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Quasi-likelihood estimation in a mixed fractional Black-Scholes model

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Abstract Let $B^H = \{B_t^H, t \geqslant 0\}$ be a fractional Brownian motion with Hurst index $0 < H < 1$, and let $B = \{B_t, t \geqslant 0\}$ be an independent Brownian motion. In this study, we investigate the parameter estimation of a mixed fractional Black-Scholes model

$$S_t^H = S_0^H + \mu \int_0^t S_s^H \mathrm{d}s + \sigma \int_0^t S_s^H \mathrm{d}(B_s + B_s^H)\,,$$

where $\sigma > 0$, $\mu \in \mathbb{R}$ are two unknown parameters. Using quasi-likelihood estimation, when the system is observed at some discrete time instants $\{t_i = ih, i = 0, 1, 2, \ldots, n\}$, we give estimations of the parameters μ and σ provided $h = h(n) \to 0$, $nh \to \infty$ and $h^{1+\gamma} n \to 1$ for some $\gamma > 0$, as $n \to \infty$. We present the asymptotic normality of the estimators based on the velocity of $nh^{1+\gamma} - 1$ tending to zero as n tends to infinity. Finally, we perform numerical calculus and simulations using factual data from the stock market to verify the effectiveness of the established estimators.

Keywords Quasi-likelihood estimation, Fractional Brownian motion, Mixed fractional Black-Scholes model, Fractional Itô integral, Asymptotic distribution

2020 Mathematics Subject Classification 60G22, 60H07, 60F05, 62M09

1. Introduction

By mixed fractional Brownian motion, we mean a linear combination of different fractional Brownian motions. In this paper, we examine the special form

$$M_t^{H,\alpha} = B_t + \alpha B_t^H, \quad t \geqslant 0,$$

where $B = \{B_t, t \geqslant 0\}$ is a Brownian motion, $B^H = \{B_t^H, t \geqslant 0\}$ is an independent fractional Brownian motion with Hurst index $H \in (0, 1)$, and $\alpha \neq 0$ is a real number. In 2001, Cheridito [10] showed that the process $M^{H,\alpha} = \{M_t^{H,\alpha}, t \geqslant 0\}$ is not a semimartingale if $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{4}]$ that it is equivalent to $\sqrt{1 + \alpha^2}$ times Brownian motion if $H = \frac{1}{2}$. However, when $H \in (\frac{3}{4}, 1)$, he showed that the process $M^{H,\alpha}$ is a weak semimartingale, which is equivalent to Brownian motion.

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As an application, that study considered the price of a European call option for an asset driven by mixed fractional Brownian motion $M^{H,\alpha}$. Meanwhile, in 2007, Bender et al [6] showed that the following mixed model is arbitrage-free with regular portfolios:

$$\mathrm{d}S_t^H = \mu S_t^H \mathrm{d}t + \sigma S_t^H \left(\mathrm{d}B_t + \alpha \mathrm{d}B_t^H\right), \quad S_0^H > 0, \tag{1.1}$$

if $\alpha > 0$ and $H \in (\frac{3}{4}, 1)$, where μ, σ are two unknown parameters. For results related to the mixed model we refer to, see, among others, Androshchuk and Mishura [2], Azmoodeh let al [3] and Cheridito [11]. It is known that a type model driven by a Black-Scholes fractional Brownian motion with $\frac{1}{2} < H < 1$ probably cannot be equipped with an economically sensible class of arbitrage-free portfolios that is rich enough for pricing purposes. However, one can define the market by using the Wick-Itô integral, such that the type of market is complete and arbitrage-free. Other studies of the Black-Scholes model associated with fractional Brownian motion include in Bender and Elliott [5], Biagini [7], Björk-Hult [8], Cheridito [11], Elliott and Chan [14, 15], Greene-Fielitz [17], Hu-Oksendal [21], Necula [24], Lo [22], Mishura [23], and Rogers [29], as well as the references therein.

Meanwhile, the classical Black-Scholes formula is one of the most important outcomes of the study of continuous time models in finance. However, the fitness of the model has been questioned based on the assumption of constant volatility, since empirical evidence shows that volatility actually depends on time in a way that is not predictable. This is sometimes identified as the reason for inaccurate predictions made by the Black-Scholes formula. The need for better ways of understanding the behavior of many natural processes has motivated the development of dynamic models of these processes that consider the influence of past events on current and future states of the system. This view is especially appropriate for the study of financial variables since predictions about their evolution rely strongly on knowledge of their past. It would be of interest, then, to use such a mixed model to study certain questions associated with the Black-Scholes model.

