Probability, Uncertainty and Quantitative Risk
Vol. 10, No. 4, 2025, 615–638

https://www.aimsciences.org/PUQR
DOI:10.3934/puqr.2025027

Convergence rate of Riccati-based discretization for linear quadratic optimal control problem of stochastic mixed systems

Yuping Rao, Yanqing Wang*

School of Mathematics and Statistics, Southwest University, Chongqing 400715, *China*

Email: ryp955797@163.com, yqwang@swu.edu.cn

Abstract This paper presents a closed-loop numerical algorithm for the quadratic optimal control problem of a linear mixed system that combines both deterministic and stochastic controls. The core idea is to numerically solve two Riccati equations by using linear quadratic theory. Based on these numerical solutions, a feedback-type discretization method for the original problem is developed, along with an analysis of its convergence rate. A significant advantage of the proposed method is that it avoids the computation of backward stochastic differential equations associated with Pontryagin's maximum principle, leading to notable improvements in computational efficiency. This work builds upon the theoretical framework established by Hu and Tang (*Probab. Uncertain. Quant. Risk*, 4 (2019), Paper No. 1) and focuses on its numerical implementation.

Keywords Convergence rate, Closed-loop control strategy, Riccati equation, Mixed deterministic and stochastic controls

2020 Mathematics Subject Classification 49N10, 93B52

1. Introduction

Let $T > 0$ and $W = \{W(t) : t \geqslant 0\}$ be a one-dimensional standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = \{\mathcal{F}_t\}_{t \geqslant 0}$ is the natural filtration of W, augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$. Our objective is to *efficiently approximate* the optimal control $u^*(\cdot) = \big(u_1^*(\cdot), u_2^*(\cdot)\big)$ that minimizes the cost functional

$$\begin{aligned}\mathcal{J}\big(u(\cdot)\big) = \mathbb{E}\Big[\int_0^T \big(\langle Qx(t), x(t)\rangle + \langle R_1 u_1(t), u_1(t)\rangle + \langle R_2 u_2(t), u_2(t)\rangle\big)\,\mathrm{d}t\Big] \\ + \mathbb{E}\big[\langle Gx(T), x(T)\rangle\big]\,,\end{aligned} \tag{1.1}$$

subject to the mixed controlled system

$$\begin{cases}\mathrm{d}x(t) = \big[Ax(t) + B_1 u_1(t) + B_2 u_2(t)\big]\,\mathrm{d}t \\ \qquad\quad + \big[Cx(t) + D_1 u_1(t) + D_2 u_2(t)\big]\,\mathrm{d}W(t), \quad \forall\, t \in [0, T]\,, \\ x(0) = x_0\,.\end{cases} \tag{1.2}$$

Received 30 November 2024; Accepted 26 October 2025; Early access 10 December 2025
*Corresponding author

Here $G, Q \in \mathbb{S}^n$, $A, C \in \mathbb{R}^{n\times n}$, $B_1, D_1 \in \mathbb{R}^{n\times \ell_1}$, $B_2, D_2 \in \mathbb{R}^{n\times \ell_2}$, $R_1 \in \mathbb{S}^{\ell_1}$, $R_2 \in \mathbb{S}^{\ell_2}$, $u(\cdot)\triangleq\big(u_1(\cdot), u_2(\cdot)\big)$ is a control variable, and the admissible control set for the system (1.2) is

$$\mathcal{U}_{ad}\triangleq L^2(0,T;\mathbb{R}^{\ell_1}) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^{\ell_2}).$$

Note that there are two controllers in (1.2): $u_1(\cdot)$ can impose a deterministic action, *i.e.*, $u_1(\cdot) \in L^2(0,T;\mathbb{R}^{\ell_1})$, while $u_2(\cdot)$ represents a stochastic action, *i.e.*, $u_2(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^{\ell_2})$. Consequently, we refer to (1.2) as a *mixed* controlled system. The mixed linear quadratic optimal control problem (MLQ problem, for short) of concern can be formally stated as follows:

Problem (MLQ) For a given $x_0 \in \mathbb{R}^n$, find $u^*(\cdot)\triangleq\big(u_1^*(\cdot), u_2^*(\cdot)\big) \in \mathcal{U}_{ad}$ such that

$$\mathcal{J}\big(u^*(\cdot)\big) = \inf_{u(\cdot)\in\mathcal{U}_{ad}} \mathcal{J}\big(u(\cdot)\big).$$

MLQ problems serve as a prototype of mixed control problems, which combine deterministic and stochastic elements, and are of particular interest in modern control theory due to their broad applications in finance, engineering, and large-scale network systems. These problems arise in scenarios where the behavior of a system is influenced by two controllers owning different information, extending the classical framework by introducing additional complexity (see *e.g.*, [6, 9, 12, 17, 23]).

The first rigorous study of Problem **(MLQ)** was conducted in [9], where the maximum principle was derived, leading to two Riccati equations that form the foundation for both open-loop and closed-loop solutions. As a special case of Problem **(MLQ)**, the linear quadratic optimal control problems governed by mean-field systems (MF-LQ problems) were firstly studied in [23], where two controllers were connected by the relation $u_1(\cdot) = \mathbb{E}\big[u_2(\cdot)\big]$. [9] addressed both the open-loop and closed-loop solvability of Problem **(MLQ)**. More precisely, the unique open-loop solvability of Problem **(MLQ)** is equivalent to the unique solvability of a coupled forward-backward stochastic differential equation (FBSDE) (see [9, Theorem 1] or Lemma 2.1). In contrast, the closed-loop solvability is characterized by the solvability of two underlying differential Riccati equations; see (2.5), (2.6) and Lemma 2.5, or [9, Section 3].

Two strategies can be employed to numerically solve Problem **(MLQ)** — the open-loop approach and the closed-loop approach. For the open-loop approach, the reader can refer to [8] for PDE systems, [2, 4, 7, 19] for SDE systems, [5, 11, 13, 14, 16] for SPDE systems, and the references therein. For the closed-loop approach, see [3, 10, 15, 21, 22] and so on.

For Problem **(MLQ)**, both previously mentioned approaches can be adopted. However, two key challenges arise when implementing the open-loop approach in the stochastic setting: (1) decoupling the forward–backward stochastic differential equations (FBSDEs) (1.2), (2.2), (2.3), and (2.4) for Problem **(MLQ)**; and (2) solving a family of backward stochastic differential equations (BSDEs) numerically (see *e.g.*, [2, Section 3] and [13, Algorithm 5.1]). Despite considerable effort, existing numerical methods for solving BSDEs remain unsatisfactory. Till now, there *do not* exist universally acknowledged, effective, and reliable methods in the literature.

In contrast, the closed-loop approach, based on the Riccati framework, offers significant advantages in MLQ problems. First, it avoids solving the coupled FBSDEs and reduces the computational burden, especially in high-dimensional settings. Second, the closed-loop approach provides robust control, ensuring stability and reliable performance while offering a practical solution to MLQ problems.

In this work, we adopt the closed-loop approach to design an efficient algorithm. For Problem

(MLQ), the theoretical backup is Lemma 2.5, and the following is the outline of the closed-loop based scheme for Problem **(MLQ)**:

(i) Discretization and error estimate for Riccati equation (2.5)

Relying on the relationship between (2.5) and a stochastic LQ problem (see Problem **(SLQ)** in Section 3), we adopt the "first discretization, then optimalization" strategy to discretize Problem **(SLQ)** and obtain Problem $\textbf{(SLQ)}_\tau$ governed by controlled stochastic difference systems. Thereby, a difference Riccati equation (3.8) corresponding to Problem $\textbf{(SLQ)}_\tau$ can be regarded as an approximation of the differential Riccati equation (2.5). This strategy has been successfully used to numerically solve stochastic LQ problems with proven convergence rates (see [18, 21]).

(ii) Discretization and error estimate for Riccati equation (2.6)

Compared to the Riccati equation (2.5), equation (2.6) depends on $P_1(\cdot)$. While we establish a relationship between (2.6) and a deterministic LQ problem (see Problem **(DLQ)** in Section 3), this derived LQ problem remains dependent on $P_1(\cdot)$. Thus, we apply the Riccati discretization from step (i) to the " first discretization, then optimization" strategy, resulting in Problem $\textbf{(DLQ)}_\tau$. The corresponding discrete Riccati equation (3.15) for Problem $\textbf{(DLQ)}_\tau$ serves as an approximation to Riccati equation (2.6). A first-order convergence rate is established in Theorem 3.10 This approach has also been successfully applied to MF-LQ problems (see [20]).

(iii) Discretization of Problem **(MLQ)** and convergence rate

Inspired by the feedback law of the optimal control to Problem **(MLQ)** stated in Lemma 2.5, we combine the Euler method and approximations of (2.5), (2.6) to propose a discrete feedback scheme

$$\begin{cases} u_{1,\tau}(t_k) = M_{1,k}\mathbb{E}[x_\tau(t_k)]\,, \\ u_{2,\tau}(t_k) = M_{2,k}\big(x_\tau(t_k) - \mathbb{E}[x_\tau(t_k)]\big) + M_{3,k}\mathbb{E}[x_\tau(t_k)], \quad k = 0,1,\ldots,N-1\,, \end{cases}$$

where $M_{i,k}$ is given in (4.2) for $i = 1,2,3\,, k = 0,1,\ldots,N-1$. Furthermore, we derive that

$$\max_{k=0,1,\ldots,N-1} \big\|u_1^*(t_k) - u_{1,\tau}(t_k)\big\| \leqslant \mathcal{C}\tau\,, \quad \sqrt{\mathbb{E}\big[\max_{k=0,1,\ldots,N-1} \|u_2^*(t_k) - u_{2,\tau}(t_k)\|^2\big]} \leqslant \mathcal{C}\sqrt{\tau}\,;$$

see Theorem 4.3.

This paper advances the study of MLQ problems by developing numerical methods that efficiently solve the underlying Riccati equations and provide a precise feedback control scheme. The remainder of this paper is organized as follows: In Section 2, we introduce some notations, review both the open-loop and closed-loop strategies for Problem **(MLQ)**, and present several frequently used results. In Section 3, we adopt the " first discretization, then optimization" strategy to discretize two underlying Riccati equations, (2.5) and (2.6), related to Problem **(MLQ)**, and prove the convergence rates. In Section 4, applying the closed-loop strategy, we propose a Riccati-based algorithm for Problem **(MLQ)** and derive the convergence rate. Finally, we validate the theoretical findings with numerical examples.

2. Preliminaries

In this section, we introduce the notations and results used throughout the study. Let $\mathbb{S}^n$ denote the set of $n \times n$ symmetric matrices, $\mathbb{S}^n_+$ the set of $n \times n$ positive semidefinite matrices, and $\mathbb{S}^n_{++}$ the set of $n \times n$ positive definite matrices. A matrix A is written as $A > 0$ if it is positive definite, and as $A \geqslant 0$ if it is positive semidefinite. For any matrix-valued function

$R(\cdot):[0,T]\to\mathbb{S}^n$, we use $R(\cdot)\gg 0$ to indicate that $R(\cdot)$ is uniformly positive definite, *i.e.*, there exists a $\delta>0$ such that $R(t)\geqslant \delta I_n$ for all $t\in[0,T]$. For any matrix A, we define $\|A\|\triangleq\sqrt{\rho(A^\top A)}$, where ρ denotes the spectral radius. Additionally,

$$L^2(0,T;\mathbb{R}^{\ell_1})\triangleq\Big\{\varphi:[0,T]\to\mathbb{R}^{\ell_1}\;\Big|\;\int_0^T\|\varphi(t)\|^2\,\mathrm{d}t<\infty\Big\},$$
$$L^2_{\mathbb{F}}(0,T;\mathbb{R}^{\ell_2})\triangleq\Big\{\varphi:[0,T]\times\Omega\to\mathbb{R}^{\ell_2}\;\Big|\;\varphi(\cdot)\text{ is }\mathbb{F}\text{-adapted and }\mathbb{E}\Big[\int_0^T\|\varphi(t)\|^2\,\mathrm{d}t\Big]<\infty\Big\}.$$

Let $\mathcal{I}_\tau=\{t_n\}_{n=0}^N\subset[0,T]$ a time partition with step size $\tau\triangleq\max_{n=0,1,\dots,N-1}\{t_{n+1}-t_n\}$, and define $\Delta_nW\triangleq W(t_n)-W(t_{n-1})$ for all $n=1,\dots,N$. Given the time partition $\mathcal{I}_\tau$, we define $\nu:[0,T)\to\mathcal{I}_\tau$ by

$$\nu(t)=t_k\qquad\forall\,t\in[t_k,t_{k+1}),\quad k=0,1,\dots,N-1\,. \tag{2.1}$$

For simplicity, we choose a uniform partition, *i.e.*, $\tau=\frac{T}{N}$, and $\tau\leqslant 1$.

Throughout this work, we make use of the following assumptions:

(H) $Q\geqslant 0,\ R_1>0,\ R_2>0,\ G\geqslant 0\,.$

The results presented in this work remain valid for general partitions and time-varying deterministic coefficients, provided that the following assumptions are satisfied:

(H1) $Q(\cdot)\geqslant 0,\ R_1(\cdot)\gg 0,\ R_2(\cdot)\gg 0,\ G\geqslant 0\,.$

(H2) $A(\cdot)$, $B_1(\cdot)$, $B_2(\cdot)$, $C(\cdot)$, $D_1(\cdot)$, $D_2(\cdot)$, $Q(\cdot)$, $R_1(\cdot)$, $R_2(\cdot)$ satisfy the Lipschitz condition.

The following presents the maximum principle for Problem **(MLQ)**; see [9, Theorem 1].

