

Optimal investment strategies for hybrid pension with common shock and risk-sharing

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Abstract This paper investigates the optimal investment problem for hybrid pension plans in a financial market with jump-diffusion risky assets, where both the contribution and the benefit are adjusted based on the plan's performance, and risks are shared across different generations. The investment in a risk-free asset and two risky assets is carried out by the managers of the pension fund. The model of risky asset is assumed to be modulated by a compound Poisson process, with the two risky asset price processes correlated through a common shock. The objective of this study is to seek the optimal investment strategies and risk-sharing arrangements for plan trustees and participants that minimize the costs associated with unstable contribution risks, unstable benefit risks, and discontinuous risks. By applying the stochastic optimal control approach, the closed-form expressions of the optimal strategy and value function are derived. Numerical examples are provided to analyze the effects of parameters on the optimal strategies. In the context of the hybrid pension plan, these strategies effectively facilitate intergenerational risk-sharing.

Keywords Hybrid pension, Intergenerational risk-sharing, Common shock, Jump-diffusion process, Hamilton–Jacobi–Bellman equation

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1. Introduction

As population aging intensifies, pension management has become a mainstream research topic worldwide. In the literature, pension schemes are categorized into three distinct types: defined benefit (DB) pension plans, defined contribution (DC) pension plans, and hybrid pension plans. In a DB plan, trustees assume the financial responsibility and accountability for managing the risks associated with the plan, while in a DC plan, participants hold individual accounts and bear the entirety of the longevity and investment risks. However, neither of these plans is ideal in response to the challenges faced. This is because although DB schemes may be optimal for

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retirees as they pay the maximum payouts, the trustee bears all contribution and financial risks. This may not result in the optimal solution for the trustees. In contrast, DC plan participants bear all contribution and financial risks. Potentially low financial returns from the investment portfolio, coupled with a meager contribution rate, may lead to insufficient pension payments upon retirement, posing an unfavorable scenario for plan members. Thus increasing attention is given to new alternatives that can provide better risk sharing for both participants and plan trustees. An optimal pension scheme should ensure ample benefits for participants while fostering equitable risk-sharing between them and the plan trustees.

Recently, hybrid pension plans that combine the features of both DB and DC pension plans have been investigated. Hybrid pension plans are widespread across the world. They strike a balance between the two "polar extremes" of purely DC and purely DB pension schemes. This middle-ground position endows them with significantly more flexibility, making them a popular choice. The main objective of the research, reported by [1], is to deepen the understanding of risk sharing and hybrid pension plans in contrast to traditional final salary and pure DC plans. Subsequently, an array of hybrid plans has been conceptualized by pension experts and thoroughly examined in the relevant literature, expanding the field's understanding and possibilities, for example, the defined ambition (DA) plan ([2] and [3]), the target benefit plan ([4]), the risk-sharing DB plan ([5]) and the national defined contribution (NDC) plans ([6] and [7]). [8] provided an overview of the types and development of hybrid pension plans around the world. Additionally, it described in-depth four different hybrid pension plans as case studies. These include the hybrid DB plans in the Netherlands, the non-financial DC plan in Sweden, cash balance plans in the United States, Canada and Japan, and the Riester plan in Germany.

The hybrid pension plans offer enhanced retirement security on a sustainable, stable, and economically viable foundation, featuring risk distribution across various age groups for optimal balance and security. We focus on the portfolio allocation decisions together with pension policies, which are intergenerational risk-sharing rules, to dynamically adjust the pension fund's financial position. Intergenerational risk-sharing can significantly enhance overall welfare. [9] proved that intergenerational risk transfer was welfare-improving for all current and future generations in a collective pension scheme compared with participants in individual schemes. [10] found that well-structured intergenerational risk sharing via collective schemes could be welfare-enhancing above the optimal individual benchmark. [11] studied intergenerational risks and cost sharing for a variety of collectively funded pension plans and found that pension schemes with a well-structured volatility-risk-adjusted component could be welfare enhancing for entry and future cohorts. Besides, there are other studies in the literature that tried to find optimal contributions and/or benefit adjustment policies within hybrid pension funds, such as [12], [13] and [14].

This paper extends previous research to a jump-diffusion model for risky asset prices, wherein the counting processes exhibit interdependence through a shared shock mechanism, and the Brownian motions are assumed to be correlated. This improved framework offers a more subtle depiction of the complex dynamics in financial markets. Indeed, this type of model offers a more realistic portrayal of the actual financial market, as it acknowledges that information frequently arrives unexpectedly, triggering sudden jumps in stock prices. Furthermore, it recognizes that certain pieces of information can generate a common shock across multiple risky assets, exemplified by fluctuations in interest rates or oil prices. Research on common shock problems have been extensively discussed in the past years, as documented in references [15], [16], [17], and [18]. Under the criterion of maximizing the expected exponential utility, [16] considered the optimal reinsurance strategy in a risk model with two dependent classes of insurance business

under the variance premium principle. [15] considered the problem of optimal reinsurance with two dependent classes of insurance risks in a regime-switching financial market. Within the mean-variance framework, [17] studied the optimal reinsurance and investment problem in a financial market with a jump-diffusion risky asset, where the insurance risk model was modulated through a compound Poisson process, and the two jump-number processes were correlated through a common shock. Then, [18] extended their work to the optimization problem of maximizing the expected utility of terminal wealth with the constraint that the optimal reinsurance strategy lies in the unit interval. Therefore, the model adopted in this study is more practical and reasonable. Undoubtedly, it renders the optimization problem more intricate and challenging.

This study has several contributions. Firstly, the model we considered is set within an overlapping generations' economy, in contrast to the classic hybrid pension model. This characteristic enables risk-sharing among all generations. Secondly, we introduce jumps into the price dynamics of risky assets and consider the dependence of shared shocks between two jump-diffusion risky assets. Such a model is more in line with the actual financial market environment. In reality, information often arrives unexpectedly, which typically results in a jump in stock prices. Moreover, certain information can trigger a common shock to risky assets. Finally, we present numerical examples to demonstrate the influence of parameters on the optimal strategies.

The rest of this paper is organized as follows. In Section 2, we introduce the financial market model and the framework for the hybrid pension plan. In Section 3, we present the optimal investment problem for the hybrid pension and derive closed-form solutions using the dynamic programming principle. In Section 4, we provide a special example to study the optimal investment problem for the hybrid pension when the single risky asset follows a jump-diffusion model. In Section 5, we analyze the numerical impacts of model parameters on the optimal strategies. Finally, our conclusions are presented in Section 6.

2. Model formulation

2.1 Financial market

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, \mathbb{P})$ be a complete filtered probability space with the filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$, that is, \mathcal{F}_t contains the information about the financial market and the information about the salary of the members at their retirement time t , which is available to the sponsor of the pension plan.

We consider that the financial market consists of a risk-free asset (bond) and two risky assets (stocks). The price process of the bond $S_0(t)$ is given by

$$dS_0(t) = rS_0(t)dt, \quad S_0(0) = 1, \quad (2.1)$$

where constant $r > 0$ denotes the interest rate of the bond. The price processes of the other two stocks $S_i(t) (i = 1, 2)$ are modeled using the following jump-diffusion processes

$$dS_i(t) = S_i(t-)[\mu_i dt + \sigma_i dW_i(t) + \alpha_i dL_i(t)], \quad S_i(0) = s_{i0}, \quad (2.2)$$

where s_{i0} is the deterministic initial price, μ_i is the appreciation rate of the stock, σ_i and α_i are the volatility coefficients for Brownian motion and the jump process, respectively. μ_i , σ_i and α_i are positive constants. Additionally, $W_1(t)$ and $W_2(t)$ are independent standard Brownian motions. Then, the jump processes are given by

$$L_1(t) = \sum_{i=1}^{M_1(t)} X_i \quad \text{and} \quad L_2(t) = \sum_{i=1}^{M_2(t)} Y_i,$$

where $M_j(t)$ is a counting process for class $j(j = 1, 2)$. We assume that $\{X_i\}_{i \geq 1}$ and $\{Y_i\}_{i \geq 1}$ are two sequences of independent random variables representing the jump amplitudes of the two risky assets' prices $S_1(t)$ and $S_2(t)$, respectively, and they are independent of the counting processes $M_1(t)$ and $M_2(t)$, respectively. Let X and Y be generic random variables with the same distributions as X_1 and Y_1 , and denote their distribution functions by $F_X(x)$ and $F_Y(y)$, respectively. Following [18], we further assume that $\{X_i, i \geq 1\}$ and $\{Y_i, i \geq 1\}$ are i.i.d. with values in $(-1, \infty)$, which guarantees the positivity of the stock prices after each jump. Moreover, the first moments are finite and given by $E[X_i] = u_1$ and $E[Y_i] = u_2$, and the second moments are finite and given by $E[X_i^2] = \beta_1^2$ and $E[Y_i^2] = \beta_2^2$. The dependence between the two counting processes is introduced via a common-shock structure. Specifically,

$$M_1(t) = N_1(t) + N_3(t) \quad \text{and} \quad M_2(t) = N_2(t) + N_3(t),$$

where $N_1(t)$, $N_2(t)$ and $N_3(t)$ are independent Poisson random processes with parameters λ_3 , λ_4 and λ_5 , respectively. Therefore, the jump processes generated by these two classes of risky assets take the following form

$$L_1(t) = \sum_{i=1}^{N_1(t)+N_3(t)} X_i \quad \text{and} \quad L_2(t) = \sum_{i=1}^{N_2(t)+N_3(t)} Y_i.$$

It is readily apparent that the interdependence between these two classes of risky assets is attributable to a common shock governed by the counting process $N_3(t)$.