In this study, we consider parameter estimation of (1.1) with $H \in (0, 1)$ and $\alpha = 1$ using quasi-likelihood estimation. We assume the integrals concerning Brownian motion and fractional Brownian motion are the Itô and fractional Itô integrals, respectively. Thus, we can give the solution of (1.1) with $H \in (0, 1)$ as follows:

$$S_t^H = S_0^H \exp\left\{\sigma\left(B_t^H + B_t\right) + \mu t - \frac{1}{2}\sigma^2\left(t^{2H} + t\right)\right\}, \quad t \geqslant 0, \tag{1.2}$$

which is called geometric mixed fractional Brownian motion (gmfBm, in shortly). Denote

$$X_t^H = \log S_t^H - \log S_0^H = \sigma\left(B_t^H + B_t\right) + \mu t - \frac{1}{2}\sigma^2\left(t^{2H} + t\right)$$

for $t \geqslant 0$ and let gmfBm be observed at some discrete time instants $\{t_i = ih, i = 0, 1, 2, \ldots, n\}$ satisfying

$$h = h(n) \to 0, \quad nh \to \infty, \quad h^{1+\gamma} n \to 1$$

for some $\gamma > 0$, as n tends to infinity. Denote $Y_{t_i}^H = X_{t_i}^H - X_{t_{i-1}}^H$. We get a quasi-likelihood function of parameter μ and σ^2 as follows:

$$\begin{aligned} L_1(\mu, \sigma^2) :&= \prod_{i=1}^n f_{Y_{t_i}^H}(Y_{t_i}^H) \\ &= \prod_{i=1}^n \frac{1}{\sigma\sqrt{2\pi(h + h^{2H})}} \exp\left\{-\frac{1}{2\sigma^2(h + h^{2H})}\left(Y_{t_i}^H - \mu h + \frac{1}{2}\sigma^2\left(t_i^{2H} - t_{i-1}^{2H} + h\right)\right)^2\right\}, \end{aligned}$$

where $f_{Y_{t_i}^H}(\cdot)$ is the density of the random variable $Y_{t_i}^H = X_{t_i}^H - X_{t_{i-1}}^H$. By using the quasi-likelihood function, we determine that the estimators $\hat{\mu}_n$ and $\hat{\theta}_n$ of μ and $\theta = \sigma^2$ satisfy the following equations:

$$\begin{cases} \hat{\mu}_n = \dfrac{1}{nh} \displaystyle\sum_{i=1}^{n} \left(Y_{t_i}^H + \dfrac{1}{2} \hat{\theta}_n \hat{t}_i \right), \\ \hat{\theta}_n = \dfrac{1}{\displaystyle\sum_{i=1}^{n} (\hat{t}_i)^2} \left(-2n\hat{h} + 2\sqrt{n^2 \hat{h}^2 + \displaystyle\sum_{i=1}^{n} (\hat{t}_i)^2 \sum_{i=1}^{n} \left(Y_{t_i}^H - \hat{\mu}_n h \right)^2} \right), \end{cases} \tag{1.3}$$

where $\hat{t}_i = t_i - t_{i-1} + t_i^{2H} - t_{i-1}^{2H}$ and $\hat{h} = h + h^{2H}$.

Our main aim is to study the asymptotic behaviors of the two estimators. Many studies have investigated parameter estimation questions associated with the classical model. Few, however, use the quasi-likelihood method for the parameter estimation of stochastic equations. Given the Gaussian nature of the samples, we can expect the estimators to contain the characteristic of the quadratic variation of the sample trajectory. This means that when proving the asymptotic behavior of the estimators, the relevant characteristics of fractional Brownian motion can be used, which is convenient for research. Of course, we can also directly use p-variations to estimate these parameters; however, this is not easy to use in the case of multiple parameters. Obtaining these asymptotic behaviors requires certain interesting properties of fractional Brownian motion.

