Probability, Uncertainty and Quantitative Risk
Vol. 10, No. 4, 2025, 569–588

https://www.aimsciences.org/PUQR
DOI:10.3934/puqr.2025025

American barrier option pricing with floating interest rate based on uncertain fractional differential equations

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Abstract Pricing barrier options pose a significant challenge in financial derivative valuation because they are activated only when the underlying asset reaches a predetermined barrier. The first-hitting time model was employed to characterize the activation process. In addition, the pricing of American barrier options with a floating interest rate is dynamically represented by an uncertain fractional differential equation. The study derives price formulas for various American barrier options, including up-and-in call, down-and-in put, up-and-output, and down-and-out call options. The proposed model enhances the accuracy of capturing the long-tail distribution and tail risk of financial markets, thereby addressing their complexity and nonlinearity. Furthermore, the predictor-corrector method is utilized to compute the numerical prices for the barrier options with floating interest rates, supplemented by illustrative numerical examples.

Keywords Uncertainty theory, Uncertain fractional differential equation, First-hitting time model, Floating interest rate, American barrier option

2020 Mathematics Subject Classification 91G20, 91G30

1. Introduction

Barrier options are derivative contracts with an additional condition that must be satisfied for the option to be exercised or specific payout features to be triggered. The barrier is typically defined in relation to the underlying asset price, often denoting a predetermined price level that the asset must reach or surpass to activate the option. Barrier options play a critical role in financial derivatives, enabling complex portfolio strategies and serving as adaptable risk management tools. Consequently, accurate pricing of barrier options is essential. The pricing of these options is a complex research area, and selecting an appropriate model is crucial for reliable price estimation. Furthermore, parameter estimation and intricate numerical computations within the model pose considerable challenges.

Received 30 March 2024; Accepted 8 September 2025; Early access 30 October 2025
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Barrier option pricing employs several common methods. The Black–Scholes model [1] and some extension methods [5–6,17] are straightforward and suitable for pricing European barrier options. However, they are less effective for complex barrier conditions, such as those in the American barrier options. Monte Carlo simulation-based methods offer high flexibility, accommodating all types of barrier options [28], but require numerous simulations to achieve accurate results, thereby incurring high computational costs. Tree-based pricing methods can approximate barrier option prices and provide dynamic visualization over time [27]. However, they struggle to effectively address complex barrier conditions and high-dimensional problems. Although many researchers use stochastic differential equations in probability theory to model price changes for barrier option pricing, their methods often fail to capture the nuances of real financial markets.

Barrier option pricing within the framework of uncertainty theory [13] is an active research area. Uncertain differential equations (UDEs) effectively account for uncertainty and risk factors in financial markets. Incorporating uncertainty into pricing models enables a more precise assessment of market volatility and risk premiums. Pricing formulas for European [13], American [2], and American barrier options [4] are derived from Liu's stock model [13]. The uncertain mean-reverting stock model [15] improves long-term price predictions, facilitating pricing formulas for lookback options [20], barrier options [19], and other derivatives. Sun and Su [18] expanded this model by incorporating floating interest rates. When UDEs lack analytic solutions, numerical methods such as Euler [23], Runge–Kutta [21], and Adams [22] are employed. The UDE framework, which is grounded in rigorous mathematics and statistics, supports accurate option pricing. This method yields precise option prices and risk indicators, enhancing investment decisions and risk management, whether through analytical or numerical solutions.

In 2015, Zhu [25] introduced the application of uncertain fractional differential equations (UFDEs) to finance, demonstrating the existence and uniqueness of their solution. Jin et al. [7–8] explored the extreme values and time integrals of UFDE solutions and applied them to the price of American options and zero-coupon bonds, respectively. Building on this, Jin et al. [9–10] investigated the first-hitting time (FHT) of UDE solutions and applied it to price Asian barrier options. Jin and Yang [11] utilized the α-path to establish the monotonicity theorem of UFDEs, deriving pricing formulas for European and American options. However, when UFDEs lack analytic solutions, the predictor-corrector method is applied to obtain numerical solutions [7–11] because of its high accuracy.

To enhance the accurate characterization of nonstationary financial data, explain random fluctuations, capture long-tail distributions, and solve the inherent tail risk in financial markets, an uncertain fractional differential model with floating interest rates is introduced. The model is designed to capture the complex and nonlinear characteristics of financial markets. This study specifically extends the UFDE and FHT models for four American barrier options in detail for the first time. The prediction-corrector method was used to solve UFDE. Compared with previous studies using numerical methods such as the Euler, Runge–Kutta, and Adams methods, the proposed method achieves higher convergence and accuracy in solving fractional differential equations, effectively enhancing the accuracy of the calculation of American barrier option pricing and supporting improved investment decision-making and risk assessment. The remainder of this study is organized as follows: Section 2 provides the essential definitions and theorems of the UFDE and the uncertain fractional FHT model. Section 3 derives pricing formulas for American barrier options using the inverse uncertainty distribution (IUD). Section 4 provides numerical examples of the American barrier option price calculation. Section 5 concludes the study.

2. Preliminary

This section introduces some relevant knowledge and theorems of UFDE in uncertainty theory, which will use in this study.

2.1 Uncertain fractional differential equation

Definition 2.1 *(Liu [13]) An uncertain process C_t is called a canonical Liu process if*
(i) $C_0 = 0$ *, and almost all sample paths are Lipschitz continuous,*
(ii) C_t *has stationary and independent increments,*
(iii) *every increment $C_{s+t} - C_s$ is a normal uncertain variable with an uncertainty distribution*

$$\Phi_t(x) = \left(1 + e^{\left(\frac{-\pi x}{\sqrt{3}t}\right)}\right)^{-1}.$$

The Wiener and Liu processes can be distinguished by the following two aspects. First, nearly all sample paths of the Wiener process are continuous but nondifferentiable, whereas those of the Liu process are Lipschitz continuous, exhibiting greater smoothness. Second, the increments of the Wiener process follow a Gaussian distribution, whereas those of the Liu process follow an uncertainty distribution.

Zhu [25] introduced two types of UFDEs based on uncertainty theory. In this study, we only focus on Caputo-type UFDE.

Definition 2.2 *(Zhu [25]) Let C_t be a canonical process. Suppose that f and $g\colon [0, +\infty] \times R \to R$ are two functions. Then*

$$\begin{cases} {}^{c}D^{p}Y_t = f(t, Y_t) + g(t, Y_t)\dfrac{\mathrm{d}C_t}{\mathrm{d}t}, \\ Y_t^{(k)}|_{t=0} = y_k, \quad k = 0, 1, \ldots, m-1. \end{cases} \tag{1}$$

For the Caputo type, it is called a UFDE. The solution of equation (1) is

$$Y_t = \sum_{k=0}^{m-1} \frac{y_k t^k}{\Gamma(k+1)} + \frac{1}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} f(\tau, Y_\tau)\mathrm{d}\tau + \frac{1}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} g(\tau, Y_\tau)\mathrm{d}C_\tau, \tag{2}$$

where $m - 1 < p \leqslant m$, m is a positive integer, and $\Gamma(p) = \int_0^{+\infty} t^{p-1}e^{-t}\mathrm{d}t$ is the gamma function.

The existence and uniqueness of the solution to the UFDE (1) is shown in [26].

Furthermore, let $f(t, Y_t) = (m_1 - a_1 Y_t)$, $g(t, Y_t) = \sigma_1$, and $k = 0$, equation (1) can be transformed into

$$\begin{cases} {}^{c}D^{p}Y_t = (m_1 - a_1 Y_t) + \sigma_1 \dfrac{\mathrm{d}C_t}{\mathrm{d}t}, \\ Y_t|_{t=0} = y_0. \end{cases} \tag{3}$$

Applying Mittag–Leffler function $E_{p,q}(y) = \sum_{k=0}^{\infty} \dfrac{y^k}{\Gamma(pk+q)}$ to the solution of (3), Lu and Zhu [14] deduced

$$\begin{aligned} Y_t =& y_0 E_{p,1}(-a_1 t^p) + \int_0^t m_1 (t-\tau)^{p-1} E_{p,p}(-a_1(t-\tau)^p)\mathrm{d}\tau \\ &+ \int_0^t \sigma_1 (t-s)^{p-1} E_{p,p}(-a_1(t-\tau)^p)\mathrm{d}C_\tau. \end{aligned} \tag{4}$$

For any p, $q > 0$ and complex number y, $E_{p,q}(y)$ is a convergent series [12,16].