Lemma 2.1 *Under the assumption* (**H**), Problem **(MLQ)** *admits a unique optimal pair* $\big(x^*(\cdot),u^*(\cdot)\big)\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)\times\mathcal{U}_{ad}$. *Moreover, the following costate equation admits a unique solution:*

$$\begin{cases}\mathrm{d}p(t)=-\big[A^\top p(t)+C^\top q(t)+Qx^*(t)\big]\,\mathrm{d}t+q(t)\,\mathrm{d}W(t),\quad\forall\,t\in[0,T]\,,\\ p(T)=Gx^*(T)\,,\end{cases} \tag{2.2}$$

and the optimal control $u^*(\cdot)=\big(u_1^*(\cdot),\,u_2^*(\cdot)\big)$ *satisfies the following optimality condition*

$$B_1^\top\mathbb{E}\big[p(t)\big]+D_1^\top\mathbb{E}\big[q(t)\big]+R_1u_1^*(t)=0,\quad\forall\,t\in[0,T]\,, \tag{2.3}$$

$$B_2^\top p(t)+D_2^\top q(t)+R_2u_2^*(t)=0,\quad\forall\,t\in[0,T]\,. \tag{2.4}$$

The above lemma provides the open-loop strategy for Problem **(MLQ)**. However, to propose an effective algorithm, our primary focus is on obtaining the closed-loop optimal control strategy, *i.e.*, the optimal control in the form of state feedback. Before proceeding, we introduce two differential Riccati equations related to Problem **(MLQ)** (omitting the time variable t; see *e.g.*, [9])

$$\begin{cases}\dot P_1+P_1A+A^\top P_1+C^\top P_1C+Q-(P_1B_2+C^\top P_1D_2)\Lambda_2^{-1}(P_1)(P_1B_2+C^\top P_1D_2)^\top=0\,,\\ P_1(T)=G\,,\end{cases} \tag{2.5}$$

and

$$\begin{cases}\dot P_2+P_2\widetilde A(P_1)+\widetilde A^\top(P_1)P_2+\widetilde Q(P_1)-P_2\widehat B(P_1)\bar\Lambda^{-1}(P_1)\widehat B^\top(P_1)P_2=0\,,\\ P_2(T)=G\,,\end{cases} \tag{2.6}$$

where for any $P \in \mathbb{S}^n_+$

$$\begin{cases} \Lambda_i(P) \triangleq R_i + D_i^\top P D_i, \quad i = 1, 2\,, \\ \widehat{\Lambda}(P) \triangleq \Lambda_1(P) - D_1^\top P D_2 \Lambda_2^{-1}(P) D_2^\top P D_1\,, \\ U(P) \triangleq P - P D_2 \Lambda_2^{-1}(P) D_2^\top P\,, \\ \widetilde{Q}(P) \triangleq Q + C^\top U(P) C - C^\top U(P) D_1 \widehat{\Lambda}^{-1}(P) D_1^\top U(P) C\,, \\ \widetilde{A}(P) \triangleq A - B_2 \Lambda_2^{-1}(P) D_2^\top P C - \big[B_1 - B_2 \Lambda_2^{-1}(P) D_2^\top P D_1\big] \widehat{\Lambda}^{-1}(P) D_1^\top U(P) C\,, \\ \widehat{B}(P) \triangleq \big(B_2, B_1 - B_2 \Lambda_2^{-1}(P) D_2^\top P D_1\big)\,, \\ \bar{\Lambda}(P) \triangleq \begin{pmatrix} \Lambda_2(P) & 0 \\ 0 & \widehat{\Lambda}(P) \end{pmatrix}. \end{cases} \tag{2.7}$$

Remark 2.2 *In essence, the Riccati equation* (2.6) *is fundamentally similar to the Riccati equation discussed in [9, (31)], differing only in terms of the notation employed.*

For any $P \in \mathbb{S}^n_+$, it is obvious that $\Lambda_i(P) \geqslant R_i > 0$ for $i = 1, 2$. In what follows, we tend to consider the positive definiteness and semi-definiteness of $\widetilde{Q}(P)$, $\widehat{\Lambda}(P)$ and $\bar{\Lambda}(P)$. To achieve this objective, we need the following auxiliary lemma.

Lemma 2.3 *Let $F \in \mathbb{S}^n_{++}$ and $R \in \mathbb{S}^m_{++}$. Then for any $D \in \mathbb{R}^{n\times m}$, the following matrix inequality holds:*

$$F^{-1} - D(R + D^\top F D)^{-1} D^\top > 0\,.$$

Proof The assertion can be derived directly through the following calculation:

$$\begin{pmatrix} F^{-1} - D(R + D^\top F D)^{-1} D^\top & 0 \\ 0 & R + D^\top F D \end{pmatrix} = W \begin{pmatrix} F^{-1} & 0 \\ 0 & R \end{pmatrix} W^\top$$

along with the fact that $\begin{pmatrix} F^{-1} & 0 \\ 0 & R \end{pmatrix} > 0$, and

$$W \triangleq \begin{pmatrix} I_n - D(R + D^\top F D)^{-1} D^\top F & -D(R + D^\top F D)^{-1} \\ D^\top F & I_m \end{pmatrix}$$

is invertible. □

Lemma 2.4 *Let the assumption* **(H)** *hold. Then for any $P \in \mathbb{S}^n_+$, it holds that*

$$\widehat{\Lambda}(P) > R_1\,, \quad \bar{\Lambda}(P) \geqslant \operatorname{diag}\{R_2, R_1\}\,, \quad \widetilde{Q}(P) > Q\,. \tag{2.8}$$

Proof By applying Lemma 2.3 with $D = P^{\frac{1}{2}} D_2$, $R = R_2$ and $F = I_n$, we have

$$\begin{aligned} \widehat{\Lambda}(P) &= \Lambda_1(P) - D_1^\top P^{\frac{1}{2}} \big[P^{\frac{1}{2}} D_2\big]\big[R_2 + (P^{\frac{1}{2}} D_2)^\top (P^{\frac{1}{2}} D_2)\big]^{-1} \big[P^{\frac{1}{2}} D_2\big]^\top P^{\frac{1}{2}} D_1 \\ &> \Lambda_1(P) - D_1^\top P^{\frac{1}{2}} P^{\frac{1}{2}}(t) D_1 \\ &= R_1\,, \end{aligned}$$

which is the first assertion.

By combining the first assertion and the definition of $\Lambda_2(\cdot)$, we get:

$$\bar{\Lambda}(P) \geqslant \text{ diag}\left\{R_2, R_1\right\},$$

which establishes the second desired assertion.

To derive the third assertion, we first apply Lemma 2.3 to derive $U(P) \geqslant 0$. Then, by the fact that $\widehat{\Lambda}(P) = R_1 + D_1^\top U(P) D_1$, and Lemma 2.3, we arrive at

$$\begin{aligned}
&C^\top U(P) D_1 \widehat{\Lambda}^{-1}(P) D_1^\top U(P) C \\
&= C^\top U^{\frac{1}{2}}(P) \big[U^{\frac{1}{2}}(P) D_1\big] \big[R_1 + D_1^\top U(P) D_1\big]^{-1} \big[D_1^\top U^{\frac{1}{2}}(P)\big] U^{\frac{1}{2}}(P) C \\
&< C^\top U(P) C,
\end{aligned}$$

and subsequently

$$\widetilde{Q}(P) > Q + C^\top U(P) C - C^\top U(P) C = Q.$$

That completes the proof. □

Relying on the Riccati equations (2.5) and (2.6), the optimal control for Problem **(MLQ)** admits the following feedback form; see [9, Theorem 2].

Lemma 2.5 *Under the assumption* (**H**)*, Riccati equations* (2.5) *and* (2.6) *admit unique positive semidefinite solutions* $P_1(\cdot)$ *and* $P_2(\cdot)$*, respectively. Moreover, the optimal control* $u^*(\cdot) = \big(u_1^*(\cdot), u_2^*(\cdot)\big)$ *of* Problem **(MLQ)** *has the following feedback form:*

$$u_1^*(t) = M_1(t)\mathbb{E}\big[x^*(t)\big],\ u_2^*(t) = M_2(t)\big(x^*(t) - \mathbb{E}\big[x^*(t)\big]\big) + M_3(t)\mathbb{E}\big[x^*(t)\big],\ \ \forall\, t \in [0, T], \tag{2.9}$$

where

$$\begin{cases}
M_1 \triangleq -\widehat{\Lambda}^{-1}(P_1)\big[B_1^\top P_2 + D_1^\top P_1 C - D_1^\top P_1 D_2 \Lambda_2^{-1}(P_1)[B_2^\top P_2 + D_2^\top P_1 C]\big], \\
M_2 \triangleq -\Lambda_2^{-1}(P_1)\big[B_2^\top P_1 + D_2^\top P_1 C\big], \\
M_3 \triangleq -\Lambda_2^{-1}(P_1)\big[B_2^\top P_2 + D_2^\top P_1 C + D_2^\top P_1 D_1 M_1\big].
\end{cases} \tag{2.10}$$

In the final part of this section, we state the following result, which will be used frequently in the sequel.

Lemma 2.6 *Let* $K(\cdot) \in C([0,T]; \mathbb{R}^{n\times n})$ *be a function that satisfies the Lipschitz condition. If* $K^{-1}(\cdot)$ *exists and is uniformly bounded, then* $K^{-1}(\cdot)$ *also satisfies the Lipschitz condition.*

Proof Without loss of generality, suppose that the Lipschitz constant of $K(\cdot)$ is L, and $\|K^{-1}(\cdot)\| \leqslant M$. Then for any $s_1, s_2 \in [0, T]$,

$$\|K^{-1}(s_1) - K^{-1}(s_2)\| \leqslant \|K^{-1}(s_1)\|\|K(s_2) - K(s_1)\|\|K^{-1}(s_2)\| \leqslant M^2 L |s_2 - s_1|,$$

which settles the assertion. □

3. Discretization of Riccati equations

In this Section, we mainly propose a discretization scheme for the Riccati equation (2.6) and prove its convergence rate, which is crucial for designing a Riccati-based algorithm for Problem **(MLQ)**. Since (2.6) is a $P_1(\cdot)$-dependent equation, we firstly review a numerical scheme for the Riccati equation (2.5) based on the "first discretization, then optimization" strategy, and then apply the same strategy to discretize the Riccati equation (2.6).

For a given uniform partition $\mathcal{I}_\tau = \{t_k\}_{k=0}^N$ on [0,T] and any $t_l \in \mathcal{I}_\tau,\ l = 0, 1, \ldots, N-1$, we define the discrete state space $\mathbb{X}_\tau(t_l, T)$ and the control space $\mathbb{U}_\tau(t_l, T) \triangleq \mathbb{U}_{1,\tau}(t_l, T) \times \mathbb{U}_{2,\tau}(t_l, T)$,

$$\begin{aligned}
\mathbb{X}_\tau(t_l, T) &\triangleq \big\{x(\cdot) \in L^2_{\mathbb{F}}(t_l, T; \mathbb{R}^n) \,\big|\, x(t) = x(t_k), \forall\, t \in [t_k, t_{k+1}), k = l, \ldots, N-1\big\}\,,\\
\mathbb{U}_{1,\tau}(t_l, T) &\triangleq \big\{u(\cdot) \in L^2(t_l, T; \mathbb{R}^{\ell_1}) \,\big|\, u(t) = u(t_k), \forall\, t \in [t_k, t_{k+1}), k = l, \ldots, N-1\big\}\,,\\
\mathbb{U}_{2,\tau}(t_l, T) &\triangleq \big\{u(\cdot) \in L^2_{\mathbb{F}}(t_l, T; \mathbb{R}^{\ell_2}) \,\big|\, u(t) = u(t_k), \forall\, t \in [t_k, t_{k+1}), k = l, \ldots, N-1\big\}\,,
\end{aligned}$$

and for any $x(\cdot) \in \mathbb{X}_\tau(t_l, T), u(\cdot) = \big(u_1(\cdot), u_2(\cdot)\big) \in \mathbb{U}_\tau(t_l, T)$,

$$\begin{aligned}
\|x(\cdot)\|_{\mathbb{X}_\tau(t_l,T)} &\triangleq \bigg(\tau \sum_{k=l}^{N-1} \mathbb{E}\left[\|x(t_k)\|^2\right]\bigg)^{\frac{1}{2}}\,,\\
\|u(\cdot)\|_{\mathbb{U}_\tau}(t_l, T) &\triangleq \bigg(\tau \sum_{k=l}^{N-1} \|u_1(t_k)\|^2 + \tau \sum_{k=l}^{N-1} \mathbb{E}\big[\|u_2(t_k)\|^2\big]\bigg)^{\frac{1}{2}}\,.
\end{aligned}$$

We now introduce a family of parameterized MLQ problems for any $t \in [0, T]$ as follows:

$$\begin{cases}
\mathrm{d}x(s) = \ [Ax(s) + B_1u_1(s) + B_2u_2(s)]\,\mathrm{d}s + [Cx(s) + D_1u_1(s)\\
\qquad\qquad + D_2u_2(s)]\mathrm{d}W(s), \ \ \forall\, s \in [t, T]\,,\\
x(t) = x_t \in \mathbb{R}^n\,,
\end{cases} \tag{3.1}$$

with the following cost functional

$$\begin{aligned}
\mathcal{J}\big(t, x_t; u(\cdot)\big) = \mathbb{E}\Big[\int_t^T \big(\langle Qx(s), x(s)\rangle + \langle R_1u_1(s), u_1(s)\rangle + \langle R_2u_2(s), u_2(s)\rangle\big)\,\mathrm{d}s\Big]\\
+ \mathbb{E}\big[\langle Gx(T), x(T)\rangle\big]\,.
\end{aligned} \tag{3.2}$$

By discretizing the equation (3.1) and the cost functional (3.2), we obtain the following discrete system equation

$$\begin{cases}
x_\tau(t_{k+1}) = \big[(I_n + \tau A)x_\tau(t_k) + \tau B_1u_{1,\tau}(t_k) + \tau B_2u_{2,\tau}(t_k)\big]\\
\qquad\qquad + \big[Cx_\tau(t_k) + D_1u_{1,\tau}(t_k) + D_2u_{2,\tau}(t_k)\big]\Delta_{k+1}W, \quad k = l, \ldots, N-1\,,\\
x_\tau(t_l) = x_{t_l} \in \mathbb{R}^n\,,
\end{cases} \tag{3.3}$$

and a discrete cost functional

$$\mathcal{J}_\tau\big(t_l, x_{t_l}; u_\tau(\cdot)\big) = \|Q^{\frac{1}{2}}x_\tau(\cdot)\|^2_{\mathbb{X}_\tau(t_l,T)} + \big\|\big(R_1^{\frac{1}{2}}u_{1,\tau}(\cdot),\ R_2^{\frac{1}{2}}u_{2,\tau}(\cdot)\big)\big\|^2_{\mathbb{U}_\tau(t_l,T)} + \mathbb{E}\big[\langle Gx_\tau(T), x_\tau(T)\rangle\big]\,. \tag{3.4}$$

Note that when $B_1 = D_1 = 0, R_1 = 0$, only $u_2(\cdot)$ resp. $u_{2,\tau}(\cdot)$ are active in the above problems. Therefore, we introduce the following two families of parameterized stochastic LQ problems:

Problem (SLQ)^t Under the condition $B_1 = D_1 = 0,\ R_1 = 0$, for any $t \in [0, T),\ x_t \in \mathbb{R}^n$, minimize $\mathcal{J}\big(t, x_t; u(\cdot)\big)$ over the admissible control space $L^2_{\mathbb{F}}(0, T; \mathbb{R}^{\ell_2})$;

Problem (SLQ)$_\tau^{t_l}$ Under the condition $B_1 = D_1 = 0,\ R_1 = 0$, for any $l = 0, 1, \ldots, N-1,\ x_{t_l} \in \mathbb{R}^n$, minimize $\mathcal{J}_\tau\big(t_l, x_{t_l}; u_\tau(\cdot)\big)$ over the admissible control space $\mathbb{U}_{2,\tau}(t_l, T)$.