2.2 Basic plan settings

We consider the hybrid pension system consisting of working-stage and retirement-stage participants, and make the following plan settings. For the population, we develop a continuous deterministic framework considering that age and time are continuous variables. In a hybrid pension plan, all members are assumed to enter the system at age x_0 and retire at age x_r . Based on actual situations, due to the decline in fertility rate and the increase in life expectancy, the mortality rate decreases over time t , while it increases with age x . According to [19], the mortality hazard rate of cohort aged x at time t is given by

$$\mu(x, t) = \mathcal{A} + \mathcal{B}\theta^{x - \frac{1}{\omega}t},$$

which is a modification of Makeham's Law because it is not only age-dependent but also time-dependent. \mathcal{A} and \mathcal{B} are both constants. \mathcal{A} represents the baseline mortality rate, while \mathcal{B} is typically associated with the rate at which mortality increases; it reflects how mortality changes as age progresses. θ serves as the growth factor for mortality with increasing age. ω is a positive parameter called the longevity parameter, which shows the longevity trend. In other words, the human life expectancy increases by one year every ω year.

Remark 2.1 *From the mortality hazard rate above, we can get that the higher ω , the slower longevity trend becomes. According to [19], for the European Union(EU)-countries, the human life expectancy will increase by about 7.5 years over the next 50 years, which means $\omega \approx 6.67$.*

Moreover, the improvement in life expectancy is largely attributed to the expansion of the maximum age. Knell analyzed 'record female life expectancy' in [20], spanning from 1840 to 2000. A distinct trend emerged from this analysis, showing an almost perfectly linear progression. This trend is characterized by the highest value of female life expectancy across all countries with available data for the period. According to [14], we assume that the maximum age $m(t)$ increases

linearly:

$$m(t) = m_0 + \xi t, \quad 0 \leq \xi \leq 1,$$

where m_0 is the initial maximum survival age, and ξ is the increase factor of the maximum survival age.

We denote the survival function by $p(x, t)$, which is the probability that a cohort entering the system at time $t - (x - x_0)$ survives to age x . The survival function is described by

$$p(x, t) = e^{-\int_0^x \mu(v, t) dv}, \quad x_0 \leq x \leq m(t),$$

and thus the survival function can be expressed as

$$\begin{aligned} p(x, t) &= e^{-\int_0^{x-x_0} \mu(x_0+u, t-(x-x_0)+u) du} \\ &= \exp \left\{ -\mathcal{A}(x-x_0) - \frac{\mathcal{B}}{\ln \theta} \left(\theta^{x-\frac{1}{\omega}t} - \theta^{(1-\frac{1}{\omega})x_0 + \frac{1}{\omega}(x-t)} \right) \right\}, \quad x_0 \leq x \leq m(t). \end{aligned} \quad (2.3)$$

Let $n(t)$ be the entry rate that describes the density of sizes of new cohorts entering working life at the age of x_0 during time t . Then, the density of those attaining age x at time t is $n(t - (x - x_0))p(x, t)$, $x > x_0$. The function $n(t)$ follows the Malthusian population growth model, which is given by $n(t) = n_0 e^{\kappa t}$, where n_0 represents the density of new entrants at age x_0 at time $t = 0$, and κ denotes the growth rate of new entrants.

Remark 2.2 $t - (x - x_0)$ is the time at which this cohort joined the pension system. The value of $t - (x - x_0)$ could be negative, meaning that a member whose age is x at time $t \geq 0$ joined the pension system $x - x_0$ years ago.

The pension trustees receive contributions from the active members, and the total number of active members who make contributions at time t is given by

$$NC(t) = \int_{x_0}^{x_r} n(t - (x - x_0))p(x, t) dx. \quad (2.4)$$

Simultaneously, the pension plan provides retirees' benefits, and the total number of retirees who receive the benefits at time t is

$$NB(t) = \int_{x_r}^{m(t)} n(t - (x - x_0))p(x, t) dx. \quad (2.5)$$

Here, for active individuals, they receive wage income, which is normalized to 1 during the working period. Then, the total contribution income can be expressed as follows:

$$TC(t) = \int_{x_0}^{x_r} n(t - (x - x_0))p(x, t)c(t) dx = NC(t)c(t), \quad (2.6)$$

and the total benefit payment is

$$TB(t) = \int_{x_r}^{m(t)} n(t - (x - x_0))p(x, t)b(t) dx = NB(t)b(t), \quad (2.7)$$

where $c(t)$ and $b(t)$ represent the contribution amount and the income amount at time t , respectively. The specific forms will be presented in the next section.

Remark 2.3 Equations (2.4) and (2.5) can be observed in some specific conditions. For instance, the model studied in [4] can be reinstated by setting the parameter $\kappa = 0$. This setting implies that the entry rate remains constant, at $n(t) = n_0$. When $\omega = \infty$, it signifies that the force of mortality is solely dependent on age, consistent with the findings presented by [12].

2.3 Hybrid pension scheme

The pension fund trustee is responsible for the asset-liability management. Assume that the dollar amounts invested in the two stocks are $\pi_1(t)$ and $\pi_2(t)$ at time t , and then the amount invested in bond is $F(t) - \pi_1(t) - \pi_2(t)$. The dynamics of the wealth process $F(t)$ can be described by

$$\begin{aligned} dF(t) &= \pi_1(t) \frac{dS_1(t)}{S_1(t)} + \pi_2(t) \frac{dS_2(t)}{S_2(t)} + [F(t) - \pi_1(t) - \pi_2(t)] \frac{dS_0(t)}{S_0(t)} + [TC(t) - TB(t)]dt \\ &= [\pi_1(t)(\mu_1 - r) + \pi_2(t)(\mu_2 - r) + rF(t) + NC(t)c(t) - NB(t)b(t)]dt + \sigma_1\pi_1(t)dW_1(t) \\ &\quad + \sigma_2\pi_2(t)dW_2(t) + \alpha_1\pi_1(t)dL_1(t) + \alpha_2\pi_2(t)dL_2(t), \quad F(0) = f_0, \end{aligned}$$

or it can be rewritten as follows:

$$\begin{aligned} dF(t) &= [\pi_1(t)(\mu_1 - r) + \pi_2(t)(\mu_2 - r) + rF(t) + NC(t)c(t) - NB(t)b(t)]dt + \sigma_1\pi_1(t)dW_1(t) \\ &\quad + \sigma_2\pi_2(t)dW_2(t) + \int_{D_1} \alpha_1\pi_1(t)z_1N_1(dz_1, dt) + \int_{D_2} \alpha_2\pi_2(t)z_2N_2(dz_2, dt) \\ &\quad + \iint_{D_3} (\alpha_1\pi_1(t)z_1 + \alpha_2\pi_2(t)z_2)N_3(dz_1, dz_2, dt), \quad F(0) = f_0, \end{aligned} \quad (2.8)$$

where $N_1(dz_1, dt)$, $N_2(dz_2, dt)$, $N_3(dz_1, dz_2, dt)$ are independent Poisson random measures, and D_1 , D_2 are measurable subsets of $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$ and D_3 is a measurable subset of $\mathbb{R}^2 \setminus E$ with

$$E = \{(x, 0) : x \in \mathbb{R}\} \cup \{(0, y) : y \in \mathbb{R}\}. \quad (2.9)$$

Next, we introduce a hybrid pension model that effectively manages pension risks and shares the distribution across different generations. Our model is inspired by the hybrid pension plan studied in [19], where the surplus (deficit) between the total contribution income and the total benefit payments increases (decreases) the accumulation of the pension fund. Meanwhile, the accumulation is dynamically allocated to two risk-free assets and a risky asset.

