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A note for discrete time feedback control for stochastic systems driven by G-Brownian motion

Bingjun Wang[1,3], Hongjun Gao[2,*], Mingxia Yuan[4]

[1]*Jinling Institute of Technology, Nanjing* 211169, *China*
[2]*School of Mathematics, Southeast University, Nanjing* 211189, *China*
[3]*Jiangsu Key Lab for NSLSCS, Nanjing Normal University, Nanjing* 210023, *China*
[4]*Nanjing Vocational Institute of Transport Technology, Nanjing* 211188, *China*

Email: wbj586@jit.edu.cn, hjgao@seu.edu.cn, millydia@163.com

Abstract In this paper, we investigate the mean square and quasi-sure exponential stabilization of stochastic differential equations driven by G-Brownian motion, leveraging discrete-time feedback observations. We introduce a discrete-time feedback control mechanism within the drift part and demonstrate the existence of a threshold $\bar{\tau} > 0$. This threshold ensures the stability of the controlled system for any discrete step size τ that is less than $\bar{\tau}$. To validate our control strategy, we present an illustrative example.

Keywords G-Brownian motion, Discrete-time feedback, Stability

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1. Introduction

Over the past decade, Peng's G-expectation framework [9–14] has revealed novel probabilistic structures with applications in financial risk assessment and uncertainty quantification. Within this framework, the G-Brownian motion and the corresponding Itô-type stochastic calculus have been established. On this foundation, numerous scholars have studied the existence, uniqueness, and properties of solutions to the following equation driven by G-Brownian motion:

$$\mathrm{d}x(t) = b(t, x(t))\mathrm{d}t + h_{ij}(t, x(t))\mathrm{d}\langle B^i, B^j\rangle_t + \sigma(t, x(t))\mathrm{d}B_t^i, \tag{1.1}$$

where $(\langle B^i, B^j\rangle_t)_{t\geqslant 0}$, $i, j = 1, 2, \ldots, d$ is the mutual variation process of the d-dimensional G-Brownian motion $(B_t)_{t\geqslant 0}$. We refer readers to [1, 3, 15, 17] and the references therein for further insights.

Stability is a critical issue in the study of stochastic differential equations. While numerous studies have explored the stability of solutions to stochastic system (1.1), we refer readers to [4, 18, 20] for further details. Notably, the work of Li et al. [6] outlined the sufficient conditions that lead to the exponential instability of the solution for stochastic system (1.1). When the solution

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*Corresponding author

of the stochastic system is unstable, the design of a control term to achieve stability is an interesting topic. Recently, the discrete-time state feedback control proposed by Mao et al. [7, 8] has emerged as a novel control strategy for the investigation of stabilization issues. Motivated by the aforementioned works, in this paper, we aim to design a discrete-time feedback control $u(t, x([t/\tau]\tau))$ in the drift term of system (1.1), such that the stochastic system driven by G-Brownian motion under control:

$$\mathrm{d}x(t) = [b(t, x(t)) + u(t, x([t/\tau]\tau))]\mathrm{d}t + h_{ij}(t, x(t))\mathrm{d}\langle B^i, B^j\rangle_t + \sigma(t, x(t))\mathrm{d}B_t^i \tag{1.2}$$

is stable. Here, $[t/\tau]$ is the integer part of t/τ. The key feature here is that the feedback control $u(t, x([t/\tau]\tau))$ is designed based on the discrete-time observations of the state $x(t)$ at times 0, τ, 2τ, ... This approach is significantly different from the stabilization by a continuous-time (regular) feedback control $u(t, x(t))$, which is based on the current state. It is more realistic and less costly in practice.

Under the classical linear expectation framework, the Lyapunov method is a powerful tool in the stability analysis of stochastic differential equations. This method substantiates the stability of system solutions by constructing a Lyapunov functional. It first establishes the H_∞ stability of the solution, then leverages this to demonstrate mean square stability. In this paper, we intend to investigate the stability of the stochastic system (1.2) by the G-Lyapunov method. However, in the process, the cancellation of the term $|x(t) - x([t/\tau]\tau)|^2$ necessitates the construction of a highly complex G-Lyapunov functional, which may possess certain imperfections. Precisely, in order to cancel the term $|x(t) - x([t/\tau]\tau)|^2$, one would need to take a G-expectation there. Unfortunately, unlike the classical linear expectation, G-expectation requires positive homogeneity (see Preliminaries), which makes the cancellation impossible to implement. Consequently, this paper proposes several modifications to address these issues. Specifically, we apply the G-Itô's formula to $e^{\lambda t}V(t, x(t))$, and choose an appropriate constant λ to offset the G-Lyapunov functional. What we need are estimates of the term $|x(t) - x([t/\tau]\tau)|^2$, rather than its elimination. Additionally, there is no need to prove the H_∞ stability. Finally, we prove that there exists a constant $\bar{\tau} > 0$, such that the controlled stochastic system (1.2) is mean square exponentially stable and is quasi-sure exponentially stable when the discrete step size $\tau < \bar{\tau}$.