Problems **(SLQ)^t** and **(SLQ)$_\tau^{t_l}$** are standard LQ problems, and their solutions can be found in [24] and [1], respectively. In fact, Problem **(SLQ)$_\tau^{t_l}$** serves as an approximation of Problem **(SLQ)^t** (see, *e.g.*, [18, 21]), and the following three lemmata hold.

Lemma 3.1 *[1, Theorem 4.3] Under the assumption* **(H)**, *for any* $l = 0, 1, \ldots, N-1$, Problem **(SLQ)$_\tau^{t_l}$** *admits a unique optimal control in the following feedback form:*

$$u^*_{2,\tau}(t_k) = -\mathcal{G}_k^{-1}\mathcal{H}_k x^*_\tau(t_k), \quad \forall\, k = l, \dots, N-1\,, \tag{3.5}$$

where

$$\mathcal{G}_k \triangleq R_2 + \tau B_2^\top P_{1,k+1} B_2 + D_2^\top P_{1,k+1} D_2\,, \qquad \mathcal{H}_k \triangleq B_2^\top P_{1,k+1}(I_n + A\tau) + D_2^\top P_{1,k+1} C\,, \tag{3.6}$$

and

$$\inf_{u_\tau(\cdot)\in\mathbb{U}_{2,\tau}(t_l,T)} \mathcal{J}_\tau(t_l, x_{t_l}; u_\tau(\cdot)) = \langle P_{1,l} x_{t_l}, x_{t_l}\rangle\,, \tag{3.7}$$

where $P_{1,\cdot} \triangleq \{P_{1,k}\}_{k=0}^N$ *satisfies the following discrete Riccati equation:*

$$\begin{cases} P_{1,k} = (I_n + A\tau)^\top P_{1,k+1}(I_n + A\tau) + \tau C^\top P_{1,k+1} C + \tau Q \\ \qquad\quad - \tau \mathcal{H}_k^\top \mathcal{G}_k^{-1} \mathcal{H}_k\,,\ \forall\, k = l, l+1, \dots, N-1, \\ P_{1,N} = G\,. \end{cases} \tag{3.8}$$

Lemma 3.2 *[21, Lemma 3.3] Let* $P_1(\cdot)$ *resp.* $P_{1,\cdot}$ *be the solutions of Riccati equations* (2.5) *resp.* (3.8)*. Then* $P_1(\cdot) \in C([0,T]; \mathbb{S}^n_+)$*,* $P_{1,\cdot} \subset \mathbb{S}^n_+$*, and there exists a constant* $\mathcal{C}$ *independent of* τ *such that*

$$\begin{cases} \sup\limits_{t\in[0,T]} \|P_1(t)\| + \max\limits_{l=0,1,\dots,N} \|P_{1,l}\| \leqslant \mathcal{C}\,, \\ \|P_1(t) - P_1(s)\| \leqslant \mathcal{C}|t-s|, \quad \forall\, t, s \in [0,T]\,. \end{cases} \tag{3.9}$$

As an application of Lemmata 2.4 and 3.2, we obtain the following property:

Corollary 3.3 *Let* $P_1(\cdot)$ *resp.* $P_{1,\cdot}$ *be the solutions of Riccati equations* (2.5) *resp.* (3.8)*. Then*

$$\sup_{t\in[0,T]} \big\|\varphi\big(P_1(t)\big)\big\| + \max_{l=0,1,\dots,N} \big\|\varphi(P_{1,l})\big\| \leqslant \mathcal{C}\,,$$

where $\varphi(\cdot) = \widetilde{Q}(\cdot)\,, \widetilde{A}(\cdot)\,, \widehat{B}(\cdot)$ *and* $\mathcal{C}$ *depend on data.*

Lemma 3.4 *[18, Theorem 4.1] Under the assumption* (**H**)*, for the solution* $P_1(\cdot)$ *of differential Riccati equation* (2.5) *and the solution* $P_{1,\cdot}$ *of the discrete Riccati equation* (3.8)*, there exists a constant* $\mathcal{C}$ *independent of* τ *such that*

$$\max_{k=0,1,\dots,N} \|P_{1,k} - P_1(t_k)\| \leqslant \mathcal{C}\tau\,. \tag{3.10}$$

To design a Riccati-based scheme for Problem **(MLQ)**, we need to solve the Riccati equation (2.6). To achieve this, we further introduce two families of parameterized deterministic LQ problems, using $\widetilde{A}(\cdot)$, $\widehat{B}(\cdot)$, $\widetilde{Q}(\cdot)$, $\bar{\Lambda}(\cdot)$ as defined in (2.7).

Problem(DLQ)^t For any given $t \in [0,T]$, $x_t \in \mathbb{R}^n$, minimize the cost functional

$$\bar{\mathcal{J}}(t, x_t; u(\cdot)) = \int_t^T \big\langle \bar{Q}(s)x(s), x(s)\big\rangle + \big\langle \bar{R}(s)u(s), u(s)\big\rangle \,\mathrm{d}s + \big\langle \bar{G}x(T), x(T)\big\rangle\,,$$

over the admissible control set $L^2(0,T;\mathbb{R}^{\ell_1})$, where $x(\cdot)$ denotes the state of the following system:

$$\begin{cases} \dot{x}(s) = \bar{A}(s)x(s) + \bar{B}(s)u(s), \quad \forall\, s \in [t,T]\,, \\ x(t) = x_t\,, \end{cases} \tag{3.11}$$

with

$$\bar{A}(\cdot)=\widetilde{A}\big(P_1(\cdot)\big),\ \ \bar{B}(\cdot)=\widehat{B}\big(P_1(\cdot)\big),\ \ \bar{G}=G\,,\ \ \bar{Q}(\cdot)=\widetilde{Q}\big(P_1(\cdot)\big),\ \ \bar{R}(\cdot)=\bar{\Lambda}\big(P_1(\cdot)\big)\,. \tag{3.12}$$

Problem(DLQ)$_\tau^{t_l}$ For a given $x_{t_l}\in\mathbb{R}^n$, minimize the cost functional

$$\bar{\mathcal{J}}_\tau(t_l,x_{t_l};u_\tau)=\tau\sum_{k=l}^{N-1}\big[\big\langle\bar{Q}_k x_\tau(t_k),x_\tau(t_k)\big\rangle+\big\langle\bar{R}_k u_\tau(t_k),u_\tau(t_k)\big\rangle\big]+\big\langle\bar{G}x_\tau(T),x_\tau(T)\big\rangle\,,$$

over the admissible control set $\mathbb{U}_{1,\tau}(t_l,T)$, where $x_\tau(\cdot)$ denotes the state of the following system:

$$\begin{cases}x_\tau(t_{k+1})=(I_n+\tau\bar{A}_k)x_\tau(t_k)+\tau\bar{B}_k u_\tau(t_k), \quad \forall\, k=l,\ldots,N-1\,,\\ x_\tau(t_l)=x_{t_l}\,,\end{cases} \tag{3.13}$$

with

$$\bar{A}_k=\widetilde{A}(P_{1,k}),\qquad \bar{B}_k=\widehat{B}(P_{1,k}),\qquad \bar{Q}_k=\widetilde{Q}(P_{1,k}),\qquad \bar{R}_k=\bar{\Lambda}(P_{1,k})\,. \tag{3.14}$$

By Lemmata 2.4 and 3.2, $\bar{Q}(\cdot)$ is positive semidefinite and $\bar{R}(\cdot)$ is uniformly positive definite. Thus, standard LQ theory implies that, for any $t\in[0,T]$, Problem **(DLQ)**^{t} admits a unique solution, and it is straightforward to verify that the corresponding differential Riccati equation is precisely (2.6). Similarly, for any $l=0,1,\ldots,N-1$, Problem **(DLQ)**$_\tau^{t_l}$ has a unique optimal control. We naturally derive the discrete Riccati equation corresponding to Problem **(DLQ)**$_\tau^{t_l}$,

$$\begin{cases}P_{2,k}=(I_n+\bar{A}_k\tau)^\top P_{2,k+1}(I_n+\bar{A}_k\tau)+\tau\bar{Q}_k-\tau\bar{\mathcal{H}}_k^\top\bar{\mathcal{G}}_k^{-1}\bar{\mathcal{H}}_k, \quad \forall\, k=l,l+1,\ldots,N-1\,,\\ P_{2,N}=\bar{G}\,,\\ \bar{\mathcal{H}}_k=\bar{B}_k^\top P_{2,k+1}(I_n+\bar{A}_k\tau), \quad \bar{\mathcal{G}}_k=\bar{R}_k+\tau\bar{B}_k^\top P_{2,k+1}\bar{B}_k\,,\end{cases} \tag{3.15}$$

which serves as an approximation of (2.6). Based on (3.15), Problem **(DLQ)**$_\tau^{t_l}$ admits a unique optimal control in the following feedback form:

$$u_\tau^*(t_k)=-\bar{\mathcal{G}}_k^{-1}\bar{\mathcal{H}}_k x_\tau^*(t_k), \quad k=l,l+1,\ldots,N-1\,. \tag{3.16}$$

The following result shows the regularity of solutions to Riccati equations (2.6) and (3.15).

Lemma 3.5 *Let $P_2(\cdot)$ resp. $P_{2,\cdot}$ be the solutions to Riccati equations* (2.6) *resp.* (3.15)*. Then there exists a constant $\mathcal{C}$ independent of τ such that*

$$\begin{cases}\sup_{t\in[0,T]}\|P_2(t)\|+\max_{l=0,1,\ldots,N}\|P_{2,l}\|\leqslant\mathcal{C}\,,\\ \|P_2(t)-P_2(s)\|\leqslant\mathcal{C}|t-s|, \quad \forall\, t,s\in[0,T]\,.\end{cases} \tag{3.17}$$

Proof (1) Let $\big(x^*(\cdot),u^*(\cdot)\big)$ and $\big(x_\tau^*(\cdot),u_\tau^*(\cdot)\big)$ be the optimal pairs of Problems **(DLQ)**^{t} and **(DLQ)**$_\tau^{t_l}$, respectively. Then

$$\inf_{u(\cdot)\in L^2(t,T;\mathbb{R}^{\ell_1})}\bar{\mathcal{J}}\big(t,x_t;u(\cdot)\big)=\bar{\mathcal{J}}\big(t,x_t;u^*(\cdot)\big)=\langle P_2(t)x_t,x_t\rangle\,, \quad \forall\, t\in[0,T]\,, \tag{3.18}$$

$$\inf_{u_\tau(\cdot)\in\mathbb{U}_{1,\tau}(t_l,T)}\bar{\mathcal{J}}_\tau\big(t_l,x_{t_l};u_\tau(\cdot)\big)=\bar{\mathcal{J}}_\tau\big(t_l,x_{t_l};u_\tau^*(\cdot)\big)=\langle P_{2,l}x_{t_l},x_{t_l}\rangle\,, \quad \forall\, l=0,\ldots,N\,. \tag{3.19}$$

For the system (3.11) of Problem **(DLQ)**^{t} , by setting $u(\cdot)=0$, we conclude that

$$x(s)=\Phi_1(s,t)x_t, \quad \forall\, s\in[t,T]\,,$$

where $\Phi_1(\cdot,t)$ is state transition matrix, *i.e.*,

$$\begin{cases} \dfrac{\mathrm{d}\Phi_1(s,t)}{\mathrm{d}s} = \bar{A}(s)\Phi_1(s,t), \quad \forall\, s \in [t,T]\,, \\ \Phi_1(t,t) = I_n\,. \end{cases}$$

The assumption $R_1 > 0, R_2 > 0$ and Lemma 2.4 imply that $\|\bar{A}(\cdot)\| \leqslant \mathcal{C}$, which, along with the stability property of ODEs, leads to

$$\sup_{s\in[t,T]} \|x(s)\| \leqslant \sup_{s\in[t,T]} \|\Phi_1(s,t)\| \cdot \|x_t\| \leqslant \mathcal{C}\|x_t\|\,. \tag{3.20}$$

Now, by applying the Cauchy-Schwarz inequality, together with (3.20) and Corollary 3.3, we deduce that

$$0 \leqslant \langle P_2(t)x_t, x_t\rangle \leqslant \bar{\mathcal{J}}(t, x_t; 0) = \int_t^T \big\langle \bar{Q}(s)x(s), x(s)\big\rangle \;\mathrm{d}s + \big\langle \bar{G}x(T), x(T)\big\rangle \leqslant \mathcal{C}\|x_t\|^2\,.$$

Subsequently,

$$\sup_{t\in[0,T]} \|P_2(t)\| \leqslant \mathcal{C}\,.$$

Similarly, we can derive $\max_{k=l,l+1,\dots,N} \|I_n + \tau\bar{A}_k\| \leqslant 1 + \mathcal{C}\tau$. Then by taking $u_\tau(\cdot) = 0$ in (3.13), we have

$$\max_{k=l,l+1,\dots,N} \|x_\tau(t_k)\| \leqslant (1+\mathcal{C}\tau)^{k-l}\|x_{t_l}\| \leqslant \mathcal{C}\|x_{t_l}\|\,. \tag{3.21}$$

Thus, by applying (3.19) and (3.21), we find that

$$\max_{l=0,1,\dots,N} \|P_{2,l}\| \leqslant \mathcal{C}\,.$$

The first assertion is proved.