Let $f(t,Y_t)=a_2Y_t$ and $g(t,Y_t)=\sigma_2$, equation (1) can be transformed into

$$\begin{cases} {}^cD^pY_t=a_2Y_t+\sigma_2\dfrac{\mathrm{d}C_t}{\mathrm{d}t}, \\ Y_t|_{t=0}=y_0, \end{cases} \tag{5}$$

the solution is

$$Y_t=y_0E_{p,1}(a_2t^p)+\int_0^t\sigma_2(t-\tau)^{p-1}E_{p,p}(a_2(t-\tau)^p)\mathrm{d}C_\tau. \tag{6}$$

Definition 2.3 *(Liu [13]) Assume that Z is a Boolean system containing elements $z_1,z_2,\dots,z_n$. A Boolean function h is called a structure function of Z if*

$$Z=1 \text{ if and only if } h(z_1,z_2,\dots,z_n)=1.$$

The reliability index is an uncertain measure of how well the system works, i.e.,

$$Rel=\mathcal{M}\{h(z_1,z_2,\dots,z_n)=1\},$$

where the Boolean system is composed of a series of elements with only two possible states. The Boolean function is defined on a Boolean system that outputs a Boolean value based on the input Boolean values. The uncertain measure $\mathcal{M}$ is a function defined on the σ-algebra $\mathcal{L}$, satisfying the following conditions: (1)$\mathcal{M}(\Gamma)=1$; (2)$\mathcal{M}(\emptyset)=0$; (3)$\forall A,B\in\mathcal{L},\mathcal{M}(A)\leqslant\mathcal{M}(B)$; (4)$\forall A_1,A_2,\dots\in\mathcal{L},\ A_1\cap A_2=\emptyset,\ \mathcal{M}(\cup A_i)=\sum\mathcal{M}(A_i)$.

2.2 The α-path of uncertain fractional differential equation

Definition 2.4 *(Lu and Zhu [14]) Consider UFDE*

$${}^cD^pY_t=f(t,Y_t)+g(t,Y_t)\frac{\mathrm{d}C_t}{\mathrm{d}t}, \tag{7}$$

The initial value is y_0. Its α-path satisfies the fractional differential equation

$${}^cD^pY_t^\alpha=f(t,Y_t^\alpha)+|g(t,Y_t^\alpha)|\Phi^{-1}(\alpha),\quad Y_0^\alpha=y_0, \tag{8}$$

where $0<\alpha<1$, $\Phi^{-1}(\alpha)=\dfrac{\sqrt{3}}{\pi}\ln\dfrac{\alpha}{1-\alpha}$.

Equation (3) has an α-path

$$X_t^\alpha=y_0E_{p,1}(-a_1t^p)+(m_1+\sigma_1\Phi^{-1}(\alpha))t^pE_{p,p+1}(-a_1t^p), \tag{9}$$

and equation (5) has an α-path

$$Y_t^\alpha=y_0E_{p,1}(a_2t^p)+\sigma_2\Phi^{-1}(\alpha)t^pE_{p,p+1}(a_2t^p). \tag{10}$$

Theorem 2.1 *(Lu and Zhu [14]) Assume that Y_t and Y_t^α are the solution and α-path of the UFDE (7), respectively, then*

$$\mathcal{M}\{Y_t\leqslant Y_t^\alpha,\ \ \forall t\}=\alpha,$$

$$\mathcal{M}\{Y_t>Y_t^\alpha,\ \ \forall t\}=1-\alpha.$$

Theorem 2.2 *(Lu and Zhu [14]) Let Y_t and Y_t^α be the solution and α-path of the UFDE(7), respectively, the solution Y_t has an IUD as follows:*

$$\Psi_t^{-1}(\alpha)=Y_t^\alpha.$$

Theorem 2.3 *(Yao [24]) For any time $\tau > 0$, and strictly increasing (decreasing) function $F(y)$, the supremum*

$$\sup_{0\leqslant t\leqslant\tau} F(Y_t)$$

has an IUD

$$\Psi_\tau^{-1}(\alpha) = \sup_{0\leqslant t\leqslant\tau} F(Y_t^\alpha) \qquad \left(\Psi_\tau^{-1}(\alpha) = \sup_{0\leqslant t\leqslant\tau} F(Y_t^{1-\alpha})\right);$$

and the infimum

$$\inf_{0\leqslant t\leqslant\tau} F(Y_t)$$

has an IUD

$$\Psi_\tau^{-1}(\alpha) = \inf_{0\leqslant t\leqslant\tau} F(Y_t^\alpha) \qquad \left(\Psi_\tau^{-1}(\alpha) = \inf_{0\leqslant t\leqslant\tau} F(Y_t^{1-\alpha})\right);$$

the time integral

$$\int_0^\tau F(Y_t)\mathrm{d}t$$

has an IUD

$$\Psi_\tau^{-1}(\alpha) = \int_0^\tau F(Y_t^\alpha)\mathrm{d}t \qquad \left(\Psi_\tau^{-1}(\alpha) = \int_0^\tau F(Y_t^{1-\alpha})\mathrm{d}t\right).$$

2.3 First-hitting time of solution

For any given level l, the time at which an uncertain process Y_t reaches l depends on its path. Thus, the FHT τ_l is an uncertain variable and is defined as follows:

$$\tau_l = \inf\{t \geqslant 0 | Y_t = l\}.$$

Let $F(Y_t)$ be a monotone function of the UFDE solution Y_t. The FHT τ_l at which Y_t hits the given level l has an uncertainty distribution $U(\tau)$ as follows [9]. When $F(Y_t)$ is strictly increased,

$$U(\tau) = \begin{cases} 1 - \inf\left\{\alpha \in (0,1) | \sup\limits_{0\leqslant t\leqslant\tau} F(Y_t^\alpha) \geqslant l\right\}, & \text{if } l > F(Y_0), \\ \sup\left\{\alpha \in (0,1) | \inf\limits_{0\leqslant t\leqslant\tau} F(Y_t^\alpha) \leqslant l\right\}, & \text{if } l < F(Y_0). \end{cases} \tag{11}$$

When $F(Y_t)$ decreases strictly,

$$U(\tau) = \begin{cases} \sup\left\{\alpha \in (0,1) | \sup\limits_{0\leqslant t\leqslant\tau} F(Y_t^\alpha) \geqslant l\right\}, & \text{if } l > F(Y_0), \\ 1 - \inf\left\{\alpha \in (0,1) | \inf\limits_{0\leqslant t\leqslant\tau} F(Y_t^\alpha) \leqslant l\right\}, & \text{if } l < F(Y_0). \end{cases} \tag{12}$$

2.4 Uncertain fractional first-hitting time model

Assume that R_t is the interest rate, Y_t is the stock price, B is the barrier level, and τ is the first time that Y_t touches off the barrier level. The R_t and Y_t are governed by the uncertain fractional FHT model

$$\begin{cases} {}^cD^{p_1}R_t = (m_1 - a_1R_t) + \sigma_1\dfrac{\mathrm{d}C_{1t}}{\mathrm{d}t}, \quad R_t|_{t=0} = r_0, \\ {}^cD^{p_2}Y_t = a_2Y_t + \sigma_2\dfrac{\mathrm{d}C_{2t}}{\mathrm{d}t}, \quad Y_t|_{t=0} = y_0, \\ \tau = \inf\{t \geqslant 0 | Y_t = B\}, \end{cases} \tag{13}$$

where $m_1,\ a_1,\ a_2,\ \sigma_1$, and σ_2 are positive real number, $0 < p_1, p_2 \leqslant 1$, C_{1t} and C_{2t} are independent Liu processes. According to equations (9) and (10), the α-path of the FHT model is given by

$$\begin{cases} R_t^\alpha = r_0E_{p_1,1}(-a_1t^{p_1}) + (m_1 + \sigma_1\Phi^{-1}(\alpha))t^{p_1}E_{p_1,p_1+1}(-a_1t^{p_1}), \\ Y_t^\alpha = y_0E_{p_2,1}(a_2t^{p_2}) + \sigma_2\Phi^{-1}(\alpha)t^{p_2}E_{p_2,p_2+1}(a_2t^{p_2}). \end{cases} \tag{14}$$

3. American barrier option pricing

We assume that the stock price and interest rate are determined by model (13). A barrier option is a financial contract that activates or expires when the underlying asset price reaches the barrier level. The barrier options are classified as knock-in or knock-out.