The rest of this paper is organized as follows. In Section 2, we introduce some preliminary results and notations in G-expectation framework. In Section 3, we prove the main theorem. Finally, in Section 4, we provide an example to validate the obtained theory.

2. Preliminaries

In this section, we introduce the notations and preliminary results that will be utilized throughout the paper. Further details can be found in [2, 5, 9, 14].

Throughout this paper, we will use the following notation. Let R^n denote the n-dimensional Euclidean space, and let $|x|$ denote the Euclidean norm of a vector x. Let $\mathbb{N}$ denote the set of integers. Denote by $C_{b,lip}(R^n)$ the space of all bounded real-valued Lipschitz continuous functions defined on R^n. Let $x \vee y = \max\{x, y\}$ and $R^+ = (0, +\infty)$, let A^T denotes the transpose of matrix A, and let $\mathbb{S}^d$ be the space of all $d \times d$-dimensional symmetric matrices.

- $\Omega_T = C_0([0, T]; R^d)$ denotes the space of R^d-valued continuous functions on $[0, T]$ with $w_0 = 0$, endowed with the distance

$$d(w^1, w^2) := \sum_{N=1}^{\infty} 2^{-N}((\max_{0\leqslant t\leqslant N} |w_t^1 - w_t^2|) \wedge 1).$$

$B_t(w) = w_t$ is the canonical process. Let $\mathcal{B}(\Omega_T)$ denote the Borel σ-algebra of Ω_T.

- $L_{ip}(\Omega_T) := \{\varphi(B_{t_1}, ..., B_{t_n}) : t_1, t_2, \ldots, t_n \in [0, T],\ \varphi \in C_{b,Lip}(R^{d\times n})\}$.

According to [14], there exists a sublinear functional $\hat{\mathbb{E}}$ such that for $X, Y \in L_{ip}(\Omega_T)$, the following properties hold:

(i) $\hat{\mathbb{E}}[X] \geqslant \hat{\mathbb{E}}[Y]$ if $X \geqslant Y$.

(ii) $\hat{\mathbb{E}}[C] = C$ for $C \in R$.

(iii) $\hat{\mathbb{E}}[X + Y] \leqslant \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$.

(iv) Positive homogeneity: $\hat{\mathbb{E}}[\lambda X] = \lambda\hat{\mathbb{E}}[X]$ for $\lambda \geqslant 0$.

The triplet $(\Omega, L_{ip}(\Omega_T), \hat{\mathbb{E}})$ is referred to as a sublinear expectation space, and $X \in L_{ip}(\Omega_T)$ is termed a random variable.

Moreover, the canonical process $(B_t)_{t\geqslant 0}$ (which called a G-Brownian motion with $G(A) = \hat{\mathbb{E}}(B_1^T AB_1)$ for $A \in \mathbb{S}^d$) satisfies:

(i) For each $s, t > 0$, $B_{t+s} - B_t$ and B_s are identically distributed and $B_{t+s} - B_t$ is independent from $(B_{t_1}, B_{t_2}, \ldots, B_{t_n})$ for each $n \in \mathbb{N}$ and $0 \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant t_n \leqslant t$;

(ii) $\hat{\mathbb{E}}[B_t] = \hat{\mathbb{E}}[-B_t] = 0$ and $\lim_{t\downarrow 0} \hat{\mathbb{E}}[|B_t|^3]t^{-1} = 0$.

In this context, $\hat{\mathbb{E}}$ is referred to as the G-expectation. We have $\underline{\sigma}^2 = -\hat{\mathbb{E}}[-|B_1^i|^2] \leqslant \hat{\mathbb{E}}[|B_1^i|^2] = \bar{\sigma}^2$, $i = 1, 2, \ldots, d$. Specifically, when $d = 1$, B_1 follows the G-normal distribution, denoted by $B_1 \sim N(\{0\}, [\underline{\sigma}^2, \bar{\sigma}^2])$.

- $L_G^p(\Omega_T)$ is the completion of $L_{ip}(\Omega_T)$ under the natural norm $\|X\|_{p,G} = (\hat{\mathbb{E}}[|X|^p])^{\frac{1}{p}}$, $p \geqslant 1$.
- $M_G^{p,0}(0, T)$ is the space of simple process: $\eta_t(w) = \sum_{k=0}^{N-1} \xi_k(w)I_{[t_k,t_{k+1})}(t)$, where $\xi_k \in L_G^p(\Omega_{t_k})$, $0 = t_0 \leqslant t_1 \leqslant \ldots \leqslant t_{N-1} \leqslant t_N = T$, $p \geqslant 1$.
- $M_G^p(0, T)$ is the completion of $M_G^{p,0}(0, T)$ under the norm $\|\eta\|_{M_G^p(0,T)} = [\int_0^T \hat{\mathbb{E}}[|\eta_t|^p]\mathrm{d}t]^{\frac{1}{p}}$.