(2) Relying on the Riccati equation (2.6), the fact that $P_1(\cdot) \in C([0,T]; \mathbb{S}^n_+)$, together with Lemma 2.4 and Corollary 3.3, we have

$$\sup_{t\in[0,T]} \Big[\big\|\widetilde{A}\big(P_1(t)\big)\big\| + \big\|\widetilde{Q}\big(P_1(t)\big)\big\| + \big\|\widehat{B}\big(P_1(t)\big)\big\| + \big\|\bar{\Lambda}^{-1}\big(P_1(t)\big)\big\|\Big] \leqslant \mathcal{C}\,,$$

which yields

$$\|P_2(t) - P_2(s)\| \leqslant \int_s^t \mathcal{C}\big[\|P_2(r)\| + \|P_2(r)\|^2\big]\mathrm{d}r \leqslant \mathcal{C}|t-s|\,.$$

That completes the proof. □

Lemma 3.6 *For any $t \in [0,T]$, suppose that $\big(x^*(\cdot), u^*(\cdot)\big)$ is the optimal pair of* Problem $\mathbf{(DLQ)}^t$. *Then there exists a constant $\mathcal{C}$ such that*

$$\begin{cases} \sup\limits_{s\in[t,T]} [\|x^*(s)\| + \|u^*(s)\|] \leqslant \mathcal{C}\|x_t\|\,, \\ \|x^*(s_1) - x^*(s_2)\| + \|u^*(s_1) - u^*(s_2)\| \leqslant \mathcal{C}|s_1 - s_2|\|x_t\|, \quad \forall\, s_1, s_2 \in [t,T]\,. \end{cases} \tag{3.22}$$

Proof (1) By substituting the feedback representation of the optimal control for Problem $\mathbf{(DLQ)}^t$

$$u^*(\cdot) = -\bar{R}^{-1}\bar{B}^\top P_2(\cdot)x^*(\cdot) = -\bar{\Lambda}^{-1}(P_1)\widehat{B}^\top(P_1)P_2(\cdot)x^*(\cdot) \tag{3.23}$$

into the system equation of Problem $\textbf{(DLQ)}^t$, we have

$$\begin{cases} \dot{x}^*(s) = \big[\widetilde{A}(P_1) - \widehat{B}(P_1)\bar{\Lambda}^{-1}(P_1)\widehat{B}^\top(P_1)P_2(s)\big]x^*(s), \quad \forall\, s \in [t,T]\,, \\ x^*(s) = x_t\,. \end{cases} \tag{3.24}$$

On the other hand, by Lemmata 2.4, 3.5 and Corollary 3.3, we know that the coefficient of (3.24) is uniformly bounded. Subsequently, the stability property of the solution to ODE (3.24) yields

$$\sup_{s\in[t,T]} \|x^*(s)\| \leqslant \mathcal{C}\|x_t\|\,. \tag{3.25}$$

Then, by applying the feedback representation (3.23) again,

$$\sup_{s\in[t,T]} \|u^*(s)\| \leqslant \sup_{s\in[t,T]} \big[\|\bar{\Lambda}^{-1}(P_1)\widehat{B}^\top(P_1)P_2(s)\| \cdot \|x_t\|\big] \leqslant \mathcal{C}\|x_t\|\,.$$

That settles the first assertion of (3.22).

(2) By applying the assertion $(3.22)_1$ and the state equation (3.11), we obtain

$$\|x^*(s_1) - x^*(s_2)\| \leqslant \int_{s_1}^{s_2} \|\bar{A}(s)x^*(s) + \bar{B}(s)u^*(s)\|\,\mathrm{d}s \leqslant \mathcal{C}|s_1 - s_2|\|x_t\|\,. \tag{3.26}$$

In what follows, we prove the $u(\cdot)$-part in $(3.22)_2$. Based on the feedback form (3.23), for any $s_1, s_2 \in [t,T]$, it follows that:

$$\begin{aligned} \|u^*(s_1) - u^*(s_2)\| \leqslant{}& \big\|\bar{\Lambda}^{-1}\big(P_1(s_1)\big) - \bar{\Lambda}^{-1}\big(P_1(s_2)\big)\big\| \cdot \big\|\widehat{B}^\top\big(P_1(s_1)\big)P_2(s_1)x^*(s_1)\big\| \\ &+ \big\|\bar{\Lambda}^{-1}\big(P_1(s_2)\big)\big\| \cdot \big\|\widehat{B}^\top\big(P_1(s_1)\big) - \widehat{B}^\top\big(P_1(s_2)\big)\big\| \cdot \big\|P_2(s_1)x^*(s_1)\big\| \\ &+ \big\|\bar{\Lambda}^{-1}\big(P_1(s_2)\big)\big\| \cdot \big\|\widehat{B}^\top\big(P_1(s_2)\big)\big\| \cdot \big\|P_2(s_1) - P_2(s_2)\big\| \cdot \|x^*(s_1)\| \\ &+ \big\|\bar{\Lambda}^{-1}\big(P_1(s_2)\big)\big\| \cdot \big\|\widehat{B}^\top\big(P_1(s_2)\big)\big\| \cdot \|P_2(s_2)\| \cdot \|x^*(s_1) - x^*(s_2)\|\,. \end{aligned} \tag{3.27}$$

Hence, to derive the Lipschitz continuity of $u^*(\cdot)$, we only need to address the same property for $\bar{\Lambda}^{-1}(P_1)$ and $\widehat{B}(P_1)$. For the former one, we have

$$\begin{aligned} &\|\bar{\Lambda}^{-1}\big(P_1(s_1)\big) - \bar{\Lambda}^{-1}\big(P_1(s_2)\big)\| \\ ={}& \left\|\begin{pmatrix} \Lambda_2^{-1}\big(P_1(s_1)\big) - \Lambda_2^{-1}\big(P_1(s_2)\big) & 0 \\ 0 & \widehat{\Lambda}^{-1}\big(P_1(s_1)\big) - \widehat{\Lambda}^{-1}\big(P_1(s_2)\big) \end{pmatrix}\right\| \\ ={}& \max\left\{\|\Lambda_2^{-1}\big(P_1(s_1)\big) - \Lambda_2^{-1}\big(P_1(s_2)\big)\|, \|\widehat{\Lambda}^{-1}\big(P_1(s_1)\big) - \widehat{\Lambda}^{-1}\big(P_1(s_2)\big)\|\right\}\,. \end{aligned}$$

Lemma 3.2 implies that

$$\|\Lambda_2\big(P_1(s_1)\big) - \Lambda_2\big(P_1(s_2)\big)\| = \|D_2^\top[P_1(s_1) - P_1(s_2)]D_2\| \leqslant \mathcal{C}|s_1 - s_2|\,,$$

which, along with $\Lambda_2(P_1) \geqslant R_2$ and Lemma 2.6, leads to

$$\|\Lambda_2^{-1}\big(P_1(s_1)\big) - \Lambda_2^{-1}\big(P_1(s_2)\big)\| \leqslant \mathcal{C}|s_1 - s_2|\,.$$

By a similar argument

$$\|\widehat{\Lambda}^{-1}\big(P_1(s_1)\big) - \widehat{\Lambda}^{-1}\big(P_1(s_2)\big)\| \leqslant \mathcal{C}|s_1 - s_2|\,.$$

Hence,

$$\|\bar{\Lambda}^{-1}\big(P_1(s_1)\big) - \bar{\Lambda}^{-1}\big(P_1(s_2)\big)\| \leqslant \mathcal{C}|s_1 - s_2|\,. \tag{3.28}$$

By the same argument and Lemma 3.2, it follows that:

$$
\begin{aligned}
&\left\|\widehat{B}\big(P_1(s_1)\big)-\widehat{B}\big(P_1(s_2)\big)\right\| \\
=&\left\|\begin{pmatrix}0 & B_2\Lambda_2^{-1}\big(P_1(s_2)\big)D_2^\top P_1(s_2)D_1-B_2\Lambda_2^{-1}\big(P_1(s_1)\big)D_2^\top P_1(s_1)D_1\end{pmatrix}\right\| \\
\leqslant&\|B_2\|\cdot\left\|\Lambda_2^{-1}\big(P_1(s_2)\big)-\Lambda_2^{-1}\big(P_1(s_1)\big)\right\|\cdot\left\|D_2^\top P_1(s_2)D_1\right\| \\
&+\left\|B_2\Lambda_2^{-1}\big(P_1(s_1)\big)D_2^\top\right\|\cdot\|P_1(s_2)-P_1(s_1)\|\cdot\|D_1\| \\
\leqslant&\,\mathcal{C}|s_1-s_2|\,.
\end{aligned}
\tag{3.29}
$$

Finally, by combining (3.26), (3.27), (3.28), (3.29) and Lemma 3.5, we conclude that

$$\|u^*(s_1)-u^*(s_2)\|\leqslant\mathcal{C}|s_1-s_2|\|x_t\|\,.$$

That completes the proof. □

Lemma 3.7 *For any* $l=0,1,\dots,N-1$*, suppose that* $\big(x_\tau^*(\cdot),u_\tau^*(\cdot)\big)$ *is the optimal pair of* Problem $\textbf{(DLQ)}_\tau^{t_l}$*. Then there exists a constant* $\mathcal{C}$ *independent of* τ *and* l *such that*

$$\max_{k=l,l+1,\dots,N}\|x_\tau^*(t_k)\|+\max_{k=l,l+1,\dots,N-1}\|u_\tau^*(t_k)\|\leqslant\mathcal{C}\|x_{t_l}\|\,.$$

Proof By applying the feedback representation of the optimal control stated in (3.16) and the state equation (3.13), we find that

$$
\begin{aligned}
x_\tau^*(t_k)=&(I_n+\tau\bar{A}_{k-1}-\tau\bar{B}_{k-1}\bar{\mathcal{G}}_{k-1}^{-1}\bar{\mathcal{H}}_{k-1})x_\tau^*(t_{k-1}) \\
=&\prod_{j=k-1}^{l}(I_n+\tau\bar{A}_j-\tau\bar{B}_j\bar{\mathcal{G}}_j^{-1}\bar{\mathcal{H}}_j)x_{t_l}\,.
\end{aligned}
\tag{3.30}
$$

Here, the product is taken in the backward order, and we adopt the convention that an empty product equals the identity matrix, *i.e.*,

$$\prod_{j=l-1}^{l}(I_n+\tau\bar{A}_j-\tau\bar{B}_j\bar{\mathcal{G}}_j^{-1}\bar{\mathcal{H}}_j)=I_n\,.$$

Based on (3.14), (3.15) and Lemmata 3.2, 3.5, we can deduce that

$$\max_{k=l,l+1,\dots,N}\|I_n+\tau\bar{A}_k-\tau\bar{B}_k\bar{\mathcal{G}}_k^{-1}\bar{\mathcal{H}}_k\|\leqslant1+\mathcal{C}\tau\,,$$

where $\mathcal{C}$ is independent of τ. Hence,

$$\max_{k=l,l+1,\dots,N}\|x_\tau^*(t_k)\|\leqslant(1+\mathcal{C}\tau)^{k-l}\|x_{t_l}\|\leqslant\mathcal{C}\|x_{t_l}\|\,.$$

Subsequently, by applying the feedback representation (3.16) again, we have

$$\max_{k=l,l+1,\dots,N-1}\|u_\tau^*(t_k)\|\leqslant\max_{k=l,l+1,\dots,N-1}\big[\|\bar{\mathcal{G}}_k^{-1}\bar{\mathcal{H}}_k\|\|x_\tau^*(t_k)\|\big]\leqslant\mathcal{C}\|x_{t_l}\|\,.$$

That completes the proof. □

Lemma 3.8 *Let* $u^*(\cdot)$ *be the optimal control of* Problem $\textbf{(DLQ)}^{t_l}$*, and* $\bar{x}_\tau(\cdot)$ *denote the solution to the discrete state equation* (3.13) *with* $u_\tau(\cdot)=u^*(\cdot)$*. Then there exists a constant* $\mathcal{C}$ *independent of* τ *and* l *such that*

$$
\begin{cases}
\displaystyle\max_{k=l,l+1,\dots,N}\left\|\bar{x}_\tau(t_k)\right\|\leqslant\mathcal{C}\left\|x_{t_l}\right\|, \\
\displaystyle\max_{k=l,l+1,\dots,N}\left\|\bar{x}_\tau(t_k)-x^*(t_k)\right\|\leqslant\mathcal{C}\tau\left\|x_{t_l}\right\|.
\end{cases}
$$

Proof (1) By taking $u_\tau(\cdot)=u^*(\cdot)$ in (3.13) and applying induction, we have

$$\begin{aligned}\bar{x}_\tau(t_{k+1}) &= (I_n+\tau\bar{A}_k)\bar{x}_\tau(t_k)+\tau\bar{B}_k u^*(t_k)\\ &= (I_n+\tau\bar{A}_k)(I_n+\tau\bar{A}_{k-1})\bar{x}_\tau(t_{k-1})+(I_n+\tau\bar{A}_k)\tau\bar{B}_{k-1}u^*(t_{k-1})+\tau\bar{B}_k u^*(t_k)\\ &= \cdots\\ &= \prod_{i=k}^{l}(I_n+\tau\bar{A}_i)x_{t_l}+\sum_{j=l}^{k}\prod_{i=k}^{j+1}(I_n+\tau\bar{A}_i)\bar{B}_j\tau u^*(t_j)\,.\end{aligned}$$

Then the uniformly bounded property of $\bar{A}.$ and $\bar{B}.$ derived in Corollary 3.3, along with Lemma 3.6, yields

$$\max_{k=l,l+1,\ldots,N}\big\|\bar{x}_\tau(t_k)\big\|\leqslant(1+\mathcal{C}\tau)^{N-l}\big\|x_{t_l}\big\|+\sum_{j=l}^{N-1}\prod_{i=1}^{j+1}(1+\mathcal{C}\tau)\mathcal{C}\tau\big\|u^*(t_i)\big\|\leqslant\mathcal{C}\big\|x_{t_l}\big\|\,,$$

which is the first assertion.

