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Uniqueness of solutions to quadratic BSDEs with locally Lipschitz generator

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Abstract We study the uniqueness of solutions of backward stochastic differential equations (BSDEs), which generator verifies $|F(t,y,z)| \leqslant \alpha_t + \beta_t|y| + \theta_t|z| + f(|y|)|z|^2$, where α_t, β_t, θ_t are positive processes and the function f is positive, continuous and increasing. The uniqueness of solutions of such BSDEs is derived in two situations, when F is locally Lipschitz and when F is jointly convex. As a byproduct: we show the existence of viscosity solutions to the associated semilinear partial differential equations, which can contain nonlinearity that has quadratic growth in the gradient of the solution.

Keywords Quadratic backward stochastic differential equations, Minimal and maximal solutions, Uniqueness of solutions, Comparison theorem, Partial differential equation, Viscosity solution

2020 Mathematics Subject Classification 60H10, 60H20, 60H30, 91G10

1. Introduction

Let $(W_t)_{0\leqslant t\leqslant T}$ be a d-dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$. Here, $(\mathcal{F}_t)_{0\leqslant t\leqslant T}$ stands for the $\mathbb{P}$-completion of filtration generated by W. Let $F : [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d \mapsto \mathbb{R}$ be a progressively measurable process. Let ξ be a $\mathbb{R}$-valued random variable which is measurable with respect to $\mathcal{F}_T$. Consider the BSDE

$$Y_t = \xi + \int_t^T F(s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s\cdot \mathrm{d}W_s, \tag{1.1}$$

the random variable ξ is called the terminal condition and F is called the generator or the driver. We denote BSDE (ξ, F) a BSDE with driver F and terminal condition ξ. The BSDE(ξ, F) is called a quadratic BSDE if F has at most a quadratic growth with respect to z. We call ξ the bounded terminal value, if $\xi \in \mathcal{S}^\infty$ (see the definition of the space in Section 2).

Nonlinear backward stochastic differential equations were first introduced in [27] and solved in the case where F is uniformly Lipschitz in both of its variables, y and z. Since then, extensive

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research has been conducted to explore various properties related to BSDEs, including the existence of solutions under conditions weaker than the Lipschitz assumption, as well as their connections to partial differential equations (PDEs), optimal control, finance, and insurance. A class of particular interest is the BSDEs with quadratic growth in the control variable z. Such BSDEs have a wide range of applications, including utility maximization problems, quenching problems (see [11, 24, 25]), and gas flow in porous media (see [18, 19]), among others. The problem of existence of solutions was first studied for bounded terminal conditions in [23], where the author demonstrated the monotone stability of quadratic BSDEs. In [6], the authors explored the existence of solutions in the case of an exponentially integrable terminal value by combining the localization method with monotone stability.

In [23], the uniqueness of the solution is established in the case where ξ is bounded and F is locally Lipschitz continuous. In [7], the uniqueness of unbounded solutions to quadratic BSDEs is studied, assuming that ξ possesses all exponential moments and that the driver F is Lipschitz continuous in y, and either convex or concave in z. This result was further strengthened in [9] by assuming that ξ admits certain exponential moments in L^p with p greater than a constant γ. The critical case $p = \gamma$ was discussed in [10]. The problem of uniqueness of the solution with a non-convex generator was addressed in [28] for Markovian BSDEs with a Lipschitz continuous generator. In [17], a BSDE with a coefficient exhibiting quadratic growth of the form $P(|y|)|z|^2$, where P is a specific polynomial form, and satisfies a Lipschitz property, was considered. In [3], the existence of solutions was established for BSDEs with an Ł^2-integrable terminal value, where the quadratic part of the generator is dominated by $f(|y|)|z|^2$, with f being globally integrable and locally bounded. Moreover, the uniqueness of the Ł^2-solution was proved in [3] for the particular case in which the generator has the form $f(|y|)|z|^2$.

In [1], the existence of solutions is established for a quadratic BSDE (QBSDE) where the quadratic part of the generator is dominated by $f(|y|)|z|^2$, with f being measurable. The case in which f is globally integrable was also considered in [1], and the existence of an L^1 -solution was established when the terminal value is merely L^1-integrable. A particular case with a generator of the form $f(|y|)|z|^2$ and a bounded terminal value was discussed in [30]. In [2], BSDEs with a singular quadratic term of the form $\delta \frac{z^2}{y}$ were studied. Moreover, the uniqueness of solutions was investigated under the assumption that the terminal value is bounded.

Other results concerning the solvability of quadratic BSDEs can be found in [4, 8, 9, 14, 20, 26]. An extensive overview of QBSDEs is provided in the recent paper [15].

In this paper, we are concerned with the problem of the uniqueness of the solution of equation (1.1) in the case of a bounded terminal value, and the generator F satisfies the following conditions:

- $|F(t,y,z)| \leqslant \alpha_t + \beta_t|y| + \theta_t|z| + f(|y|)|z|^2$, where α_t, β_t, θ_t are some positive processes.
- For every $(t, y, y', z, z') \in [0,T] \times \mathbb{R}^2 \times \mathbb{R}^{2d}$, and every $\delta \in (0,2)$

$$\begin{aligned} |F(t,y,z) - F(t,y',z')| \leqslant\ & \kappa_y \left[(1 + |z|^\delta + |z'|^\delta)|y - y'|\right] \\ & + \kappa_z \left[1 + (f(|y|) + f(|y'|))(|z| + |z'|)\right]|z - z'|. \end{aligned}$$

- For every (t, ω), $F(t, \omega, ., .)$ is jointly convex.

Under the second condition, we borrow the technique used in [17]. This method allows us to obtain uniqueness in the space of bounded solutions, such that the stochastic integral $\int_0^{\cdot} Z_s \cdot \mathrm{d}W_s$ is a BMO martingale. Our uniqueness result generalizes the one obtained in [17]. As a consequence, we also prove the existence of viscosity solutions for the corresponding quadratic PDEs. When the driver F is jointly convex, we derive a dual convex representation for the component Y of the solution, which further ensures uniqueness. Moreover, we demonstrate that the comparison principle for solutions follows from the uniqueness and domination arguments. We note that the comparison of solutions has been established in the specific case $F(t,y,z)=k|z|+f(y)|z|^2$ in [1, 3, 30].

The paper is organized as follows. In Section 2, we provide definitions, notations, and preliminaries, and recall the so-called domination method as presented in [1]. In Section 3, we establish the uniqueness and comparison of solutions in the case of a locally Lipschitz generator, and prove the existence of viscosity solutions to the associated quadratic PDEs. Section 4 is dedicated to the uniqueness and comparison of solutions, as well as the existence of viscosity solutions when F is jointly convex.

2. Preliminaries

The following notations will be used.

- $\mathcal{C}$:= the space of continuous and $\mathcal{F}_t$-adapted processes.
- $\mathcal{S}^p$:= the space of continuous, $\mathcal{F}_t$-adapted processes φ such that
$$\|\varphi\|_{\mathcal{S}^p} := \mathbb{E}\left(\sup_{0\leqslant t\leqslant T}|\varphi_t|^p\right)^{\frac{1}{p}} < \infty.$$
- $\mathcal{S}^\infty$:= the space of bounded measurable processes.
- $\mathcal{L}^2$:= the space of $\mathcal{F}_t$-adapted processes φ satisfying $\int_0^T|\varphi_s|^2\mathrm{d}s<+\infty$ $\mathbb{P}$-a.s.
- $\mathcal{M}^p$:= the space of $\mathcal{F}_t$-adapted processes φ satisfying
$$\|\varphi\|_{\mathcal{M}^p} := \mathbb{E}\left[\left(\int_0^T|\varphi_s|^2\mathrm{d}s\right)^{\frac{p}{2}}\right]^{\frac{1}{p}} < +\infty.$$
- $\mathcal{B}$:= the space of uniformly integrable martingales M with $M_0=0$ satisfying
$$\|M\|_{\mathcal{B}} := \sup_{\tau}||\mathbb{E}\big(\langle M\rangle_T-\langle M\rangle_\tau/\mathcal{F}_\tau\big)||^{\frac{1}{2}}_{\mathcal{S}^\infty} < \infty,$$
where the supremum is taken over all stopping times $\tau\in[0,T]$.
- BMO:= the space of $\mathcal{F}_t$-adapted processes Z such that $\int_0^\cdot Z_s\cdot\mathrm{d}W_s$ is in the space $\mathcal{B}$.

For the sake of completeness, we briefly recall the domination argument and provide its proof. This technique is used in [1] to directly derive the existence of solutions to BSDE (1.1), without relying on a priori estimates or approximations. Here, it is used to derive a comparison result when the uniqueness of solutions holds.

Lemma 2.1 *(The domination argument [1]). Let F be continuous in (y,z) for a.e. (t,ω). Assume that the* BSDE(ξ,F) *satisfies the following domination conditions.*

There exist two BSDEs with parameters (ξ_1,F_1) and (ξ_2,F_2) such that:

(D1) $\xi_1\leqslant\xi\leqslant\xi_2$

(D2) BSDE(ξ_1,F_1) *and* BSDE(ξ_2,F_2), *respectively, admit solutions (Y^1,Z^1) and (Y^2,Z^2), satisfying:*

(a) $Y^1\leqslant Y^2$,

(b) *for every (t,ω), $y\in[Y^1_t(\omega),\ Y^2_t(\omega)]$ and $z\in\mathbb{R}^d$,*

(i) $F_1(t,y,z)\leqslant F(t,y,z)\leqslant F_2(t,y,z)$

(ii) $|F(t,\omega,y,z)|\leqslant\eta_t(\omega)+C_t(\omega)|z|^2$

where C and η are $\mathcal{F}_t$-adapted processes such that C is continuous and η satisfies for a.s ω, $\int_0^T|\eta_s(\omega)|\mathrm{d}s<\infty$.

Then, the BSDE(ξ,F) *admits a solution (Y,Z) such that $Y^1\leqslant Y\leqslant Y^2$. Moreover, the* BSDE$(\xi,F)$ *has maximal and minimal solutions, denoted Y^M and Y^m, respectively, such that*

$Y^1 \leqslant Y^m \leqslant Y^M \leqslant Y^2$.

Proof Using Theorem 3.2 in [13] with $L = Y^1$ and $U = Y^2$, there exists a process (Y, Z, K^+, K^-) such that (Y, Z) belongs to $\mathcal{C} \times \mathcal{L}^2$ and (Y, Z, K^+, K^-) satisfies the subsequent system, for $t \in [0,\ T]$,

$$\begin{cases} Y_t = \xi + \displaystyle\int_t^T F(s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \cdot \mathrm{d}W_s \\ \qquad + \displaystyle\int_t^T \mathrm{d}K_s^+ - \int_t^T \mathrm{d}K_s^-, \\ \forall\, t \leqslant T, \quad Y_t^1 \leqslant Y_t \leqslant Y_t^2, \\ \displaystyle\int_0^T (Y_t - Y_t^1)\mathrm{d}K_t^+ = \int_0^T (Y_t^2 - Y_t)\mathrm{d}K_t^- = 0, \text{ a.s.}, \\ K_0^+ = K_0^- = 0, \quad K^+, K^- \text{ are continuous nondecreasing}, \\ \mathrm{d}K^+ \perp \mathrm{d}K^-. \end{cases} \tag{2.1}$$

Moreover, equation (2.1) has a minimal solution and a maximal solution.

It remains to prove that $\mathrm{d}K^+ = \mathrm{d}K^- = 0$. Since Y_t^2 is a solution to the BSDE(ξ_2, F_2), applying Tanaka's formula to $(Y_t^2 - Y_t)^+$ we get that

$$\begin{aligned}(Y_t^2 - Y_t)^+ = (Y_0^2 - Y_0)^+ &+ \int_0^t 1_{\{Y_s^2 > Y_s\}}[F(s, Y_s, Z_s) - F_2(s, Y_s^2, Z_s^2)]\mathrm{d}s \\ &+ \int_0^t 1_{\{Y_s^2 > Y_s\}}(\mathrm{d}K_s^+ - \mathrm{d}K_s^-) + \int_0^t 1_{\{Y_s^2 > Y_s\}}(Z_s^2 - Z_s) \cdot \mathrm{d}W_s \\ &+ L_t^0(Y^2 - Y),\end{aligned}$$

where $L_\cdot^0(Y^2 - Y)$ stands for the local time of the process $(Y^2 - Y)$ at 0.