Let $\eta \in M_G^p(0, T)$ for $p \geqslant 2$, then stochastic integral with respect to the d-dimensional G-Brownian motion $(B_t)_{t\geqslant 0}$ can be well defined as $\int_0^t \eta_s\,\mathrm{d}B_s^i$, $i = 1, 2, \ldots, d$. Similarly, for $\eta \in M_G^p(0, T)$ with $p \geqslant 1$, the following integrals can also be defined: $\int_0^t \eta_s\,\mathrm{d}s$ and $\int_0^t \eta_s\,\mathrm{d}\langle B^i, B^j\rangle_s$, where $(\langle B^i, B^j\rangle_t)_{t\geqslant 0}$ denotes the quadratic variation process of the d-dimensional G-Brownian motion $(B_t)_{t\geqslant 0}$, $i, j = 1, 2, \ldots, d$.

Lemma 2.1 *([14]) There exists a weakly compact subset $\mathcal{P} \subset \mathcal{M}(\Omega_T)$, the set of probability measures on $(\Omega_T, \mathcal{B}(\Omega_T))$, such that*

$$\hat{\mathbb{E}}[\xi] = \sup_{P\in\mathcal{P}} E_P(\xi) \quad \textit{for all} \quad \xi \in L_G^1(\Omega_T).$$

Definition 2.1 *The capacity $\mathbb{C}$ associated with $\hat{\mathbb{E}}$ is defined by*

$$\mathbb{C}(A) := \sup_{P\in\mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega_T).$$

If $\mathbb{C}(A) = 0$ for $A \in \mathcal{B}(\Omega_T)$, we call A a polar set. If a property holds outside a polar set, we say that the property holds quasi-surely (q.s.).

Remark 2.1 *When there exists a unique probability measure $P \in \mathcal{P}$, the G-expectation $\hat{\mathbb{E}}$ reduces to the classical linear expectation E. Consequently, the G-Brownian motion simplifies to* *the classical Brownian motion. In this case, the volatility parameters become equal, i.e.,* $\underline{\sigma}^2 = \bar{\sigma}^2$*, and the quadratic variation of the Brownian motion is given by* $\langle B \rangle_t = t$ *.*

Lemma 2.2 *(Markov's inequality) Let* $X \in L_G^p(\Omega_T)$, $p \geqslant 1$*. Then, for each constant* $M > 0$,

$$\mathbb{C}(|X| > M) \leqslant \frac{\hat{\mathbb{E}}[|X|^p]}{M^p}.$$

Lemma 2.3 *([3]) For* $p \geqslant 2$ *and* $\eta \in M_G^p(0,T)$*, there exists a positive constant* C_p'' *depending on* p *such that*

$$\hat{\mathbb{E}}\left[\sup_{0\leqslant u\leqslant T}\left|\int_0^u \eta_r \mathrm{d}B_r^i\right|^p\right] \leqslant C_p'' T^{\frac{p}{2}-1}\int_0^T \hat{\mathbb{E}}|\eta_r|^p \mathrm{d}r.$$

Lemma 2.4 *([3]) For* $p \geqslant 1$ *and* $\eta \in M_G^p(0,T)$*, there exists a positive constant* C_p' *depending on* p *such that*

$$\hat{\mathbb{E}}\left[\sup_{0\leqslant u\leqslant T}\left|\int_0^u \eta_r \mathrm{d}\langle B^i, B^j\rangle_r\right|^p\right] \leqslant C_p' T^{p-1}\int_0^T \hat{\mathbb{E}}|\eta_r|^p \mathrm{d}r.$$

3. Exponential stability of the solutions

Consider an n-dimensional stochastic differential equation (recall that we use Einstein summation convention) given by

$$\mathrm{d}x(t) = b(t,x(t))\,\mathrm{d}t + h_{ij}(t,x(t))\,\mathrm{d}\langle B^i, B^j\rangle_t + \sigma_i(t,x(t))\,\mathrm{d}B_t^i, \tag{3.1}$$

with the initial condition $x(0) = x_0 \in R^n$. Here, $b,\ h_{ij},\ \sigma_i : R^+ \times R^n \to R^n$ are functions satisfying $b(\cdot,x),\ h_{ij}(\cdot,x),\ \sigma_i(\cdot,x) \in M_G^2([0,T];R^n)$, where $M_G^2([0,T];R^n)$ denotes the space of n-dimensional stochastic processes such that each component belongs to $M_G^2(0,T)$.