Consider the target contribution c and target benefit b set at time 0, within a finite time interval spanning from 0 to T , where T represents the terminal time. For each age cohort, denoted by x , the target liability equals the difference between the present values of risk-free benefits and the yet-to-be-paid risk-free contributions.

$$L(x) = \begin{cases} \int_0^T (NB(t)be^{\tau t} - NC(t)ce^{\tau t})e^{-r(t-(x-x_0))}dt, & x_r \geq x \geq x_0, \\ \int_0^T NB(t)be^{\tau t}e^{-r(t-(x-x_0))}dt, & m(t) \geq x \geq x_r, \end{cases} \quad (2.10)$$

where τ is the target instantaneous growth rate of the target contribution and benefit. The target liabilities of the fund can be calculated as the sum of the liabilities for each age cohort.

$$\begin{aligned} L &= \int_{x_0}^{x_r} \int_0^T (NB(t)be^{\tau t} - NC(t)ce^{\tau t})e^{-r(t-(x-x_0))}dtdx \\ &\quad + \int_{x_r}^{m(t)} \int_0^T NB(t)be^{\tau t}e^{-r(t-(x-x_0))}dtdx. \end{aligned} \quad (2.11)$$

Given the static age composition of the fund and the fixed target benefit level, the target liability L remains constant over time. In fact, when the fund is fully invested in the risk-free asset, the actual liability coincides exactly with the target liability L . If the fund is invested in risky assets, such as equity investments, there may arise a surplus or deficit in funding relative to the target liability. The fund surplus is defined as the difference between assets and the target liability level:

$$SP(t) = F(t) - L. \quad (2.12)$$

It is worth noting that hybrid pension schemes may accrue surpluses or deficits, resulting in intergenerational transfers and thus facilitating the sharing of intergenerational risks.

Besides the adjustment of the plan members' target liabilities (2.11) by linking them to the fund assets (2.8), the hybrid pension model presented in this paper dynamically and simultaneously adjusts both the current total pension contributions made by active members and the benefit payments due to retirees. These adjustments are implicitly influenced by the financial risks faced by the pension fund, which are reflected in its surplus level or funding ratio.

To achieve our goal mentioned above, we define a simple policy for cohort contribution $c(t)$ and benefit $b(t)$, where the contribution per cohort is a function of the target contribution rate level $ce^{\tau t}$ and the funding residual per active cohort $\frac{SP(t)}{NC(t)}$, and the benefit per cohort is a function of the target benefit rate level $be^{\tau t}$ and the funding residual per retire cohort $\frac{SP(t)}{NB(t)}$,

$$c(t) = ce^{\tau t} - \alpha \frac{SP(t)}{NC(t)} = ce^{\tau t} - \lambda_1(t), \quad (2.13)$$

$$b(t) = be^{\tau t} + \beta \frac{SP(t)}{NB(t)} = be^{\tau t} + \lambda_2(t), \quad (2.14)$$

where $\lambda_1(t) = \alpha \frac{A(t)-L}{NC(t)}$, $\lambda_2(t) = \beta \frac{A(t)-L}{NB(t)}$ represent the adjustments to the pension contributions and benefit payments, respectively. The slope coefficients α and β are the so-called spread parameters for contribution income and benefit payment, respectively, which change with the adjustment of the hybrid pension system and thus reflect the speed of absorbing the funding imbalances. The funding imbalance is immediately and fully absorbed when $\alpha = 1$ or $\beta = 1$. More specifically, a fraction α of the funding residual is shared among employees and a fraction β of the funding residual is shared among retirees. A decrease in the values of α and β signifies that a portion of the funding surplus is deferred to future generations, thereby facilitating a greater degree of risk sharing across generations. In simpler terms, the lower the values of α and β , the higher the extent to which risks are shared between generations.

The two adjusted processes $\{c(t); t \geq 0\}$ and $\{b(t); t \geq 0\}$ describe the pension system over time. They consist of decisions on contribution and benefit payment for each cohort at time t . When the underlying fund value experiences gains at time t , indicated by a positive fund surplus level $SP(t)$, which indicates an asset investment outperformance, we anticipate a positive adjustment in both $\lambda_1(t)$ and $\lambda_2(t)$ policies. This adjustment aims to augment the "promised" benefit payments while concurrently reducing the "normal" required contributions at that particular time t . When the underlying fund surplus incurs a deficit due to investment losses, we in turn anticipate negative policy adjustments to both $\lambda_1(t)$ and $\lambda_2(t)$. Here, increasing the contributions and decreasing the benefits can rebalance the budget. Overall, high and low asset returns of the fund are spread over active members and retirees, which implies an intergenerational risk transfer or sharing.

Definition 1 (*Admissible strategy*)

A strategy $\pi = \{(\pi_1(t), \pi_2(t), \lambda_1(t), \lambda_2(t)), t \in [0, T]\}$ is said to be admissible if

- (i) π is progressively measurable with respect to $\{\mathcal{F}_t\}_{t \geq 0}$;
- (ii) $\forall t \in [0, T], \pi_1(t) \geq 0, \pi_2(t) \geq 0, \lambda_1(t) \geq 0, \lambda_2(t) \geq 0$, and $E[\int_0^T (|\pi_1(t)|^2 + |\pi_2(t)|^2) dt] < \infty$;
- (iii) (π, F^π) is a unique strong solution to the stochastic differential equation (2.8).

The set composed of all admissible strategy processes is defined as Π which is not empty.

In the following section, we set a continuous-time optimal investment problem for the hybrid pension plan.

3. Optimal investment problem for the hybrid pension

3.1 Optimization problem

Based on the context above, the hybrid pension plan we consider encompasses both active members who contribute to the fund and retired members who receive pension benefits. Given that contributions and benefits are subject to changes over time, the trustees of the plan are required to meticulously formulate asset-allocation strategies, as well as contribution and benefit strategies, that strike a reasonable balance between the interests of both active contributing members and retired beneficiaries. Since the pension plan we are considering is a collective endeavor encompassing various generations, the pension trustees should prioritize the stable and sustainable operation of the plan, while ensuring that investment risks are shared intergenerationally.

To achieve this goal, in our hybrid pension model, the pension trustees seek a robust optimal strategy, π , that minimizes the expected discounted cost of unstable contribution risk, unstable benefit risk, and discontinuity risk over the time horizon $[t, T]$. The model setting is consistent with [4] and [12]. We propose a cost function that consists of three parts. The first part is the deviation between the actual contribution and the target contribution. The second part is the deviation between the actual benefit and the target benefit. The third component is introduced to capture the plan's discontinuity risk, which stems from intergenerational transfers that are either too high or too low. We therefore specify a terminal fund goal, given by $f_0 e^{rT}$, and measure the discontinuity risk as the squared deviation between the actual terminal value of the fund, $F(T)$, and this terminal fund goal. Under such circumstances, a welfare cost arises due to the discontinuous accumulation risk. Hence, the objective is to minimize the deviations within the finite time horizon $[t, T]$. The objective function can be expressed as follows:

$$V(t, f) = \min_{\pi \in \Pi} \mathbb{E} \left\{ \int_t^T \left[\gamma_1 (c(u) - ce^{\tau t})^2 e^{-ru} + \gamma_2 (b(u) - be^{\tau t})^2 e^{-ru} \right] du + \gamma_3 (F(T) - f_0 e^{rT})^2 e^{-rT} \middle| F(t) = f \right\}, \quad (3.1)$$

where Π is a set of all admissible strategies of $\pi = \{(\pi_1(t), \pi_2(t), \lambda_1(t), \lambda_2(t)), t \in [0, T]\}$. The non-negative constants γ_1 , γ_2 , and γ_3 serve as weight parameters, indicating the relative importance of the cost associated with unstable contribution risk, the cost of unstable benefit risk, and the cost of discontinuous risk within the overall cost function, respectively.

Thus, the optimal investment problem for the hybrid pension plan is established. The pension trustees aim to find the optimal strategies for asset allocation and for both benefit and contribution adjustments that minimize the deviations within the finite time horizon.

3.1.1 Optimal investment strategies

In this section, using the standard stochastic control approach, we solve the optimal investment problem for the hybrid pension and derive closed-form expressions of the optimal policy.