Since $(Y_t^2 - Y_t)^+ = (Y_t^2 - Y_t)$, we identify the terms of $(Y_t^2 - Y_t)^+$ with those of $(Y_t^2 - Y_t)$ to derive that:

$$(Z_s - Z_s^2)1_{\{Y_s^2 = Y_s\}} = 0 \quad \text{for } a.e.\ (s, \omega),$$

and

$$\int_0^t 1_{\{Y_s^2 = Y_s\}}(\mathrm{d}K_s^+ - \mathrm{d}K_s^-) = L_t^0(Y^2 - Y) + \int_0^t 1_{\{Y_s^2 = Y_s\}}[F_2(s, Y_s^2, Z_s^2) - F(s, Y_s, Z_s)]\mathrm{d}s.$$

Since $\int_0^t 1_{\{Y_s^2 = Y_s\}}\mathrm{d}K_s^+ = 0$, it follows that

$$0 \leqslant L_t^0(Y^2 - Y) + \int_0^t 1_{\{Y_s^2 = Y_s\}}[F_2(s, Y_s^2, Z_s^2) - F(s, Y_s, Z_s)]\mathrm{d}s = -\int_0^t 1_{\{Y_s^2 = Y_s\}}\mathrm{d}K_s^- \leqslant 0.$$

Thus, $\int_0^t 1_{\{Y_s^2 = Y_s\}}\mathrm{d}K_s^- = 0$, which implies that $\mathrm{d}K^- = 0$. Arguing similarly, we get that $\mathrm{d}K^+ = 0$. Hence, (Y, Z) is a solution to the (non-reflected) BSDE (ξ, F). □

We consider the following assumption:

$(\mathcal{A}1)$ The mapping $(y, z) \to F(t, y, z)$ is continuous for $a.e$ (t, ω), and for any $(y, z) \in \mathbb{R} \times \mathbb{R}^d$ it holds that:

$$|F(t, y, z)| \leqslant \alpha_t + \beta_t |y| + f(|y|)|z|^2,$$

where α_t, β_t are some $(\mathcal{F}_t)$-adapted processes which are positive and f is a real valued function which is continuous, increasing and positive on $\mathbb{R}_+$.

Next, we present the main lines of the existence result of solutions for BSDE (1.1) obtained in [1] under the assumption $(\mathcal{A}1)$.

We set

$$g(t,y,z) := \alpha_t + \beta_t|y| + f(|y|)|z|^2. \tag{2.2}$$

Theorem 2.2 *(Existence). Let* $(\mathcal{A}1)$ *be satisfied. Assume, moreover, that* ξ, $\int_0^T \alpha_s \mathrm{d}s$, *and* $\int_0^T \beta_s \mathrm{d}s$ *are bounded. Then, the BSDE (1.1) admits a solution* $(Y,Z) \in \mathcal{S}^\infty \times BMO$, *such that* $Y^{-g} \leqslant Y \leqslant Y^g$, *where* Y^g *and* Y^{-g} *are solutions of* $\mathrm{BSDE}(\xi^+, g)$ *and* $\mathrm{BSDE}(-\xi^-, -g)$, *respectively.*

Proof The idea consists in using Lemma 2.1 and [1, Proposition 3.1]. We consider the two solutions constructed in [1, Proposition 3.1], which satisfy $Y^{-g} \leqslant Y^g$. Then, Lemma 2.1 applied to $\xi_1 = -\xi^-$, $\xi_2 = \xi^+$, $F_1 = -g$ and $F_2 = g$ yields the existence of a solution (Y,Z).

Since ξ, $\int_0^T \alpha_s \mathrm{d}s$, and $\int_0^T \beta_s \mathrm{d}s$ are bounded, then a standard computation shows that Y is bounded. We shall show that $\mathbb{E}\left(\int_0^T |Z_s|^2 \mathrm{d}s\right) < \infty$. Put

$$K(y) := \int_0^y \exp\left(-2\int_0^z f(r)\mathrm{d}r\right)\mathrm{d}z.$$

Then, the function

$$v(x) := \int_0^x K(y)\exp\left(2\int_0^y f(r)\mathrm{d}r\right)\mathrm{d}y \tag{2.3}$$

satisfies the differential equation $\frac{1}{2}v''(x) - f(x)v'(x) = \frac{1}{2}$ on $\mathbb{R}$ and has the following properties: v and v' are positive on $\mathbb{R}_+$ and the function $x \longmapsto v(|x|)$ belongs to $\mathcal{C}^2(\mathbb{R})$. For $N > 0$, we define $\tau_N := \inf\{t > 0 : \int_0^t |v'(Y_s)|^2|Z_s|^2 \mathrm{d}s \geqslant N\} \wedge T$. Itô's formula applied to $v(|Y_t|)$ gives that for every $t \in [0,T]$,

$$\begin{aligned} v(|Y_0|) = v(|Y_{t\wedge\tau_N}|) &+ \int_0^{t\wedge\tau_N}\left[\mathrm{sgn}(Y_s)v'(|Y_s|)F(s,Y_s,Z_s) - \frac{1}{2}v''(|Y_s|)|Z_s|^2\right]\mathrm{d}s \\ &- \int_0^{t\wedge\tau_N} \mathrm{sgn}(Y_s)v'(|Y_s|)Z_s \cdot \mathrm{d}W_s\,. \end{aligned}$$

Using assumption $(\mathcal{A}1)$ and the fact that $\frac{1}{2}v''(x) - f(x)v'(x) = \frac{1}{2}$ on $\mathbb{R}$, we get

$$\frac{1}{2}\int_0^{t\wedge\tau_N} |Z_s|^2\,\mathrm{d}s \leqslant v(|Y_{t\wedge\tau_N}|) + \int_0^{t\wedge\tau_N} (\alpha_s + \beta_s|Y_s|)v'(|Y_s|)\,\mathrm{d}s - \int_0^{t\wedge\tau_N} \mathrm{sgn}(Y_s)v'(|Y_s|)Z_s \cdot \mathrm{d}W_s\,.$$

Since $\int_0^T \alpha_s \mathrm{d}s$, $\int_0^T \beta_s \mathrm{d}s$, and Y are bounded, we conclude by Fatou's lemma that

$$\mathbb{E}\int_0^T |Z_s|^2\,\mathrm{d}s < \infty.$$

Now, we prove that the process $(\int_0^t Z_s \cdot \mathrm{d}W_s)_{0\leqslant t\leqslant T}$ is a BMO martingale. By Itô's formula, we obtain that for any $\mathcal{F}_t$-stopping time $\tau \leqslant T$,

$$\begin{aligned} v(|Y_T|) = v(|Y_\tau|) &+ \int_\tau^T \left[\frac{1}{2}v''(|Y_s|)|Z_s|^2 - \mathrm{sgn}(Y_s)v'(|Y_s|)F(s,Y_s,Z_s)\right]\mathrm{d}s \\ &+ \int_\tau^T \mathrm{sgn}(Y_s)v'(|Y_s|)Z_s \cdot \mathrm{d}W_s\,. \end{aligned}$$

Using the fact that Y is bounded and $Z \in \mathcal{M}^2$, then passing to the conditional expectation, we obtain

$$\mathbb{E}\left(\int_\tau^T |Z_s|^2 \mathrm{d}s/\mathcal{F}_\tau\right) \leqslant C + \mathbb{E}\left(\int_\tau^T [(\alpha_s + \beta_s|Y_s|)v'(|Y_s|)]\mathrm{d}s/\mathcal{F}_\tau\right),$$

where C is a positive constant. Again, using the boundedness of Y, $\int_0^T \alpha_s \mathrm{d}s$, and $\int_0^T \beta_s \mathrm{d}s$, we get that $(\int_0^t Z_s \cdot \mathrm{d}W_s)_{0\leqslant t\leqslant T}$ is a BMO martingale. □

3. Uniqueness of solutions when the generator is locally Lipschitz

This section presents the uniqueness and comparison of solutions for the BSDE (1.1) under the following assumption:

($\mathcal{A}2$) There exist constants $c, \kappa_y, \kappa_z > 0$ such that for each $(t, \omega, y, y', z, z') \in [0,T] \times \Omega \times \mathbb{R}^2 \times \mathbb{R}^d \times \mathbb{R}^d$, $\| \int_0^T F(t,0,0)\mathrm{d}t\|_{\mathcal{S}^\infty} < c$, and for every $\delta \in (0,2)$

$$\begin{aligned}|F(t,y,z) - F(t,y',z')| \leqslant{}& \kappa_y \left[(1 + |z|^\delta + |z'|^\delta)|y - y'|\right] \\ &+ \kappa_z \left[1 + (f(|y|) + f(|y'|))(|z| + |z'|)\right]|z - z'|,\end{aligned}$$

where f is a real valued function which is continuous, increasing and positive on $\mathbb{R}_+$.

Remark 3.1 *One can readily notice that, under assumption ($\mathcal{A}2$), the generator F satisfies assumption ($\mathcal{A}1$). In the sequel, the function g will denote the function that majorizes F, as endowed by assumption ($\mathcal{A}2$).*

The following theorem establishes the uniqueness of the solution to the BSDE (1.1).

Theorem 3.2 *(Uniqueness). Let assumption ($\mathcal{A}2$) hold. Assume, moreover, that ξ is bounded. Then, the BSDE (1.1) admits a unique solution $(Y,Z) \in \mathcal{S}^\infty \times BMO$ such that $Y^{-g} \leqslant Y \leqslant Y^g$, where Y^g and Y^{-g} are solutions of* BSDE(ξ^+, g) *and* BSDE$(-\xi^-, -g)$*, respectively.*

Proof Let (Y,Z) and (Y',Z') be two solutions of the BSDE (1.1) in the space $\mathcal{S}^\infty \times BMO$ such that $Y^{-g} \leqslant Y,\ Y' \leqslant Y^g$. We set $\delta Y_t = Y_t - Y'_t$ and $\delta Z_t = Z_t - Z'_t$ and define the processes:

$$\begin{aligned}\Gamma_t &:= \frac{F(t,Y_t,Z_t) - F(t,Y'_t,Z_t)}{Y_t - Y'_t} 1_{\{Y_t - Y'_t \neq 0\}}, \\ e_t &:= \exp\left(\int_0^t \Gamma_s \mathrm{d}s\right), \\ \Lambda_t &:= \frac{F(t,Y'_t,Z_t) - F(t,Y'_t,Z'_t)}{|Z_t - Z'_t|^2}(Z_t - Z'_t) 1_{\{|Z_t - Z'_t| \neq 0\}}.\end{aligned} \tag{3.1}$$

For a local martingale M, we denote its stochastic exponential by:

$$\mathcal{E}(M) := \exp\left(M_T - \frac{1}{2}\langle M\rangle_T\right). \tag{3.2}$$

Using assumption ($\mathcal{A}2$), we get

$$|\Lambda_t| \leqslant \kappa_z(1 + 2f(|Y'_t|)(|Z_t| + |Z'_t|)).$$

Hence, for every stopping time $\tau \in [0,T]$, we obtain

$$\mathbb{E}\left(\int_\tau^T |\Lambda_t|^2 \mathrm{d}t/\mathcal{F}_\tau\right) \leqslant CT + C\mathbb{E}\left(\int_\tau^T |Z_t|^2 \mathrm{d}t + \int_\tau^T |Z'_t|^2 \mathrm{d}t/\mathcal{F}_\tau\right) < \infty, \tag{3.3}$$

where C is a positive constant depending on k_z, f and $\|Y'\|_{\mathcal{S}^\infty}$. Since $\int_0^\cdot Z_s \cdot \mathrm{d}W_s$ and $\int_0^\cdot Z'_s \cdot \mathrm{d}W_s$ are BMO martingales, $\int_0^\cdot \Lambda_s \cdot \mathrm{d}W_s$ is also a BMO martingale.

Furthermore, the BMO properties ensure that the stochastic exponential $\mathcal{E}\left(\int_0^\cdot \Lambda_s \cdot \mathrm{d}W_s\right)$ is a true martingale, and there exists $p > 1$ such that $\mathcal{E}\left(\int_0^\cdot \Lambda_s \cdot \mathrm{d}W_s\right) \in L^p$.

We now consider a new probability $\mathbb{Q} = \mathcal{E}\left(\int_0^\cdot \Lambda_s \cdot \mathrm{d}W_s\right)\mathbb{P}$. By Girsanov's theorem, the process defined by $\tilde{W}_t := W_t - \int_0^t \Lambda_s \mathrm{d}s$ is a $\mathbb{Q}$-Brownian motion.