(2) Setting $e_k=\bar{x}_\tau(t_k)-x^*(t_k)$, we find that

$$\begin{aligned}\|e_{k+1}-e_k\| =&\ \|\tau\bar{A}_k\bar{x}_\tau(t_k)+\tau\bar{B}_k u^*(t_k)-[x^*(t_{k+1})-x^*(t_k)]\|\\ \leqslant&\ \int_{t_k}^{t_{k+1}}\Big[\big\|\bar{A}_k\big\|\cdot\big\|x^*(t_k)-\bar{x}(t_k)\big\|+\big\|\bar{A}(s)-\bar{A}_k\big\|\cdot\big\|x^*(s)\big\|\\ &+\big\|\bar{A}_k\big\|\cdot\big\|x^*(s)-x^*(t_k)\big\|+\big\|\bar{B}(s)-\bar{B}_k\big\|\cdot\big\|u^*(s)\big\|\\ &+\big\|\bar{B}_k\big\|\cdot\big\|u^*(s)-u^*(t_k)\big\|\Big]\,\mathrm{d}s\\ =:&\ \int_{t_k}^{t_{k+1}}\sum_{j=1}^{5}I_j(s)\,\mathrm{d}s\,.\end{aligned}$$

For terms $I_2(\cdot)$ and $I_4(\cdot)$, by Lemmata 3.2 and 2.4, we have that $\Lambda_1(P_1(\cdot)),\Lambda_1(P_{1,\cdot})$, $\Lambda_2^{-1}(P_1(\cdot))$, $\Lambda_2^{-1}(P_{1,\cdot})$ and $\widehat{\Lambda}^{-1}(P_1(\cdot)),\widehat{\Lambda}^{-1}(P_{1,\cdot})$ are uniformly bounded. Combining this with the matrix identity

$$K^{-1}-L^{-1}=K^{-1}(L-K)L^{-1}\,,$$

valid for any invertible matrices K and L, and using (3.12), (3.14), $(3.9)_2$ and Lemma 3.4, we obtain

$$\big\|\bar{A}(s)-\bar{A}_k\big\|+\big\|\bar{B}(s)-\bar{B}_k\big\|\leqslant\mathcal{C}\tau\,.$$

Hence, we conclude by $(3.22)_1$ in Lemma 3.6 that

$$I_2(s)+I_4(s)\leqslant\mathcal{C}\tau\big\|x_{t_l}\big\|\,.$$

For terms $I_3(\cdot)$ and $I_5(\cdot)$, Lemma 3.6 leads to

$$I_3(s)+I_5(s)\leqslant\max_{k=l,l+1,\ldots,N}\Big[\big(\big\|\bar{A}_k\big\|+\big\|\bar{B}_k\big\|\big)\cdot\big(\big\|u^*(s)-u^*(t_k)\big\|+\big\|x^*(s)-x^*(t_k)\big\|\big)\Big]\leqslant\mathcal{C}\tau\big\|x_{t_l}\big\|\,.$$

By combining these estimates, we arrive at

$$\|e_{k+1}\|\leqslant(1+\mathcal{C}\tau)\|e_k\|+\mathcal{C}\tau^2\big\|x_{t_l}\big\|\,,$$

which, together with discrete Gronwall's inequality, yields

$$\max_{k=l,l+1,\ldots,N}\|e_k\|\leqslant\mathcal{C}\tau\big\|x_{t_l}\big\|\,.$$

That completes the proof. □

With a similar procedure to that in the proof of Lemma 3.8, we derive the following result:

Lemma 3.9 *Let $u_\tau^*(\cdot)$ be the optimal control of* Problem $\textbf{(DLQ)}_\tau^{t_l}$*, and $\widehat{x}(\cdot)$ denote the solution to the state equation* (3.11) *with $u(\cdot)=u_\tau^*(\cdot)$. Then there exists a constant $\mathcal{C}$ independent of τ such that*

$$\begin{cases} \sup\limits_{t\in[t,T]} \big\|\widehat{x}(t)\big\| \leqslant \mathcal{C}\big\|x_{t_l}\big\|\,, \\ \max\limits_{k=l,l+1,\ldots,N} \big\|\widehat{x}(t_k)-x_\tau^*(t_k)\big\| \leqslant \mathcal{C}\tau\big\|x_{t_l}\big\|\,. \end{cases}$$

The following estimate on the difference between $P_2(\cdot)$ and $P_{2,\cdot}$ is the main result of this section, which will be used to design an algorithm for Problem **(MLQ)**; see Theorem 4.3 and Algorithm 1.

Theorem 3.10. *Let $P_2(\cdot)$ and $P_{2,\cdot}$ be the solutions of Riccati equations* (2.6) *and* (3.15)*. Then there is a constant $\mathcal{C}$ independent of τ such that*

$$\max_{l=0,1,\ldots,N} \big\|P_2(t_l)-P_{2,l}\big\| \leqslant \mathcal{C}\tau\,.$$

Proof Let $\big(x^*(\cdot),u^*(\cdot)\big)$ and $\big(x_\tau^*(\cdot),u_\tau^*(\cdot)\big)$ be the optimal pairs of Problems $\textbf{(DLQ)}^{t_l}$ and $\textbf{(DLQ)}_\tau^{t_l}$, respectively. Then, there are two possible cases.

Case (1) $\langle P_2(t_l)x_{t_l},x_{t_l}\rangle \leqslant \langle P_{2,l}x_{t_l},x_{t_l}\rangle$

In this case, combining with Lemmata 3.2, 3.6, and 3.8, we arrive at

$$\begin{aligned} &\langle P_{2,l}x_{t_l},x_{t_l}\rangle-\langle P_2(t_l)x_{t_l},x_{t_l}\rangle \\ &=\bar{\mathcal{J}}_\tau\big(t_l,x_{t_l};u_\tau^*(\cdot)\big)-\bar{\mathcal{J}}\big(t_l,x_{t_l};u^*(\cdot)\big) \\ &\leqslant \bar{\mathcal{J}}_\tau\big(t_l,x_{t_l};u^*\big(\nu(\cdot)\big)\big)-\bar{\mathcal{J}}\big(t_l,x_{t_l};u^*(\cdot)\big) \\ &\leqslant \sum_{k=l}^{N-1}\int_{t_k}^{t_{k+1}} \Big[\|\bar{Q}_k-\bar{Q}(t)\|\cdot\|\bar{x}_\tau(t_k)\|^2+\big(\|\bar{Q}(t)\bar{x}_\tau(t_k)\|+\|\bar{Q}(t)x^*(t)\|\big)\cdot\|\bar{x}_\tau(t_k)-x^*(t)\| \\ &\quad+\|\bar{R}_k-\bar{R}(t)\|\cdot\|u^*(t_k)\|^2+\big(\|\bar{R}(t)u^*(t_k)\|+\|\bar{R}(t)u^*(t)\|\big)\cdot\|u^*(t_k)-u^*(t)\|\Big]\,\mathrm{d}t \\ &\quad+\big(\|\bar{G}\bar{x}_\tau(T)\|+\|\bar{G}x^*(T)\|\big)\cdot\|\bar{x}_\tau(T)-x^*(T)\| \\ &\leqslant \mathcal{C}\tau\|x_{t_l}\|^2\,. \end{aligned}$$

Case (2) $\langle P_2(t_l)x_{t_l},x_{t_l}\rangle \geqslant \langle P_{2,l}x_{t_l},x_{t_l}\rangle$

In this case, by taking the argument of Case (1) and combining with Lemmata 3.2, 3.6, and 3.9, we have

$$\langle P_2(t_l)x_{t_l},x_{t_l}\rangle-\langle P_{2,l}x_{t_l},x_{t_l}\rangle \leqslant \mathcal{C}\tau\|x_{t_l}\|^2\,.$$

Hence, a combination of these two cases yields

$$\big|\big\langle (P_2(t_l)-P_{2,l})x_{t_l},x_{t_l}\big\rangle\big| \leqslant \mathcal{C}\tau\|x_{t_l}\|^2\,,$$

which leads to the desired assertion. □

4. Discretization of Problem (MLQ) and its convergence rate

Based on $P_{1,\cdot}$ and $P_{2,\cdot}$, which are approximations of $P_1(\cdot)$ and $P_2(\cdot)$ respectively, and inspired by Lemma 2.5, we propose the following feedback-type approximation for optimal control $u^*(\cdot)$ of Problem **(MLQ)**:

$$\begin{cases} u_{1,\tau}(t_k) = M_{1,k}\mathbb{E}[x_\tau(t_k)]\,, \\ u_{2,\tau}(t_k) = M_{2,k}\big(x_\tau(t_k) - \mathbb{E}[x_\tau(t_k)]\big) + M_{3,k}\mathbb{E}[x_\tau(t_k)], \quad k = 0, 1, \ldots, N-1\,, \end{cases} \tag{4.1}$$

where

$$\begin{cases} M_{1,k} \triangleq -\widehat{\Lambda}^{-1}(P_{1,k})\big[B_1^\top P_{2,k} + D_1^\top P_{1,k}C - D_1^\top P_{1,k}D_2\Lambda_2^{-1}(P_{1,k})[B_2^\top P_{2,k} + D_2^\top P_{1,k}C]\big]\,, \\ M_{2,k} \triangleq -\Lambda_2^{-1}(P_{1,k})\big[B_2^\top P_{1,k} + D_2^\top P_{1,k}C\big]\,, \\ M_{3,k} \triangleq -\Lambda_2^{-1}(P_{1,k})\big[B_2^\top P_{2,k} + D_2^\top P_{1,k}C + D_2^\top P_{1,k}D_1M_{1,k}\big]\,, \end{cases} \tag{4.2}$$

and $P_{1,\cdot}, P_{2,\cdot}$ solve difference Riccati equations (3.8) and (3.15), respectively. Before deriving the convergence of the proposed control in (4.1), we present some auxiliary results.

Lemma 4.1 *For $\{M_i(\cdot)\}_{i=1}^3$ and $\{M_{i,\cdot}\}_{i=1}^3$ defined in* (2.10) *and* (4.2)*, respectively, it holds that*

$$\begin{cases} \sup\limits_{t\in[0,T]} \max\limits_{i=1,2,3} \|M_i(t)\| + \max\limits_{k=0,1,\ldots,N-1} \max\limits_{i=1,2,3} \|M_{i,k}\| \leqslant \mathcal{C}\,, \\ \max\limits_{k=0,1,\ldots,N-1} \sup\limits_{t\in[t_k,t_{k+1})} \max\limits_{i=1,2,3} \|M_i(t) - M_{i,k}\| \leqslant \mathcal{C}\tau\,, \end{cases} \tag{4.3}$$

where $\mathcal{C}$ is independent of τ.

Proof The first assertion can be directly derived by Lemmata 2.4, 3.2, and 3.5. Below, we prove the second assertion. Set

$$\begin{aligned} \Phi(t) &= B_1^\top P_2(t) + D_1^\top P_1(t)C - D_1^\top P_1(t)D_2\Lambda_2^{-1}\big(P_1(t)\big)[B_2^\top P_2(t) + D_2^\top P_1(t)C]\,, \\ \Phi_k &= B_1^\top P_{2,k} + D_1^\top P_{1,k}C - D_1^\top P_{1,k}D_2\Lambda_2^{-1}(P_{1,k})[B_2^\top P_{2,k} + D_2^\top P_{1,k}C]\,. \end{aligned}$$

A direct calculation leads to

$$\begin{aligned} \Phi(t) - \Phi(t_k) = {} & B_1^\top\big[P_2(t) - P_{2,k}\big] + D_1^\top\big[P_1(t) - P_{1,k}\big]C \\ & - D_1^\top\big[P_1(t) - P_{1,k}\big]D_2\Lambda_2^{-1}\big(P_1(t)\big)[B_2^\top P_2(t) + D_2^\top P_1(t)C] \\ & - D_1^\top P_{1,k}D_2\Lambda_2^{-1}\big(P_1(t)\big)D_2^\top\big[P_{1,k} - P_1(t)\big]D_2\Lambda_2^{-1}(P_{1,k})[B_2^\top P_2(t) + D_2^\top P_1(t)C] \\ & - D_1^\top P_{1,k}D_2\Lambda_2^{-1}(P_{1,k})\Big(B_2^\top\big[P_2(t) - P_{2,k}\big] + D_2^\top\big[P_1(t) - P_{1,k}\big]C\Big)\,. \end{aligned}$$

Then Lemmata 3.2, 3.4, and 3.5, together with Theorem 3.10, imply that

$$\begin{aligned} &\sup_{t\in[0,T]} \|\Phi(t)\| + \max_{k=0,1,\ldots,N} \|\Phi_k\| \leqslant \mathcal{C}\,, \\ &\max_{k=0,1,\ldots,N-1} \sup_{t\in[t_k,t_{k+1})} \|\Phi(t) - \Phi(t_k)\| \leqslant \mathcal{C}\tau\,. \end{aligned}$$

Similarly, we can derive

$$\max_{k=0,1,\ldots,N-1} \sup_{t\in[t_k,t_{k+1})} \big\|\widehat{\Lambda}^{-1}\big(P_1(t)\big) - \widehat{\Lambda}^{-1}(P_{1,k})\big\| \leqslant \mathcal{C}\tau\,.$$

Subsequently, these three estimates and Lemma 2.4 yield

$$\begin{aligned} &\max_{k=0,1,\ldots,N-1} \sup_{t\in[t_k,t_{k+1})} \big\|M_1(t) - M_{1,k}\big\| \\ \leqslant{} & \max_{k=0,1,\ldots,N-1} \sup_{t\in[t_k,t_{k+1})} \big\|\widehat{\Lambda}^{-1}\big(P_1(t)\big) - \widehat{\Lambda}^{-1}(P_{1,k})\big\| \times \max_{k=0,1,\ldots,N} \|\Phi_k\| \\ & + \sup_{t\in[0,T]} \big\|\widehat{\Lambda}^{-1}\big(P_1(t)\big)\big\| \times \max_{k=0,1,\ldots,N-1} \sup_{t\in[t_k,t_{k+1})} \|\Phi(t) - \Phi(t_k)\| \\ \leqslant{} & \mathcal{C}\tau\,. \end{aligned}$$

Following a similar procedure, we have

$$\max_{k=0,1,\ldots,N-1}\ \sup_{t\in[t_k,t_{k+1})}\Big[\big\|M_2(t)-M_{2,k}\big\|+\big\|M_3(t)-M_{3,k}\big\|\Big]\leqslant \mathcal{C}\tau\,.$$

This completes the proof. □

Lemma 4.2 *Suppose that* $\big(x^*(\cdot),u^*(\cdot)\big)$ *is the optimal pair of* Problem **(MLQ)**, *and* $x_\tau(\cdot)$ *is the state of* (3.3) *with* $l=0$, *where* $u_\tau(\cdot)$ *is given by* (4.1). *Then*

$$\begin{cases}\sup\limits_{t\in[0,T]}\big\|\mathbb{E}\big[x^*(t)\big]\big\|+\max\limits_{k=0,1,\ldots,N}\big\|\mathbb{E}\big[x_\tau(t_k)\big]\big\|\leqslant \mathcal{C}\|x_0\|\,,\\ \max\limits_{k=0,1,\ldots,N-1}\ \sup\limits_{t\in[t_k,t_{k+1})}\big\|\mathbb{E}[x^*(t)]-\mathbb{E}[x^*(t_k)]\big\|\leqslant \mathcal{C}\tau\|x_0\|\,,\\ \mathbb{E}\Big[\sup\limits_{t\in[0,T]}\|x^*(t)\|^2\Big]+\mathbb{E}\Big[\max\limits_{k=0,1,\ldots,N}\big\|x_\tau(t_k)\big\|^2\Big]\leqslant \mathcal{C}\|x_0\|^2\,,\\ \mathbb{E}\Big[\big\|x^*(t)-x^*(s)\big\|^2\Big]\leqslant \mathcal{C}|t-s|\|x_0\|^2,\quad \forall\, t,s\in[0,T]\,.\end{cases}\tag{4.4}$$