The variational operator on any function $V(t, f)$ is defined as follows:

$$\begin{aligned} \mathcal{A}^\pi V(t, f) = & V_t + \frac{1}{2} (\pi_1^2 \sigma_1^2 + \pi_2^2 \sigma_2^2) V_{ff} + (\gamma_1 \lambda_1^2 + \gamma_2 \lambda_2^2) e^{-rt} \\ & + [\pi_1 (\mu_1 - r) + \pi_2 (\mu_2 - r) + rf + NC(t)(ce^{\tau t} - \lambda_1) - NB(t)(be^{\tau t} + \lambda_2)] V_f \\ & + \lambda_3 E[V(t, f + \alpha_1 \pi_1 X) - V(t, f)] + \lambda_4 E[V(t, f + \alpha_2 \pi_2 Y) - V(t, f)] \\ & + \lambda_5 E[V(t, f + \alpha_1 \pi_1 X + \alpha_2 \pi_2 Y) - V(t, f)], \end{aligned} \quad (3.2)$$

where V_t, V_f, V_{ff} are partial derivatives of $V(t, f)$.

According to the principle of stochastic dynamic programming, the Hamilton–Jacobi–Bellman (HJB) equation can be derived as follows:

$$\min_{\pi \in \Pi} \mathcal{A}^\pi V(t, f) = 0. \quad (3.3)$$

We present the optimal asset allocation, contribution and benefit adjustment policies for the optimal investment problem (3.1) in the following theorem.

Theorem 3.1 *For the optimal investment problem (3.1), the optimal asset allocation policy, contribution and benefit adjustment policies are given, respectively, by*

$$\pi_1^*(t, f) = g_0(f + Q(t)), \quad (3.4)$$

$$\pi_2^*(t, f) = g_1(f + Q(t)), \quad (3.5)$$

$$\lambda_1^*(t, f) = \frac{\gamma_3}{2\gamma_1} NC(t)P(t)(f + Q(t)), \quad (3.6)$$

$$\lambda_2^*(t, f) = \frac{\gamma_3}{2\gamma_2} NB(t)P(t)(f + Q(t)), \quad (3.7)$$

as we denote that

$$g_0 = \frac{cc_4 - c_2c_3}{c_1c_2 - c^2}, \quad g_1 = \frac{cc_3 - c_1c_4}{c_1c_2 - c^2},$$

where

$$\begin{aligned} c_1 &= 2\sigma_1^2 + 2\lambda_3\alpha_1^2\beta_1^2 + 2\lambda_5\alpha_1^2\beta_1^2, & c_2 &= 2\sigma_2^2 + 2\lambda_4\alpha_2^2\beta_2^2 + 2\lambda_5\alpha_2^2\beta_2^2, \\ c_3 &= \mu_1 - r + 2\alpha_1u_1(\lambda_3 + \lambda_5), & c_4 &= \mu_2 - r + 2\alpha_2u_2(\lambda_4 + \lambda_5), \\ c &= 2\alpha_1\alpha_2u_1u_2. \end{aligned}$$

The corresponding value function is given by

$$V(t, f) = \gamma_3 e^{-rt} P(t)(f + Q(t))^2, \quad (3.8)$$

where the expressions of $P(t)$ and $Q(t)$ are given below. We denote that

$$\begin{aligned} g_2(t) &= \gamma_3 \left(\frac{NC(t)^2}{\gamma_1} + \frac{NB(t)^2}{\gamma_2} \right), & g_3(t) &= NC(t)ce^{\tau t} - NB(t)be^{\tau t}, \\ g_4 &= \alpha_1^2\beta_1^2g_0^2 + \alpha_2^2\beta_2^2g_1^2 + 2\alpha_1\alpha_2u_1u_2g_0g_1 + 2\alpha_1u_1g_0 + 2\alpha_2u_2g_1. \end{aligned} \quad (3.9)$$

Then, $P(t)$ and $Q(t)$ can be expressed as follows:

$$\begin{aligned} P(t) &= \frac{1}{\frac{g_2(t)}{4\phi} + (1 - \frac{g_2(t)}{4\phi})e^{-\phi(T-t)}}, \\ Q(t) &= -\frac{g_3(t)}{2r}(1 - e^{r(T-t)}) - f_0e^{rt}, \end{aligned} \quad (3.10)$$

where

$$\begin{aligned} \phi &= g_0(\mu_1 - r) + g_1(\mu_2 - r) + g_0^2\sigma_1^2 + g_1^2\sigma_2^2 + (2\alpha_1g_0u_1 + \alpha_1^2g_0^2\beta_1^2)\lambda_3 \\ &\quad + (2\alpha_2g_1u_2 + \alpha_2^2g_1^2\beta_2^2)\lambda_4 + g_4\lambda_5. \end{aligned} \quad (3.11)$$

Proof To solve (3.1), we conjecture that the solution takes the following form:

$$\begin{aligned} V(t, f) &= \gamma_3 e^{-rt} P(t)(f + Q(t))^2, \\ P(T) &= 1, \quad Q(T) = -f_0e^{rT}, \end{aligned} \quad (3.12)$$

whose partial derivatives are

$$\begin{aligned} V_t &= \gamma_3 e^{-rt} [(P_t - rP(t))(f + Q(t))^2 + 2P(t)Q_t(f + Q(t))], \\ V_f &= 2\gamma_3 e^{-rt} P(t)(f + Q(t)), \\ V_{ff} &= 2\gamma_3 e^{-rt} P(t). \end{aligned} \quad (3.13)$$

Besides, from (3.12), we get

$$\begin{cases} E[V(t, f + \alpha_1 \pi_1 X) - V(t, f)] \\ = \gamma_3 e^{-rt} P(t) [2\alpha_1 \pi_1 u_1 (f + Q(x)) + \alpha_1^2 \pi_1^2 \beta_1^2], \\ E[V(t, f + \alpha_2 \pi_2 Y) - V(t, f)] \\ = \gamma_3 e^{-rt} P(t) [2\alpha_2 \pi_2 u_2 (f + Q(x)) + \alpha_2^2 \pi_2^2 \beta_2^2], \\ E[V(t, f + \alpha_1 \pi_1 X + \alpha_2 \pi_2 Y) - V(t, f)] \\ = \gamma_3 e^{-rt} P(t) [\alpha_1^2 \pi_1^2 \beta_1^2 + \alpha_2^2 \pi_2^2 \beta_2^2 + 2\alpha_1 \alpha_2 \pi_1 \pi_2 u_1 u_2 \\ + 2(\alpha_1 \pi_1 u_1 + \alpha_2 \pi_2 u_2)(f + Q(x))]. \end{cases} \quad (3.14)$$

Substituting (3.13) and (3.14) into (3.2), we have

$$\begin{aligned} \mathcal{A}^\pi V(t, f) &= [(P_t - rP(t))(f + Q(t))^2 + 2P(t)Q_t(f + Q(t))] + P(t)(\pi_1^2 \beta_1^2 + \pi_2^2 \beta_2^2) \\ &+ \frac{\gamma_1 \lambda_1^2 + \gamma_2 \lambda_2^2}{\gamma_3} + P(t)(f + Q(t))[rf + \pi_1(\mu_1 - r) + \pi_2(\mu_2 - r)] \\ &+ NC(t)(ce^{rt} - \lambda_1) - NB(t)(be^{rt} + \lambda_2)] \\ &+ P(t)\lambda_3[2\alpha_1 \pi_1 u_1 (f + Q(t)) + \alpha_1^2 \pi_1^2 \beta_1^2] + P(t)\lambda_4[2\alpha_2 \pi_2 u_2 (f + Q(t)) + \alpha_2^2 \pi_2^2 \beta_2^2] \\ &+ P(t)\lambda_5[\alpha_1^2 \pi_1^2 \beta_1^2 + \alpha_2^2 \pi_2^2 \beta_2^2 + 2\alpha_1 \alpha_2 \pi_1 \pi_2 u_1 u_2 + 2(\alpha_1 \pi_1 u_1 + \alpha_2 \pi_2 u_2)(f + Q(x))]. \end{aligned} \quad (3.15)$$

To obtain the optimal investment strategies $\pi_1^*(t, f)$ and $\pi_2^*(t, f)$, first, we let

$$\begin{aligned} h(\pi_1, \pi_2) &= P(t)(\pi_1^2 \beta_1^2 + \pi_2^2 \beta_2^2) + P(t)(f + Q(t))[\pi_1(\mu_1 - r) + \pi_2(\mu_2 - r)] \\ &+ P(t)\lambda_3[2\alpha_1 \pi_1 u_1 (f + Q(t)) + \alpha_1^2 \pi_1^2 \beta_1^2] + P(t)\lambda_4[2\alpha_2 \pi_2 u_2 (f + Q(t)) + \alpha_2^2 \pi_2^2 \beta_2^2] \\ &+ P(t)\lambda_5[\alpha_1^2 \pi_1^2 \beta_1^2 + \alpha_2^2 \pi_2^2 \beta_2^2 + 2\alpha_1 \alpha_2 \pi_1 \pi_2 u_1 u_2 + 2(\alpha_1 \pi_1 u_1 + \alpha_2 \pi_2 u_2)(f + Q(x))]. \end{aligned}$$