Uniqueness of Y We have

$$\delta Y_t = \int_t^T F(s, Y_s, Z_s) - F(s, Y'_s, Z'_s)\mathrm{d}s - \int_t^T \delta Z_s \cdot \mathrm{d}W_s\,.$$

Using Itô's formula, we get

$$e_t \delta Y_t = \int_t^T e_s \left(F(s, Y_s, Z_s) - F(s, Y'_s, Z'_s)\right)\mathrm{d}s - \int_t^T e_s \delta Z_s \cdot \mathrm{d}W_s - \int_t^T e_s \Gamma_s \delta Y_s \mathrm{d}s\,.$$

Rewriting the previous equation under the probability $\mathbb{Q}$, we obtain:

$$\begin{aligned} e_t \delta Y_t &= \int_t^T e_s \left(F(s, Y_s, Z_s) - F(s, Y'_s, Z'_s) - \Gamma_s \delta Y_s\right)\mathrm{d}s - \int_t^T e_s \delta Z_s \cdot \mathrm{d}\tilde{W}_s - \int_t^T e_s \delta Z_s \cdot \Lambda_s \mathrm{d}s \\ &= \int_t^T e_s \left(F(s, Y_s, Z_s) - F(s, Y'_s, Z'_s) - \Gamma_s \delta Y_s - \delta Z_s \cdot \Lambda_s\right)\mathrm{d}s - \int_t^T e_s \delta Z_s \cdot \mathrm{d}\tilde{W}_s\,. \end{aligned}$$

On the other hand, since

$$F(s, Y_s, Z_s) - F(s, Y'_s, Z'_s) - \Gamma_s \delta Y_s - \delta Z_s \cdot \Lambda_s = 0\,,$$

it follows that

$$e_t \delta Y_t + \int_t^T e_s \delta Z_s \cdot \mathrm{d}\tilde{W}_s = 0, \quad \mathbb{Q}\text{-}a.s\,. \tag{3.4}$$

Next, we claim that $\int_t^T e_s \delta Z_s \cdot \mathrm{d}\tilde{W}_s$ is a true martingale under $\mathbb{Q}$. Indeed,

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}}\left[\int_0^T |e_s|^2 |\delta Z_s|^2 \mathrm{d}s\right] \\ &\leqslant \mathbb{E}\left[\mathcal{E}\left(\int_0^\cdot \Lambda_s \cdot \mathrm{d}W_s\right) \sup_{0\leqslant s\leqslant T} |e_s|^2 \int_0^T |\delta Z_s|^2 \mathrm{d}s\right] \\ &\leqslant \left[\mathbb{E}\left[\mathcal{E}\left(\int_0^\cdot \Lambda_s \cdot \mathrm{d}W_s\right)^p\right]\right]^{\frac{1}{p}} \left[\mathbb{E}\left[\sup_{0\leqslant s\leqslant T} |e_s|^{2q} \left(\int_0^T |\delta Z_s|^2 \mathrm{d}s\right)^q\right]\right]^{\frac{1}{q}} \\ &\leqslant \left[\mathbb{E}\left[\mathcal{E}\left(\int_0^\cdot \Lambda_s \cdot \mathrm{d}W_s\right)^p\right]\right]^{\frac{1}{p}} \left[\mathbb{E}\left[\sup_{0\leqslant s\leqslant T} |e_s|^{2qr}\right]\right]^{\frac{1}{qr}} \left[\mathbb{E}\left[\left(\int_0^T |\delta Z_s|^2 \mathrm{d}s\right)^{qr'}\right]\right]^{\frac{1}{qr'}}\,. \end{aligned}$$

Since $\int_0^\cdot Z_s \cdot \mathrm{d}W_s$ is a BMO martingale, it follows that

$$\mathbb{E}\left[\sup_{0\leqslant s\leqslant T} |e_s|^{2qr}\right] \leqslant \mathbb{E}\left[\exp\left(2qr\kappa_y \int_0^T 1 + 2|Z_s|^\delta \mathrm{d}s\right)\right] < \infty\,.$$

Since $\int_0^\cdot \delta Z_s \cdot \mathrm{d}W_s$ is also a BMO martingale, for every $q \geqslant 2$, we have $\mathbb{E}\left[\left(\int_0^T |\delta Z_s|^2 \mathrm{d}s\right)^q\right]^{\frac{1}{q}} < \infty$.

It follows that

$$\mathbb{E}_{\mathbb{Q}}\left[\int_0^T |e_s|^2|\delta Z_s|^2\mathrm{d}s\right] < +\infty\,.$$

Passing to the conditional expectation with respect to $\mathcal{F}_t$ in (3.4), we obtain that $e_t\delta Y_t = 0$ $\mathbb{Q}$-*a.s.* and $\mathbb{P}$-a.s. Taking into account the continuity of e_t and δY_t, we deduce that there exists a set $A \subset \Omega$ with $\mathbb{P}(A) = 0$ such that

$$\text{for all } \ t \in [0,T], \quad \omega \in \Omega\backslash A \quad e_t(\omega) > 0\,.$$

It follows that for any $t \in [0,T]$, $Y_t = Y'_t$ $\mathbb{P}$-a.s.

Uniqueness of Z Using equation (3.4) and the Itô isometry, we get $\mathbb{E}_{\mathbb{Q}}\left(\int_0^T |e_s|^2|\delta Z_s|^2\mathrm{d}s\right) = 0$, which implies that $|\delta Z_s| = 0\,\mathrm{d}s \otimes \mathrm{d}\mathbb{P}$-a.e. Hence, $Z_s = Z'_s$ $\mathrm{d}s \otimes \mathrm{d}\mathbb{P}$-a.e. The proof is complete. ☐

Remark 3.3 *A result analogous to Theorem 3.2 is presented in Theorem 2.3 of [16] for multi-dimensional quadratic BSDEs. Notably, Theorem 2.3 establishes a local uniqueness property, while global uniqueness is addressed in Theorem 2.6 under an additional assumption on the driver.*

We now present a comparison of solutions, which will be used to establish the existence of viscosity solutions.

Proposition 3.4 *Let* (Y^1, Z^1) *and* (Y^2, Z^2) *be two solutions of* $\mathrm{BSDE}(\xi_1, F_1)$ *and* $\mathrm{BSDE}(\xi_2, F_2)$*, respectively, assumed to belong to* $\mathcal{S}^\infty \times BMO$*, and such that* $Y^{-g} \leqslant Y^1, Y^2 \leqslant Y^g$*. We assume, moreover, that the conditions of Theorem 3.2 are satisfied, and that*

$$\xi_1 \leqslant \xi_2 \quad \textit{and} \quad F_1(t, Y_t^2, Z_t^2) \leqslant F_2(t, Y_t^2, Z_t^2) \quad \mathrm{d}t \times \mathrm{d}\mathbb{P}\text{-a.e.}$$

Then, for every $t \in [0,T]$*, we have* $Y_t^1 \leqslant Y_t^2$ $\mathbb{P}$-a.s.
Furthermore, if $\xi_1 < \xi_2$ *or* $F_1(t, Y_t^2, Z_t^2) < F_2(t, Y_t^2, Z_t^2)$ *on a set of positive* $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}$*-measure then* $Y_0^1 < Y_0^2$*.*

Proof We set $\delta Y_t = Y_t^1 - Y_t^2$, $\delta Z_t = Z_t^1 - Z_t^2$, $\delta\xi = \xi_1 - \xi_2$, and $\delta F_t = F_1(t, Y_t^2, Z_t^2) - F_2(t, Y_t^2, Z_t^2)$ for each $t \in [0,T]$. We define

$$\Gamma_t := \frac{F_1(t, Y_t^1, Z_t^1) - F_1(t, Y_t^2, Z_t^1)}{Y_t^1 - Y_t^2} 1_{\{Y_t^1 - Y_t^2 \neq 0\}}, \quad e_t := \exp\left(\int_0^t \Gamma_s \mathrm{d}s\right),$$
$$\Lambda_t := \frac{F_1(t, Y_t^2, Z_t^1) - F_1(t, Y_t^2, Z_t^2)}{|Z_t^1 - Z_t^2|^2}\left(Z_t^1 - Z_t^2\right) 1_{\{|Z_t^1 - Z_t^2| \neq 0\}}.$$

We proceed as in the proof of Theorem 3.2 to show that there exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$, and a $\mathbb{Q}$-Brownian motion $W^{\mathbb{Q}}$ such that

$$\begin{aligned} e_t\delta Y_t &= e_T\delta\xi - \int_t^T e_s\delta Z_s \cdot \mathrm{d}W_s^{\mathbb{Q}} + \int_t^T e_s\delta F_s\mathrm{d}s, \\ &= \mathbb{E}_{\mathbb{Q}}\left[e_T\delta\xi + \int_t^T e_s\delta F_s\,\mathrm{d}s/\mathcal{F}_t\right]. \end{aligned}$$

Using the facts that $e_t > 0$, $\delta\xi \leqslant 0$, and $\delta F_t \leqslant 0$, we get $e_t\delta Y_t \leqslant 0$, which implies that $\delta Y_t = Y_t^1 - Y_t^2 \leqslant 0$ $\mathbb{Q}$-a.s and also $\mathbb{P}$-a.s.

Taking $t = 0$ and using the fact that $\delta\xi < 0$ or $\delta F_t < 0$, we obtain that $\delta Y_0 = Y_0^1 - Y_0^2 < 0$. This completes the proof. ☐

3.1 Existence of viscosity solutions for quadratic PDEs with locally Lipschitz nonlinearities

In this subsection, we apply the uniqueness result for BSDEs to demonstrate the existence of a viscosity solution to quadratic partial differential equations. We place our BSDE in a Markovian framework and proceed to prove the associated nonlinear Feynman-Kac formula. For $(t,x) \in [0,T] \times \mathbb{R}^d$, we introduce the system below:

$$\begin{cases} X_s^{t,x} = x + \displaystyle\int_t^s b(r, X_r^{t,x})\mathrm{d}r + \int_t^s \sigma(r, X_r^{t,x})\mathrm{d}W_r, \\ Y_s^{t,x} = h(X_T^{t,x}) + \displaystyle\int_s^T F(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\mathrm{d}r - \int_s^T Z_r^{t,x} \cdot \mathrm{d}W_r. \end{cases} \tag{3.5}$$

We consider the following PDE:

$$\begin{cases} \dfrac{\partial u}{\partial t} + Lu(t,x) + F(t,x,u(t,x),\sigma^t \nabla_x u(t,x)) = 0, \\ u(T,x) = h(x), \end{cases} \tag{3.6}$$

where L is the infinitesimal generator of $X^{t,x}$ defined by

$$L := \sum_{i=1}^d b_i(t,x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^d (\sigma\sigma^t)_{i,j}(t,x)\frac{\partial^2}{\partial x_i \partial x_j}. \tag{3.7}$$

We make the following assumptions:

$(\mathcal{A}'2)$ The map $(t,x) \mapsto F(t,x,0,0)$ is bounded, F is continuous, and for every $(t,x,y,y',z,z') \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^2 \times \mathbb{R}^d \times \mathbb{R}^d$ and every $\delta \in (0,2)$

$$\begin{aligned} |F(t,x,y,z) - F(t,x,y',z')| \leqslant\ & \kappa_y \left[\left(1 + |z|^\delta + |z'|^\delta\right)\right] |y - y'| \\ & + \kappa_z \left[1 + \left(f(|y|) + f(|y'|)\right)(|z| + |z'|)\right] |z - z'|, \end{aligned}$$

where f is a real valued function which is continuous, increasing, and positive on $\mathbb{R}_+$.

$(\mathcal{A}3)$ $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ are continuous functions, and there exists $c > 0$ such that: $|b(t,x)| + |\sigma(t,x)| \leqslant c(1+|x|), \quad$ for every $(t,x) \in [0,T] \times \mathbb{R}^d$.

$(\mathcal{A}4)$ The forward SDE of system (3.5) satisfies the pathwise uniqueness property.

$(\mathcal{A}5)$ The function $h : \mathbb{R}^d \to \mathbb{R}$ is continuous and bounded.

It was shown in [1] (see the proofs of Theorem 3.1 and Corollary 3.2) that the uniform bounds for the norms of $(Y^{t,x}, Z^{t,x})$ depend only on the bounds of ξ, $\int_0^T \alpha_s \mathrm{d}s$, and $\int_0^T \beta_s \mathrm{d}s$. The following lemma states this result.