Define the indicator function as below

$$I_B(y) = \begin{cases} 1, & \text{if } y \geqslant B, \\ 0, & \text{if } y < B, \end{cases} \tag{15}$$

where B is the barrier level.

3.1 American knock-in barrier option

The knock-in barrier option includes the up-and-in and down-and-in barrier options. This section examines the option of the American up-and-in call barrier. Let X be the strike price, T and B be the expiration date and barrier level, respectively, and $Y_0 < B$. The stock price and interest rate are determined by the model (13), which captures long-term dynamics. Let f_{ui}^c denote the option price, then

$$f_{ui}^c = E\left[\sup_{0\leqslant t\leqslant T} e^{-\int_0^t R_s\mathrm{d}s} I_B\left(\sup_{0\leqslant \mu\leqslant t} Y_\mu\right) \cdot (Y_t - X)^+\right]. \tag{16}$$

Theorem 3.1 *Under the FHT model (13), and at the barrier level B , the reliability index of the American up-and-in call barrier option is*

$$Rel = \mathcal{M}\left\{\sup_{0\leqslant t\leqslant T} Y_t \geqslant B\right\} = 1 - \beta, \tag{17}$$

where

$$\beta = \inf\left\{\alpha | \sup_{0\leqslant t\leqslant T} Y_t^\alpha \geqslant B\right\}$$

and

$$Y_t^\alpha = y_0E_{p_2,1}(a_2t^{p_2}) + \sigma_2\Phi^{-1}(\alpha)t^{p_2}E_{p_2,p_2+1}(a_2t^{p_2}).$$

Proof The American up-and-in call barrier option is valid which relies on

$$I_B\left(\sup_{0\leqslant \mu\leqslant t} Y_\mu\right) = 1$$

holds if and only if for arbitrary given $t \in [0,T]$, which implies

$$I_B\left(\sup_{0\leqslant t\leqslant T} Y_t\right)=1.$$

The condition is equivalent to the uncertain events $\mathcal{M}\{\sup_{0\leqslant t\leqslant T} Y_t \geqslant B\}$, which can also be described as $\{\tau < T\}$. Hence

$$Rel=\mathcal{M}\left\{\sup_{0\leqslant t\leqslant T} Y_t \geqslant B\right\}=\mathcal{M}\{\tau<T\}.$$

Considering the uncertainty distribution of the FHT that satisfies equation (11),

$$\mathcal{M}\{\tau<T\}=U(T)=1-\inf\left\{\alpha\in(0,1)|\sup_{0\leqslant t\leqslant T} Y_t^\alpha \geqslant B\right\},$$

when $Y_0 < B$. Let $\beta=\inf\left\{\alpha\in(0,1)|\sup_{0\leqslant t\leqslant T} Y_t^\alpha \geqslant B\right\}$, therefore

$$\mathcal{M}\{\tau<T\}=U(T)=1-\beta$$

where Y_t^α is the α-path of the model (13), which can be seen in equation (14),

$$Y_t^\alpha=y_0E_{p_2,1}(a_2t^{p_2})+\sigma_2\Phi^{-1}(\alpha)t^{p_2}E_{p_2,p_2+1}(a_2t^{p_2}).$$

Thus, Theorem 3.1 is proved. □

Theorem 3.2 *The price of the American up-and-in call barrier option with a strike price X and an expiration date T for model (13) is*

$$f_{ui}^c=\int_\beta^1 \sup_{0\leqslant t\leqslant T} e^{-\int_0^t R_s^{1-\alpha}\mathrm{d}s}(Y_t^\alpha-X)^+\mathrm{d}\alpha$$

where R_t^α and Y_t^α are the α-path of the UFDE (13),

$$R_t^\alpha=r_0E_{p_1,1}(-a_1t^{p_1})+(m_1+\sigma_1\Phi^{-1}(\alpha))t^{p_1}E_{p_1,p_1+1}(-a_1t^{p_1}),$$

$$Y_t^\alpha=y_0E_{p_2,1}(a_2t^{p_2})+\sigma_2\Phi^{-1}(\alpha)t^{p_2}E_{p_2,p_2+1}(a_2t^{p_2}),$$

and $\beta=inf\{\alpha|\sup_{0\leqslant t\leqslant T} Y_t^\alpha \geqslant B\}$.

Proof At first, to certify the variable

$$Z_t=\sup_{0\leqslant t\leqslant T} e^{-\int_0^t R_s\mathrm{d}s}I_B\left(\sup_{0\leqslant\mu\leqslant t} Y_\mu\right)\cdot(Y_t-X)^+$$

has an α-path

$$Z_t^\alpha=\sup_{0\leqslant t\leqslant T} e^{-\int_0^t R_s^{1-\alpha}\mathrm{d}s}I_B\left(\sup_{0\leqslant\mu\leqslant t} Y_\mu^\alpha\right)\cdot(Y_t^\alpha-X)^+.$$

For a sample path $Z_t(\gamma)$, let

$$\begin{aligned}
\Xi_1^+&=\{\gamma|Z_t\leqslant Z_t^\alpha,\ \ t\in[0,T]\},\ \ \Xi_1^-=\{\gamma|Z_t>Z_t^\alpha,\ \ t\in[0,T]\},\\
\Xi_2^+&=\{\gamma|Y_t\leqslant Y_t^\alpha,\ \ t\in[0,T]\},\ \ \Xi_2^-=\{\gamma|Y_t>Y_t^\alpha,\ \ t\in[0,T]\},\\
\Xi_3^+&=\left\{\gamma|R_t\geqslant R_t^{1-\alpha},\ \ t\in[0,T]\right\},\ \ \Xi_3^-=\left\{\gamma|R_t<R_t^{1-\alpha},\ \ t\in[0,T]\right\}.
\end{aligned}$$

Since Z_t is monotonic, we obtain

$$\begin{aligned}\Xi_1^+ = &\left\{\sup_{0\leqslant t\leqslant T} e^{-\int_0^t R_s \mathrm{d}s} I_B\left(\sup_{0\leqslant \mu\leqslant t} Y_\mu\right)\cdot (Y_t - X)^+\right.\\ &\left.\leqslant \sup_{0\leqslant t\leqslant T} e^{-\int_0^t R_s^{1-\alpha} \mathrm{d}s} I_B\left(\sup_{0\leqslant \mu\leqslant t} Y_\mu^\alpha\right)\cdot (Y_t^\alpha - X)^+\right\}\\ &\supset \left\{Y_t \leqslant Y_t^\alpha, R_t \geqslant R_t^{1-\alpha},\ \ t\in[0,T]\right\}.\end{aligned}$$

Similarly,

$$\begin{aligned}\Xi_1^- = &\left\{\sup_{0\leqslant t\leqslant T} e^{-\int_0^t R_s \mathrm{d}s} I_B\left(\sup_{0\leqslant \mu\leqslant t} Y_\mu\right)\cdot (Y_t - X)^+\right.\\ &\left.> \sup_{0\leqslant t\leqslant T} e^{-\int_0^t R_s^{1-\alpha} \mathrm{d}s} I_B\left(\sup_{0\leqslant \mu\leqslant t} Y_\mu^\alpha\right)\cdot (Y_t^\alpha - X)^+\right\}\\ &\supset \left\{Y_t > Y_t^\alpha, R_t < R_t^{1-\alpha},\ \ t\in[0,T]\right\}.\end{aligned}$$

According to the independence of Y_t and R_t, we obtain

$$\begin{aligned}\mathcal{M}\{\Xi_1^+\} &\geqslant \mathcal{M}\left\{Y_t \leqslant Y_t^\alpha, R_t \geqslant R_t^{1-\alpha},\ \ t\in[0,T]\right\}\\ &= \mathcal{M}\{\Xi_2^+\}\wedge \mathcal{M}\{\Xi_3^+\}\\ &= \alpha.\end{aligned}$$

Likewise,

$$\begin{aligned}\mathcal{M}\{\Xi_1^-\} &\geqslant \mathcal{M}\left\{Y_t > Y_t^\alpha, R_t < R_t^{1-\alpha},\quad t\in[0,T]\right\}\\ &= \mathcal{M}\{\Xi_2^-\}\wedge \mathcal{M}\{\Xi_3^-\}\\ &= 1-\alpha.\end{aligned}$$

Secondly, we have

$$\mathcal{M}\{\Xi_1^+\} + \mathcal{M}\{\Xi_1^-\} = 1$$

by the duality axiom of uncertain measure. Hence

$$\mathcal{M}\{\Xi_1^+\} = \alpha,\ \ \mathcal{M}\{\Xi_1^-\} = 1-\alpha.$$

This proves that Y_t^α is an α-path of the uncertain variable Y_t.