Differentiating $h(\pi_1, \pi_2)$ with respect to π_1 and π_2 respectively, we obtain

$$\begin{cases} \frac{\partial h(\pi_1, \pi_2)}{\partial \pi_1} = (2\sigma_1^2 + 2\lambda_3 \alpha_1^2 \beta_1^2 + 2\lambda_5 \alpha_1^2 \beta_1^2) \pi_1 \\ \quad + 2\alpha_1 \alpha_2 u_1 u_2 \pi_2 + (f + Q(t))[\mu_1 - r + 2\alpha_1 u_1 (\lambda_3 + \lambda_5)], \\ \frac{\partial h(\pi_1, \pi_2)}{\partial \pi_2} = (2\sigma_2^2 + 2\lambda_4 \alpha_2^2 \beta_2^2 + 2\lambda_5 \alpha_2^2 \beta_2^2) \pi_2 \\ \quad + 2\alpha_1 \alpha_2 u_1 u_2 \pi_1 + (f + Q(t))[\mu_2 - r + 2\alpha_2 u_2 (\lambda_4 + \lambda_5)], \\ \frac{\partial^2 h(\pi_1, \pi_2)}{\partial \pi_1^2} = 2\sigma_1^2 + 2\lambda_3 \alpha_1^2 \beta_1^2 + 2\lambda_5 \alpha_1^2 \beta_1^2, \\ \frac{\partial^2 h(\pi_1, \pi_2)}{\partial \pi_2^2} = 2\sigma_2^2 + 2\lambda_4 \alpha_2^2 \beta_2^2 + 2\lambda_5 \alpha_2^2 \beta_2^2, \\ \frac{\partial^2 h(\pi_1, \pi_2)}{\partial \pi_1 \pi_2} = 2\alpha_1 \alpha_2 u_1 u_2. \end{cases} \quad (3.16)$$

Second, we ensure that $h(\pi_1, \pi_2)$ is a convex function with respect to π_1 and π_2 .

To prove that $h(\pi_1, \pi_2)$ is a convex function with respect to π_1 and π_2 , it is sufficient to prove that the Hessian matrix of $h(\pi_1, \pi_2)$ is positive definite. The Hessian matrix is

$$H = \begin{pmatrix} \frac{\partial^2 h(\pi_1, \pi_2)}{\partial \pi_1^2} & \frac{\partial^2 h(\pi_1, \pi_2)}{\partial \pi_1 \partial \pi_2} \\ \frac{\partial^2 h(\pi_1, \pi_2)}{\partial \pi_2 \partial \pi_1} & \frac{\partial^2 h(\pi_1, \pi_2)}{\partial \pi_2^2} \end{pmatrix} = \begin{pmatrix} 2\sigma_1^2 + 2\lambda_3\alpha_1^2\beta_1^2 + 2\lambda_5\alpha_1^2\beta_1^2 & 2\alpha_1\alpha_2u_1u_2 \\ 2\alpha_1\alpha_2u_1u_2 & 2\sigma_2^2 + 2\lambda_4\alpha_2^2\beta_2^2 + 2\lambda_5\alpha_2^2\beta_2^2 \end{pmatrix}.$$

Next, we can split the matrix into the following form

$$H = \begin{pmatrix} 2\sigma_1^2 & 2\alpha_1\alpha_2u_1u_2 \\ 2\alpha_1\alpha_2u_1u_2 & 2\sigma_2^2 \end{pmatrix} + \begin{pmatrix} 2\lambda_3\alpha_1^2\beta_1^2 + 2\lambda_5\alpha_1^2\beta_1^2 & 0 \\ 0 & 2\lambda_4\alpha_2^2\beta_2^2 + 2\lambda_5\alpha_2^2\beta_2^2 \end{pmatrix}.$$

The second half is a positive definite matrix, while the first half can be derived from the Cauchy–Schwarz inequality, as stated in [17]. The details are as follows.

According to the Cauchy–Schwarz inequality

$$\left(\sum_{i=1}^n x_i y_i \right)^2 \leq \left(\sum_{i=1}^n x_i^2 \right) \left(\sum_{i=1}^n y_i^2 \right),$$

we let $x_1 = \sigma_1, y_1 = \alpha_1\alpha_2u_1, x_2 = \sigma_2, y_2 = \alpha_1\alpha_2u_2$, and we can get $|\sigma_1^2\sigma_2^2 - \alpha_1^2\alpha_2^2u_1^2u_2^2| \geq 0$. So the first half is a positive definite matrix.

Thus, the Hessian matrix is a positive definite matrix. Therefore, the minimizer $(\pi_1^*(t, f), \pi_2^*(t, f))$ is the solution of the equations

$$\begin{cases} (2\sigma_1^2 + 2\lambda_3\alpha_1^2\beta_1^2 + 2\lambda_5\alpha_1^2\beta_1^2)\pi_1 + 2\alpha_1\alpha_2u_1u_2\pi_2 = -(f + Q(t))[\mu_1 - r + 2\alpha_1u_1(\lambda_3 + \lambda_5)], \\ (2\sigma_2^2 + 2\lambda_4\alpha_2^2\beta_2^2 + 2\lambda_5\alpha_2^2\beta_2^2)\pi_2 + 2\alpha_1\alpha_2u_1u_2\pi_1 = -(f + Q(t))[\mu_2 - r + 2\alpha_2u_2(\lambda_4 + \lambda_5)]. \end{cases}$$

That is

$$\begin{cases} \pi_1^*(t, f) = (f + Q(t))\left(\frac{cc_4 - c_2c_3}{c_1c_2 - c^2}\right) = g_0(f + Q(t)), \\ \pi_2^*(t, f) = (f + Q(t))\left(\frac{cc_3 - c_1c_4}{c_1c_2 - c^2}\right) = g_1(f + Q(t)). \end{cases} \quad (3.17)$$

Besides, differentiating (3.15) with respect to λ_1 and λ_2 respectively, we get

$$\lambda_1^*(t, f) = \frac{\gamma_3}{2\gamma_1} NC(t)P(t)(f + Q(t)), \quad (3.18)$$

$$\lambda_2^*(t, f) = \frac{\gamma_3}{2\gamma_2} NB(t)P(t)(f + Q(t)). \quad (3.19)$$

Denote

$$g_2(t) = \gamma_3 \left(\frac{NC(t)^2}{\gamma_1} + \frac{NB(t)^2}{\gamma_2} \right), \quad g_3(t) = NC(t)ce^{\tau t} - NB(t)be^{\tau t}, \\ g_4 = \alpha_1^2\beta_1^2g_0^2 + \alpha_2^2\beta_2^2g_1^2 + 2\alpha_1\alpha_2u_1u_2g_0g_1 + 2\alpha_1u_1g_0 + 2\alpha_2u_2g_1.$$

Substituting (3.17), (3.18) and (3.19) into (3.15), we have

$$\begin{aligned}
\min_{\pi \in \Pi} \mathcal{A}^\pi V(t, f) = & \gamma_3 e^{-rt} \left\{ \left(P_t + \left(g_0(\mu_1 - r) + g_1(\mu_2 - r) + (2\alpha_1 g_0 u_1 + \alpha_1^2 g_0^2 \beta_1^2) \lambda_3 \right. \right. \right. \\
& + (2\alpha_2 g_1 u_2 + \alpha_2^2 g_1^2 \beta_2^2) \lambda_4 + g_0^2 \sigma_1^2 + g_1^2 \sigma_2^2 + g_4 \lambda_5 \Big) P(t) - \frac{g_2(t)}{4} P(t)^2 \Big) f^2 \\
& + 2 \left[\left(P_t + \left(g_0(\mu_1 - r) + g_1(\mu_2 - r) + (2\alpha_1 g_0 u_1 + \alpha_1^2 g_0^2 \beta_1^2) \lambda_3 \right. \right. \right. \\
& + (2\alpha_2 g_1 u_2 + \alpha_2^2 g_1^2 \beta_2^2) \lambda_4 + g_0^2 \sigma_1^2 + g_1^2 \sigma_2^2 + g_4 \lambda_5 \Big) P(t) - \frac{g_2(t)}{4} P(t)^2 \Big) Q(t) \\
& + \frac{P(t)}{2} (2Q_t - 2rQ(t) + g_3(t)) \Big] f \\
& + \left(P_t + \left(g_0(\mu_1 - r) + g_1(\mu_2 - r) + (2\alpha_1 g_0 u_1 + \alpha_1^2 g_0^2 \beta_1^2) \lambda_3 \right. \right. \\
& + (2\alpha_2 g_1 u_2 + \alpha_2^2 g_1^2 \beta_2^2) \lambda_4 + g_0^2 \sigma_1^2 + g_1^2 \sigma_2^2 + g_4 \lambda_5 \Big) P(t) - \frac{g_2(t)}{4} P(t)^2 \Big) Q(t)^2 \\
& \left. + P(t) Q(t) (2Q_t - 2rQ(t) + g_3(t)) \right\}. \tag{3.20}
\end{aligned}$$

