{\{\tau>T\}}\Bigg]\Bigg] \\
= & \max_{(c,p,\boldsymbol{\pi})\in\Pi(t)} \mathbb{E}_t\Bigg[\int_t^{T} S(s,t)\alpha_1 \mathrm{e}^{-\rho(s-t)} U_1\left(c(s)-h(s)\right)\mathrm{d}s \\
& + \int_t^T \alpha_2 \mathrm{e}^{-\rho(s-t)} U_2\left(X(s)+\frac{p(s)}{\zeta(s)}\right) f(s,t)\mathrm{d}s + \alpha_3 \mathrm{e}^{-\rho(T-t)} U_3\left(X(T)\right)S(T,t)\Bigg] \\
= & \max_{(c,p,\boldsymbol{\pi})\in\Pi(t)} \mathbb{E}_t\Bigg[\int_t^{T} S(s,t)\alpha_1 \mathrm{e}^{-\int_t^s \rho \mathrm{d}\mu} U_1\left(c(s)-h(s)\right)\mathrm{d}s \\
& + \int_t^T \alpha_2 \mathrm{e}^{-\int_t^s \rho \mathrm{d}\mu} U_2\left(X(s)+\frac{p(s)}{\zeta(s)}\right) f(s,t)\mathrm{d}s + \alpha_3 \mathrm{e}^{-\int_t^T \rho \mathrm{d}\mu} U_3\left(X(T)\right)S(T,t)\Bigg] \\
= & \max_{(c,p,\boldsymbol{\pi})\in\Pi(t)} \mathbb{E}_t\Bigg[\int_t^{T} \mathrm{e}^{-\int_t^s (\rho+\lambda(u))\mathrm{d}u}\Bigg(\alpha_1 U_1\left(c(s)-h(s)\right) \\
& + \alpha_2 \lambda(s) U_2\left(X(s)+\frac{p(s)}{\zeta(s)}\right)\Bigg)\mathrm{d}s + \alpha_3 \mathrm{e}^{-\int_t^T (\rho+\lambda(u))\mathrm{d}u} U_3\left(X(T)\right)\Bigg],
\end{aligned}
$$

then, we can obtain (2.10). □

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