Lemma 3.5 *Let assumptions $(\mathcal{A}'2)$ and $(\mathcal{A}3) - (\mathcal{A}5)$ be satisfied. Then,*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \|Y^{t,x}\|_{S^\infty} + \|\int_0^\cdot Z_s^{t,x} \cdot \mathrm{d}W_s\|_{BMO} < \infty. \tag{3.8}$$

Proposition 3.6 *Let $(\mathcal{A}'.2)$ and $(\mathcal{A}3) - (\mathcal{A}5)$ be satisfied. Let $(X^{t,x}, Y^{t,x}, Z^{t,x})$ be the unique solution of the system (3.5). Then, the function $(t,x) \mapsto u(t,x) := Y_t^{t,x}$ is continuous.*

Proof Let $(t_n, x_n)_{n\in\mathbb{N}}$ be a sequence that converges to (t,x) in $[0,T] \times \mathbb{R}^d$, and consider the system (3.5) with initial condition (t_n, x_n). The following holds:

$$Y_s^{t_n,x_n} = h(X_T^{t_n,x_n}) + \int_s^T 1_{[t_n,T]}(r) F(r, X_r^{t_n,x_n}, Y_r^{t_n,x_n}, Z_r^{t_n,x_n})\mathrm{d}r - \int_s^T Z_r^{t_n,x_n} \cdot \mathrm{d}W_r.$$

We define

$$\begin{aligned}
\Gamma_r^n &= \frac{F(r,X_r^{t_n,x_n},Y_r^{t,x},Z_r^{t,x})-F(r,X_r^{t_n,x_n},Y_r^{t_n,x_n},Z_r^{t,x})}{Y_r^{t,x}-Y_r^{t_n,x_n}}1_{\{Y_r^{t,x}-Y_r^{t_n,x_n}\neq 0\}}1_{[t,T]}(r),\\
e_r^n &= \exp\left(\int_0^r \Gamma_v^n \mathrm{d}v\right),\\
\Lambda_r^n &= \frac{F(r,X_r^{t_n,x_n},Y_r^{t_n,x_n},Z_r^{t,x})-F(r,X_r^{t_n,x_n},Y_r^{t_n,x_n},Z_r^{t_n,x_n})}{|Z_r^{t,x}-Z_r^{t_n,x_n}|^2}\\
&\quad\times\left(Z_r^{t,x}-Z_r^{t_n,x_n}\right)1_{\{|Z_r^{t,x}-Z_r^{t_n,x_n}|\neq 0\}}1_{[t_n,T]}(r)\,.
\end{aligned}\tag{3.9}$$

Clearly,

$$|\Lambda_r^n| \leqslant k_z\left(1+2f(|Y_r^{t_n,x_n}|)\left(|Z_r^{t,x}|+|Z_r^{t_n,x_n}|\right)\right),$$

and

$$\|\int_0^{\cdot}\Lambda_s^n\cdot \mathrm{d}W_s\|_{BMO}\leqslant C\sqrt{T}+C\|\int_0^{\cdot}Z_s^{t,x}\cdot\mathrm{d}W_s\|_{BMO}+C\|\int_0^{\cdot}Z_s^{t_n,x_n}\cdot\mathrm{d}W_s\|_{BMO},$$

where C is a constant depending only on k_z, f, and $\|Y^{t_n,x_n}\|_{S^\infty}$. Hence, $\int_0^{\cdot}\Lambda_r^n\cdot\mathrm{d}W_r$ is a BMO martingale, $\mathbb{Q}^n=\mathcal{E}\left(\int_0^{\cdot}\Lambda_s^n\cdot\mathrm{d}W_s\right)\mathbb{P}$ is a probability measure equivalent to $\mathbb{P}$, and $W_t^{\mathbb{Q}^n}=W_t-\int_0^t\Lambda_s^n\mathrm{d}s$ is a $\mathbb{Q}^n$-Brownian motion.

Put $\delta Y_s^n := Y_s^{t,x}-Y_s^{t_n,x_n}$ and $\delta Z_s^n := Z_s^{t,x}-Z_s^{t_n,x_n}$. We have:

$$\begin{aligned}
\delta Y_s^n &= g(X_T^{t,x})-g(X_T^{t_n,x_n})\\
&\quad+\int_s^T 1_{[t,T]}(r)F(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})-1_{[t_n,T]}(r)F(r,X_r^{t_n,x_n},Y_r^{t_n,x_n},Z_r^{t_n,x_n})\mathrm{d}r\\
&\quad-\int_s^T\left(Z_r^{t,x}-Z_r^{t_n,x_n}\right)\cdot\mathrm{d}W_r\,.
\end{aligned}\tag{3.10}$$

Using Itô's formula and rewriting the previous equation under the probability $\mathbb{Q}^n$, we get:

$$\begin{aligned}
e_s^n\delta Y_s^n &= e_T^n\delta Y_T^n\\
&\quad+\int_s^T e_r^n\left(1_{[t,T]}(r)F(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})-1_{[t_n,T]}(r)F(r,X_r^{t_n,x_n},Y_r^{t_n,x_n},Z_r^{t_n,x_n})\right)\mathrm{d}r\\
&\quad-\int_s^T e_r^n\left(\Gamma_r^n\delta Y_r^n+\delta Z_r^n\cdot\Lambda_r^n\right)\mathrm{d}r-\int_s^T e_r^n\delta Z_r^n\cdot\mathrm{d}W_r^{\mathbb{Q}^n}\,.
\end{aligned}\tag{3.11}$$

We shall compute the quantity

$$1_{[t,T]}(r)F(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})-1_{[t_n,T]}(r)F(r,X_r^{t_n,x_n},Y_r^{t_n,x_n},Z_r^{t_n,x_n})-\Gamma_r^n\delta Y_r^n-\delta Z_r^n\cdot\Lambda_r^n,$$

which appears in the previous equation.

We have

$$\begin{aligned}
&1_{[t,T]}(r)F(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})-1_{[t_n,T]}(r)F(r,X_r^{t_n,x_n},Y_r^{t_n,x_n},Z_r^{t_n,x_n})-\Gamma_r^n\delta Y_r^n-\delta Z_r^n\cdot\Lambda_r^n\\
&=1_{[t,T]}(r)\left(F(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})-F(r,X_r^{t_n,x_n},Y_r^{t,x},Z_r^{t,x})\right)\\
&\quad+\left(1_{[t,T]}(r)-1_{[t_n,T]}(r)\right)F(r,X_r^{t_n,x_n},Y_r^{t_n,x_n},Z_r^{t,x})\,.
\end{aligned}$$

Arguing as above, we show that $\int_0^{\cdot}e_r^n\delta Z_r^n\cdot\mathrm{d}W_r^{\mathbb{Q}^n}$ is a $\mathbb{Q}^n$-martingale.

We define

$$\mathcal{E}_t := \exp\left(\int_0^t\Lambda_r^n\cdot\mathrm{d}W_r-\frac{1}{2}\int_0^t|\Lambda_r^n|^2\mathrm{d}r\right),$$

and

$$\begin{aligned} R^n_r := {} & 1_{[t,T]}(r)\left(F(r, X^{t,x}_r, Y^{t,x}_r, Z^{t,x}_r) - F(r, X^{t_n,x_n}_r, Y^{t,x}_r, Z^{t,x}_r)\right) \\ & + \left(1_{[t,T]}(r) - 1_{[t_n,T]}(r)\right) F(r, X^{t_n,x_n}_r, Y^{t_n,x_n}_r, Z^{t,x}_r)\,. \end{aligned}$$

We take the conditional expectation with respect to $\mathcal{F}_s$ in equation (3.11) to derive

$$\begin{aligned} |\mathbb{E}_{\mathbb{Q}^n}\left(e^n_s \delta Y^n_s/\mathcal{F}_s\right)| &= \left|\mathbb{E}_{\mathbb{Q}^n}\left(e^n_T \delta Y^n_T + \int_s^T e^n_r R^n_r \mathrm{d}r/\mathcal{F}_s\right)\right|, \\ |\mathbb{E}_{\mathbb{Q}^n}\left(\delta Y^n_s/\mathcal{F}_s\right)| &\leqslant \mathbb{E}_{\mathbb{Q}^n}\left((e^n_s)^{-1} e^n_T\,|\delta Y^n_T| + (e^n_s)^{-1}\sup_{s\leqslant r\leqslant T} e^n_r \int_s^T |R^n_r|\,\mathrm{d}r/\mathcal{F}_s\right), \\ |\delta Y^n_s|\,.\,|\mathbb{E}\left(\mathcal{E}_T/\mathcal{F}_s\right)| &\leqslant \mathbb{E}\left(\mathcal{E}_T \tilde{e^n}\left(|\delta Y^n_T| + \int_s^T |R^n_r|\,\mathrm{d}r\right)/\mathcal{F}_s\right), \\ |\delta Y^n_s| &\leqslant \mathcal{E}_s^{-1}\mathbb{E}\left(\mathcal{E}_T \tilde{e^n}\left(|\delta Y^n_T| + \int_s^T |R^n_r|\,\mathrm{d}r\right)/\mathcal{F}_s\right), \end{aligned}$$

where $\tilde{e^n} := \sup_{0\leqslant s\leqslant r\leqslant T}\exp\left(\int_s^r \Gamma_v\,\mathrm{d}v\right)$. Since $\int_0^{\cdot} \Lambda^n_s \cdot \mathrm{d}W_s$ is a BMO martingale, it follows from the second point of Lemma A.1 that $\mathcal{E}_T \in L^{p'}$, for some $p' > 1$. Applying Hölder's inequality, we obtain:

$$\begin{aligned} |\delta Y^n_s| &\leqslant \mathcal{E}_s^{-1}\mathbb{E}\left(\mathcal{E}_T \tilde{e^n}\left(|\delta Y^n_T| + \int_s^T R^n_r \mathrm{d}r\right)/\mathcal{F}_s\right), \\ |\delta Y^n_s| &\leqslant \mathcal{E}_s^{-1}\mathbb{E}\left(\mathcal{E}_T^{p'}/\mathcal{F}_s\right)^{\frac{1}{p'}}\left[\mathbb{E}\left((\tilde{e^n})^p\left(|\delta Y^n_T| + \int_s^T |R^n_r|\,\mathrm{d}r\right)^p/\mathcal{F}_s\right)\right]^{\frac{1}{p}}. \end{aligned}$$

Again, by the second assertion of Lemma A.1, it follows that:

$$|\delta Y^n_s| \leqslant C_{p'}\left[\mathbb{E}\left((\tilde{e^n})^p\left(|\delta Y^n_T| + \int_s^T |R^n_r|\,\mathrm{d}r\right)^p/\mathcal{F}_s\right)\right]^{\frac{1}{p}},$$

where $C_{p'}$ is constant depending on p' and $\|\int_0^{\cdot} \Lambda^n_s \cdot dW_s\|_{BMO}$. Applying Jensen's inequality for concave functions and Doob's inequality, we obtain:

$$\begin{aligned} \sup_{0\leqslant s\leqslant T}|\delta Y^n_s|^2 &\leqslant C^2_{p'}\left[\sup_{0\leqslant s\leqslant T}\left[\mathbb{E}\left((\tilde{e^n})^p\left(|\delta Y^n_T| + \int_0^T |R^n_r|\,\mathrm{d}r\right)^p/\mathcal{F}_s\right)\right]^2\right]^{\frac{1}{p}}, \\ \mathbb{E}\left[\sup_{0\leqslant s\leqslant T}|\delta Y^n_s|^2\right] &\leqslant C^2_{p'}\mathbb{E}\left[\left[\sup_{0\leqslant s\leqslant T}\mathbb{E}\left((\tilde{e^n})^r\left(|\delta Y^n_T| + \int_0^T |R^n_r|\,\mathrm{d}r\right)^p/\mathcal{F}_s\right)\right]^2\right]^{\frac{1}{p}}, \\ \mathbb{E}\left[\sup_{0\leqslant s\leqslant T}|\delta Y^n_s|^2\right] &\leqslant C^2_{p'}\left[\mathbb{E}\left[\sup_{0\leqslant s\leqslant T}\mathbb{E}\left((\tilde{e^n})^p\left(|\delta Y^n_T| + \int_0^T |R^n_r|\,\mathrm{d}r\right)^p/\mathcal{F}_s\right)\right]^2\right]^{\frac{1}{p}}, \\ \mathbb{E}\left[\sup_{0\leqslant s\leqslant T}|\delta Y^n_s|^2\right] &\leqslant C^2_{p'}C\left[\mathbb{E}\left((\tilde{e^n})^{2p}\left(|\delta Y^n_T| + \int_0^T |R^n_r|\,\mathrm{d}r\right)^{2p}\right)\right]^{\frac{1}{p}}. \end{aligned}$$

Again, by applying Hölder's inequality with q and q', we obtain:

$$\begin{aligned}\mathbb{E}\left(\sup_{0\leqslant s\leqslant T}|\delta Y_s^n|^2\right) &\leqslant C_{p'}^2 C\left[\mathbb{E}\left((\tilde{e^n})^{2p}\left(|\delta Y_T^n|+\int_0^T|R_r^n|\,\mathrm{d}r\right)^{2p}\right)\right]^{\frac{1}{p}},\\ &\leqslant C_{p'}^2 C\left[\mathbb{E}(\tilde{e^n})^{2pq'}\right]^{\frac{1}{pq'}}\left[\mathbb{E}\left(|\delta Y_T^n|+\int_0^T|R_r^n|\,\mathrm{d}r\right)^{2pq}\right]^{\frac{1}{pq}}.\end{aligned}$$

By the second part of point 4 in Lemma A.1, we find that:

$$\mathbb{E}\left(\sup_{0\leqslant s\leqslant T}|\delta Y_s^n|^2\right)\leqslant C_{p'}^2 CC'\left[\mathbb{E}\left(|\delta Y_T^n|+\int_0^T|R_r^n|\,\mathrm{d}r\right)^{2pq}\right]^{\frac{1}{pq}},$$

where C' is a constant that depends only on δ, p, q' and $\|\int_0^{\cdot} Z_s^{t,x}\cdot\mathrm{d}W_s\|_{BMO}$.