According to (3.3), we let the coefficients of f^2 , f be zero, which leads to the following system of differential equations:

$$\begin{aligned}
P_t + [g_0(\mu_1 - r) + g_1(\mu_2 - r) + g_0^2 \sigma_1^2 + g_1^2 \sigma_2^2 + (2\alpha_1 g_0 u_1 + \alpha_1^2 g_0^2 \beta_1^2) \lambda_3 \\
+ (2\alpha_2 g_1 u_2 + \alpha_2^2 g_1^2 \beta_2^2) \lambda_4 + g_4 \lambda_5] P(t) - \frac{g_2(t)}{4} P(t)^2 = 0, \tag{3.21} \\
2Q_t - 2rQ(t) + g_3(t) = 0.
\end{aligned}$$

The differential equations (3.21) with boundary conditions in (3.12) can be easily solved; thus, this theorem is verified. \square

4. Special Case

In this section, we discuss a specific case in our model by examining the optimal investment problem for hybrid pension funds within a jump-diffusion model. In this specific case, we assume that the pension trustees' investment assets consist of a risk-free asset and a risky asset. The price process of the risky asset $S_3(t)$ follows a jump-diffusion model.

$$dS_3(t) = S_3(t) \left[\mu_3 dt + \sigma_3 dW_3(t) + d \left(\sum_{i=1}^{N(t)} Z_i \right) \right], \quad S_3(0) = s_3 > 0, \tag{4.1}$$

where $s_3 > 0$ is the initial price of the risky asset; μ_3 and σ_3 are positive constants and represent the expected return and the volatility of the risky asset, respectively; $N(t)$ represents the number of jumps of the risky asset price within the time interval $[0, t]$ and follows the homogeneous Poisson process with the intensity λ ; Z_i is the i th jump amplitude of the risky asset price, and Z_i , $i = 1, 2, \dots$, are independent and identically distributed random variables with the distribution function $F_Z(z)$. Here, Z is a generic random variable. Similar to what is presented in [18], the jump sizes Z_i are assumed to be an i.i.d sequence with values in $(-1, \infty)$. Furthermore, there are finite first-order moment $E[Z_i] = u_3$ and second-order moment $E[Z_i^2] = \beta_3^2$. Similar to [21], we assume that $\{W_3(t)\}_{t \in [0, T]}$, $\{N(t)\}_{t \in [0, T]}$ and $\{Z_i, i \geq 1\}$ are independent.

Then, we define the following compound Poisson process $\sum_{i=1}^{N(t)} Z_i$ using the Poisson random measures $N(\cdot, \cdot)$ on $\Omega \times [0, T] \times [-1, \infty)$:

$$\sum_{i=1}^{N(t)} Z_i = \int_0^t \int_{-1}^{\infty} zN(ds, dz), \quad \forall t \in [0, T].$$

In this case, if $\nu(dt, dz) = \lambda dt dF_Z(z)$, we have

$$\mathbb{E} \left[\sum_{i=1}^{N(t)} Z_i \right] = \int_0^t \int_{-1}^{\infty} z\nu(ds, dz), \quad \forall t \in [0, T],$$

and $\nu(\cdot, \cdot)$ is the compensator of the random measure $N(\cdot, \cdot)$. Therefore, the compensated measure $\tilde{N} = N(\cdot, \cdot) - \nu(\cdot, \cdot)$ is related to the compound Poisson process $\sum_{i=1}^{N(t)} Z_i$ as follows:

$$\int_0^t \int_{-1}^{\infty} z\tilde{N}(ds, dz) = \sum_{i=1}^{N(t)} Z_i - \mathbb{E} \left(\sum_{i=1}^{N(t)} Z_i \right), \quad \forall t \in [0, T].$$

To get a higher yield, let the amount of money invested in the risky asset be denoted by $\pi(t)$, and the amount invested in the risk-free asset be denoted by $A(t) - \pi(t)$, where $A(t)$ is the size of the pension fund for each surviving member at time t . Therefore, the dynamics of the wealth process $A(t)$ can be described by

$$\begin{aligned} dA(t) &= \pi(t) \frac{dS_3(t)}{S_3(t)} + [A(t) - \pi(t)] \frac{dS_0(t)}{S_0(t)} + [TC(t) - TB(t)]dt \\ &= [\pi(t)(\mu_3 - r) + rA(t) + NC(t)c(t) - NB(t)b(t)]dt + \sigma_3\pi(t)dW_3(t) \\ &\quad + \int_{-1}^{\infty} \pi(t)zN(dt, dz), \quad A(0) = a_0. \end{aligned} \tag{4.2}$$

The optimal investment problem is the same as (3.1) and is given by

$$\begin{aligned} V(t, a) &= \min_{\pi \in \Pi} \mathbb{E} \left\{ \int_t^T \left[\gamma_1(c(u) - ce^{\tau t})^2 e^{-ru} + \gamma_2(b(u) - be^{\tau t})^2 e^{-ru} \right] du \right. \\ &\quad \left. + \gamma_3(A(T) - a_0 e^{rT})^2 e^{-rT} \middle| A(t) = a \right\}. \end{aligned} \tag{4.3}$$

Then, the variational operator on any function $V(t, a)$ is defined as follows:

$$\begin{aligned} \mathcal{A}^\pi V(t, a) &= V_t + \frac{1}{2} \pi^2 \sigma_3^2 V_{aa} + (\gamma_1 \lambda_1^2 + \gamma_2 \lambda_2^2) e^{-rt} \\ &\quad + [\pi(\mu_3 - r) + ra + NC(t)(ce^{\tau t} - \lambda_1) - NB(t)(be^{\tau t} + \lambda_2)]V_a \\ &\quad + \lambda E[V(t, a + \pi Z) - V(t, a)], \end{aligned} \tag{4.4}$$

where V_t, V_a, V_{aa} are partial derivatives of $V(t, a)$.

Theorem 4.1 *For the optimal investment problem (4.3), the optimal asset allocation policy, contribution and benefit adjustment policy are given, respectively, by*

$$\pi^*(t, a) = -g_5(a + Q(t)), \tag{4.5}$$

$$\lambda_1^*(t, a) = \frac{\gamma_3}{2\gamma_1} NC(t)P(t)(a + Q(t)), \tag{4.6}$$

$$\lambda_2^*(t, a) = \frac{\gamma_3}{2\gamma_2} NB(t)P(t)(a + Q(t)), \tag{4.7}$$

where

$$g_5 = \frac{\mu_3 - r + 2\lambda u_3}{2(\sigma_3^2 + \lambda\beta_3^2)}, \quad (4.8)$$

and the corresponding value function is given by

$$V(t, a) = \gamma_3 e^{-rt} P(t) (a + Q(t))^2, \quad (4.9)$$

where $P(t)$ and $Q(t)$ are given by

$$\begin{aligned} P(t) &= \frac{1}{\frac{g_2(t)}{4\Phi} + (1 - \frac{g_2(t)}{4\Phi})e^{-\Phi(T-t)}}, \\ Q(t) &= -\frac{g_3(t) - \frac{g_2(t)}{2}}{r} (1 - e^{\frac{r}{2}(T-t)}) - f_0 e^{rt}, \end{aligned} \quad (4.10)$$

where

$$\begin{aligned} g_2(t) &= \gamma_3 \left(\frac{NC(t)^2}{\gamma_1} + \frac{NB(t)^2}{\gamma_2} \right), \quad g_3(t) = NC(t)ce^{\tau t} - NB(t)be^{\tau t}, \\ \Phi &= -g_5(\mu_3 - r) + \sigma_3^2 g_5^2 + \lambda g_5^2 \beta_3^2 - 2\lambda g_5 u_3. \end{aligned} \quad (4.11)$$

Proof We try to speculate on the solution in the following form:

$$\begin{aligned} V(t, a) &= \gamma_3 e^{-rt} P(t) (a + Q(t))^2, \\ P(T) &= 1, \quad Q(T) = -a_0 e^{rT}. \end{aligned} \quad (4.12)$$