Let $C^{1,2}([0,+\infty)\times R^n, R^+)$ denote the space of positive functions $V(t,x)$ for which the first and second partial derivatives V_t, V_x, V_{xx} are continuous on $[0,+\infty)\times R^n$, and V_{xx} satisfies the local Lipschitz condition. Here,

$$\begin{aligned}
V_t(t,x) &:= \frac{\partial V(t,x)}{\partial t},\\
V_x(t,x) &:= \left(\frac{\partial V(t,x)}{\partial x_1}, \frac{\partial V(t,x)}{\partial x_2}, \dots, \frac{\partial V(t,x)}{\partial x_n}\right)^T,\\
V_{xx}(t,x) &:= \left(\frac{\partial^2 V(t,x)}{\partial x_i \partial x_j}\right)_{n\times n}.
\end{aligned}$$

Remark 3.1 *If the coefficients* $b,\ h_{ij},\ \sigma_i$ *satisfy all the necessary conditions outlined in Li et al. [6], then stochastic differential equation (3.1) admits a unique solution* $x(t)$*. Furthermore, for some positive constants* $C_1,\ C_2,\ \lambda$*, assume there exists a function* $V(t,x) \in C^{1,2}([0,+\infty)\times R^n, R^+)$ *such that*

$$C_1|x|^2 \leqslant V(t,x) \leqslant C_2|x|^2$$

and

$$V_t(t,x) + V_x^T(t,x)b(t,x) + G\left(2V_x^T(t,x)h_{ij}(t,x) + \sigma_i^T(t,x)V_{xx}(t,x)\sigma_j(t,x)\right) \geqslant \lambda V(t,x).$$

According to Theorem 5.10 in [6], the solution $x(t)$ *of equation (3.1) is mean square exponentially unstable.*

We aim to employ a discrete-time feedback control strategy of the form $u(t, x([t/\tau]\tau))$ to stabilize the stochastic system (3.1). Here, $\tau > 0$ represents the time interval between consecutive state observations, and $[t/\tau] = \sup\{k \in \mathbb{N}; k \leqslant t/\tau\}$ denotes the integer part. For notational convenience, let $\delta_t = [t/\tau]\tau$. Consequently, we consider the following discrete-time feedback control system:

$$\mathrm{d}x(t) = [b(t, x(t)) + u(t, x(\delta_t))]\mathrm{d}t + h_{ij}(t, x(t))\mathrm{d}\langle B^i, B^j\rangle_t + \sigma_i(t, x(t))\mathrm{d}B_t^i, \tag{3.2}$$

with the initial condition $x(0) = x_0 \in R^n$.

Let $V(t, x) \in C^{1,2}([0, +\infty) \times R^n, R^+)$, and define $\mathcal{L}$ as

$$\begin{aligned}\mathcal{L}V(t, x) := {} & V_t(t, x) + V_x^T(t, x)[b(t, x) + u(t, x)] \\ & + G\left(2V_x^T(t, x)h_{ij}(t, x) + \sigma_i^T(t, x)V_{xx}(t, x)\sigma_j(t, x)\right).\end{aligned} \tag{3.3}$$

In the subsequent analysis, we make the following assumptions:

(H1) For any $x, y \in R^n$, there exist positive constants M and M_1 such that

$$|b(t, x) - b(t, y)| \vee |h_{ij}(t, x) - h_{ij}(t, y)| \vee |\sigma_i(t, x) - \sigma_i(t, y)| \leqslant M|x - y|,$$

$$|u(t, x) - u(t, y)| \leqslant M_1|x - y|.$$

(H2) There exist constants $\lambda > 0$, $\lambda_1 > 0$, and $0 < C_1 < C_2$ such that

$$\mathcal{L}V(t, x) + \lambda_1|V_x(t, x)|^2 \leqslant -\lambda V(t, x),$$

$$C_1|x|^2 \leqslant V(t, x) \leqslant C_2|x|^2.$$

Remark 3.2 *To ensure stability for $t \geqslant 0$, we assume that $b(t, 0) = u(t, 0) = h_{ij}(t, 0) = \sigma_i(t, 0) = 0$. Under Assumption (H1), and following the results from [6], we obtain that the stochastic system (3.2) admits a unique solution, denoted by $x(t)$.*

Lemma 3.1 *([16]) Let assumption (H1) hold. Set $L(\tau) = [8M^2(\tau^2 + \bar{\sigma}^4\tau^2 + \bar{\sigma}^2\tau) + 4\tau^2M_1^2]\cdot e^{8M^2(\tau^2+\bar{\sigma}^4\tau^2+\bar{\sigma}^2\tau)}$ for $\tau \geqslant 0$. If τ is sufficiently small such that $2L(\tau) < 1$, then for all $t \geqslant 0$, the solution $x(t)$ of stochastic system (3.2) satisfies*

$$\hat{\mathbb{E}}[|x(t) - x(\delta_t)|^2] \leqslant \frac{2L(\tau)}{1 - 2L(\tau)}\hat{\mathbb{E}}[|x(t)|^2].$$

Theorem 3.1 *Suppose assumptions (H1) and (H2) are satisfied. Then, there exists a positive constant $\bar{\tau}$ such that for $\tau \in (0, \bar{\tau})$, the solution $x(t)$ of stochastic system (3.2) is mean square exponentially stable,*

$$\hat{\mathbb{E}}[|x(t)|^2] \leqslant \frac{C_2}{C_1}|x_0|^2e^{-\gamma t},$$

where $\gamma = \lambda - \frac{M_1^2L(\tau)}{4C_1\lambda_1(1-2L(\tau))} > 0$.