On the other hand, by assumption $(\mathcal{A}'.2)$, we have:

$$\begin{aligned}|R_r^n| &= |1_{[t,T]}(r)\left(F(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})-F(r,X_r^{t_n,x_n},Y_r^{t,x},Z_r^{t,x})\right)\\ &\quad+\left(1_{[t,T]}(r)-1_{[t_n,T]}(r)\right)F(r,X_r^{t_n,x_n},Y_r^{t_n,x_n},Z_r^{t,x})|\\ &\leqslant\left|F(r,X_r^{t,x},0,0)\right|+2\left|F(r,X_r^{t_n,x_n},0,0)\right|+c_1\|Y^{t_n,x_n}\|_{S^\infty}+c_2\|Y^{t,x}\|_{S^\infty}+c_3|Z_r^{t,x}|^2,\end{aligned}$$

where c_1, c_2, and c_3 are positive constants that are independent of n.

Using the continuity and boundedness of the functions h and $(t,x)\mapsto F(t,x,.,.)$, the convergence of (X^{t_x,x_n}) (see Proposition B.1), and Lebesgue's dominated convergence theorem, we conclude that

$$\lim_{n\to+\infty}\mathbb{E}\left[\left(|\delta Y_T^n|+\int_0^T R_r^n\mathrm{d}r\right)^{2pq}\right]=\lim_{n\to+\infty}\mathbb{E}\left[\left(|h(X_T^{t,x})-h(X_T^{t_n,x_n})|+\int_0^T R_r^n\mathrm{d}r\right)^{2pq}\right]=0.$$

It follows that

$$\lim_{n\to+\infty}\mathbb{E}\left(\sup_{0\leqslant s\leqslant T}|Y_s^{t,x}-Y_s^{t_n,x_n}|^2\right)=0.$$

Since,

$$\begin{aligned}|Y_{t_n}^{t_n,x_n}-Y_t^{t,x}|^2 &\leqslant 2\mathbb{E}|Y_{t_n}^{t_n,x_n}-Y_{t_n}^{t,x}|^2+2\mathbb{E}|Y_{t_n}^{t,x}-Y_t^{t,x}|^2\\ &\leqslant 2\mathbb{E}\left(\sup_{0\leqslant s\leqslant T}|Y_s^{t_n,x_n}-Y_s^{t,x}|^2\right)+2\mathbb{E}|Y_{t_n}^{t,x}-Y_t^{t,x}|^2.\end{aligned}$$

The result follows by taking the limit as $n\to\infty$. □

We now state and prove the main result of this subsection, which establishes the existence of a viscosity solution to the quadratic PDE (3.6).

Theorem 3.7 *Let assumptions $(\mathcal{A}'2)$, $(\mathcal{A}3)$, and $(\mathcal{A}5)$ be satisfied. Then, the function $(t,x)\mapsto u(t,x)=Y_t^{t,x}$ is a viscosity solution to the PDE* (3.6).

Proof We will show only that u is a viscosity subsolution. Let $\varphi\in\mathcal{C}^{1,2}\left([0,T]\times\mathbb{R}^d\right)$ and let $(\bar{t},\bar{x})\in[0,T]\times\mathbb{R}^d$ be a local maximum of $(u-\varphi)$. We assume that $u(\bar{t},\bar{x})=\varphi(\bar{t},\bar{x})$ and

$$\frac{\partial\varphi}{\partial t}(\bar{t},\bar{x})+L\varphi(\bar{t},\bar{x})+F(\bar{t},\bar{x},u(\bar{t},\bar{x}),\sigma^t\nabla\varphi(\bar{t},\bar{x}))<0.$$

It follows that there exists $\alpha>0$ such that, for each $(t,x)\in[\bar{t},\bar{t}+\alpha]\times B(\bar{x},\alpha)$, we have

$u(t,x) \leqslant \varphi(t,x)$ and

$$\frac{\partial \varphi}{\partial t}(t,x) + L\varphi(t,x) + F(t,x,u(t,x),\sigma^t \nabla\varphi(t,x)) < 0. \tag{3.12}$$

Define

$$\tau = \inf\{t \geqslant \bar{t} : |X_t^{\bar{t},\bar{x}} - \bar{x}| \geqslant \alpha\} \wedge (\bar{t}+\alpha),$$

and

$$(\tilde{Y}_t, \tilde{Z}_t) = (Y_{t\wedge\tau}^{\bar{t},\bar{x}}, 1_{[\bar{t},\tau]} Z_{t\wedge\tau}^{\bar{t},\bar{x}}), \quad t \in [\bar{t}, \bar{t}+\alpha].$$

As shown in [23], the Markov property of $X^{t,x}$ and the uniqueness of the backward component imply that

$$\tilde{Y}_t = u(\tau, X_\tau^{\bar{t},\bar{x}}) + \int_t^{\bar{t}+\alpha} 1_{[\bar{t},\tau]}(s) F(s, X_s^{\bar{t},\bar{x}}, u(s, X_s^{\bar{t},\bar{x}}), \tilde{Z}_s)\mathrm{d}s - \int_t^{\bar{t}+\alpha} \tilde{Z}_s \cdot \mathrm{d}W_s.$$

Using Itô's formula, we show that the process

$$(\hat{Y}_t, \hat{Z}_t) := \left(\varphi(t, X_{s\wedge\tau}^{\bar{t},\bar{x}}), 1_{[\bar{t},\tau]}(s)\sigma^t \nabla\varphi(t, X_{t\wedge\tau}^{\bar{t},\bar{x}})\right), \quad t \in [\bar{t}, \bar{t}+\alpha],$$

solves the BSDE

$$\hat{Y}_t = \varphi(\tau, X_\tau^{\bar{t},\bar{x}}) - \int_t^{\bar{t}+\alpha} 1_{[\bar{t},\tau]}(s) \left(\frac{\partial\varphi}{\partial t} + L\varphi\right)(s, X_s^{\bar{t},\bar{x}})\mathrm{d}s - \int_t^{\bar{t}+\alpha} \hat{Z}_s \cdot \mathrm{d}W_s.$$

Since $\varphi(t,x)$ and $\left(\frac{\partial\varphi}{\partial t} + L\varphi\right)(t,x)$ are continuous functions in a compact neighbourhood around $(\bar{t},\bar{x})$, then there are bounded. Consequently, $\hat{Y}$ is bounded, and we can show that $\left(\int_{\bar{t}}^{\cdot} \hat{Z}_s \cdot \mathrm{d}W_s\right)$ is a BMO martingale on $[\bar{t}, \bar{t}+\alpha]$. Since $\tilde{Y} - \hat{Y}$ satisfies the following BSDE

$$\begin{aligned}\tilde{Y}_t - \hat{Y}_t = {} & u(\tau, X_\tau^{\bar{t},\bar{x}}) - \varphi(\tau, X_\tau^{\bar{t},\bar{x}}) - \int_t^{\bar{t}+\alpha} (\tilde{Z}_s - \hat{Z}_s)\cdot \mathrm{d}W_s \\ & + \int_t^{\bar{t}+\alpha} 1_{[\bar{t},\tau]}(s) \left(F(s, X_s^{\bar{t},\bar{x}}, u(s, X_s^{\bar{t},\bar{x}}), \tilde{Z}_s) + \left(\frac{\partial\varphi}{\partial t} + L\varphi\right)(s, X_s^{\bar{t},\bar{x}})\right)\mathrm{d}s,\end{aligned}$$

it follows that

$$\begin{aligned}\tilde{Y}_t - \hat{Y}_t = {} & u(\tau, X_\tau^{\bar{t},\bar{x}}) - \varphi(\tau, X_\tau^{\bar{t},\bar{x}}) - \int_t^{\bar{t}+\alpha} (\tilde{Z}_s - \hat{Z}_s)\cdot \mathrm{d}W_s \\ & + \int_t^{\bar{t}+\alpha} 1_{[\bar{t},\tau]}(s) \left(F(s, X_s^{\bar{t},\bar{x}}, u(s, X_s^{\bar{t},\bar{x}}), \tilde{Z}_s) - F(s, X_s^{\bar{t},\bar{x}}, u(s, X_s^{\bar{t},\bar{x}}), \hat{Z}_s)\right)\mathrm{d}s \\ & + \int_t^{\bar{t}+\alpha} 1_{[\bar{t},\tau]}(s) \left(F(s, X_s^{\bar{t},\bar{x}}, u(s, X_s^{\bar{t},\bar{x}}), \hat{Z}_s) + \left(\frac{\partial\varphi}{\partial t} + L\varphi\right)(s, X_s^{\bar{t},\bar{x}})\right)\mathrm{d}s \\ = {} & u(\tau, X_\tau^{\bar{t},\bar{x}}) - \varphi(\tau, X_\tau^{\bar{t},\bar{x}}) - \int_t^{\bar{t}+\alpha} \left(\tilde{Z}_s - \hat{Z}_s\right)\cdot (\mathrm{d}W_s - \Lambda_s \mathrm{d}s) \\ & + \int_t^{\bar{t}+\alpha} 1_{[\bar{t},\tau]}(s) \left(F(s, X_s^{\bar{t},\bar{x}}, u(s, X_s^{\bar{t},\bar{x}}), \hat{Z}_s) + \left(\frac{\partial\varphi}{\partial t} + L\varphi\right)(s, X_s^{\bar{t},\bar{x}})\right)\mathrm{d}s,\end{aligned}$$

where

$$\Lambda_s = 1_{[\bar{t},\tau]}(s) \frac{F(s, X_s^{\bar{t},\bar{x}}, u(s, X_s^{\bar{t},\bar{x}}), \tilde{Z}_s) - F(s, X_s^{\bar{t},\bar{x}}, u(s, X_s^{\bar{t},\bar{x}}), \hat{Z}_s)}{|\tilde{Z}_s - \hat{Z}_s|^2} (\tilde{Z}_s - \hat{Z}_s) 1_{\{|\tilde{Z}_s - \hat{Z}_s|^2 \neq 0\}}.$$

As previously, $\int_{\bar{t}}^{\cdot} \Lambda_r \cdot \mathrm{d}W_r$ is a BMO martingale, and $\tilde{W}_s = W_s - \int_{\bar{t}}^{s} \Lambda_r \mathrm{d}r$ is a Brownian motion.

It follows that

$$\begin{aligned}\tilde{Y}_t - \hat{Y}_t &= u(\tau, X_\tau^{\bar{t},\bar{x}}) - \varphi(\tau, X_\tau^{\bar{t},\bar{x}}) - \int_t^{\bar{t}+\alpha} \left(\tilde{Z}_s - \hat{Z}_s\right) \cdot \mathrm{d}\tilde{W}_s \\ &+ \int_t^{\bar{t}+\alpha} 1_{[\bar{t},\tau]} \left(F(s, X_s^{\bar{t},\bar{x}}, u(s, X_s^{\bar{t},\bar{x}}), \hat{Z}_s) + \left(\frac{\partial \varphi}{\partial t} + L\varphi\right)(s, X_s^{\bar{t},\bar{x}}) \right) \mathrm{d}s.\end{aligned}$$

Taking the expectation with respect to $\mathcal{F}_{\bar{t}}$ and following the argument used in the proof of Proposition 3.4 for $t = \bar{t}$, and considering that $u \leqslant \varphi$ and the strictness of inequality (3.12), we obtain $\tilde{Y}_{\bar{t}} < \hat{Y}_{\bar{t}}$, and thus $u(\bar{t}, \bar{x}) < \varphi(\bar{t}, \bar{x})$. This contradicts the assumption that $u(\bar{t}, \bar{x}) = \varphi(\bar{t}, \bar{x})$, and therefore, u is a viscosity subsolution. □

4. Uniqueness of solutions with convex generators

In this section, we will establish the uniqueness and comparison results for the solutions of the BSDE (1.1). We will focus on the case where the function F exhibits quadratic growth and is jointly convex in the variables (y, z). Subsequently, we will apply these results to solve the corresponding quadratic semilinear PDE.

4.1 Uniqueness of solutions of the BSDE (1.1) with a convex generator

We introduce the following assumptions:

$(\mathcal{H}1)$ F is continuous in (y, z) for $a.e$ (t, ω), and fulfilling for all $(y, z) \in \mathbb{R} \times \mathbb{R}^d$:

$$|F(t, y, z)| \leqslant \alpha_t + \beta_t|y| + \theta_t|z| + f(|y|)|z|^2,$$

where α_t, β_t, θ_t are some $(\mathcal{F}_t)$-adapted processes which are positive, and f is a real valued function which is continuous, increasing, and positive on $\mathbb{R}_+$.

$(\mathcal{H}2)$ For every $(t, \omega) \in [0, T] \times \Omega$, $F(t, \omega, ., .)$ is jointly convex.