Moreover, according to Theorem 3.1, $\sup_{0\leqslant t\leqslant T} Y_t^\beta = B$. Since $\sup_{0\leqslant t\leqslant T} Y_t^\alpha$ is an increasing function of α, $\sup_{0\leqslant t\leqslant T} Y_t^\alpha \geqslant \sup_{0\leqslant t\leqslant T} Y_t^\beta = B$ holds when $\alpha\geqslant\beta$ is satisfied.

Therefore, the pricing formula is

$$\begin{aligned}f_{ui}^c &= \int_0^1 \sup_{0\leqslant t\leqslant T} e^{-\int_0^t R_s^{1-\alpha}\mathrm{d}s} I_B\left(\sup_{0\leqslant \mu\leqslant t} Y_\mu^\alpha\right)(Y_t^\alpha - X)^+ \mathrm{d}\alpha\\ &= \int_\beta^1 \sup_{0\leqslant t\leqslant T} e^{-\int_0^t R_s^{1-\alpha}\mathrm{d}s}(Y_t^\alpha - X)^+\mathrm{d}\alpha.\end{aligned}$$

□

For the American down-and-in put barrier option, the stock price and interest rate are determined using the FHT model (13). Let X be the strike price, T and B be the expiration date and barrier level, respectively, and $Y_0 > B$. Let f_{di}^p denote the option price, then

$$f_{di}^p = E\left[\sup_{0\leqslant t\leqslant T} e^{-\int_0^t R_s \mathrm{d}s}\left(1 - I_B\left(\inf_{0\leqslant \mu\leqslant t} Y_\mu\right)\right)\cdot (X - Y_t)^+\right]. \tag{18}$$

Theorem 3.3 *Under the FHT model (13), and at the barrier level B, the reliability index of the American down-and-in put barrier option is given by the following:*

$$Rel = \mathcal{M}\left\{\inf_{0\leqslant t\leqslant T} Y_t \leqslant B\right\} = \beta, \tag{19}$$

where

$$\beta = \sup\left\{\alpha | \inf_{0\leqslant t\leqslant T} Y_t^\alpha \leqslant B\right\}$$

and

$$Y_t^\alpha = y_0 E_{p_2,1}(a_2 t^{p_2}) + \sigma_2 \Phi^{-1}(\alpha) t^{p_2} E_{p_2,p_2+1}(a_2 t^{p_2}).$$

Proof The American down-and-in put barrier option is valid which relies on

$$I_B\left(\inf_{0\leqslant \mu\leqslant t} Y_\mu\right) = 0$$

holds if and only if for arbitrary given $t \in [0, T]$, which implies

$$I_B\left(\inf_{0\leqslant t\leqslant T} Y_t\right) = 0.$$

This is equivalent to the uncertain events $\mathcal{M}\{\inf_{0\leqslant t\leqslant T} Y_t \leqslant B\}$, which can also be described as $\{\tau < T\}$. Hence,

$$Rel = \mathcal{M}\left\{\inf_{0\leqslant t\leqslant T} Y_t \leqslant B\right\} = \mathcal{M}\{\tau < T\}.$$

Considering the uncertainty distribution of the FHT that satisfies equation (11),

$$\mathcal{M}\{\tau < T\} = U(T) = \sup\left\{\alpha \in (0,1) | \inf_{0\leqslant t\leqslant T} Y_t^\alpha \leqslant B\right\}$$

when $Y_0 > B$. Let $\beta = \sup\{\alpha \in (0,1) | \inf_{0\leqslant t\leqslant T} Y_t^\alpha \leqslant B\}$, therefore

$$\mathcal{M}\{\tau < T\} = U(T) = \beta,$$

where Y_t^α is the α-path of the FHT model (13), which is expressed in equation (14),

$$Y_t^\alpha = y_0 E_{p_2,1}(a_2 t^{p_2}) + \sigma_2 \Phi^{-1}(\alpha) t^{p_2} E_{p_2,p_2+1}(a_2 t^{p_2}).$$

Thus, Theorem 3.3 is proved. □

Theorem 3.4 *The price of the American down-and-in put barrier option with a strike price X and an expiration date T for the model (13) is*

$$f_{di}^p = \int_0^\beta \sup_{0\leqslant t\leqslant T} e^{-\int_0^t R_s^\alpha \mathrm{d}s} (X - Y_t^\alpha)^+ \mathrm{d}\alpha$$

where R_t^α and Y_t^α are the α-path of the UFDE (13),

$$R_t^\alpha = r_0 E_{p_1,1}(-a_1 t^{p_1}) + (m_1 + \sigma_1 \Phi^{-1}(\alpha)) t^{p_1} E_{p_1,p_1+1}(-a_1 t^{p_1}),$$

$$Y_t^\alpha = y_0 E_{p_2,1}(a_2 t^{p_2}) + \sigma_2 \Phi^{-1}(\alpha) t^{p_2} E_{p_2,p_2+1}(a_2 t^{p_2}),$$

and $\beta = \sup\{\alpha | \inf_{0\leqslant t\leqslant T} Y_t^\alpha \leqslant B\}$.

Proof At first, to certify that the following variable

$$Z_t = \sup_{0\leqslant t\leqslant T} e^{-\int_0^t R_s \mathrm{d}s}\left(1 - I_B\left(\inf_{0\leqslant \mu\leqslant t} Y_\mu\right)\right) \cdot (X - Y_t)^+$$

has an α-path

$$Z_t^\alpha = \sup_{0\leqslant t\leqslant T} e^{-\int_0^t R_s^{1-\alpha}\mathrm{d}s}\left(1 - I_B\left(\inf_{0\leqslant\mu\leqslant t} Y_\mu^{1-\alpha}\right)\right)\cdot(X - Y_t^{1-\alpha})^+.$$

Let

$$\begin{aligned}
\Xi_1^+ &= \{\gamma|Z_t \leqslant Z_t^\alpha,\ \ t\in[0,T]\},\ \ \Xi_1^- = \{\gamma|Z_t > Z_t^\alpha,\ \ t\in[0,T]\},\\
\Xi_2^+ &= \left\{\gamma|Y_t \geqslant Y_t^{1-\alpha},\ \ t\in[0,T]\right\},\ \ \Xi_2^- = \left\{\gamma|Y_t < Y_t^{1-\alpha},\ \ t\in[0,T]\right\},\\
\Xi_3^+ &= \left\{\gamma|R_t \geqslant R_t^{1-\alpha},\ \ t\in[0,T]\right\},\ \ \Xi_3^- = \left\{\gamma|R_t < R_t^{1-\alpha},\ \ t\in[0,T]\right\}.
\end{aligned}$$

Since Z_t is monotonic, we derive

$$\begin{aligned}
\Xi_1^+ = &\left\{\sup_{0\leqslant t\leqslant T} e^{-\int_0^t R_s\mathrm{d}s}\left(1 - I_B\left(\inf_{0\leqslant\mu\leqslant t} Y_\mu\right)\right)\cdot(X - Y_t)^+\right.\\
&\left.\leqslant \sup_{0\leqslant t\leqslant T} e^{-\int_0^t R_s^{1-\alpha}\mathrm{d}s}\left(1 - I_B\left(\inf_{0\leqslant\mu\leqslant t} Y_\mu^{1-\alpha}\right)\right)\cdot(X - Y_t^{1-\alpha})^+\right\}\\
&\supset \left\{Y_t \geqslant Y_t^{1-\alpha}, R_t \geqslant R_t^{1-\alpha},\ \ t\in[0,T]\right\}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\Xi_1^- = &\left\{\sup_{0\leqslant t\leqslant T} e^{-\int_0^t R_s\mathrm{d}s}\left(1 - I_B\left(\inf_{0\leqslant\mu\leqslant t} Y_\mu\right)\right)\cdot(X - Y_t)^+\right.\\
&\left.> \sup_{0\leqslant t\leqslant T} e^{-\int_0^t R_s^{1-\alpha}\mathrm{d}s}\left(1 - I_B\left(\inf_{0\leqslant\mu\leqslant t} Y_\mu^{1-\alpha}\right)\right)\cdot(X - Y_t^{1-\alpha})^+\right\}\\
&\supset \left\{Y_t < Y_t^{1-\alpha}, R_t < R_t^{1-\alpha},\ \ t\in[0,T]\right\}.
\end{aligned}$$