Proof (1) For the state equation (3.1), by applying the optimal feedback control (2.9) and then taking expectations on both sides of the equation, we have

$$\mathbb{E}\big[x^*(t)\big]=x_0+\int_0^t (A+B_1M_1+B_2M_3)\mathbb{E}\big[x^*(s)\big]\,\mathrm{d}s,\quad \forall\, t\in[0,T]\,.$$

Then, the uniformly bounded property of M_1, M_3 derived in Lemma 4.1 and Gronwall's inequality imply that

$$\sup_{t\in[0,T]}\big\|\mathbb{E}\big[x^*(t)\big]\big\|\leqslant \mathcal{C}\|x_0\|\,.$$

For the discrete equation (3.3), the optimal feedback control (4.1) yields

$$\begin{cases}\mathbb{E}\big[x_\tau(t_{k+1})\big]=\big[(I_n+\tau A)+\tau B_1M_{1,k}+\tau B_2M_{3,k}\big]\mathbb{E}\big[x_\tau(t_k)\big]=:\Psi_k\mathbb{E}[x_\tau(t_k)]\,,\\ \mathbb{E}[x_\tau(0)]=x_0\,.\end{cases}\tag{4.5}$$

By taking into account Lemma 4.1, we have $\max_{k=0,1,\ldots,N-1}\|\Psi_k\|\leqslant 1+\mathcal{C}\tau$. Consequently,

$$\max_{k=0,1,\ldots,N}\big\|\mathbb{E}\big[x_\tau(t_k)\big]\big\|\leqslant (1+\mathcal{C}\tau)^N\|x_0\|\leqslant \mathcal{C}\|x_0\|\,.$$

(2) By $(4.4)_1$ and Lemma 4.1, it is straightforward to derive the following

$$\begin{aligned}&\max_{k=0,1,\ldots,N-1}\ \sup_{t\in[t_k,t_{k+1})}\big\|\mathbb{E}[x^*(t)]-\mathbb{E}[x^*(t_k)]\big\|\\ \leqslant\ &\max_{k=0,1,\ldots,N-1}\ \sup_{t\in[t_k,t_{k+1})}\int_{t_k}^t\|A+B_1M_1+B_2M_3\|\cdot\big\|\mathbb{E}[x^*(s)]\big\|\,\mathrm{d}s\\ \leqslant\ &\mathcal{C}\tau\|x_0\|\,,\end{aligned}$$

that is $(4.4)_2$.

(3) By substituting the feedback representations (2.9) and (4.1) into the state equations (3.1) and (3.3), respectively, we have

$$\begin{cases}\mathrm{d}x^*(t)=\Big[(A+B_2M_2)x^*(t)+\big[B_1M_1-B_2M_2+B_2M_3\big]\mathbb{E}\big[x^*(t)\big]\Big]\,\mathrm{d}t\\ \qquad\qquad+\Big[(C+D_2M_2)x^*(t)+\big[D_1M_1-D_2M_2+D_2M_3\big]\mathbb{E}\big[x^*(t)\big]\Big]\,\mathrm{d}W(t),\quad \forall\, t\in[0,T]\,,\\ x^*(0)=x_0\,,\end{cases}$$

and

$$\begin{cases} x_\tau(t_{k+1}) = \big[I_n + \tau A + \tau B_2 M_{2,k}\big] x_\tau(t_k) \\ \qquad\qquad + \tau\big[B_1 M_{1,k} - B_2 M_{2,k} + B_2 M_{3,k}\big]\mathbb{E}\big[x_\tau(t_k)\big] + \Big[\big[C + D_2 M_{2,k}\big] x_\tau(t_k) \\ \qquad\qquad + \big[D_1 M_{1,k} - D_2 M_{2,k} + D_2 M_{3,k}\big]\mathbb{E}\big[x_\tau(t_k)\big]\Big]\Delta_{k+1} W, \quad \forall\, k = 0, 1, \ldots, N-1\,, \\ x_\tau(0) = x_0\,. \end{cases} \tag{4.6}$$

Then, Lemma 4.1, $(4.4)_1$ and the stability property of SDEs yield

$$\mathbb{E}\big[\sup_{t\in[0,T]} \|x^*(t)\|^2\big] \leqslant \mathcal{C}\Big[\|x_0\|^2 + \int_0^T \big\|\mathbb{E}\big[x^*(t)\big]\big\|^2\,\mathrm{d}t\Big] \leqslant \mathcal{C}\|x_0\|^2\,.$$

Based on the difference equation of $x_\tau(\cdot)$, Lemma 4.1, $(4.4)_1$ and discrete Gronwall's inequality, we can deduce for any $k = 0, 1, \ldots, N-1$ that

$$\begin{aligned} &\mathbb{E}\big[\big\|x_\tau(t_{k+1})\big\|^2\big] \\ \leqslant\ &\mathbb{E}\Big\{\big\|\big[I_n + \tau A + \tau B_2 M_{2,k}\big] x_\tau(t_k)\big\|^2 + \tau^2\big\|\big[B_1 M_{1,k} - B_2 M_{2,k} + B_2 M_{3,k}\big]\mathbb{E}\big[x_\tau(t_k)\big]\big\|^2 \\ &+ 2\tau\big\|I_n + \tau A + \tau B_2 M_{2,k}\big\|\big\|B_1 M_{1,k} - B_2 M_{2,k} + B_2 M_{3,k}\big\|\big\|\mathbb{E}\big[x_\tau(t_k)\big]\big\|^2 \\ &+ \tau\big\|\big[C + D_2 M_{2,k}\big] x_\tau(t_k)\big\|^2 + \tau\big\|\big[D_1 M_{1,k} - D_2 M_{2,k} + D_2 M_{3,k}\big]\mathbb{E}\big[x_\tau(t_k)\big]\big\|^2 \\ &+ 2\tau\big\|C + D_2 M_{2,k}\big\|\big\|D_1 M_{1,k} - D_2 M_{2,k} + D_2 M_{3,k}\big\|\big\|\mathbb{E}\big[x_\tau(t_k)\big]\big\|^2\Big\} \\ \leqslant\ &(1 + \mathcal{C}\tau)\mathbb{E}\big[\big\|x_\tau(t_k)\big\|^2\big] + \mathcal{C}\tau\big\|\mathbb{E}\big[x_\tau(t_k)\big]\big\|^2 \\ \leqslant\ &\mathcal{C}\|x_0\|^2\,, \end{aligned} \tag{4.7}$$

where $\mathcal{C}$ is independent of τ and k.

Then, the following representation of $x_\tau(\cdot)$ to (4.6)

$$\begin{aligned} x_\tau(t_{k+1}) = x_0 &+ \sum_{j=0}^{k}\Big[\tau\big[A + B_2 M_{2,j}\big] x_\tau(t_j) + \tau\big[B_1 M_{1,j} - B_2 M_{2,j} + B_2 M_{3,j}\big]\mathbb{E}\big[x_\tau(t_j)\big]\Big] \\ &+ \sum_{j=0}^{k}\Big[\big[C + D_2 M_{2,j}\big] x_\tau(t_j) + \big[D_1 M_{1,j} - D_2 M_{2,j} + D_2 M_{3,j}\big]\mathbb{E}\big[x_\tau(t_j)\big]\Big]\Delta_{j+1} W \end{aligned}$$

and Lemma 4.1, $(4.4)_1$, Doob's inequality, (4.7), lead to

$$\begin{aligned} &\mathbb{E}\big[\max_{k=0,1,\ldots,N-1} \big\|x_\tau(t_{k+1})\big\|^2\big] \\ \leqslant\ &\mathcal{C}\Big[\|x_0\|^2 + \tau\sum_{j=0}^{N}\mathbb{E}\big[\|x_\tau(t_j)\|^2\big] + \tau\sum_{j=0}^{N}\big\|\mathbb{E}\big[x_\tau(t_j)\big]\big\|^2\Big] \\ \leqslant\ &\mathcal{C}\|x_0\|^2\,. \end{aligned}$$

This establishes $(4.4)_3$.

(4) $(4.4)_4$ can be directly derived by $(4.4)_1$ and $(4.4)_3$. □

The following are the convergence rates for the Riccati-based discretization of Problem **(MLQ)**.

Theorem 4.3 *Suppose that* $\big(x^*(\cdot), u^*(\cdot)\big)$ *is the optimal pair of* Problem **(MLQ)**, *where* $u^*(\cdot) = \big(u_1^*(\cdot), u_2^*(\cdot)\big)$, *and* $x_\tau(\cdot)$ *is the state of the discrete system* (3.3) *with the feedback control*

$u_\tau(\cdot)=\big(u_{1,\tau}(\cdot),u_{2,\tau}(\cdot)\big)$ *given in* (4.1)*. Then the following error estimates hold:*

$$\begin{cases}\max\limits_{k=0,1,\ldots,N}\big\|\mathbb{E}[x^*(t_k)]-\mathbb{E}[x_\tau(t_k)]\big\|+\max\limits_{k=0,1,\ldots,N-1}\big\|u_1^*(t_k)-u_{1,\tau}(t_k)\big\|\leqslant \mathcal{C}\tau\|x_0\|\,,\\ \mathbb{E}\big[\max\limits_{k=0,1,\ldots,N}\|x^*(t_k)-x_\tau(t_k)\|^2\big]+\mathbb{E}\big[\max\limits_{k=0,1,\ldots,N-1}\|u_2^*(t_k)-u_{2,\tau}(t_k)\|^2\big]\leqslant \mathcal{C}\tau\|x_0\|^2\,,\end{cases}\tag{4.8}$$

where $\mathcal{C}$ is a constant independent of τ.

Proof Since the proof is long, we divide it into three steps.

(1) In this step, we aim to prove the first assertion of (4.8). We begin by adopting a similar technique as used in the proof of Lemma 3.8. Setting $\widehat{e}_k=\mathbb{E}[x^*(t_k)]-\mathbb{E}[x_\tau(t_k)]$, based on state equations (3.1) and (3.3), we conclude that:

$$\begin{aligned}&\|\widehat{e}_{k+1}-\widehat{e}_k\|\\ \leqslant&\int_{t_k}^{t_{k+1}}\big[\big\|\big(A+B_1M_1+B_2M_3\big)\mathbb{E}[x^*(t)]-\big(A+B_1M_{1,k}+B_2M_{2,k}\big)\mathbb{E}[x_\tau(t_k)]\big\|\big]\,\mathrm{d}t\\ \leqslant&\int_{t_k}^{t_{k+1}}\Big[\big\|A+B_1M_1+B_2M_3\big\|\cdot\big\|\mathbb{E}\big[x^*(t)\big]-\mathbb{E}\big[x^*(t_k)\big]\big\|+\big\|A+B_1M_1+B_2M_3\big\|\cdot\|\widehat{e}_k\|\\ &+\big\|B_1M_1+B_2M_3-B_1M_{1,k}-B_2M_{3,k}\big\|\cdot\big\|\mathbb{E}\big[x_\tau(t_k)\big]\big\|\Big]\,\mathrm{d}t\\ =:&\int_{t_k}^{t_{k+1}}\sum_{i=1}^{3}I_i(t)\,\mathrm{d}t\,.\end{aligned}$$

Lemmata 4.1 and 4.2 lead to

$$\sup_{t\in[t_k,t_{k+1})}\big[I_1(t)+I_3(t)\big]\leqslant\mathcal{C}\tau\|x_0\|\,.$$

Thus,

$$\|\widehat{e}_{k+1}\|\leqslant(1+\mathcal{C}\tau)\|\widehat{e}_k\|+\mathcal{C}\tau^2\|x_0\|\,,$$

which, together with discrete Gronwall's inequality, leads to

$$\max_{k=0,1,\ldots,N}\|\widehat{e}_k\|\leqslant\mathcal{C}\tau\|x_0\|\,.$$

This proves the $x(\cdot)$-part of the assertion $(4.8)_1$.