We put

$$\tilde{g}(t, y, z) := \alpha_t + \beta_t|y| + \theta_t|z| + f(|y|)|z|^2.$$

According to [1], under the assumption $(\mathcal{H}1)$, and that ξ, β, $\int_0^T \alpha_s \mathrm{d}s$, and $\int_0^T \theta_s^2 \mathrm{d}s$ are bounded, the BSDE (1.1) admits a solution (Y, Z) such that $Y^{-\tilde{g}} \leqslant Y \leqslant Y^{\tilde{g}}$, where $Y^{\tilde{g}}$ and $Y^{-\tilde{g}}$ are, respectively, solutions to BSDE$(\xi^+, \tilde{g})$ and BSDE$(-\xi^-, -\tilde{g})$.

We denote by F^* the convex conjugate of F, defined as follows:

$$F^*(t, \omega, b, a) := \sup_{(y,z) \in \mathbb{R} \times \mathbb{R}^d} \left(by + a \cdot z - F(t, \omega, y, z)\right).$$

In the following proposition, we provide a representation of the solution Y of the BSDE (1.1) using the convex conjugate of F. This representation will serve to establish the uniqueness result.

Proposition 4.1 *Let $(\mathcal{H}1)$ and $(\mathcal{H}2)$ be satisfied. Furthermore, assume that ξ, β, $\int_0^T \alpha_s \mathrm{d}s$, and $\int_0^T \theta_s^2 \mathrm{d}s$ are bounded. Then, every solution (Y, Z) of the BSDE (1.1), in $\mathcal{S}^\infty \times BMO$ and satisfying $Y^{-\tilde{g}} \leqslant Y \leqslant Y^{\tilde{g}}$, admits the following convex dual representation:*

$$Y_t = ess \sup_{a,b} \mathbb{E}_{\mathbb{Q}^a} \left[e^{\int_t^T b_s \, \mathrm{d}s} \xi - \int_t^T e^{\int_t^u b_s \, \mathrm{d}s} F^*(u, b_u, a_u) \mathrm{d}u | \mathcal{F}_t \right], \tag{4.1}$$

where the supremum is taken over progressively measurable processes $a : [0, T] \times \Omega \mapsto \mathbb{R}^d$ and $b : [0, T] \times \Omega \mapsto \mathbb{R}$ such that $\int_0^\cdot a_s \cdot \mathrm{d}W_s$ is in the space $\mathcal{B}$, and $|b| \leqslant \sup_{t \in [0,T]} |\beta_t|$.

Proof Let (Y,Z) be a solution of the BSDE (1.1) in $\mathcal{S}^\infty \times BMO$ such that $Y^{-\tilde{g}} \leqslant Y \leqslant Y^{\tilde{g}}$. Let b be a real-valued progressively measurable process satisfying $|b| \leqslant \sup_{t\in[0,T]} |\beta_t|$, and a an $\mathbb{R}^d$-valued progressive process such that $\int_0^\cdot a_s \cdot \mathrm{d}W_s$ is a BMO martingale. By Itô's formula, we have:

$$Y_t = e^{\int_t^T b_s \ \mathrm{d}s}\xi - \int_t^T e^{\int_t^u b_s \ \mathrm{d}s}\left(b_u Y_u - F(u,Y_u,Z_u)\right)\mathrm{d}u - \int_t^T e^{\int_t^u b_s \ \mathrm{d}s} Z_u \cdot \mathrm{d}W_u.$$

Since $\int_0^\cdot a_s \cdot \mathrm{d}W_s$ is a BMO martingale, we consider the equivalent probability

$$\mathbb{Q}^a := \exp\left(\int_0^T a_s \cdot \mathrm{d}W_s - \frac{1}{2}\int_0^T |a_s|^2 \ \mathrm{d}s\right)\mathbb{P}.$$

It follows that the process $B_t = W_t - \int_0^t a_u \, \mathrm{d}u$ is a Brownian motion. We now rewrite the previous equation under the probability measure $\mathbb{Q}^a$,

$$\begin{aligned}
Y_t &= e^{\int_t^T b_s \ \mathrm{d}s}\xi - \int_t^T e^{\int_t^u b_s \ \mathrm{d}s}\left(b_u Y_u + a_u \cdot Z_u - F(u,Y_u,Z_u)\right)\mathrm{d}u - \int_t^T e^{\int_t^u b_s \ \mathrm{d}s} Z_u \cdot \mathrm{d}B_u \\
Y_t &\geqslant e^{\int_t^T b_s \ \mathrm{d}s}\xi - \int_t^T e^{\int_t^u b_s \ \mathrm{d}s} F^*(u,b_u,a_u)\mathrm{d}u - \int_t^T e^{\int_t^u b_s \ \mathrm{d}s} Z_u \cdot \mathrm{d}B_u \\
Y_t &\geqslant ess\sup_{a,b} \mathbb{E}_{\mathbb{Q}^a}\left[e^{\int_t^T b_s \ \mathrm{d}s}\xi - \int_t^T e^{\int_t^u b_s \ \mathrm{d}s} F^*(u,b_u,a_u)\mathrm{d}u \middle| \mathcal{F}_t\right].
\end{aligned}$$

By the convexity of F, there exist $\bar{a}(t,\omega)$ and $\bar{b}(t,\omega)$, which can and will be chosen to be progressively measurable (see [12, 29]) such that:

$$F(t,Y_t,Z_t) = \bar{b}_t Y_t + \bar{a}_t \cdot Z_t - F^*(t,\bar{b}_t,\bar{a}_t). \tag{4.2}$$

Let us show that $|\bar{b}| \leqslant \|\sup_{t\in[0,T]} \beta_t\|_{\mathcal{S}^\infty}$.

If $|\bar{b}| > \|\sup_{t\in[0,T]} \beta_t\|_{\mathcal{S}^\infty}$, then for any y, z it holds that

$$F^*(t,\bar{b}_t,\bar{a}_t) \geqslant \bar{b}_t y + \bar{a}_t \cdot z - \alpha_t - \beta_t|y| - \theta_t|z| - f(|y|)|z|^2.$$

Taking $z = 0$ and $y = n\bar{b}_t$ yields

$$F^*(t,\bar{b}_t,\bar{a}_t) \geqslant n|\bar{b}_t|(|\bar{b}_t| - \beta_t) - \alpha_t,$$

tending n to infinity, we obtain a contradiction with the fact that $F^*(t,\bar{b}_t,\bar{a}_t)$ is finite. Thus, $|\bar{b}| \leqslant \|\sup_{t\in[0,T]} |\beta_t|\,\|_{\mathcal{S}^\infty}$.

We consider a sequence of stopping time given by:

$$\tau_n := \inf\left\{s \geqslant t, \ \int_t^s |Z_u|^2 \mathrm{d}u \geqslant n\right\} \wedge T.$$

Since $\int_0^\cdot Z_s \cdot \mathrm{d}B_s$ is a BMO martingale, the sequence τ_n increases to T as n goes to infinity. We will show that $\int_t^{T\wedge\tau_n} |\bar{a}_s|^2 \mathrm{d}s$ is bounded. By the definition of F^* and assumption $(\mathcal{H}1)$ we have:

$$F^*(t,b,a) \geqslant -\alpha_t + a \cdot z - \theta_t|z| - f(0)|z|^2.$$

It follows that

$$F^*(t,b,a) \geqslant \max_{z\in\mathbb{R}^d}\{-\alpha_t - \theta_t^2 + a \cdot z - |z|^2 - f(0)|z|^2\} = -\alpha_t - \theta_t^2 + \frac{|a|^2}{4(1+f(0))}. \tag{4.3}$$

Let ϵ be a constant such that $0<\epsilon<\frac{1}{4(1+f(0))}$. We use the previous inequality to derive:

$$\begin{aligned}\left(\frac{1}{4(1+f(0))}-\epsilon\right)|\bar{a}_t|^2 &\leqslant F^*(t,\bar{b}_t,\bar{a}_t)+\alpha_t+\theta_t^2-\epsilon|\bar{a}_t|^2,\\ &\leqslant -F(t,Y_t,Z_t)+\bar{b}_tY_t+\bar{a}_t\cdot Z_t+\alpha_t+\theta_t^2-\epsilon|\bar{a}_t|^2,\\ &\leqslant 2\alpha_t+\theta_t^2+2\beta_t|Y_t|+\theta_t|Z_t|+f(|Y_t|)|Z_t|^2+\bar{a}_t\cdot Z_t-\epsilon|\bar{a}_t|^2,\\ &\leqslant 2\alpha_t+\theta_t^2+2\beta_t|Y_t|+\theta_t|Z_t|+f(|Y_t|)|Z_t|^2+\sup_{a_t}\left\{a_t\cdot Z_t-\epsilon|a_t|^2\right\},\\ &\leqslant 2\alpha_t+\theta_t^2+2\beta_t|Y_t|+\theta_t|Z_t|+f(|Y_t|)|Z_t|^2+\frac{1}{4\epsilon}|Z_t|^2,\\ &\leqslant C\alpha_t+C\theta_t^2+C\beta_t|Y_t|+C\theta_t|Z_t|+Cf(|Y_t|)|Z_t|^2+C|Z_t|^2, \qquad (4.4)\end{aligned}$$

where C is some positive constant. Hence, we conclude that $\int_t^{T\wedge\tau_n}|\bar{a}_s|^2\mathrm{d}s$ is bounded. Consequently, $\int_0^{\cdot}a_s^n\cdot\mathrm{d}W_s$ is a BMO martingale, where $a^n:=a1_{[0,\tau_n]}$ and $\mathbb{Q}^{a^n}$ is a probability measure equivalent to $\mathbb{P}$. By Itô's formula applied to $e^{\int_t^u\bar{b}_s\mathrm{d}s}Y_u$ and Girsanov's transformation, we can derive:

$$\begin{aligned}Y_t=e^{\int_t^{T\wedge\tau_n}\bar{b}_s\,\mathrm{d}s}Y_{T\wedge\tau_n}&-\int_t^{T\wedge\tau_n}e^{\int_t^u\bar{b}_s\,\mathrm{d}s}\left(\bar{b}_uY_u+\bar{a}_u\cdot Z_u-F(u,Y_u,Z_u)\right)\mathrm{d}u\\ &-\int_t^{T\wedge\tau_n}e^{\int_t^u\bar{b}_s\,\mathrm{d}s}Z_u\cdot\mathrm{d}B_u\\ Y_t\leqslant\mathbb{E}_{\mathbb{Q}^{a^n}}&\left[e^{\int_t^Tb_s^n\,\mathrm{d}s}Y_{T\wedge\tau_n}-\int_t^Te^{\int_t^ub_s^n\,\mathrm{d}s}F^*(u,b_u^n,a_u^n)\mathrm{d}u|\mathcal{F}_t\right],\end{aligned}$$

where the last inequality follows from equality (4.2) and the fact $F^*(t,b^n,a^n)\leqslant 1_{[0,\tau_n]}F^*(t,b,a)$. Now, due to the continuity of Y, it follows that the sequence $Y_{T\wedge\tau_n}$ converges to Y_T $\mathbb{P}$-a.s. Taking $n\to+\infty$, we obtain:

$$Y_t\leqslant\sup_{a,b}\mathbb{E}_{\mathbb{Q}^a}\left[e^{\int_t^Tb_s\,\mathrm{d}s}\xi-\int_t^Te^{\int_t^ub_s\,\mathrm{d}s}F^*(u,b_u,a_u)\mathrm{d}u|\mathcal{F}_t\right].$$

This completes the proof. □

We now state and prove the uniqueness result for solutions to the BSDE (1.1).

Theorem 4.2 *Let conditions* ($\mathcal{H}1$) *and* ($\mathcal{H}2$) *be satisfied. Assume moreover that* ξ, β, $\int_0^T\alpha_s\mathrm{d}s$, *and* $\int_0^T\theta_s^2\mathrm{d}s$ *are bounded. Then, for every solution* (Y,Z), (Y',Z') *of the BSDE* (1.1), *in* $\mathcal{S}^\infty\times BMO$ *and satisfying* $Y^{-\tilde{g}}\leqslant Y,Y'\leqslant Y^{\tilde{g}}$, *the processes* Y *and* Y' *are indistinguishable, and* $Z=Z'$ $\mathrm{d}t\otimes\mathbb{P}$-a.e.

Proof Consider (Y,Z) and (Y',Z') two solutions of BSDE (1.1). By combining the representation (4.1) with the continuity of the paths of Y and Y', we obtain the uniqueness of the component Y. The uniqueness of Z follows by applying Itô's formula to $|Y_t-Y_t'|^2$. □

The following theorem provides a comparison of solutions to the BSDE (1.1). It will be proved by combining the uniqueness of solutions with the domination argument. The proof employs a different technique than the one used to establish Proposition 3.4.