According to the independence of Y_t and R_t, we obtain

$$\begin{aligned}
\mathcal{M}\{\Xi_1^+\} &\geqslant \mathcal{M}\left\{Y_t \geqslant Y_t^{1-\alpha},\quad R_t \geqslant R_t^{1-\alpha},\quad t\in[0,T]\right\}\\
&= \mathcal{M}\{\Xi_2^+\}\wedge\mathcal{M}\{\Xi_3^+\}\\
&= \alpha.
\end{aligned}$$

Likewise,

$$\begin{aligned}
\mathcal{M}\{\Xi_1^-\} &\geqslant \mathcal{M}\left\{Y_t < Y_t^{1-\alpha},\quad R_t < R_t^{1-\alpha},\quad t\in[0,T]\right\}\\
&= \mathcal{M}\{\Xi_2^-\}\wedge\mathcal{M}\{\Xi_3^-\}\\
&= 1-\alpha.
\end{aligned}$$

In addition, $\mathcal{M}\{\Xi_1^+\} + \mathcal{M}\{\Xi_1^-\} = 1$ by the duality axiom of uncertain measure. Hence,

$$\mathcal{M}\{\Xi_1^+\} = \alpha,\ \ \mathcal{M}\{\Xi_1^-\} = 1-\alpha.$$

This proves that Y_t^α is an α-path of the uncertain variable Y_t.

Moreover, according to Theorem 3.3, we have $\inf_{0\leqslant t\leqslant T} Y_t^\beta = B$. Since $\inf_{0\leqslant t\leqslant T} Y_t^\alpha$ is an increasing function of α, $\inf_{0\leqslant t\leqslant T} Y_t^\alpha \leqslant \inf_{0\leqslant t\leqslant T} Y_t^\beta = B$ holds when $\alpha\leqslant\beta$ is satisfied.

Therefore, the formula is

$$\begin{aligned}
f_{di}^p &= \int_0^1 \sup_{0\leqslant t\leqslant T} e^{-\int_0^t R_s^{1-\alpha}\mathrm{d}s}\left(1 - I_B\left(\inf_{0\leqslant\mu\leqslant t} Y_\mu^{1-\alpha}\right)\right)(X - Y_t^{1-\alpha})^+\mathrm{d}\alpha\\
&= \int_0^\beta \sup_{0\leqslant t\leqslant T} e^{-\int_0^t R_s^{\alpha}\mathrm{d}s}(X - Y_t^\alpha)^+\mathrm{d}\alpha.
\end{aligned}$$

□

3.2 American knock-out barrier option

The American up-and-out put barrier option was analyzed. Let $Y_0 < B$, the stock price and interest rate are determined by an uncertain fractional FHT model (13). Let f_{uo}^p denote the option price, then

$$f_{uo}^p = E\left[\sup_{0\leqslant t\leqslant T} e^{-\int_0^t R_s \mathrm{d}s}\left(1 - I_B\left(\sup_{0\leqslant \mu\leqslant t} Y_\mu\right)\right)\cdot (X - Y_t)^+\right]. \tag{20}$$

Theorem 3.5 *Under the FHT model (13), and at the barrier level B , the reliability index of the American up-and-out put barrier option is given by*

$$Rel = \mathcal{M}\{\sup_{0\leqslant t\leqslant T} Y_t < B\} = \beta, \tag{21}$$

where

$$\beta = \inf\left\{\alpha | \sup_{0\leqslant t\leqslant T} Y_t^\alpha \geqslant B\right\}$$

and

$$Y_t^\alpha = y_0 E_{p_2,1}(a_2 t^{p_2}) + \sigma_2 \Phi^{-1}(\alpha) t^{p_2} E_{p_2,p_2+1}(a_2 t^{p_2}).$$

Proof The American up-and-out put barrier option is valid if the condition

$$I_B\left(\sup_{0\leqslant \mu\leqslant t} Y_\mu\right) = 0$$

holds if and only if for an arbitrary given $t \in [0, T]$, which implies

$$I_B\left(\sup_{0\leqslant t\leqslant T} Y_t\right) = 0.$$

This is equivalent to the uncertain events $\mathcal{M}\{\sup_{0\leqslant t\leqslant T} Y_t < B\}$ which can also be described as $\{\tau \geqslant T\}$. Hence

$$Rel = \mathcal{M}\left\{\sup_{0\leqslant t\leqslant T} Y_t < B\right\} = \mathcal{M}\{\tau \geqslant T\}.$$

Considering the uncertainty distribution of the FHT that satisfies equation (11),

$$\mathcal{M}\{\tau \geqslant T\} = 1 - U(T) = \inf\left\{\alpha \in (0,1) | \sup_{0\leqslant t\leqslant T} Y_t^\alpha \geqslant B\right\}$$

when $Y_0 < B$. Let $\beta = \inf\left\{\alpha \in (0,1) | \sup_{0\leqslant t\leqslant T} Y_t^\alpha \geqslant B\right\}$, therefore

$$\mathcal{M}\{\tau \geqslant T\} = 1 - U(T) = \beta,$$

where Y_t^α is the α-path of the FHT model (13), which expressed in equation(14),

$$Y_t^\alpha = y_0 E_{p_2,1}(a_2 t^{p_2}) + \sigma_2 \Phi^{-1}(\alpha) t^{p_2} E_{p_2,p_2+1}(a_2 t^{p_2}).$$

Thus, Theorem 3.5 is proved. □

Theorem 3.6 *The price of the American up-and-out put barrier option with a strike price X and an expiration date T for model (13) is*

$$f_{uo}^p = \int_0^\beta \sup_{0\leqslant t\leqslant T} e^{-\int_0^t R_s^\alpha \mathrm{d}s}(X - Y_t^\alpha)^+ \mathrm{d}\alpha$$

where R_t^α and Y_t^α are the α-path of the UFDE (13),

$$R_t^\alpha = r_0 E_{p_1,1}(-a_1 t^{p_1}) + (m_1 + \sigma_1 \Phi^{-1}(\alpha)) t^{p_1} E_{p_1,p_1+1}(-a_1 t^{p_1}),$$

$$Y_t^\alpha = y_0 E_{p_2,1}(a_2 t^{p_2}) + \sigma_2 \Phi^{-1}(\alpha) t^{p_2} E_{p_2,p_2+1}(a_2 t^{p_2}),$$

and $\beta = \inf\{\alpha| \sup_{0\leqslant t\leqslant T} Y_t^\alpha \geqslant B\}$.

Proof Similar to Theorem 3.4, the uncertain variable is given by the following

$$Y_t = \sup_{0\leqslant t\leqslant T} e^{-\int_0^t R_s \mathrm{d}s} \left(1 - I_B\left(\sup_{0\leqslant \mu\leqslant t} Y_\mu\right)\right) \cdot (X - Y_t)^+$$

has an α-path

$$Y_t^\alpha = \sup_{0\leqslant t\leqslant T} e^{-\int_0^t R_s^{1-\alpha} \mathrm{d}s} \left(1 - I_B\left(\sup_{0\leqslant \mu\leqslant t} Y_\mu^{1-\alpha}\right)\right) \cdot (X - Y_t^{1-\alpha})^+.$$

According to Theorem 3.5, we obtain $\sup_{0\leqslant t\leqslant T} Y_t^\beta = B$. Since $\sup_{0\leqslant t\leqslant T} Y_t^\alpha$ is an increasing function of α, $\sup_{0\leqslant t\leqslant T} Y_t^\alpha < \sup_{0\leqslant t\leqslant T} Y_t^\beta = B$ holds when $\alpha < \beta$ is satisfied.