The partial derivatives of $V(t, a)$ are

$$\begin{aligned} V_t &= \gamma_3 e^{-rt} [(P_t - rP(t))(a + Q(t))^2 + 2P(t)Q_t(a + Q(t))], \\ V_a &= 2\gamma_3 e^{-rt} P(t)(a + Q(t)), \\ V_{aa} &= 2\gamma_3 e^{-rt} P(t). \end{aligned} \quad (4.13)$$

From (4.12), we get

$$E[V(t, a + \pi Z) - V(t, a)] = \gamma_3 e^{-rt} P(t) [2\pi u_3(a + Q(t)) + \pi^2 \beta_3^2]. \quad (4.14)$$

Substituting (4.13) and (4.14) back into (4.4), we have

$$\begin{aligned} \mathcal{A}^\pi V(t, a) &= (P_t - rP(t))(a^2 + 2Q(t)a + Q(t)^2) + 2P(t)Q_t(a + Q(t)) \\ &\quad + P(t)(a + Q(t))[\pi(\mu_3 - r) + ra + NC(t)(ce^{\tau t} - \lambda_1) - NB(t)(be^{\tau t} + \lambda_2)] \\ &\quad + P(t)\pi^2 \sigma_3^2 + \lambda P(t)[2\pi\mu_3(a + Q(t)) + \pi^2 \sigma_3^2] + \frac{\gamma_1 \lambda_1^2 + \gamma_2 \lambda_2^2}{\gamma_3}. \end{aligned} \quad (4.15)$$

According to the first-order optimality conditions, we get

$$\pi^*(t, a) = -(a + Q(t)) \frac{\mu_3 - r + 2\lambda u_3}{2(\sigma_3^2 + \lambda\beta_3^2)} = -g_5(a + Q(t)), \quad (4.16)$$

$$\lambda_1^*(t, a) = \frac{\gamma_3}{2\gamma_1} NC(t)P(t)(a + Q(t)), \quad (4.17)$$

$$\lambda_2^*(t, a) = \frac{\gamma_3}{2\gamma_2} NB(t)P(t)(a + Q(t)). \quad (4.18)$$

Assuming that $g_2(t) = \gamma_3 \left(\frac{NC(t)^2}{\gamma_1} + \frac{NB(t)^2}{\gamma_2} \right)$, $g_3(t) = NC(t)ce^{\tau t} - NB(t)be^{\tau t}$, and inserting (4.16), (4.17) and (4.18) into (4.4), we obtain

$$\begin{aligned} \min_{\pi \in \Pi} \mathcal{A}^\pi V(t, a) &= \gamma_3 e^{-\tau t} \left\{ [P_t + (-g_5(\mu_3 - r) + \sigma_3^2 g_5^2 + \lambda g_5^2 \beta_3^2 - 2\lambda g_5 u_3)P(t) + \frac{g_2(t)}{4}P(t)^2]a^2 \right. \\ &\quad + 2[(P_t + (-g_5(\mu_3 - r) + \sigma_3^2 g_5^2 + \lambda g_5^2 \beta_3^2 - 2\lambda g_5 u_3)P(t) + \frac{g_2(t)}{4}P(t)^2)Q(t) \\ &\quad + (g_3(t) - \frac{g_2(t)}{2})P(t) - rP(t)Q(t) + 2P(t)Q_t]a \\ &\quad + [(P_t + (-g_5(\mu_3 - r) + \sigma_3^2 g_5^2 + \lambda g_5^2 \beta_3^2 - 2\lambda g_5 u_3)P(t) + \frac{g_2(t)}{4}P(t)^2)Q(t) \\ &\quad \left. + (g_3(t) - \frac{g_2(t)}{2})P(t) - rP(t)Q(t) + 2P(t)Q_t]Q(t) \right\}. \end{aligned} \quad (4.19)$$

According to the principle of stochastic dynamic programming, the HJB equation can be derived as follows:

$$\min_{\pi \in \Pi} \mathcal{A}^\pi V(t, a) = 0. \quad (4.20)$$

Let the coefficients of a^2 , a be zero, which leads to the following system of differential equations

$$\begin{aligned} P_t + (-g_5(\mu_3 - r) + \sigma_3^2 g_5^2 + \lambda g_5^2 \beta_3^2 - 2\lambda g_5 u_3)P(t) + \frac{g_2(t)}{4}P(t)^2 &= 0, \\ \left(g_3(t) - \frac{g_2(t)}{2} \right) - rQ(t) + 2Q_t &= 0. \end{aligned} \quad (4.21)$$

The differential equations (4.21) with boundary conditions in (4.12) can be easily solved; thus, this theorem is verified. \square

5. Numerical illustrations

In Section 3, we have established the explicit expression of the optimal investment allocation and adjustment strategies of both the contribution and benefit for each cohort in the hybrid pension model. In this section, we present some numerical examples to illustrate the impact of important parameters on the optimal results.

The parameter settings in the numerical simulation are mainly in line with those in the literature [19] and [22]. For the age and time-dependent survival function, we set $\mathcal{A} = 2.2 * 10^{-4}$, $\mathcal{B} = 2.7 * 10^{-6}$, $\theta = 1.124$. Meanwhile, we set $\omega = 4$, which translates to an increase in the average life expectancy by one year every four years.

In the pension market aspect, the age of new entrants into the pension system is denoted as $x_0 = 25$, while the retirement age is $x_r = 65$. At the initial time $t = 0$, the number of new entrants is $n_0 = 10$, and the initial fund accumulation stands at $f_0 = 3000$. The time horizon is $T = 20$. Additionally, the target contribution is $c = 0.1$, and the target benefit is $b = 0.7$. This setup outlines the key parameters influencing the pension system's dynamics. The target growth is $\tau = 0.02$. We assume that the weight parameters of $\gamma_1 = 4$, $\gamma_2 = 4$, and $\gamma_3 = 1$ and we vary these values to analyze their impact.

In the financial market aspect, as shown in Table 1, the parameters for risky assets, such as the risk-free rate r and the expected value of the jump amplitude μ_i , are reasonable and consistent with the real-world financial market conditions.

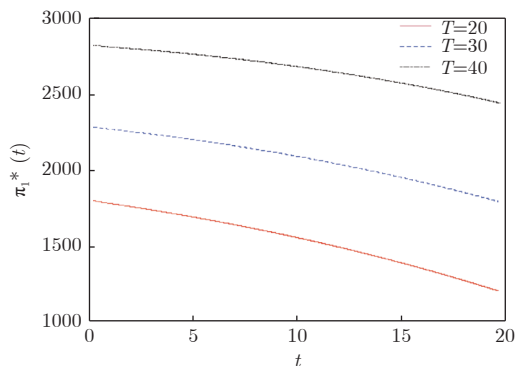
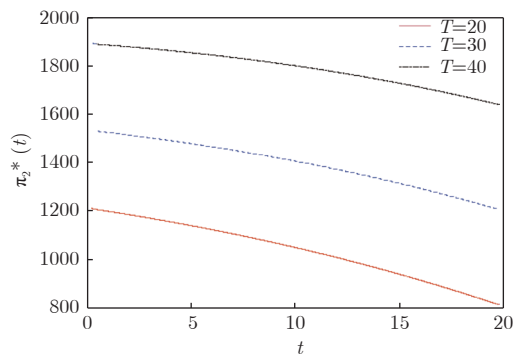
Table 1 Parameter settings used in the numerical simulation

r	μ_1	μ_2	σ_1	σ_2	α_1	α_2
0.01	0.15	0.12	0.4	0.3	0.1	0.1
u_1	u_2	β_1	β_2	λ_3	λ_4	λ_5
0.5	0.5	1	0.5	0.12	0.48	0.2

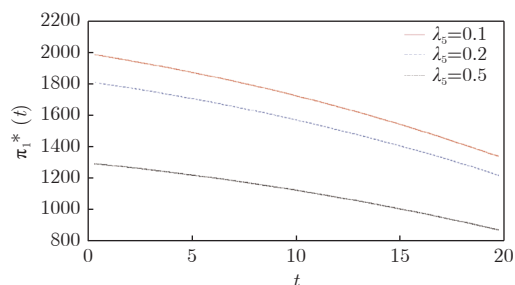
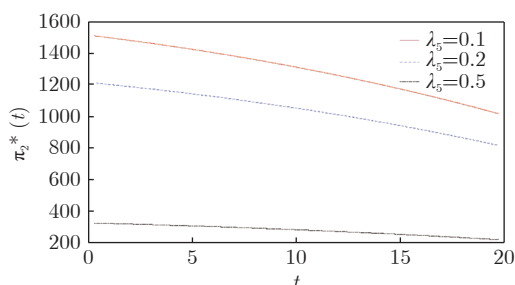
5.1 Effects of parameters on optimal investment strategies

We first perform numerical analysis on the optimal investment strategies by varying the value of one parameter while keeping all other parameters constant. In this subsection, $\pi_1^*(t)$ and $\pi_2^*(t)$ denote the optimal dollar amounts invested in the two risky assets.