Proof For $\lambda > 0$, applying G-Itô's formula to $e^{\lambda t}V(t, x(t))$, we derive that

$$\begin{aligned}e^{\lambda t}V(t, x(t)) = {} & V(0, x(0)) + \int_0^t e^{\lambda r}[\lambda V(r, x(r)) + \mathcal{L}V(r, x(r)) \\ & + V_x^T(r, x(r))(u(r, x(\delta_r)) - u(r, x(r)))]\mathrm{d}r + P_t + M_t,\end{aligned} \tag{3.4}$$

where

$$P_t = \int_0^t e^{\lambda r}[V_x^T(r,x(r))h_{ij}(r,x(r)) + \frac{1}{2}\sigma_i^T(r,X(r))V_{xx}(r,x(r))\sigma_j(r,X(r))]\mathrm{d}\langle B^i, B^j\rangle_r$$
$$- \int_0^t e^{\lambda r}G(2V_x^T(r,x(r))h_{ij}(r,x(r)) + \sigma_i^T(r,X(r))V_{xx}(r,x(r))\sigma_j(r,X(r)))\mathrm{d}r$$

and

$$M_t = \int_0^t e^{\lambda r}V_x^T(r,x(r))\sigma_i(r,X(r))\mathrm{d}B_r^i$$

are G-martingales.

By average inequality ($ab \leqslant \lambda_1 a^2 + \frac{b^2}{4\lambda_1}$) and assumption (H1), we can obtain

$$e^{\lambda t}V(t,x(t)) \leqslant V(0,x(0)) + \int_0^t e^{\lambda r}[\lambda V(r,x(r)) + \mathcal{L}V(r,x(r)) + \lambda_1|V_x(r,x(r))|^2 + \frac{M_1^2}{4\lambda_1}|x(\delta_r) - x(r)|^2]\mathrm{d}r + P_t + M_t. \tag{3.5}$$

By assumption (H2), we further derive

$$e^{\lambda t}V(t,x(t)) \leqslant V(0,x_0) + \int_0^t e^{\lambda r}\frac{M_1^2}{4\lambda_1}|x(\delta_r) - x(r)|^2\mathrm{d}r + P_t + M_t. \tag{3.6}$$

Taking G-expectation on both sides of (3.6) and by (H2), we get

$$e^{\lambda t}\hat{\mathbb{E}}[|x(t)|^2] \leqslant \frac{1}{C_1}e^{\lambda t}\hat{\mathbb{E}}[V(t,x(t))] \leqslant \frac{1}{C_1}\hat{\mathbb{E}}[V(0,x_0)] + \frac{M_1^2}{4C_1\lambda_1}\int_0^t e^{\lambda r}\hat{\mathbb{E}}[|x(\delta_r) - x(r)|^2]\mathrm{d}r$$
$$\leqslant \frac{C_2}{C_1}|x_0|^2 + \frac{M_1^2}{4C_1\lambda_1}\int_0^t e^{\lambda r}\hat{\mathbb{E}}[|x(\delta_r) - x(r)|^2]\mathrm{d}r. \tag{3.7}$$

According to Lemma 3.1, we have

$$e^{\lambda t}\hat{\mathbb{E}}[|x(t)|^2] \leqslant \frac{C_2}{C_1}|x_0|^2 + \frac{M_1^2 L(\tau)}{2C_1\lambda_1(1-2L(\tau))}\int_0^t e^{\lambda r}\hat{\mathbb{E}}[|x(r)|^2]\mathrm{d}r. \tag{3.8}$$

Then, by Gronwall's inequality, we obatin

$$e^{\lambda t}\hat{\mathbb{E}}[|x(t)|^2] \leqslant \frac{C_2}{C_1}|x_0|^2 e^{\frac{M_1^2 L(\tau)}{2C_1\lambda_1(1-2L(\tau))}t}. \tag{3.9}$$

Moreover, it follows that

$$\hat{\mathbb{E}}[|x(t)|^2] \leqslant \frac{C_2}{C_1}|x_0|^2 e^{\left[\frac{M_1^2 L(\tau)}{2C_1\lambda_1(1-2L(\tau))} - \lambda\right]t}. \tag{3.10}$$