Theorem 4.3 (Comparison theorem). *We consider generators* F_1 *and* F_2 *satisfying assumptions* ($\mathcal{H}1$) *and* ($\mathcal{H}2$). *Moreover, assume that* ξ_1, ξ_2, β, $\int_0^T\alpha_s\mathrm{d}s$, *and* $\int_0^T{\theta_s}^2\mathrm{d}s$ *are bounded. Let* $\left(Y^1,Z^1\right)$, $\left(Y^2,Z^2\right)$ *be the respective solutions of* BSDE(ξ_1, F_1) *and* BSDE(ξ_2, F_2), *which are* *assumed to belong to* $\in \mathcal{S}^\infty \times BMO$.

We further assume that $\xi_1 \leqslant \xi_2$, $F_1(t,y,z) \leqslant F_2(t,y,z)$ $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}$-*a.e. Then,* $Y_t^1 \leqslant Y_t^2$, $0 \leqslant t \leqslant T$, $\mathbb{P}$-*a.s.*

If, moreover, $F_1(t,Y_t^2,Z_t^2) < F_2(t,Y_t^2,Z_t^2)$ *on a set* A *of positive* $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}$-*measure, then we have* $Y_t^1 < Y_t^2$ $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}$-*a.e. on the set* A.

Proof We have

$$Y_t^1 = \xi_1 + \int_t^T F_1(s, Y_s^1, Z_s^1)\,\mathrm{d}s - \int_t^T Z_s^1 \cdot \mathrm{d}W_s,$$
$$Y_t^2 = \xi_2 + \int_t^T F_2(s, Y_s^2, Z_s^2)\,\mathrm{d}s - \int_t^T Z_s^2 \cdot \mathrm{d}W_s.$$

We set

$$h(t,y,z) := -\alpha_t - \beta_t|y| - \theta_t|z| - f(|y|)|z|^2 \quad \text{and} \quad g(t,y,z) := \alpha_t + \beta_t|y| + \theta_t|z| + f(|y|)|z|^2.$$

We notice that

$$h(t,y,z) \leqslant F_1(t,y,z) \leqslant F_2(t,y,z) \leqslant g(t,y,z). \tag{4.5}$$

The BSDE $(-\xi_1^-,\, h)$ has a solution (Y^h, Z^h), and the BSDE (ξ_2^+, g) has a solution (Y^g, Z^g). We apply the domination argument, Lemma 2.1, to conclude that the BSDE$(\xi_2,\, F_2)$ has a solution $(\tilde{Y}, \tilde{Z})$ such that:

$$Y_t^h \leqslant \tilde{Y}_t \leqslant Y_t^g, \quad \forall t \in [0,T].$$

By the uniqueness of the solution, we obtain:

$$Y_t^h \leqslant Y_t^2 \leqslant Y_t^g,\ \forall t \in [0,T].$$

We apply the domination argument once again, using the BSDE $(\xi_2,\, F_2)$ and the BSDE $(-\xi_1^-,\, h)$ to show that the BSDE $(\xi_1,\, F_1)$ admits a solution $(\hat{Y}, \hat{Z})$ such that:

$$Y_t^h \leqslant \hat{Y}_t \leqslant Y_t^2,\ t \in [0,T].$$

By the uniqueness property, it follows that:

$$Y_t^1 \leqslant Y_t^2,\ t \in [0,T].$$

We now prove the second part of Theorem 4.3. According to Theorem 3.2 in [13], with $L = Y^h$ and $U = Y^2$, the reflected BSDE(ξ_1, F_1, Y^h, Y^2) has a solution (Y, Z, K^+, K^-) satisfying the following system:

$$\begin{cases} Y_t = \xi_1 + \displaystyle\int_t^T F_1(s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \cdot \mathrm{d}W_s + \int_t^T \mathrm{d}K_s^+ - \int_t^T \mathrm{d}K_s^-, \\ \forall\ t \leqslant T, \quad Y_t^h \leqslant Y_t \leqslant Y_t^2, \\ \displaystyle\int_0^T (Y_t - Y_t^h)\mathrm{d}K_t^+ = \int_0^T (Y_t^2 - Y_t)\mathrm{d}K_t^- = 0, \text{ a.s.,} \\ K_0^+ = K_0^- = 0, \quad K^+, K^- \text{ are continuous nondecreasing,} \\ \mathrm{d}K^+ \perp \mathrm{d}K^-. \end{cases} \tag{4.6}$$

For all $0 \leqslant r \leqslant t \leqslant T$, we have

$$\begin{aligned} Y_t^2 - Y_t = Y_r^2 - Y_r - \int_r^t \left[F_2(s, Y_s^2, Z_s^2) - F_1(s, Y_s, Z_s)\right] \, \mathrm{d}s + \int_r^t \left(Z_s^2 - Z_s\right) \cdot \mathrm{d}W_s \\ + \int_r^t \mathrm{d}K_s^+ - \int_r^t \mathrm{d}K_s^- . \end{aligned}$$

Tanaka's formula shows that

$$\begin{aligned} \left(Y_t^2 - Y_t\right)^+ = \left(Y_r^2 - Y_r\right)^+ - \int_r^t 1_{\{Y_s^2 > Y_s\}} \left[F_2(s, Y_s^2, Z_s^2) - F_1(s, Y_s, Z_s)\right] \, \mathrm{d}s \\ + \int_r^t 1_{\{Y_s^2 > Y_s\}} \left(Z_s^2 - Z_s\right) \cdot \mathrm{d}W_s + \int_r^t 1_{\{Y_s^2 > Y_s\}} \left(\mathrm{d}K_s^+ - \mathrm{d}K_s^-\right) \\ + \frac{1}{2} L_t^0 \left(Y^2 - Y\right) - \frac{1}{2} L_r^0 \left(Y^2 - Y\right). \end{aligned}$$

It follows that

$$\begin{aligned} 0 = - \int_r^t 1_{\{Y_s^2 = Y_s\}} \left[F_2(s, Y_s^2, Z_s^2) - F_1(s, Y_s, Z_s)\right] \, \mathrm{d}s \\ + \int_r^t 1_{\{Y_s^2 = Y_s\}} \left(Z_s^2 - Z_s\right) \cdot \mathrm{d}W_s + \int_r^t 1_{\{Y_s^2 = Y_s\}} \left(\mathrm{d}K_s^+ - \mathrm{d}K_s^-\right) \\ - \frac{1}{2} L_t^0 \left(Y^2 - Y\right) + \frac{1}{2} L_r^0 \left(Y^2 - Y\right). \end{aligned}$$

Hence,

$$1_{\{Y_s^2 = Y_s\}} \left(Z_s^2 - Z_s\right) = 0 \quad \mathrm{d}s \times \mathrm{d}\mathbb{P}\text{-}a.e.$$

It follows that

$$\begin{aligned} 0 = - \int_r^t 1_{\{Y_s^2 = Y_s\}} \left[F_2(s, Y_s^2, Z_s^2) - F_1(s, Y_s^2, Z_s^2)\right] \, \mathrm{d}s + \int_r^t 1_{\{Y_s^2 = Y_s\}} \left(\mathrm{d}K_s^+ - \mathrm{d}K_s^-\right) \\ - \frac{1}{2} L_t^0 \left(Y^2 - Y\right) + \frac{1}{2} L_r^0 \left(Y^2 - Y\right). \end{aligned}$$

Since $\int_r^t 1_{\{Y_s^2 = Y_s\}} \mathrm{d}K_s^+ = 0$, it holds that

$$\begin{aligned} 0 \leqslant \int_r^t 1_{\{Y_s^2 = Y_s\}} \left[F_2(s, Y_s^2, Z_s^2) - F_1(s, Y_s^2, Z_s^2)\right] \, \mathrm{d}s + \frac{1}{2} L_t^0 \left(Y^2 - Y\right) - \frac{1}{2} L_r^0 \left(Y^2 - Y\right) \\ = - \int_r^t 1_{\{Y_s^2 = Y_s\}} \mathrm{d}K_s^- \leqslant 0. \end{aligned} \tag{4.7}$$

Hence, we conclude that $\mathrm{d}K^- = \mathrm{d}K^+ = 0$, which means that (Y, Z) is a solution of the BSDE(ξ_1, F_1). By the uniqueness of solutions, we obtain that $(Y^1, Z^1) = (Y, Z)$. From equation (4.7), we get:

$$\int_r^t 1_{\{Y_s^2 = Y_s^1\}} \left[F_2(s, Y_s^2, Z_s^2) - F_1(s, Y_s^2, Z_s^2)\right] \, \mathrm{d}s = 0.$$

We conclude that $Y_s^1 < Y_s^2$ on the set A.

4.2 Application to quadratic PDEs with convex nonlinearities

In this subsection, we use the uniqueness result for BSDEs obtained in the previous subsection to study the solvability of parabolic PDEs with convex nonlinearities. We establish the existence of a viscosity solution to the following PDE:

$$\begin{cases} \dfrac{\partial u}{\partial t} + Lu(t,x) + F(t, u(t,x), \sigma^t \nabla_x u(t,x)) = 0, \\ u(T,x) = h(x), \end{cases} \tag{4.8}$$

where L is the operator given by (3.7).

We consider the following forward-backward SDE:

$$\begin{cases} X_s^{t,x} = x + \displaystyle\int_t^s b(r, X_r^{t,x})\mathrm{d}r + \int_t^s \sigma(r, X_r^{t,x})\mathrm{d}W_r, \\ Y_s^{t,x} = h(X_T^{t,x}) + \displaystyle\int_s^T F(r, Y_r^{t,x}, Z_r^{t,x})\mathrm{d}r - \int_s^T Z_r^{t,x} \cdot \mathrm{d}W_r. \end{cases} \tag{4.9}$$

We introduce the following assumptions:

($\mathcal{H}3$) The functions $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ are continuous, and there exists $c > 0$ such that:

$$|b(t,x)| + |\sigma(t,x)| \leqslant c(1 + |x|), \quad \text{for every } (t,x) \in [0,T] \times \mathbb{R}^d.$$

($\mathcal{H}4$) Pathwise uniqueness holds for the forward SDE of system (4.9).

($\mathcal{H}5$) The mapping $h : \mathbb{R}^d \to \mathbb{R}$ is continuous and bounded.

Under assumptions ($\mathcal{H}1$) − ($\mathcal{H}5$), the system (4.9) has a unique solution $(X^{t,x}, Y^{t,x}, Z^{t,x})$. The next proposition states the continuous dependence of the solution $Y^{t,x}$ on the parameters (t,x). The proof follows similarly to that of [2, Lemma 4.3].

Proposition 4.4 *Let the hypotheses ($\mathcal{H}1$) − ($\mathcal{H}5$) be satisfied, and assume that β, $\int_0^T \alpha_s \mathrm{d}s$, and $\int_0^T \theta_s^2 \mathrm{d}s$ are bounded. Then, the function $(t,x) \to u(t,x) := Y_t^{t,x}$ is continuous.*

Proof Let (t^n, x^n) be a sequence converging to (t,x) and assume, without loss of generality, that $t^n \downarrow t$. Proposition 4.1 yields that for each $(t,x) \in [0,T] \times \mathbb{R}^d$, the solution $(Y^{t,x}, Z^{t,x})$ of the BSDE $(h(X^{t,x}), F)$ can be represented as follows:

$$Y_t^{t,x} = ess\sup_{a,b} \mathbb{E}_{\mathbb{Q}^a}\left[e^{\int_t^T b_s \ \mathrm{d}s} h(X_T^{t,x}) - \int_t^T e^{\int_t^u b_s \ \mathrm{d}s} F^*(u, b_u, a_u)\mathrm{d}u | \mathcal{F}_t \right], \tag{4.10}$$

where the supremum is taken over progressively measurable processes $a : [0,T] \times \Omega \to \mathbb{R}^d$ and $b : [0,T] \times \Omega \to \mathbb{R}$ such that a is in BMO, and $|b| \leqslant \sup_{0\leqslant t\leqslant T} |\beta_t|$. Since the quantity $Y_t^{t,x}$ is deterministic, the equality (4.10) becomes:

$$Y_t^{t,x} = \sup_{a,b} \mathbb{E}_{\mathbb{Q}^a}\left[e^{\int_t^T b_s \,\mathrm{d}s} h(X_T^{t,x}) - \int_t^T e^{\int_t^u b_s \,\mathrm{d}s} F^*(u, b_u, a_u)\,\mathrm{d}u \right]. \tag{4.11}$$

By the definition of F^* and assumption ($\mathcal{H}1$), we have for every a, b:

$$F^*(t, b, a) \geqslant -\alpha_t.$$

Furthermore, by equality (4.11), for each $n \in \mathbb{N}$, a, and b, we get that:

$$Y_{t^n}^{t^n,x^n} \geqslant \mathbb{E}_{\mathbb{Q}^a}\left[e^{\int_{t^n}^T b_s\,\mathrm{d}s} h\left(X_T^{t^n,x^n}\right) - \int_{t^n}^T e^{\int_{t^n}^u b_s\,\mathrm{d}s}(F^*(u,b_u,a_u)+\alpha_u) - e^{\int_{t^n}^u b_s\,\mathrm{d}s}\alpha_u\,\mathrm{d}u\right].$$

By using the Beppo-Lévy theorem and the Lebesgue dominated convergence theorem, it holds that $\liminf_{n\to\infty} Y_{t^n}^{t^n,x^n} \geqslant Y_t^{t,x}$.