Therefore, the formula is

$$\begin{aligned} f_{uo}^p &= \int_0^1 \sup_{0\leqslant t\leqslant T} e^{-\int_0^t R_s^{1-\alpha} \mathrm{d}s} \left(1 - I_B\left(\sup_{0\leqslant \mu\leqslant t} Y_\mu^{1-\alpha}\right)\right) (X - Y_t^{1-\alpha})^+ \mathrm{d}\alpha \\ &= \int_0^\beta \sup_{0\leqslant t\leqslant T} e^{-\int_0^t R_s^\alpha \mathrm{d}s} (X - Y_t^\alpha)^+ \mathrm{d}\alpha. \end{aligned}$$

We analyzed the American down-and-out call barrier option under the UFDE (13). Let $Y_0 > B$ and f_{do}^c denote the option price, then

$$f_{do}^c = E\left[\sup_{0\leqslant t\leqslant T} e^{-\int_0^t R_s \mathrm{d}s} I_B\left(\inf_{0\leqslant \mu\leqslant t} Y_\mu\right) \cdot (Y_t - X)^+\right]. \tag{22}$$

□

Theorem 3.7 *Under the FHT model (13), and at barrier level B, the reliability index of American down-and-out call barrier option is*

$$Rel = \mathcal{M}\{\inf_{0\leqslant t\leqslant T} Y_t > B\} = 1 - \beta, \tag{23}$$

where

$$\beta = \sup\left\{\alpha| \inf_{0\leqslant t\leqslant T} Y_t^\alpha \leqslant B\right\}$$

and

$$Y_t^\alpha = y_0 E_{p_2,1}(a_2 t^{p_2}) + \sigma_2 \Phi^{-1}(\alpha) t^{p_2} E_{p_2,p_2+1}(a_2 t^{p_2}).$$

Proof The American down-and-out call barrier option is valid which relies on

$$I_B\left(\inf_{0\leqslant \mu\leqslant t} Y_\mu\right) = 1$$

holds if and only if for an arbitrary given $t \in [0, T]$, which implies

$$I_B\left(\inf_{0\leqslant t\leqslant T} Y_t\right) = 1.$$

This is equivalent to the uncertain events $\mathcal{M}\left\{\sup_{0\leqslant t\leqslant T} Y_t > B\right\}$, which can also be described as

$\{\tau \geqslant T\}$. Hence,

$$Rel = \mathcal{M}\left\{\inf_{0\leqslant t\leqslant T} Y_t > B\right\} = \mathcal{M}\{\tau \geqslant T\}.$$

Considering the uncertainty distribution of the FHT that satisfies equation (12),

$$\mathcal{M}\{\tau \geqslant T\} = 1 - U(T) = 1 - \sup\left\{\alpha \in (0,1) \middle| \inf_{0\leqslant t\leqslant T} Y_t^\alpha \leqslant B\right\}$$

when $Y_0 > B$. Let $\beta = \sup\{\alpha \in (0,1) | \inf_{0\leqslant t\leqslant T} Y_t^\alpha \leqslant B\}$, therefore

$$\mathcal{M}\{\tau \geqslant T\} = 1 - U(T) = 1 - \beta$$

where Y_t^α is the α-path of the FHT model (13), which can be seen in equation (14),

$$Y_t^\alpha = y_0 E_{p_2,1}(a_2 t^{p_2}) + \sigma_2 \Phi^{-1}(\alpha) t^{p_2} E_{p_2,p_2+1}(a_2 t^{p_2}).$$

Thus, Theorem 3.7 is proved. □

Theorem 3.8 *The price of American down-and-out call barrier option with a strike price X and an expiration date T for model (13) is*

$$f_{\mathrm{do}}^c = \int_\beta^1 \sup_{0\leqslant t\leqslant T} e^{-\int_0^t R_s^{1-\alpha}\mathrm{d}s}(Y_t^\alpha - X)^+ \mathrm{d}\alpha$$

where R_t^α and Y_t^α are the α-path of the UFDE (13),

$$R_t^\alpha = r_0 E_{p_1,1}(-a_1 t^{p_1}) + (m_1 + \sigma_1 \Phi^{-1}(\alpha)) t^{p_1} E_{p_1,p_1+1}(-a_1 t^{p_1}),$$

$$Y_t^\alpha = y_0 E_{p_2,1}(a_2 t^{p_2}) + \sigma_2 \Phi^{-1}(\alpha) t^{p_2} E_{p_2,p_2+1}(a_2 t^{p_2}),$$

and $\beta = \sup\{\alpha | \inf_{0\leqslant t\leqslant T} Y_t^\alpha \leqslant B\}$.

Proof At first, we can certify that the following variable

$$Z_t = \sup_{0\leqslant t\leqslant T} e^{-\int_0^t R_s \mathrm{d}s} I_B\left(\inf_{0\leqslant\mu\leqslant t} Y_\mu\right)\cdot (Y_t - X)^+$$

has an α-path

$$Z_t^\alpha = \sup_{0\leqslant t\leqslant T} e^{-\int_0^t R_s^{1-\alpha}\mathrm{d}s} I_B\left(\inf_{0\leqslant\mu\leqslant t} Y_\mu^\alpha\right)\cdot (Y_t^\alpha - X)^+,$$

whose proof is similar to Theorem 3.2.

Moreover, since $\inf_{0\leqslant t\leqslant T} Y_t^\alpha$ is an increasing function of α, we have $\inf_{0\leqslant t\leqslant T} Y_t^\beta = B$ according to Theorem 3.7. Hence, only when $\alpha > \beta$ is satisfied can $\inf_{0\leqslant t\leqslant T} Y_t^\alpha > \inf_{0\leqslant t\leqslant T} Y_t^\beta = B$ holds.

Therefore, the formula is

$$\begin{aligned} f_{\mathrm{do}}^c &= \int_0^1 \sup_{0\leqslant t\leqslant T} e^{-\int_0^t R_s^{1-\alpha}\mathrm{d}s} I_B\left(\inf_{0\leqslant\mu\leqslant t} Y_\mu^\alpha\right)(Y_t^\alpha - X)^+ \mathrm{d}\alpha \\ &= \int_\beta^1 \sup_{0\leqslant t\leqslant T} e^{-\int_0^t R_s^{1-\alpha}\mathrm{d}s}(Y_t^\alpha - X)^+ \mathrm{d}\alpha. \end{aligned}$$

□

In this section, the proposed method is used to calculate the prices of the American up-and-in put, down-and-in call, up-and-out call, and down-and-out put barrier options. For example, in the case of the down-and-out put barrier option, the IUD of

$$I_B\left(\inf_{0\leqslant t\leqslant T} Y_t\right)\cdot (X - Y_T)^+$$

cannot be derived. The other four types of barrier options will be investigated in future work.

4. Numerical simulation

The numerical method is essential because it is difficult to get the analytical solution of UFDE. This study needs to solve the fractional differential equation to obtain the IUD Y_t^α and R_t^α. The predictor-corrector scheme proposed by Diethelm et al. [3] is a proven, highly accurate, and convergent method for solving fractional differential equations. Jin et al. [7] reported a maximum absolute error of approximately 10^{-3} between the analytic and numerical solutions. Therefore, we employ the predictor-corrector scheme in our following computation.

We designed the algorithm to calculate the American up-and-in call option, applying a similar approach to the other three barrier option types.

Step 1 Set the parameters of option $y_0,\ r_0,\ a_1,\ m_1,\ \sigma_1,\ a_2,\ \sigma_2,\ p_1,\ p_2,\ X,\ B,\ T$.

Step 2 Let $0 < t_1 < \cdots < t_n < \cdots < T$. Applying predictor-corrector method, calculate Y_t^α and β, which is the minimum value of α that satisfies

$$\sup_{0\leqslant t\leqslant T} Y_t^\alpha \geqslant B.$$

Step 3 Let $\alpha = \alpha + \Delta\alpha$, and $\Delta\alpha = 0.01$.

Step 4 For a fixed $\alpha,\ t_n$, applying predictor-corrector method, calculate $R_t^{1-\alpha}$ and

$$\exp\left(-\int_0^{t_n} R_s^{1-\alpha}\mathrm{d}s\right)(Y_{t_n}^\alpha - K)^+ = \exp\left(-\int_0^{t_n} R_s^{1-\alpha}\mathrm{d}s\right)\max(Y_{t_n}^\alpha - X,\ 0).$$

Step 5 Calculate

$$Z_\alpha = \max_{0\leqslant t_n\leqslant T}\left(\exp\left(-\int_0^{t_n} R_s^{1-\alpha}\mathrm{d}s\right)\max(Y_{t_n}^\alpha - X,\ 0)\right).$$

Step 6 Return to Step 3, while $\alpha + \Delta\alpha < 1$.