The rest of this paper is organized as follows. In Section 2, we present some preliminaries for fractional Brownian motion and the mixed Black-Scholes model. Section 3 and Section 4 give the strong consistency and asymptotic normality of the estimator $\hat{\theta}_n$, respectively. We discuss the two asymptotic behaviors in two cases where the parameter μ is known and unknown. To prove the two asymptotic behaviors, we need two results for fractional Brownian motion. When the observation interval is finite, these results are known, but when the observation length nh tends to infinity, these results are also valid. In Section 5, we discuss the asymptotic behavior of the estimator $\hat{\mu}_n$. In Section 6, we perform numerical validation and conduct empirical analyses of the estimators of $\hat{\mu}_n$ and $\hat{\theta}_n$. Finally, in Appendix 7.1 and 7.2, we prove some of the lemmas used in Section 3 and Section 4. This study is the first to introduce the quasi-likelihood function into parameter estimation for the Black-Scholes formula driven by mixed fractional Brownian motion. Additionally, we provide the asymptotic behavior of the estimators over an infinite interval. Numerical simulations demonstrate the effectiveness of this method for parameter estimation. We compare it with the classical Black-Scholes formula, highlighting the advantages and applicability of our findings for financial data modeling and empirical analysis.

2. Preliminaries

In this section, we briefly recall some basic results for fractional Brownian motion and mixed fractional Brownian motion. For more on this material, see Bender [4], Biagini et al [7], Cheridito-Nualart [12], Gradinaru et al [18], Hu [20], Mishura [23], Nourdin [26], Nualart [28], and Tudor [30], as well as the references therein.

2.1 Fractional Brownian motion

A zero mean Gaussian process $B^H = \{B_t^H, 0 \leqslant t \leqslant T\}$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$ is called fBm with Hurst index $H \in (0, 1)$ if $B_0^H = 0$ and

$$\mathrm{E}\left[B_t^H B_s^H\right] = \frac{1}{2}\left[t^{2H} + s^{2H} - |t-s|^{2H}\right]$$

for $t, s \geqslant 0$. Let $\mathcal{H}$ be the completion of the linear space $\mathcal{E}$ generated by the indicator functions $1_{[0,t]}, t \in [0,T]$ with respect to the inner product

$$\langle 1_{[0,s]}, 1_{[0,t]}\rangle_{\mathcal{H}} = \frac{1}{2}\left[t^{2H} + s^{2H} - |t-s|^{2H}\right].$$

The application

$$\mathcal{E} \ni \varphi \mapsto B^H(\varphi) := \int_0^T \varphi(s)\mathrm{d}B_s^H$$

is an isometry from $\mathcal{E}$ to the Gaussian space generated by B^H and can be extended to $\mathcal{H}$. Denote by $\mathcal{S}$ the set of smooth functionals of the form

$$F = f(B^H(\varphi_1), B^H(\varphi_2), \ldots, B^H(\varphi_n)),$$

where $f \in C_b^\infty(\mathbb{R}^n)$ (f and all its derivatives are bounded) and $\varphi_i \in \mathcal{H}$. The *derivative operator* D^H (the Malliavin derivative) of a functional F of the form above is defined as

$$D^H F = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(B^H(\varphi_1), B^H(\varphi_2), \ldots, B^H(\varphi_n))\varphi_j.$$

The derivative operator D^H is then a closable operator from $L^2(\Omega)$ into $L^2(\Omega;\mathcal{H})$. We denote by $\mathbb{D}^{1,2}$ the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,2} := \sqrt{E|F|^2 + E\|D^H F\|_{\mathcal{H}}^2}.$$

The *divergence integral* δ^H is the adjoint of the derivative operator D^H. That is, a random variable u in $L^2(\Omega;\mathcal{H})$ belongs to the domain of the divergence operator δ^H, denoted by $\mathrm{Dom}(\delta^H)$, if

$$\mathrm{E}\left|\langle D^H F, u\rangle_{\mathcal{H}}\right| \leqslant c\|F\|_{L^2(\Omega)}$$

for every $F \in \mathbb{D}^{1,2}$. In this case $\delta^H(u)$ is defined by the duality relationship

$$\mathrm{E}\left[F\delta^H(u)\right] = \mathrm{E}\langle D^H F, u\rangle_{\mathcal{H}} \tag{2.1}$$

for any $F \in \mathbb{D}^{1,2}$. We use the notation

$$\delta^H(u) = \int_0^T u_s \mathrm{d}B_s^H$$

to express the Skorokhod integral of a process u. The indefinite Skorokhod integral is defined as $\int_0^t u_s \mathrm{d}B_s^H = \delta^H(u 1_{[0,t]})$. If a process u is adapted, the integral is defined as the fractional Itô integral and the Itô formula

$$f(B_t^H, t) = f(0,0) + \int_0^t \frac{\partial}{\partial x} f(B_s^H, s)\mathrm{d}B_s^H + \int_0^t \frac{\partial}{\partial t} f(B_s^H, s)\mathrm{d}s + H\int_0^t \frac{\partial^2}{\partial x^2} f(B_s^H, s)s^{2H-1}\mathrm{d}s$$

holds for all $t \in [0,T]$ and $f \in C^{2\times 1}(\mathbb{R} \times \mathbb{R}_+)$.