On the other hand, let a^n, b^n be such that:

$$Y_{t^n}^{t^n,x^n} \leqslant \mathbb{E}_{\mathbb{Q}^{a^n}}\left[e^{\int_{t^n}^T b_s^n\,\mathrm{d}s} h\left(X_T^{t^n,x^n}\right) - \int_{t^n}^T e^{\int_{t^n}^u b_s^n\,\mathrm{d}s} F^*(u,b_u^n,a_u^n)\,\mathrm{d}u\right] + \frac{1}{n}. \tag{4.12}$$

Given that h and $Y_{t_n}^{t^n,x^n}$ are bounded, there exists a positive constant $C \geqslant 0$ such that:

$$\mathbb{E}_{\mathbb{Q}^{a^n}}\left[\int_{t^n}^T e^{\int_{t^n}^u b_s^n\,\mathrm{d}s} F^*(u,b_u^n,a_u^n)\,\mathrm{d}u\right] \leqslant C.$$

Arguing as in the proof of inequality (4.3), we obtain:

$$F^*(t,b_t^n,a_t^n) \geqslant \max_{z\in\mathbb{R}^d}\{-\alpha_t - \theta_t^2 + a_t^n\cdot z - |z|^2 - f(0)|z|^2\} = -\alpha_t - \theta_t^2 + \frac{|a_t^n|^2}{4(1+f(0))}. \tag{4.13}$$

Hence, there exists a positive constant C such that $\mathbb{E}_{\mathbb{Q}^{a^n}}\left[\int_{t^n}^T \frac{1}{2}|a_u^n|^2\,\mathrm{d}u\right] \leqslant C$. According to Girsanov's theorem, we get that $\mathbb{E}\left[\mathcal{E}_{t^n,T}^{a^n}\log\left(\mathcal{E}_{t^n,T}^{a^n}\right)\right] \leqslant C$, where $\mathcal{E}_{t^n,T}^{a^n} := \exp\left(\int_{t^n}^T a_u^n\cdot\mathrm{d}W_u - \frac{1}{2}\int_{t^n}^T |a_u^n|^2\,\mathrm{d}u\right)$. Thus, by the criterion of de la Vallée Poussin, $(\mathcal{E}_{t^n,T}^{a^n})_n$ is uniformly integrable, and therefore there exists $K \in L^1$ such that $(\mathcal{E}_{t^n,T}^{a^n})_n$ converges weakly to K. Since the sequence $\left(h\left(X_T^{t^n,x^n}\right) - h\left(X_T^{t,x}\right)\right)_n$ is uniformly bounded and converges to 0 in L^2, we use [5, Lemma 2.8] to get

$$\limsup_{n\to\infty}\mathbb{E}\left[\mathcal{E}_{t^n,T}^{a^n} e^{\int_{t^n}^T b_s^n\,\mathrm{d}s}\left|h\left(X_T^{t^n,x^n}\right) - h\left(X_T^{t,x}\right)\right|\right] \leqslant C\lim_{n\to\infty}\mathbb{E}\left[\mathcal{E}_{t^n,T}^{a^n}\left|h\left(X_T^{t^n,x^n}\right) - h\left(X_T^{t,x}\right)\right|\right] = 0,$$

where C is a positive constant.

Consequently, for every $\varepsilon > 0$, there exists n large enough such that

$$\begin{aligned}\mathbb{E}_{Q^{a^n}}\left[e^{\int_{t^n}^T b_s^n\,\mathrm{d}s} h\left(X_T^{t^n,x^n}\right)\right] &= \mathbb{E}\left[\mathcal{E}_{t^n,T}^{a^n} e^{\int_{t^n}^T b_s^n\,\mathrm{d}s} h\left(X_T^{t^n,x^n}\right)\right] \\ &\leqslant \mathbb{E}\left[\mathcal{E}_{t^n,T}^{a^n} e^{\int_{t^n}^T b_s^n\,\mathrm{d}s} h\left(X_T^{t,x}\right)\right] + \varepsilon \\ &= \mathbb{E}_{Q^{a^n}}\left[e^{\int_{t^n}^T b_s^n\,\mathrm{d}s} h\left(X_T^{t,x}\right)\right] + \varepsilon.\end{aligned} \tag{4.14}$$

The previous inequality and (4.12) allow us to show that

$$\begin{aligned}Y_{t^n}^{t^n,x^n} &\leqslant \mathbb{E}_{\mathbb{Q}^{a^n}}\left[e^{\int_{t^n}^T b_s^n\,\mathrm{d}s} h\left(X_T^{t,x}\right) - \int_{t^n}^T e^{\int_{t^n}^u b_s^n\,\mathrm{d}s} F^*(u,b_u^n,a_u^n)\,\mathrm{d}u\right] + \frac{1}{n} + \varepsilon \\ &\leqslant \sup_{a,b}\mathbb{E}_{\mathbb{Q}^a}\left[e^{\int_{t^n}^T b_s\,\mathrm{d}s} h\left(X_T^{t,x}\right) - \int_{t^n}^T e^{\int_{t^n}^u b_s\,\mathrm{d}s} F^*(u,b_u,a_u)\,\mathrm{d}u\right] + \frac{1}{n} + \varepsilon.\end{aligned}$$

We successively let $n \to +\infty$ and $\varepsilon \to 0$ to get $\limsup_{n\to\infty} Y_{t^n}^{t^n,x^n} \leqslant Y_t^{t,x}$, which finishes the proof of the continuity. □

We proceed to present and prove the main result of this subsection.

Theorem 4.5 *Let* $(\mathcal{H}1)-(\mathcal{H}5)$ *be satisfied. Moreover, assume that* β*,* $\int_0^T \alpha_s \mathrm{d}s$*, and* $\int_0^T \theta_s^2 \mathrm{d}s$ *are bounded. Then, the function* u *defined by* $u(t,x) = Y_t^{t,x}$ *is a viscosity solution of the PDE* (4.8).

Proof We will show that u is a viscosity subsolution. The proof that u is a supersolution can be carried out in a similar manner. Let $\varphi \in \mathcal{C}^{1,2}\left([0,T]\times\mathbb{R}^d\right)$ and let $(\bar t, \bar x) \in [0,T]\times\mathbb{R}^d$ be a local maximum of $(u-\varphi)$. We assume that:

$$u(\bar t,\bar x) = \varphi(\bar t,\bar x) \ \text{ and } \ \frac{\partial\varphi}{\partial t}(\bar t,\bar x) + L\varphi(\bar t,\bar x) + F(\bar t, u(\bar t,\bar x), \sigma^t\nabla\varphi(\bar t,\bar x)) < 0.$$

It follows that there exists $\alpha > 0$ such that, for each $(t,x)\in[\bar t,\bar t+\alpha]\times B(\bar x,\alpha)$, we have $u(t,x)\leqslant\varphi(t,x)$, and

$$\frac{\partial\varphi}{\partial t}(t,x) + L\varphi(t,x) + F(t,u(t,x),\sigma^t\nabla\varphi(t,x)) < 0. \tag{4.15}$$

Define

$$\tau = \inf\{t\geqslant\bar t : |X^{\bar t,\bar x} - \bar x|\geqslant\alpha\}\wedge(\bar t+\alpha),$$

and

$$(\tilde Y_t, \tilde Z_t) = (Y^{\bar t,\bar x}_{t\wedge\tau}, 1_{[\bar t,\tau]}Z^{\bar t,\bar x}_{t\wedge\tau}), \quad t\in[\bar t,\bar t+\alpha].$$

We have

$$\tilde Y_t = u(\tau, X^{\bar t,\bar x}_\tau) + \int_t^{\bar t+\alpha} 1_{[\bar t,\tau]}(s)F(s,u(s,X^{\bar t,\bar x}_s),\tilde Z_s)\mathrm{d}s - \int_t^{\bar t+\alpha}\tilde Z_s\cdot\mathrm{d}W_s.$$

By Itô's formula, the process

$$(\hat Y_t,\hat Z_t) := \left(\varphi(t,X^{\bar t,\bar x}_{s\wedge\tau}), 1_{[\bar t,\tau]}(s)(\sigma^t\nabla\varphi)(t,X^{\bar t,\bar x}_{t\wedge\tau})\right),\ t\in[\bar t,\bar t+\alpha],$$

solves the BSDE

$$\hat Y_t = \varphi(\tau,X^{\bar t,\bar x}_\tau) - \int_t^{\bar t+\alpha} 1_{[\bar t,\tau]}(s)\left(\frac{\partial\varphi}{\partial t}+L\varphi\right)(s,X^{\bar t,\bar x}_s)\mathrm{d}s - \int_t^{\bar t+\alpha}\hat Z_s\cdot\mathrm{d}W_s. \tag{4.16}$$

Now we need to apply the comparison theorem to conclude that $\bar Y_{\bar t} < \hat Y_{\bar t}$. We note that, although the generator $1_{[\bar t,\tau]}(s)\left(\frac{\partial\varphi}{\partial t}+L\varphi\right)(s,x)$ is not necessarily convex, the comparison Theorem 4.3 can still be applied. This is because the convexity property is primarily used to guarantee the uniqueness of solutions to the BSDE (4.16) (as shown in the proof of Theorem 4.3). However, the uniqueness of solutions to this BSDE is already ensured by Theorem 3.2.

We have $u(\tau,X^{\bar t,\bar x}_\tau)\leqslant\varphi(\tau,X^{\bar t,\bar x}_\tau)$ and

$$1_{[\bar t,\tau]}(s)F(s,u(s,X^{\bar t,\bar x}_s),\hat Z_s) < -1_{[\bar t,\tau]}(s)\left(\frac{\partial\varphi}{\partial t}+L\varphi\right)(s,X^{\bar t,\bar x}_s).$$

We deduce, with the help of the comparison Theorem 4.3, that $u(\bar t,\bar x) = \bar Y_{\bar t} < \hat Y_{\bar t} = \varphi(\bar t,\bar x)$, which contradicts our assumptions. Therefore, u is a subsolution. □

Appendix

A. BMO martingales properties

We recall some properties of BMO martingales, for more details, we refer to [22].

Lemma A.1 1. *For a BMO martingale M, the stochastic exponential* $\mathcal{E}(M_T) := \exp\left(M_T - \frac{1}{2}\langle M\rangle_T\right)$ *has expectation* 1.

2. *For every BMO martingale M, there exists $p > 1$ such that $\mathcal{E}(M) \in L^p$. Moreover, there exists a constant c, depending only on p and the BMO norm of M, such that for any $t \in [0, T]$, it holds that*

$$\mathbb{E}\left(\mathcal{E}(M)_T^p / \mathcal{F}_t\right) \leqslant c\,\mathcal{E}(M)_t^p. \tag{A.1}$$

3. *If $\|M\|_{\mathcal{B}} < 1$, then for every stopping time $\tau \in [0, T]$*

$$\mathbb{E}\left[\exp\left(\langle M\rangle_T - \langle M\rangle_\tau\right)/\mathcal{F}_\tau\right] < \frac{1}{1 - \|M\|_{\mathcal{B}}^2}. \tag{A.2}$$

4. *If $\left(\int_0^t Z_s \cdot \mathrm{d}W_s\right)_{0 \leqslant t \leqslant T}$ is a BMO martingale, then for any $q \geqslant 1$ it holds that*

$$\mathbb{E}\left[\left(\int_0^T |Z_s|^2 \mathrm{d}s\right)^q\right] \leqslant q!\|\int_0^\cdot Z_s \cdot \mathrm{d}W_s\|_{\mathcal{B}}^{2q}.$$

Furthermore, for each $q \geqslant 1$ and for any $\varepsilon \in (0, 2)$

$$\mathbb{E}\left[\exp\left(q\int_0^T |Z_s|^\varepsilon \mathrm{d}s\right)\right] \leqslant C^* < \infty, \tag{A.3}$$

where C^* depends on q, ε and $\|\int_0^\cdot Z_s \cdot \mathrm{d}W_s\|_{\mathcal{B}}^2$.

B. Continuity of $(t, x) \to X^{t,x}$

We recall a continuity result for SDE solutions with respect to their initial conditions, proved in [21].

Proposition B.1 *Assume that* $(\mathcal{A}3)$ *and* $(\mathcal{A}4)$ *are satisfied, and let* (t_n, x_n) *be a sequence in* $[0, T] \times \mathbb{R}^d$ *converging to* (t, x). *Let* X^{t_n,x_n} *and* $X^{t,x}$ *be the unique solutions of the forward SDE in system* (3.5) *with initial conditions* (t_n, x_n) *and* (t, x), *respectively. Then,* X^{t_n,x_n} *converges to* $X^{t,x}$ *in* $\mathcal{S}^2$.

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