Step 7 Calculate the price

$$f_{ui}^c = \sum_{i=1}^{\frac{1-\beta}{\Delta\alpha}} Z_\alpha \Delta\alpha.$$

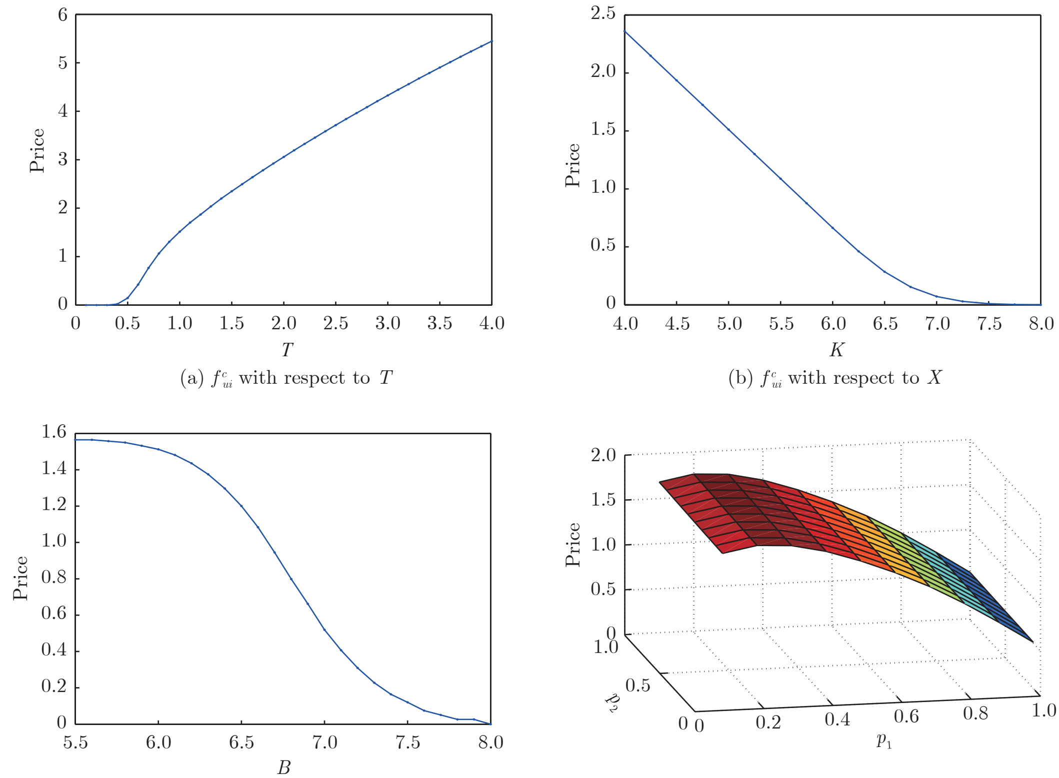
Example 1 *Under the model (13), the parameters are*

$$r_0 = 0.03,\ \ m_1 = 0.10,\ \ a_1 = 0.50,\ \ \sigma_1 = 0.04,\ \ p_1 = 0.50,$$

$$y_0 = 4.0,\ \ a_2 = 0.40,\ \ \sigma_2 = 0.30,\ \ p_2 = 0.50,\ \ X = 5,\ \ T = 1.$$

The price of the American up-and-in call barrier option at barrier level $B = 6$ is $f_{ui}^c = 1.5127$.

Figure 1 shows the price f_{ui}^c of the American up-and-in call barrier option under the model of (13) for various parameters. Figure 1(a) shows that the price of the American barrier option increases with the expiration date of T. This is because the American barrier option provides more profit opportunities. Figures 1(b) and 1(c) show that the price of the American barrier option decreases with respect to the strike price X and the barrier level B, reflecting a reduced



(a) f_{ui}^c with respect to T

(b) f_{ui}^c with respect to X

(c) f_{ui}^c with respect to B

(d) f_{ui}^c with respect to p_1 and p_2

Figure 1 f_{ui}^c with different parameters

likelihood of profit. This price dynamic is consistent with the change in option pricing under the Black-Scholes model. Figure 1(d) presents the option price changes with the orders p_1 and p_2, with p_1 having a significant impact on the price.

Example 2 *Under the model (13), the parameters are*

$$r_0 = 0.03,\ \ m_1 = 0.10,\ \ a_1 = 0.5,\ \ \sigma_1 = 0.04,\ \ p_1 = 0.80,$$

$$y_0 = 0.50,\ \ a_2 = 0.05,\ \ \sigma_2 = 0.50,\ \ p_2 = 0.80,\ \ X = 3,\ \ T = 0.50.$$

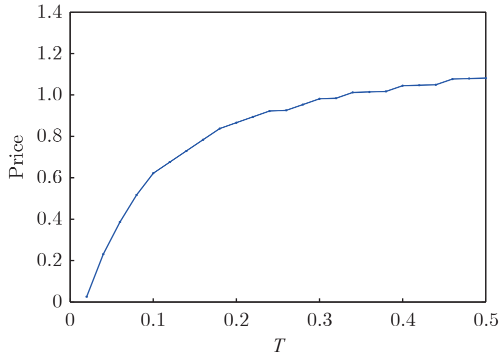
The price of the American down-and-in put barrier option at barrier level $B = 0.45$ *is* $f_{di}^p = 1.0812.$

Figure 2 shows the price f_{di}^p of the American down-and-in put barrier option under different parameters. Figures 2(a), (b), and (c) show that the price f_{di}^p increases with the expiration date T, strike price X, and barrier level B. A higher barrier level B or longer expiration date T increases the likelihood of the barrier being reached, enhancing the opportunity for profit. Figure 2(d) shows the variation in the option price with the orders p_1 and p_2, with p_1 having a significant impact on the price.

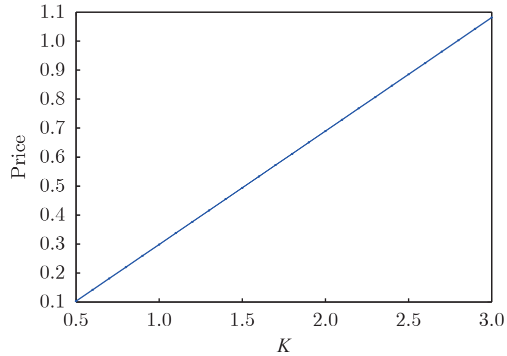
Example 3 *Under the model (13), the parameters are*

$$r_0 = 0.03,\ \ m_1 = 0.10,\ \ a_1 = 0.50,\ \ \sigma_1 = 0.04,\ \ p_1 = 0.50,$$

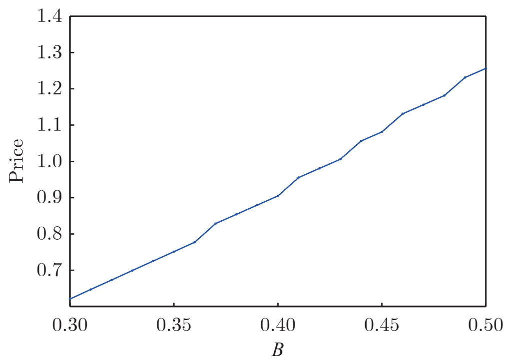
$$y_0 = 4.0,\ \ a_2 = 0.40,\ \ \sigma_2 = 0.50,\ \ p_2 = 0.50,\ \ X = 9,\ \ T = 1.$$



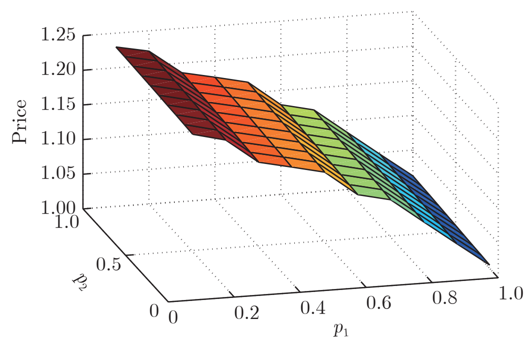
(a) f_{di}^p with respect to T



(b) f_{di}^p with respect to X



(c) f_{di}^p with respect to B



(d) f_{di}^p with respect to p_1 and p_2

Figure 2 f_{di}^p with different parameters

The price of the American up-and-out put barrier option at barrier level $B = 6$ *is* $f_{uo}^p = 0.3414$.

Figure 3 shows the price f_{uo}^p of the American up-and-out put barrier option under different parameters, as modelled in (13). Figure 3(a) shows that the price of the American barrier option is inversely proportional to the expiration date T. Figures 3(b) and (c) show that the price of the American barrier option is directly proportional to the strike price X and barrier level B. A higher barrier level or shorter expiration date reduces the likelihood of the barrier level being reached, thereby increasing the opportunity for profit. Figure 3(d) shows the variation in option price with the orders p_1 and p_2. It also can be seen that the option price changes obviously corresponding to the order p_1.