Next, we use the feedback laws (2.9) and (4.1) to derive the $u(\cdot)$-part of $(4.8)_1$, which is shown by the following calculation and Lemmata 4.1, 4.2

$$\begin{aligned}&\big\|u_1^*(t_k)-u_{1,\tau}(t_k)\big\|\\ \leqslant&\|M_1\|\cdot\big\|\mathbb{E}\big[x^*(t_k)\big]-\mathbb{E}\big[x_\tau(t_k)\big]\big\|+\|M_1-M_{1,k}\|\cdot\big\|\mathbb{E}\big[x_\tau(t_k)\big]\big\|\\ \leqslant&\mathcal{C}\tau\|x_0\|\,.\end{aligned}$$

(2) In this step, we claim that

$$\max_{k=0,1,\ldots,N}\mathbb{E}\big[\|x^*(t_k)-x_\tau(t_k)\|^2\big]\leqslant\mathcal{C}\tau\|x_0\|^2\,.\tag{4.9}$$

Set $\bar{e}_k=x^*(t_k)-x_\tau(t_k)$, and we have

$$
\begin{aligned}
&\bar{e}_{k+1}-\bar{e}_k\\
=&\int_{t_k}^{t_{k+1}}\Big\{\big[A+B_2M_2\big]\big[x^*(t)-x^*(t_k)\big]+\big[A+B_2M_2\big]\bar{e}_k+B_2\big[M_2-M_{2,k}\big]x_\tau(t_k)\\
&+\big[B_1M_1-B_2M_2+B_2M_3\big]\mathbb{E}\big[x^*(t)-x^*(t_k)\big]\quad+\big[B_1M_1-B_2M_2+B_2M_3\big]\widehat{e}_k\\
&+\Big[B_1\big[M_1-M_{1,k}\big]-B_2\big[M_2-M_{2,k}\big]+B_2\big[M_3-M_{3,k}\big]\Big]\mathbb{E}\big[x_\tau(t_k)\big]\Big\}\,\mathrm{d}t\\
&+\int_{t_k}^{t_{k+1}}\Big\{\big[C+D_2M_2\big]\big[x^*(t)-x^*(t_k)\big]+\big[C+D_2M_2\big]\bar{e}_k+D_2\big[M_2-M_{2,k}\big]x_\tau(t_k)\\
&+\big[D_1M_1-D_2M_2+D_2M_3\big]\mathbb{E}\big[x^*(t)-x^*(t_k)\big]+\big[D_1M_1-D_2M_2+D_2M_3\big]\widehat{e}_k\\
&+\Big[D_1\big[M_1-M_{1,k}\big]-D_2\big[M_2-M_{2,k}\big]+D_2\big[M_3-M_{3,k}\big]\Big]\mathbb{E}\big[x_\tau(t_k)\big]\Big\}\,\mathrm{d}W(t)\\
=:&\int_{t_k}^{t_{k+1}}\sum_{j=1}^{6}J_{j,k}(t)\,\mathrm{d}t+\int_{t_k}^{t_{k+1}}\sum_{j=1}^{6}K_{j,k}(t)\,\mathrm{d}W(t)\,.
\end{aligned}
\tag{4.10}
$$

By multiplying $\bar{e}_{k+1}$ on both sides and then taking expectations, we have

$$
\begin{aligned}
&\mathbb{E}\big[\langle\bar{e}_{k+1}-\bar{e}_k,\bar{e}_{k+1}\rangle\big]\\
&=\frac{1}{2}\mathbb{E}\big[\|\bar{e}_{k+1}\|^2-\|\bar{e}_k\|^2+\|\bar{e}_{k+1}-\bar{e}_k\|^2\big]\\
&=\mathbb{E}\Big[\int_{t_k}^{t_{k+1}}\sum_{j=1}^{6}\langle J_{j,k}(t),\bar{e}_{k+1}\rangle\ \mathrm{d}t\Big]+\mathbb{E}\Big[\int_{t_k}^{t_{k+1}}\sum_{j=1}^{6}\langle K_{j,k}(t),\bar{e}_{k+1}\rangle\ \mathrm{d}W(t)\Big]\,.
\end{aligned}
\tag{4.11}
$$

For $\mathbb{E}\Big[\int_{t_k}^{t_{k+1}}\langle J_{1,k}(t),\bar{e}_{k+1}\rangle\ \mathrm{d}t\Big]$, Lemma 4.2 and the Cauchy-Schwarz inequality lead to

$$
\begin{aligned}
\mathbb{E}\Big[\int_{t_k}^{t_{k+1}}\langle J_{1,k}(t),\bar{e}_{k+1}\rangle\ \mathrm{d}t\Big]&\leqslant\mathcal{C}\mathbb{E}\Big[\int_{t_k}^{t_{k+1}}\|x^*(t)-x^*(t_k)\|^2+\|\bar{e}_{k+1}\|^2\,\mathrm{d}t\Big]\\
&\leqslant\mathcal{C}\tau^2\|x_0\|^2+\mathcal{C}\tau\mathbb{E}\big[\big\|\bar{e}_{k+1}\big\|^2\big]\,.
\end{aligned}
$$

Using the same approach for the remaining terms, we get

$$
\mathbb{E}\Big[\int_{t_k}^{t_{k+1}}\sum_{j=1}^{6}\langle J_{j,k}(t),\bar{e}_{k+1}\rangle\ \mathrm{d}t\Big]\leqslant\mathcal{C}\tau^2\|x_0\|^2+\mathcal{C}\tau\mathbb{E}\big[\|\bar{e}_k\|^2+\|\bar{e}_{k+1}\|^2\big]\,.
\tag{4.12}
$$

For $\mathbb{E}\Big[\int_{t_k}^{t_{k+1}}\langle K_{1,k}(t),\bar{e}_{k+1}\rangle\,\mathrm{d}W(t)\Big]$, we can arrive at

$$
\begin{aligned}
\mathbb{E}\Big\langle\int_{t_k}^{t_{k+1}}K_{1,k}(t)\,\mathrm{d}W(t),\bar{e}_{k+1}\Big\rangle&=\mathbb{E}\Big\langle\int_{t_k}^{t_{k+1}}K_{1,k}(t)\,\mathrm{d}W(t),\bar{e}_{k+1}-\bar{e}_k\Big\rangle\\
&\leqslant\frac{1}{12}\mathbb{E}\big[\big\|\bar{e}_{k+1}-\bar{e}_k\big\|^2\big]+\mathcal{C}\mathbb{E}\Big[\int_{t_k}^{t_{k+1}}\big\|x^*(t)-x^*(t_k)\big\|^2\mathrm{d}t\Big]\\
&\leqslant\frac{1}{12}\mathbb{E}\big[\big\|\bar{e}_{k+1}-\bar{e}_k\big\|^2\big]+\mathcal{C}\tau^2\|x_0\|^2\,.
\end{aligned}
$$

Similarly, for the remaining terms, we have

$$
\mathbb{E}\Big[\int_{t_k}^{t_{k+1}}\sum_{j=1}^{6}\langle K_{j,k}(t),\bar{e}_{k+1}\rangle\ \mathrm{d}W(t)\Big]\leqslant6\times\frac{1}{12}\mathbb{E}\big[\big\|\bar{e}_{k+1}-\bar{e}_k\big\|^2\big]+\mathcal{C}\tau^2\|x_0\|^2+\mathcal{C}\tau\mathbb{E}\big[\|\bar{e}_k\|^2\big]\,,
\tag{4.13}
$$

where $\mathcal{C}$ is independent of τ.

A combination of estimates (4.11)–(4.13) then leads to

$$\mathbb{E}\big[\|\bar{e}_{k+1}\|^2\big] \leqslant (1+\mathcal{C}\tau)\mathbb{E}\big[\|\bar{e}_k\|^2\big] + \mathcal{C}\tau^2\|x_0\|^2\,,$$

which, together with discrete Gronwall's inequality, yields

$$\max_{k=0,1,\ldots,N} \mathbb{E}\big[\|\bar{e}_k\|^2\big] \leqslant \mathcal{C}\tau\|x_0\|^2\,. \tag{4.14}$$

That completes the proof of (4.9).

(3) We now proceed to prove the $(4.8)_2$. Based on (4.10) in step (2), we know that

$$\bar{e}_{k+1} =: \sum_{\ell=0}^{k}\Big[\int_{t_\ell}^{t_{\ell+1}}\sum_{j=1}^{6} J_{j,\ell}(t)\mathrm{d}t + \int_{t_\ell}^{t_{\ell+1}}\sum_{j=1}^{6} K_{j,\ell}(t)\mathrm{d}W(t)\Big].$$

Subsequently, by applying Doob's inequality and the Itô isometry, we arrive at

$$\begin{aligned}
&\mathbb{E}\big[\max_{k=0,\ldots,N-1}\|\bar{e}_{k+1}\|^2\big]\\
&\leqslant \mathcal{C}\mathbb{E}\Big[\max_{k=0,\ldots,N-1}\Big\|\sum_{\ell=0}^{k}\int_{t_\ell}^{t_{\ell+1}}\sum_{j=1}^{6} J_{j,\ell}(t)\mathrm{d}t\Big\|^2 + \max_{k=0,\ldots,N-1}\Big\|\sum_{\ell=0}^{k}\int_{t_\ell}^{t_{\ell+1}}\sum_{j=1}^{6} K_{j,\ell}(t)\mathrm{d}W(t)\Big\|^2\Big]\\
&\leqslant \mathcal{C}\sum_{\ell=0}^{N-1}\Big\{\mathbb{E}\Big[\int_{t_\ell}^{t_{\ell+1}}\sum_{j=1}^{6}\|J_{j,\ell}(t)\|^2\mathrm{d}t\Big] + \mathbb{E}\Big[\int_{t_\ell}^{t_{\ell+1}}\sum_{j=1}^{6}\|K_{j,\ell}(t)\|^2\mathrm{d}t\Big]\Big\}\,.
\end{aligned}$$

Based on Lemmata 4.1, 4.2 and $(4.8)_1$, (4.9), it follows that

$$\sum_{j=1}^{2}\mathbb{E}[\|J_{j,\ell}(t)\|^2] + \sum_{j=1}^{2}\mathbb{E}[\|K_{j,\ell}(t)\|^2] \leqslant \mathcal{C}\tau\|x_0\|^2\,,$$

and

$$\sum_{j=3}^{6}\mathbb{E}[\|J_{j,\ell}(t)\|^2] + \sum_{j=3}^{6}\mathbb{E}[\|K_{j,\ell}(t)\|^2] \leqslant \mathcal{C}\tau^2\|x_0\|^2\,.$$

Thus, we obtain

$$\mathbb{E}\big[\max_{k=0,1,\ldots,N}\|\bar{e}_k\|^2\big] \leqslant \mathcal{C}\tau\|x_0\|^2. \tag{4.15}$$

Finally, by applying feedback laws (2.9), (4.1) and Lemmata 4.1, 4.2, as well as estimates (4.14), (4.15), we have

$$\begin{aligned}
&\mathbb{E}\Big[\max_{k=0,\ldots,N-1}\|u_2^*(t_k) - u_{2,\tau}(t_k)\|^2\Big]\\
&= \mathbb{E}\Big\{\max_{k=0,\ldots,N-1}\Big\|\big[M_{2,k} - M_2(t_k)\big]\big(x_\tau(t_k) - \mathbb{E}[x_\tau(t_k)]\big) + M_2(t_k)\big[\big(x_\tau(t_k) - x^*(t_k)\big)\\
&\quad - \big(\mathbb{E}[x_\tau(t_k)] - \mathbb{E}[x^*(t_k)]\big)\big] + \big[M_{3,k} - M_3(t_k)\big]\mathbb{E}[x_\tau(t_k)] + M_3(t_k)\big(\mathbb{E}[x_\tau(t_k)] - \mathbb{E}[x^*(t_k)]\big)\Big\|^2\Big\}\\
&\leqslant \mathcal{C}\Big[\max_{k=0,\ldots,N-1,i=2,3}\|M_{i,k} - M_i(t_k)\|^2\mathbb{E}\big[\max_{k=0,\ldots,N}\|x_\tau(t_k)\|^2\big] + \mathbb{E}\big[\max_{k=0,\ldots,N}\|\bar{e}_k\|^2\big] + \max_{k=0,\ldots,N}\|\hat{e}_k\|^2\Big]\\
&\leqslant \mathcal{C}\tau\|x_0\|^2\,.
\end{aligned}$$

That completes the proof. □

Remark 4.4 *The feedback control given in* (4.1) *is not necessarily the optimal control for the discretization version of* Problem **(MLQ)**. *However, Theorem 4.3 suggests that this control can approximate the optimal control of* Problem **(MLQ)** *to a certain degree.*

Based on (4.3), we propose the following Riccati-based algorithm to solve Problem **(MLQ)** numerically.

Algorithm 1 The explicit Euler algorithm for Problem **(MLQ)** based on Riccati equations

Fix the time partition $\mathcal{I}_\tau$ and step size τ.

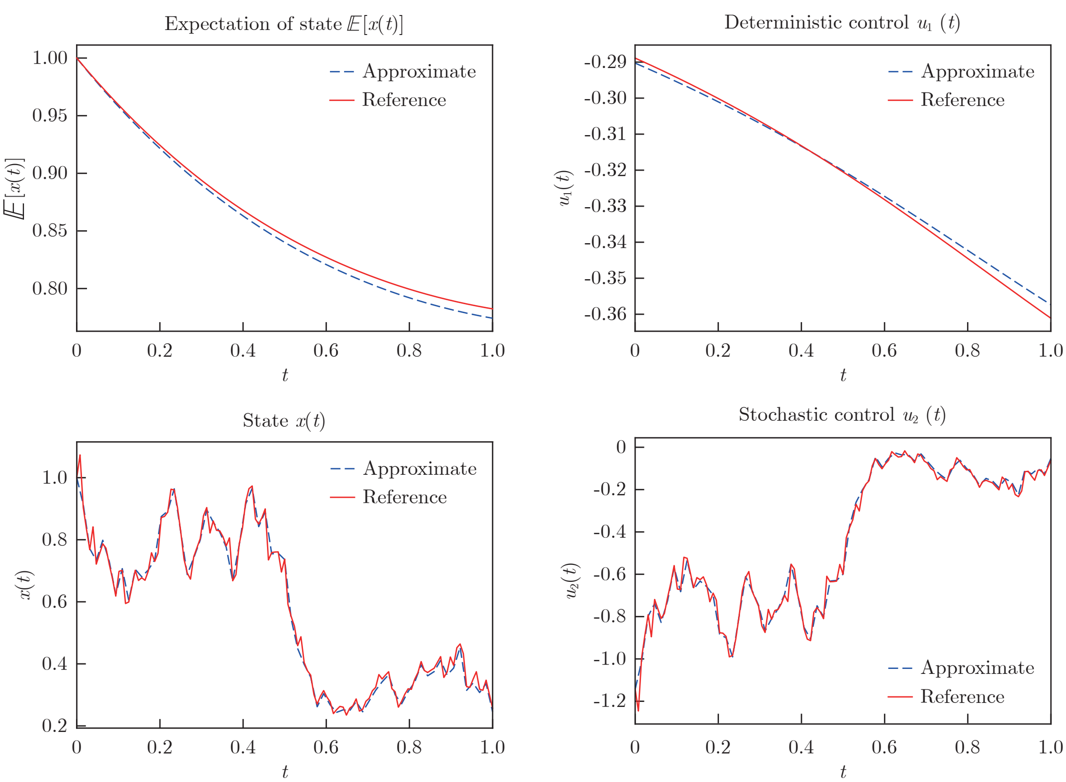
1. Calculate $\{P_{1,k}\}_{k=0}^N$ and $\{P_{2,k}\}_{k=0}^N$ by using difference Riccati equations (3.8) and (3.15).

2. Determine $\big\{\mathbb{E}\big[x_\tau(t_k)\big]\big\}_{k=0}^N$ and $\{u_{1,k}\}_{k=0}^{N-1}$ by using the feedback law (4.1) and discrete state equation (3.3). Specifically, compute $\{\mathbb{E}[x_\tau(t_k)]\}_{k=0}^N$ by (4.5), and then apply (4.1) to derive $\{u_{1,k}\}_{k=0}^{N-1}$.