According to Lemma 3.1, it is evident that $L(\tau)$ is continuous and increasing for $\tau > 0$, with $L(0) = 0$ and $\lim_{\tau\to+\infty} L(\tau) = +\infty$. Hence, there exists a positive constant τ^* such that $L(\tau^*) = \frac{1}{2}$. Let $J(\tau) := \frac{M_1^2 L(\tau)}{2C_1\lambda_1(1-2L(\tau))}$. A straightforward computation reveals that $J'(\tau) > 0$, indicating that $J(\tau)$ is continuous and increasing on $0 < \tau < \tau^*$, with $J(0) = 0$ and $\lim_{\tau\to(\tau^*)^-} J(\tau) = +\infty$. Consequently, there exists a positive constant $\tau^{**} < \tau^*$ such that $J(\tau^{**}) = \lambda$. Let $\bar{\tau} := \tau^{**}$, then for $\tau \in (0, \bar{\tau})$, we have $\frac{M_1^2 L(\tau)}{2C_1\lambda_1(1-2L(\tau))} < \lambda$ and $L(\tau) < \frac{1}{2}$.

Let $-\gamma := \frac{M_1^2 L(\tau)}{2C_1\lambda_1(1-2L(\tau))} - \lambda$, then $\gamma > 0$ and

$$\hat{\mathbb{E}}[|x(t)|^2] \leqslant \frac{C_2}{C_1}|x_0|^2 e^{-\gamma t}, \tag{3.11}$$

which implies that that $x(t)$ is mean square exponentially stable.

Theorem 3.2 *Suppose the conditions of Theorem 3.1 are satisfied. For $\tau < \bar{\tau}$, the solution of stochastic system (3.2) is quasi-sure exponentially stable, i.e.,*

$$\limsup_{t\to\infty} \frac{\ln|x(t)|}{t} \leqslant -\frac{\gamma}{4}, \quad q.s.$$

Proof Let $\tau \in (0,\bar{\tau})$. For $l\tau \leqslant t \leqslant u \leqslant (l+1)\tau$, where $l = 0,1,2,\ldots,$ consider $x(l\tau)$ as the initial value at time $l\tau$. By applying the elementary inequality $(|a+b+c+d|^p \leqslant 4^{p-1}(|a|^p + |b|^p + |c|^p + |d|^p))$ and using assumption (H1), Lemma 2.3 and Lemma 2.4, we obtain

$$\begin{aligned}
\hat{\mathbb{E}}[\sup_{l\tau\leqslant t\leqslant u} |x(t)|^2] &\leqslant 4\hat{\mathbb{E}}[|x(l\tau)|^2] + 4\hat{\mathbb{E}}\left[\sup_{l\tau\leqslant t\leqslant u}\left|\int_{l\tau}^t [b(r,x(r)) + u(r,x(\delta_r))]\mathrm{d}r\right|^2\right] \\
&\quad + 4\hat{\mathbb{E}}\left[\sup_{l\tau\leqslant t\leqslant u}\left|\int_{l\tau}^t h_{ij}(r,x(r))\mathrm{d}\langle B^i,B^j\rangle_r\right|^2\right] + 4\hat{\mathbb{E}}\left[\sup_{l\tau\leqslant t\leqslant u}\left|\int_{l\tau}^t \sigma_i(r,x(r))\mathrm{d}B_r^i\right|^2\right] \\
&\leqslant 4\hat{\mathbb{E}}[|x(l\tau)|^2] + 8\tau\int_{l\tau}^u \hat{\mathbb{E}}[|b(r,x(r))|^2 + |u(r,x(\delta_r))|^2]\mathrm{d}r \\
&\quad + 4C_2'\tau\int_{l\tau}^u \hat{\mathbb{E}}[|h_{ij}(r,x(r))|^2]\mathrm{d}r + 4C_2''\int_{l\tau}^u \hat{\mathbb{E}}[|\sigma_i(r,x(r))|^2]\mathrm{d}r \\
&\leqslant 4\hat{\mathbb{E}}[|x(l\tau)|^2] + [8M^2\tau + 8M_1^2\tau + 4M^2C_2'\tau + 4M^2C_2'']\int_{l\tau}^u \hat{\mathbb{E}}\left[\sup_{l\tau\leqslant t\leqslant s}|x(t)|^2\right]\mathrm{d}s,
\end{aligned} \tag{3.12}$$

where C_2'', C_2' are constants in Lemma 2.3, Lemma 2.4 respectively.

Applying Gronwall's inequality, we have

$$\hat{\mathbb{E}}\left[\sup_{l\tau\leqslant t\leqslant(l+1)\tau}|x(t)|^2\right] \leqslant \widetilde{C}\hat{\mathbb{E}}[|x(l\tau)|^2], \tag{3.13}$$

where $\widetilde{C} = 4e^{[8M^2\tau+8M_1^2\tau+4M^2C_2'\tau+4M^2C_2'']\tau}$.