2.2 Mixed fraction Black-Scholes model

Let $B = \{B_t, t \geqslant 0\}$ and $B^H = \left\{B_t^H, t \geqslant 0\right\}$ be mutually independent Brownian motion and fractional Brownian motion with Hurst index $H \in (0,1)$ defined on the same probability space $(\Omega, \mathcal{F}, P)$, respectively. The linear combination

$$M_t^H(\alpha,\beta) = aB_t + bB_t^H, \quad t \geqslant 0$$

is called mixed fractional Brownian motion, where $(a,b) \neq (0,0)$. Cheridito [10] showed that the process M^H is a weak semimartingale which is equivalent to the product of a constant and a Brownian motion if $a \neq 0$ and $\frac{3}{4} < H < 1$. Thus, a market based on mixed fractional Brownian motion is complete and arbitrage-free.

Consider geometric mixed fractional Brownian motion (gmfBm, for short)

$$S_t^H = S_0^H \exp\left\{\mu t - \frac{1}{2}\sigma^2\left(a^2 t + b^2 t^{2H}\right) + \sigma\left(aB_t + bB_t^H\right)\right\}$$

with $t \geqslant 0$ and $0 < H < 1$, where $\sigma > 0$ and $\mu \in \mathbb{R}$. Clearly, when $b = 0$, the process is geometric Brownian motion, and when $a = 0$ it is called the geometric fractional Brownian motion. Then, based on the Itô formula [1] we show that it is the solution of the equation

$$S_t^H = S_0^H + \mu\int_0^t S_s^H \mathrm{d}s + a\sigma\int_0^t S_s^H \mathrm{d}B_s + b\sigma\int_0^t S_s^H \mathrm{d}B_s^H \tag{2.2}$$

with $S_0^H > 0$ and $0 < H < 1$. This model is called the mixed fractional Black-Scholes model, where the integral with respect B is the classical Itô integral and the integral $\int_0^t S_s^H \mathrm{d}B_s$ is the fractional Itô integral. See Mishura [23] and Hu-Øksendal [21] for additional research associated with this model.

For simplicity, throughout this paper, we let $a = b = 1$ and H be known. The known Hurst index H means that the nature of the noise is basically clear, which is reasonable for the likelihood estimation of the model but not perfect. However, because of length considerations, we assume that the Hurst index H is known. We will consider in future research the case in which the Hurst index H is unknown and where parameters a and b are unknown. Denote $f(t) = \beta t - \frac{1}{2}\sigma^2 t^{2H}$ $(t \geqslant 0)$ with $\beta = \mu - \frac{1}{2}\sigma^2$ and

$$\begin{aligned} X_t^H &:= \log S_t^H - \log S_0^H \\ &= \sigma\left(B_t + B_t^H\right) + \mu t - \frac{1}{2}\sigma^2\left(t + t^{2H}\right) \\ &= \sigma\left(B_t + B_t^H\right) + f(t). \end{aligned} \tag{2.3}$$

Now, let gmfBm $S^H = \{S_t^H, t \geqslant 0\}$ be observed at some discrete time instants $\{t_i = ih, i = 0, 1, 2, \ldots, n\}$ satisfying the following conditions:

(C1) $h = h(n) \downarrow 0$ and $nh \to +\infty$ as $n \to \infty$;
(C2) There exists $\gamma > 0$ such that $nh^{1+\gamma} \longrightarrow 1$ as $n \to \infty$.

For the sample $\{X_{t_i}^H - X_{t_{i-1}}^H, i = 1, 2, \ldots n\}$ we consider the logarithm quasi-likelihood function

$$\log L_1(\mu,\theta) = -\frac{n}{2}\log\theta - \frac{n}{2}\log 2\pi\hat{h} - \frac{1}{2\theta\hat{h}}\sum_{i=1}^{n}\left(X_{t_i}^H - X_{t_{i-1}}^H - \mu h + \frac{1}{2}\theta\hat{t}_i\right)^2 \tag{2.4}$$