3. Obtain $\{x_\tau(t_k)\}_{k=0}^N$ and $\{u_{2,\tau}(t_k)\}_{k=0}^{N-1}$ by employing (4.1), (3.3) and $\big\{\mathbb{E}\big[x_\tau(t_k)\big]\big\}_{k=0}^N$: Firstly, solve (4.6) to obtain $\{x_\tau(t_k)\}_{k=0}^N$, and then compute $\{u_{2,\tau}(t_k)\}_{k=0}^{N-1}$ by using (4.1).

In what follows, we present three examples to demonstrate the effectiveness of the Algorithm 1. Example 4.5 involves a time-invariant system with $\mathbb{R}^1$-valued state, whereas Example 4.6 deals with a time-varying system. For each example, a figure consisting of six subfigures is provided. The first four subfigures show: (1) expectation of reference and approximate state, (2) reference and approximate deterministic control $u_1(\cdot)$, (3) one path of reference and approximate state, and (4) one path of reference and approximate stochastic control $u_2(\cdot)$. The remaining two subfigures show the convergence rates of $u_1(\cdot)$ and $\mathbb{E}[x(\cdot)]$ in (5), with $\tau_{\texttt{ref}} = 2^{-7}$, and of $u_2(\cdot)$ and $x(\cdot)$ in (6), with $\tau_{\texttt{ref}} = 2^{-7}$ and $M = 1000$ Monte Carlo samples, respectively. Example 4.7 considers a system with a 100-dimensional state and demonstrates the convergence rates of both the state and control. These rates are consistent with the theoretical results presented in Theorem 4.3.

Example 4.5 *Let* $n = \ell_1 = \ell_2 = 1$, $T = 1$, $x_0 = 1$, $A = 1$, $B_1 = 1$, $B_2 = 1$, $C = 0$, $D_1 = 1$, $D_2 = \frac{1}{2}$, $Q = 0$, $R_1 = \frac{1}{2}$, $R_2 = 1$, $G = 1$. *The numerical approximations of the optimal tuple* $\big(x(\cdot), \big(u_1(\cdot), u_2(\cdot)\big)\big)$ *and their convergence rates to the optimal tuple are presented in Figure 1.*



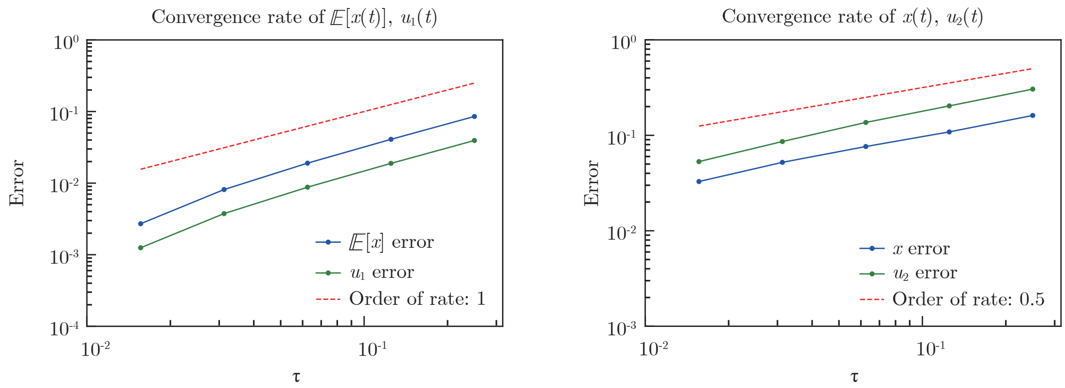
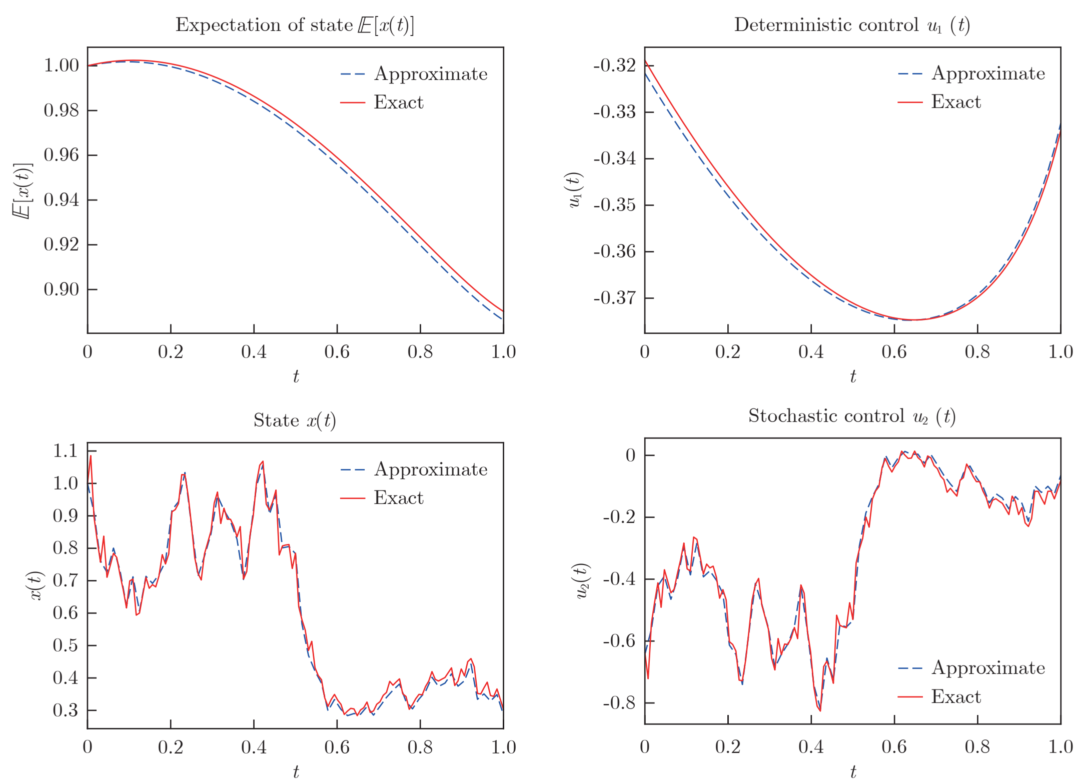


Figure 1 Numerical results for Example 4.5

Example 4.6 *Let* $n = \ell_1 = \ell_2 = 1$, $T = 1$, $x_0 = 1$, $A = \frac{1}{2}$, $B_1 = 1$, $B_2 = 1$, $C = \frac{t}{2}$, $D_1 = 1$, $D_2 = 1$, $Q = 5$, $R_1 = 1$, $R_2 = \frac{1}{2}$, $G = 1$. *The numerical approximations of the optimal tuple* $\big(x(\cdot), \big(u_1(\cdot), u_2(\cdot)\big)\big)$, *together with their convergence rates, are presented in Figure 2.*

Example 4.7 *Let* $n = 100$, $\ell_1 = \ell_2 = 3$, $T = 1, Q = I_{100}$, $R_1 = I_3$, $R_2 = 2I_3$, *and* $G = I_{100}$. *The remaining matrices, with dimensions determined by the context, are randomly generated from a uniform distribution on* $[10^{-9}, 1 - 10^{-9}]$. *In Figure 3, the convergence rates of* $u_{1,\tau}(\cdot)$, $\mathbb{E}[x_\tau(\cdot)]$, $x_\tau(\cdot)$ *and* $u_{2,\tau}(\cdot)$ *are presented, with* $\tau_{\text{ref}} = 2^{-7}$ *and* $M = 1000$ *Monte Carlo samples.*



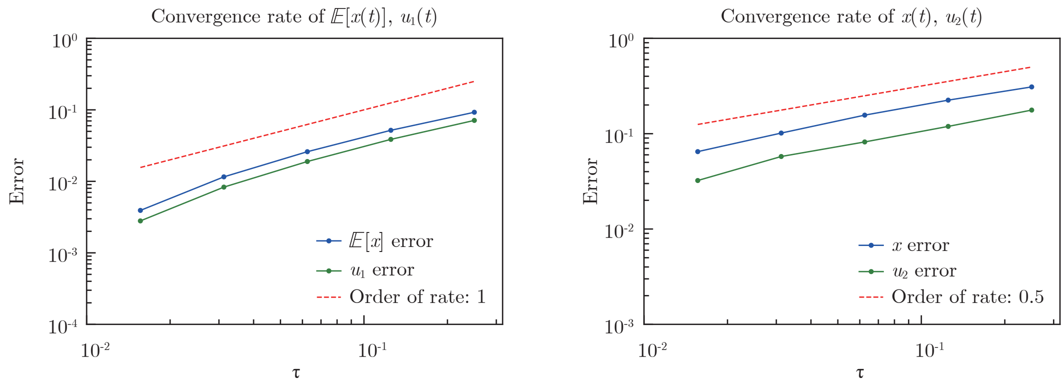


Figure 2 Numerical results for Example 4.6

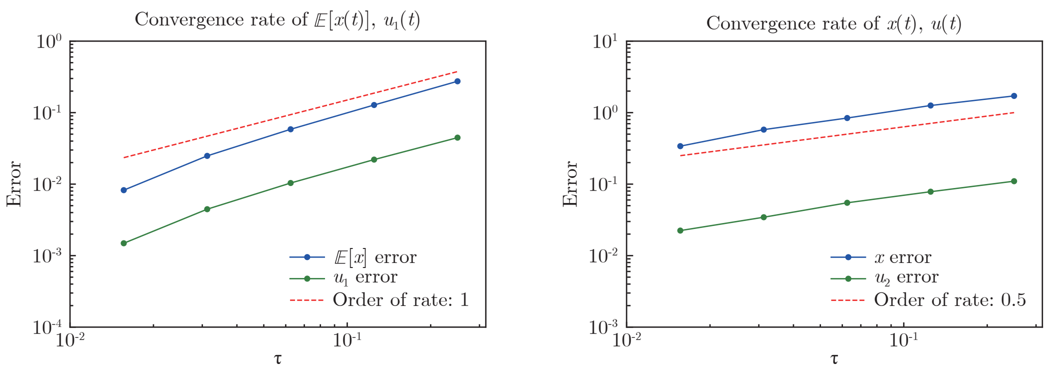


Figure 3 Numerical results for Example 4.7

Acknowledgements

This work is supported by the Graduate Research Innovation Project of Southwest University (Grant No. SWUS24080)

References

[1] Ait Rami, M., Chen, X. and Zhou, X. Y., Discrete-time indefinite LQ control with state and control dependent noises, Journal of Global Optimization, 2002, 23: 245–265

[2] Archibald, R., Bao, F. and Yong, J., A stochastic gradient descent approach for stochastic optimal control, East Asian Journal of Applied Mathematics, 2020, 10: 635–658

[3] Benner, P., Stillfjord, T. and Trautwein, C., A linear implicit Euler method for the finite element discretization of a controlled stochastic heat equation, IMA Journal of Numerical Analysis, 2022, 42: 2118–2150

[4] Du, N., Shi, J. and Liu, W., An effective gradient projection method for stochastic optimal control, International Journal of Numerical Analysis and Modeling, 2013, 10: 757–774

[5] Dunst, T. and Prohl, A., The forward-backward stochastic heat equation: Numerical analysis and simulation, SIAM Journal on Scientific Computing, 2016, 38: A2725–A2755

[6] Elliott, R., Li, X. and Ni, Y.-H., Discrete time mean-field stochastic linear-quadratic optimal control problems, Automatica, 2013, 49: 3222–3233

[7] Gong, B., Liu, W., Tang, T., Zhao, W. and Zhou, T., An efficient gradient projection method for stochastic optimal control problems, SIAM Journal on Numerical Analysis, 2017, 55: 2982–3005

[8] Hinze, M., Pinnau, R., Ulbrich, M. and Ulbrich, S., Optimization with PDE constraints, Springer, New York, 2009.

[9] Hu, Y. and Tang, S., Mixed deterministic and random optimal control of linear stochastic systems with quadratic costs, Probability, Uncertainty and Quantitative Risk, 2019, 4: 1–15

[10] Kröller, M. and Kunisch, K., Convergence rates for the feedback operators arising in the linear quadratic regulator problem governed by parabolic equations, SIAM Journal on Numerical Analysis, 1991, 28: 1350–1385

[11] Li, B. and Zhou, Q., Discretization of a distributed optimal control problem with a stochastic parabolic equation driven by multiplicative noise, Journal of Scientific Computing, 2021, 87: 68

[12] Liang, X. and Xu, J., Control for networked control systems with remote and local controllers over unreliable communication channel, Automatica, 2018, 98: 86–94

[13] Prohl, A. and Wang, Y., Strong rates of convergence for a space-time discretization of the backward stochastic heat equation, and of a linear-quadratic control problem for the stochastic heat equation, ESAIM: Control, Optimisation and Calculus of Variations, 2021, 27: 54

[14] Prohl, A. and Wang, Y., Strong error estimates for a space-time discretization of the linear-quadratic control problem with the stochastic heat equation with linear noise, IMA Journal of Numerical Analysis, 2022, 42: 3386–3429

[15] Prohl, A. and Wang, Y., Convergence with rates for a Riccati-based discretization of SLQ problems with SPDEs, IMA Journal of Numerical Analysis, 2024, 44: 3393–3434

[16] Prohl, A. and Wang, Y., Numerical methods for optimal control problems with SPDEs, arXiv: 2411.11239, 2024.

[17] Sun, Y., Li, H. and Zhang, H., Decentralized LQ control and stabilization for Itô systems, Automatica, 2024, 167: 111739

[18] Wang, Y., Stochastic linear quadratic optimal control problem: From discrete to continuous time, Chinese Journal of Contemporary Mathematics, 2018, 39: 387–406

[19] Wang, Y., Error analysis of a discretization for stochastic linear quadratic control problems governed by SDEs, IMA Journal of Mathematical Control and Information, 2021, 38: 1148–1173

[20] Wang, Y., Convergence rates of a discrete feedback control arising in mean-field linear quadratic optimal control problems(in Chinese), Science China Mathematics, 2023, 53: 1145–1162

[21] Wang, Y., Error analysis of the feedback controls arising in the stochastic linear quadratic control problems, Journal of Systems Science and Complexity, 2023, 36: 1540–1559

[22] Wang, Y., Strong error estimates for the space-time discretization of a stochastic linear quadratic control problem with control in the diffusion, SIAM Journal on Control and Optimization, 2025, 63: 3328–3355

[23] Yong, J., Linear-quadratic optimal control problems for mean-field stochastic differential equations, SIAM Journal on Control and Optimization, 2013, 51: 2809–2838

[24] Yong, J. and Zhou, X. Y., Stochastic controls: Hamiltonian systems and HJB equations, Springer, New York, 1999.