By Theorem 3.1, for all $l = 0,1,2,\ldots,$ we have

$$\hat{\mathbb{E}}\left[\sup_{l\tau\leqslant t\leqslant(l+1)\tau}|x(t)|^2\right] \leqslant \frac{\tilde{C}C_2}{C_1}|x_0|^2 e^{-l\gamma\tau}. \tag{3.14}$$

By Markov's inequality (Lemma 2.2), we obtain

$$\mathbb{C}\left(\sup_{l\tau\leqslant t\leqslant(l+1)\tau}|x(t)|^2 \geqslant e^{-\frac{1}{2}\gamma l\tau}\right) \leqslant \frac{\tilde{C}C_2}{C_1}|x_0|^2 e^{-\frac{1}{2}l\gamma\tau}.$$

Using the Borel-Cantelli lemma, we deduce that there exists a $l_0(w)$, such that for $l > l_0(w)$,

$$\sup_{l\tau\leqslant t\leqslant(l+1)\tau}|x(t)|^2 \leqslant e^{-\frac{1}{2}\gamma l\tau},\ q.s.$$

Then for $l\tau \leqslant t \leqslant (l+1)\tau$,

$$\frac{\ln|x(t)|}{t} = \frac{\ln|x(t)|^2}{2t} \leqslant \frac{\ln|x(t)|^2}{2l\tau} \leqslant \frac{\ln\sup_{l\tau\leqslant t\leqslant(l+1)\tau}|x(t)|^2}{2l\tau} \leqslant -\frac{\gamma}{4}. \tag{3.15}$$

By taking the limit superior (limsup) of inequality (3.15), we obtain a quasi-sure exponential estimate, that is

$$\limsup_{t\to\infty}\frac{\ln|x(t)|}{t}\leqslant -\frac{\gamma}{4},\quad q.s.$$

4. Example

Consider the following one-dimensional stochastic system driven by G-Brownian motion:

$$\mathrm{d}x(t)=b(t,x(t))\mathrm{d}t+h(t,x(t))\mathrm{d}\langle B\rangle_t+\sigma(t,x(t))\mathrm{d}B_t,\tag{4.1}$$

with initial value $x(0)=x_0$ and $B_1\sim N(\{0\},[\frac{1}{2},1])$.

Let

$$b(t,x)=\sin x,\quad h(t,x)=\frac{2}{1+t^2}x,\quad \sigma(t,x)=\frac{2t}{\sqrt{1+t^2}}x.$$

Then it is easy to verify that b,h,σ satisfy the assumption (H1) with $M=2$.

Define the Lyapunov function $V(t,x)=|x|^2$. The derivative calculation yields

$$\begin{aligned}&V_t(t,x)+V_x(t,x)b(t,x)+G(2V_x(t,x)h(t,x)+\sigma(t,x)V_{xx}(t,x)\sigma(t,x))\\ &=2x\sin x+G\left(\frac{8|x|^2}{1+t^2}+\frac{8t^2}{1+t^2}|x|^2\right)\\ &=2x\sin x+4|x|^2\\ &\geqslant 4|x|^2.\end{aligned}$$

Hence, by Remark 3.1, the system (4.1) is mean square exponentially unstable.

Consider the following stochastic differential equation driven by G-Brownian motion with discrete time feedback control:

$$\mathrm{d}x(t)=[b(t,x(t))+u(t,x([t/\tau]\tau))]\mathrm{d}t+h(t,x(t))\mathrm{d}\langle B\rangle_t+\sigma(t,x(t))\mathrm{d}B_t,\tag{4.2}$$

where $u(t,x)=kx,\ k=-6$. Then we have

$$\begin{aligned}\mathcal{L}V(t,x)&=2x[b(t,x)+u(t,x)]+G\left(\frac{8|x|^2}{1+t^2}+\frac{8t^2}{1+t^2}|x|^2\right)\\ &=2x\sin x-12|x|^2+4|x|^2\\ &\leqslant -6|x|^2.\end{aligned}\tag{4.3}$$

Selecting $\lambda_1=0.5$, then the assumption (H2) becomes

$$\mathcal{L}V(t,x)+\lambda_1|V_x(t,x)|^2\leqslant -6|x|^2+2|x|^2=-4|x|^2.\tag{4.4}$$

Thus, assumptions (H1) and (H2) are satisfied with $M_1=6,\ \lambda=4$.

Set $L(\tau)<\frac{1}{11}$, which implies $\tau<0.01174$ and $J(\tau)<4$. Choosing $\bar{\tau}=0.01174$, by Theorem 3.2, for $0<\tau<0.01174$, the control system (4.2) is quasi-sure exponentially stable.

5. Conclusion

In this study, we explore discrete-time feedback control for stochastic systems influenced by G-Brownian motion. We introduce a novel control mechanism in the drift component, ensuring the stability of the system. We analyze two stability criteria: mean square exponential stability and quasi-sure exponential stability. We demonstrate that there exists a critical discrete observation time $\bar{\tau}>0$, beyond which the system remains stable if the discrete step size τ is less than $\bar{\tau}$. This finding is validated through a practical example, effectively illustrating our theoretical findings.