In Figures 1 and 2, we examine the sensitivity of $\pi_1^*(t)$ and $\pi_2^*(t)$ to the parameter T . We find that the optimal amounts $\pi_1^*(t)$ and $\pi_2^*(t)$ invested in the risky assets increase as T increases. That is, when the time interval we considered is relatively long, larger amounts are allocated to the risky asset since the pension system needs more funds to remain sustainable.

**Figure 1** Effect of T on $\pi_1^*(t)$ **Figure 2** Effect of T on $\pi_2^*(t)$

From Figures 3 and 4, we can see that the optimal portfolio strategies $\pi_1^*(t)$ and $\pi_2^*(t)$ decrease as the value of λ_5 increases. This is expected, because a greater value of λ_5 implies a greater expected number of the jumps, and thus, the investor would rather invest less money to the risky asset.

**Figure 3** Effect of λ_5 on $\pi_1^*(t)$ **Figure 4** Effect of λ_5 on $\pi_2^*(t)$

In Figure 5(a), the parameters are set to $\lambda_3 = 0.12$ and $\lambda_4 = 0.48$, whereas in Figure 5(b), the parameters are set to $\lambda_3 = 0.60$ and $\lambda_4 = 0.48$. We find that when $\lambda_3 < \lambda_4$, the value of $\pi_1^*(t)$ is larger than that of $\pi_2^*(t)$, and vice versa. This implies that the values of the optimal portfolio strategies are sensitive to the counting processes.

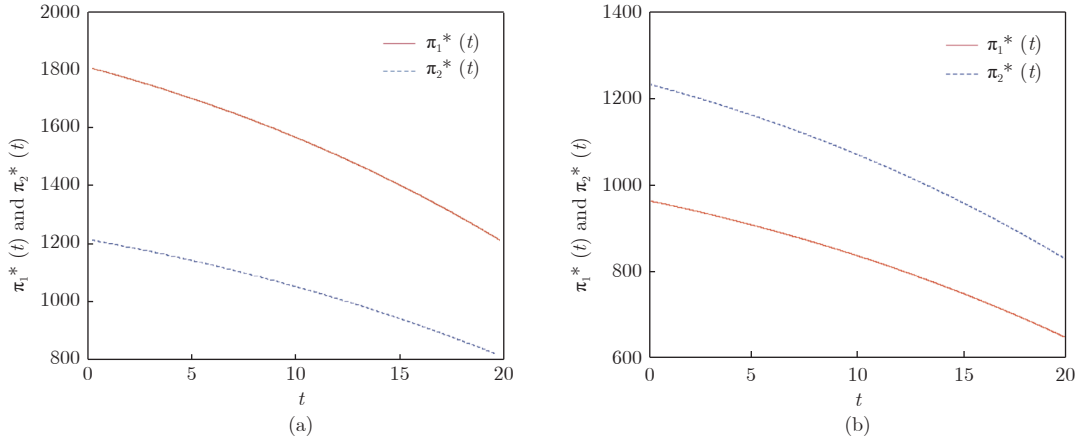


Figure 5 Effect of λ_3 and λ_4 on $\pi_1^*(t)$ and $\pi_2^*(t)$

5.2 Effects of parameters on optimal adjustment strategies

This subsection examines the effect of changes in the weighting factors T , γ_1 , and γ_2 on the optimal pension adjustment strategies $c^*(t)$ and $b^*(t)$ at a certain time point. The parameters $c^*(t)$ and $b^*(t)$ have the following relationships with $\lambda_1^*(t)$ and $\lambda_2^*(t)$.

$$c^*(t) = ce^{\tau t} - \alpha \frac{SP(t)}{NC(t)} = ce^{\tau t} - \lambda_1^*(t), \tag{5.1}$$

$$b^*(t) = be^{\tau t} + \beta \frac{SP(t)}{NB(t)} = be^{\tau t} + \lambda_2^*(t). \tag{5.2}$$

We first investigate the impact of the time interval T on the optimal adjustment strategies. In Figures 6 and 7, a longer time interval leads to lower contribution and higher benefit since there future uncertainty in the remote future and we have to put sufficient funds to the pension system to meet the budget.

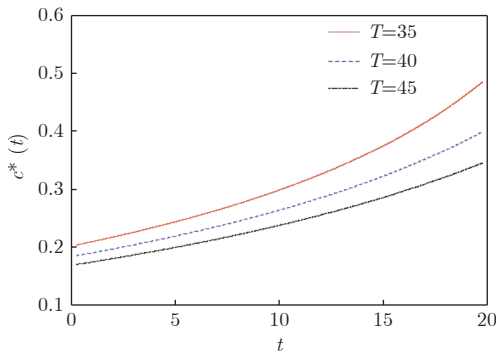


Figure 6 Effect of T on $c^*(t)$

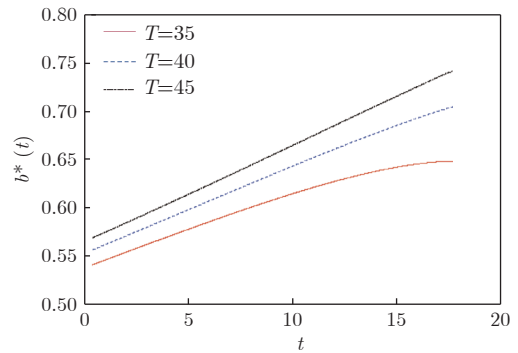


Figure 7 Effect of T on $b^*(t)$

Finally, we investigate the impacts of the weight parameters of unstable contribution γ_1 and benefit risk γ_2 on the optimal policies. In Figures 8 and 9, when γ_1 is high and γ_2 is low, this situation implies a more pronounced focus on alleviating the risk of unstable contributions rather than that of unstable benefits. Notably, both the contribution and benefit levels maintain relatively high. This underscores the fact that by ensuring a stable and sufficient level of contribution, the benefit naturally tends to be satisfactory, demonstrating the inherent balance and interdependence between these two factors.

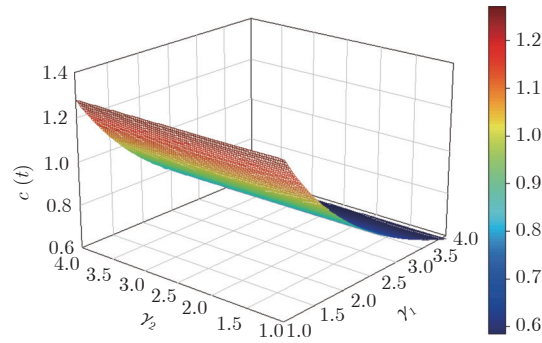


Figure 8 Effect of γ_1 and γ_2 on $c^*(t)$

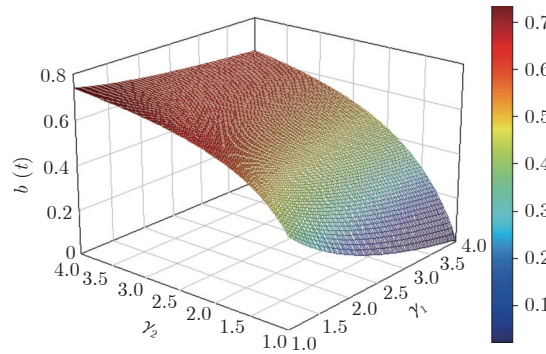


Figure 9 Effect of γ_1 and γ_2 on $b^*(t)$

6. Conclusion

This study has made a significant contribution to the understanding of optimal investment strategies and risk-sharing mechanisms within hybrid pension plans operating in a complex financial environment characterized by a jump-diffusion risky asset price model. By modeling the pension fund's investments across a risk-free asset and two correlated risky assets, modulated by a compound Poisson process, the research has successfully captured the dynamics of both stable and discontinuous risks inherent in such systems. By adjusting contributions and benefits contingent upon plan performance and facilitating risk sharing across generations, the proposed framework has demonstrated its effectiveness in minimizing the costs associated with unstable contributions, benefits, and discontinuous events. The application of stochastic optimal control techniques has enabled the derivation of closed-form expressions for the optimal investment strategies and value functions, providing valuable insights into how plan trustees and participants can navigate the uncertainties of the financial market to achieve long-term stability and sustainability. The numerical analysis further highlights the adaptability of the optimal strategies, which dynamically adjust to changes in fund performance, reinforcing the potential for intergenerational risk-sharing and resilience within hybrid pension plans.

Acknowledgements

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