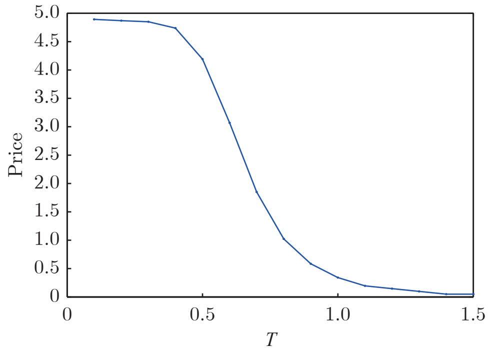
Example 4 *Under the model (13), the parameters are*

$$r_0 = 0.03, \quad m_1 = 0.10, \quad a_1 = 0.50, \quad \sigma_1 = 0.04, \quad p_1 = 0.80,$$

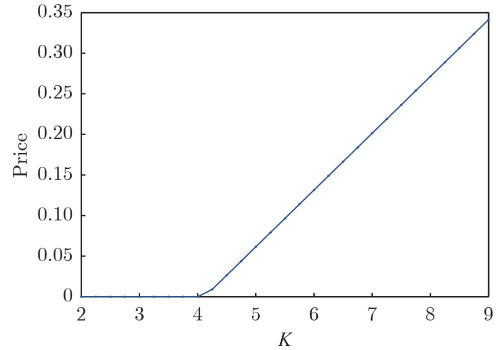
$$y_0 = 0.50, \quad a_2 = 0.05, \quad \sigma_2 = 0.50, \quad p_2 = 0.80, \quad X = 0.40, \quad T = 1.$$

The price of the American down-and-out call barrier option at barrier level $B = 0.45$ *is* $f_{do}^c = 0.2524$.

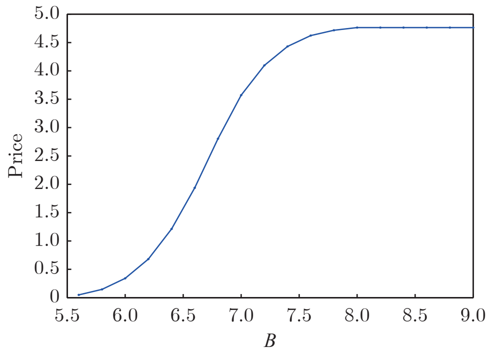
Figure 4 shows the price f_{do}^c of the American down-and-out call barrier option under various parameters, as modelled in (13). Figure 4(a) shows that the price of the American barrier option increases with the expiration date T. Figures 4(b) and (c) show that the price of the American barrier option is inversely proportional to the strike price X and barrier level B. Figure 4(d)



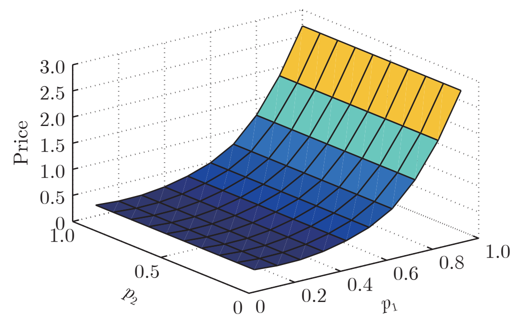
(a) f_{uo}^{p} with respect to T



(b) f_{uo}^{p} with respect to X

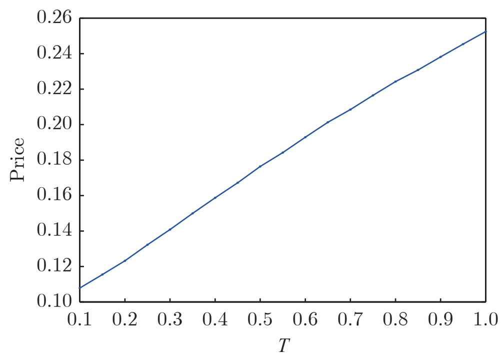


(c) f_{uo}^{p} with respect to B

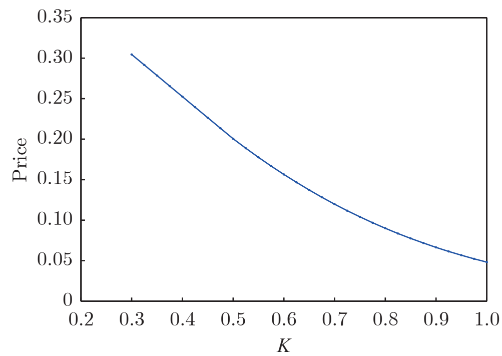


(d) f_{uo}^{p} with respect to p_1 and p_2

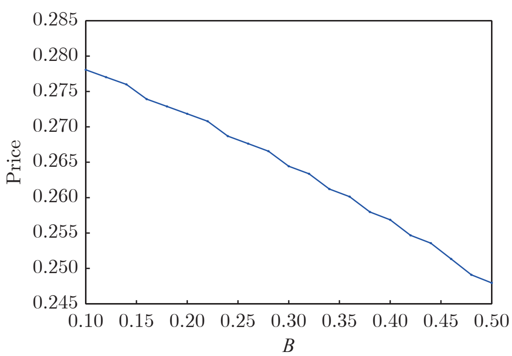
Figure 3 f_{uo}^{p} with different parameters



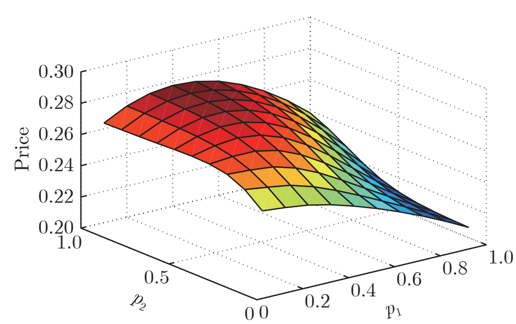
(a) f_{do}^{c} with respect to T



(b) f_{do}^{c} with respect to X



(c) f_{do}^{c} with respect to B



(d) f_{do}^{c} with respect to p_1 and p_2

Figure 4 f_{do}^{c} with different parameters shows the variation in option price with the orders p_1 and p_2, with p_1 having a more significant impact on the price than p_2.

5. Data application

Due to the difficulty in obtaining barrier option data in the market, we constructed a barrier option based on the 50ETF option. We selected the 50ETF data from January 19, 2018, and assumed a barrier level of 3.15. Based on the historical data of interest rates and underlying assets, we use the least squares method to estimate the parameters as follows:

$$m_1 = 0.01, \quad a_1 = 0.39, \quad \sigma_1 = 0.01, \quad p_1 = 0.01,$$

$$a_2 = 0.01, \quad \sigma_2 = 0.04, \quad p_2 = 0.62.$$

With an initial interest rate of $r_0 = 0.057$, an initial underlying asset price of $y_0 = 3.113$, and $X = 2.9$, we considered the expiration dates of January 24, February 28, March 28, June 27, and September 26, 2018, corresponding to $T = 5, 40, 68, 159, 238$ days. To demonstrate the effectiveness of the model, we predicted option prices for longer maturities with expiration dates of October 26, November 26, and December 26, 2018. Figure 5 shows the trend of the up-and-in call American barrier option price across different expiration dates, demonstrating an increase with longer maturities T, which is consistent with the conclusion in Example 1.

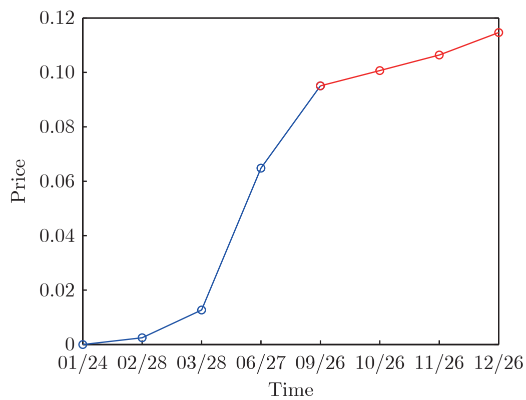


Figure 5 The barrier option price with different expiration date

6. Conclusion

This study examines the pricing of American barrier options with floating interest rates, using the FHT model as its foundation. By leveraging the uncertain fractional differential equation, a more precise depiction of market uncertainty was obtained. This study derives pricing formulas for four types of American barrier options: up-and-in call, down-and-in put, up-and-out put, and down-and-out call options. In addition, a numerical algorithm based on the predictor-corrector method was developed, complemented by illustrative numerical examples. Future research will broaden the scope by examining the pricing dynamics of four additional barrier options.

Acknowledgements

The work is supported by the National Natural Science Foundation of China (Grant No. 62403156) and Shandong Provincial Natural Science Foundation, China (Grant No. ZR2022QA055).

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