While the recent work [19] (which improves the results in Ren et al. [16]) also achieves stabilization under discrete-time feedback for G-SDEs, it employs a comparison method. This method necessitates first establishing the stability of a reference equation and then proving that the controlled system behaves similarly under the G-framework. Our method offers a simplification. By working directly with the Lyapunov functional $V(t,x)$, we provide a direct and self-contained stability proof under G-expectation. This direct Lyapunov approach, bypassing the need for a reference system, is arguably more straightforward and intuitive than the comparison technique used in [19], while arriving at the same core conclusion regarding the existence of a step size $\bar{\tau}$.

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References

[1] Bai, X. and Lin, Y., On the existence and uniqueness of solutions to stochastic differential equations driven by G-Brownian motion with integral-Lipschitz coefficients, Acta Mathematicae Applicatae Sinica, English Series, 2014, 30(3): 589–610.

[2] Denis, L., Hu, M. and Peng, S., Function spaces and capacity related to a sublinear expectation: Application to G-Brownian motion paths, Potential Analysis, 2011, 34(2): 139–161.

[3] Gao, F., Pathwise properties and homeomorphic flows for stochastic differential equations driven by G-Brownian motion, Stochastic Processes and their Applications, 2009, 119(10): 3356–3382.

[4] Hu, L., Ren, Y. and Xu, T., p-moment stability of solutions to stochastic differential equations driven by G-Brownian motion, Applied Mathematics and Computation, 2014, 230: 231–237.

[5] Li, X. and Peng, S., Stopping times and related Itô's calculus with G-Brownian motion, Stochastic Processes and their Applications, 2011, 121(7): 1492–1508.

[6] Li, X., Lin, X. and Lin, Y., Lyapunov-type conditions and stochastic differential equations driven by G-Brownian motion, Journal of Mathematical Analysis and Applications, 2016, 439(1): 235–255.

[7] Mao, X., Almost sure exponential stabilization by discrete-time stochastic feedback control, IEEE Transactions on Automatic Control, 2016, 61(6): 1619–1624.

[8] Mao, X., Liu, W., Hu, L., Luo, Q. and Lu, J., Stabilization of hybrid stochastic differential equations by feedback control based on discrete-time state observations, Systems & Control Letters, 2014, 73: 88–95.

[9] Peng, S., G-expectations, G-Brownian motion and related stochastic calculus of Itô's type, In: Benth, F. E., Di Nunno, G., Lindstrøm, T., Øksendal, B., Zhang, T.(eds.), Stochastic Analysis and Applications, 2007: 541–567.

[10] Peng, S., Filtration consistent nonlinear expectations and evaluations of contingent claims, Acta Mathematicae Applicatae Sinica, English Series, 2004, 20(1): 1–24.

[11] Peng, S., Nonlinear expectations and nonlinear Markov chains, Chinese Annals of Mathematics, Series B, 2005, 26(2): 159–184.

[12] Peng, S., Multi-dimensional G-Brownian motion and related stochastic calculus under G-expectation, Stochastic Processes and their Applications, 2008, 118(12): 2223–2253.

[13] Peng, S., Survey on normal distributions, central limit theorem, Brownian motion and the related stochastic calculus under sublinear expectation, Science in China Series A: Mathematics, 2009, 52(7): 1391–1411.

[14] Peng, S., Nonlinear Expectations and Stochastic Calculus under Uncertainty, Springer, Berlin, 2019.

[15] Qiao, H., The cocycle property of stochastic differential equations driven by G-Brownian motion, Chinese Annals of Mathematics, Series B, 2015, 36(1): 147–160.

[16] Ren, Y., Yin, W. and Sakthivel, R., Stabilization of stochastic differential equations driven by G-Brownian motion with feedback control based on discrete-time state observation, Automatica, 2018, 95: 146–151.

[17] Wang, B., Gao, H., Li, M. and Yuan, M., Reflected forward-backward stochastic differential equations driven by G-Brownian motion with continuous monotone coefficients, Qualitative Theory of Dynamical Systems, 2022, 21(1): 1–34.

[18] Yin, W., Cao, J. and Ren, Y., Quasi-sure exponential stabilization of stochastic systems induced by G-Brownian motion with discrete time feedback control, Journal of Mathematical Analysis and Applications, 2019, 474(1): 276–289.

[19] Yin, W., Cao, J., Ren, Y. and Zheng, G., Improved result on stabilization of G-SDEs by feedback control based on discrete-time observations, SIAM Journal on Control and Optimization, 2021, 50(4): 1927–1950.

[20] Zhu, Q. and Huang, T., Stability analysis for a class of stochastic delay nonlinear systems driven by G-Brownian motion, Systems & Control Letters, 2020, 140: 1–9.