with $\theta = \sigma^2$, where $\hat{t}_i = t_i - t_{i-1} + t_i^{2H} - t_{i-1}^{2H} = h + t_i^{2H} - t_{i-1}^{2H}$ and $\hat{h} = h + h^{2H}$. Then, by taking the partial derivatives of (2.4) with respect to μ and θ, respectively, we derive the quasi-likelihood equations as follows:

$$\frac{\partial \ln L_1}{\partial\mu}(\mu,\theta) = \frac{h}{\theta\hat{h}}\sum_{i=1}^{n}\left(X_{t_i}^H - X_{t_{i-1}}^H - \mu h + \frac{1}{2}\theta\hat{t}_i\right) = 0, \tag{2.5}$$

$$\frac{\partial \ln L_1}{\partial \theta}(\mu,\theta) = -\frac{n}{2\theta} - \frac{1}{8\hat{h}}\sum_{i=1}^{n}\hat{t}_i^2 + \sum_{i=1}^{n}\frac{(X_{t_i}^H - X_{t_{i-1}}^H - \mu h)^2}{2\theta^2\hat{h}} = 0. \tag{2.6}$$

Thus, by solving equations (2.5) and (2.6) we show that the quasi-likelihood estimators $(\hat{\mu}_n, \hat{\theta}_n)$ satisfy the equations

$$\begin{cases} \hat{\mu}_n = \dfrac{1}{nh}\displaystyle\sum_{i=1}^{n}\left(X_{t_i}^H - X_{t_{i-1}}^H + \frac{1}{2}\hat{\theta}_n\hat{t}_i\right), \\ \hat{\theta}_n = \dfrac{-2n\hat{h} + 2\sqrt{n^2\hat{h}^2 + \displaystyle\sum_{i=1}^{n}(\hat{t}_i)^2\sum_{i=1}^{n}\left(X_{t_i}^H - X_{t_{i-1}}^H - \hat{\mu}_n h\right)^2}}{\displaystyle\sum_{i=1}^{n}(\hat{t}_i)^2} \end{cases} \tag{2.7}$$

for every $n \geqslant 1$ and $0 < H < 1$. If both μ and θ are known, this system of equations directly gives estimates of μ and θ. However, if both μ and θ are unknown, we can also obtain estimates of μ and θ by solving the system of equations as follows. By simplifying the first equation in (2.7), we get

$$\hat{\mu}_n = \frac{1}{nh}X_{t_n}^H + \frac{1}{2nh}\sum_{i=1}^{n}\hat{t}_i\hat{\theta}_n = \frac{1}{nh}X_{t_n}^H + \beta_n\hat{\theta}_n, \tag{2.8}$$

where $\beta_n = \frac{1}{2}(1 + (nh)^{2H-1})$. Substituting (2.8) into (2.7), and performing the necessary computations and simplifications, we obtain the equation

$$\rho_n(\hat{\theta}_n)^2 + 4n\hat{h}\alpha_n\theta_n = 4\alpha_n\sum_{i=1}^{n}\left(X_{t_i}^H - X_{t_{i-1}}^H - \frac{1}{n}X_{t_n}^H\right)^2, \tag{2.9}$$

where $\rho_n = \alpha_n - 4n\beta_n^2h^2$ with $\alpha_n = \sum\limits_{i=1}^{n}(\hat{t}_i)^2$. When $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ we solve this quadratic equation (2.9) to yield the explicit estimator of θ by substituting it into (2.8), we obtain the estimator $(\hat{\mu}_n, \hat{\theta}_n)$ of (μ, θ):

$$\begin{cases} \hat{\mu}_n = \dfrac{1}{nh}X_{t_n}^H + \dfrac{\beta_n}{\rho_n}\left(-2n\hat{h} + 2\sqrt{n^2\hat{h}^2 + \rho_n\displaystyle\sum_{i=1}^{n}\left(X_{t_i}^H - X_{t_{i-1}}^H - \frac{1}{n}X_{t_n}\right)^2}\right), \\ \hat{\theta}_n = \dfrac{1}{\rho_n}\left(-2n\hat{h} + 2\sqrt{n^2\hat{h}^2 + \rho_n\displaystyle\sum_{i=1}^{n}\left(X_{t_i}^H - X_{t_{i-1}}^H - \frac{1}{n}X_{t_n}\right)^2}\right) \end{cases} \tag{2.10}$$

for every $n \geqslant 1$. When $H = \frac{1}{2}$, we have $\rho_n = 0$ and the random variables

$$X_{t_i}^H - X_{t_{i-1}}^H \sim N\left((\mu - \sigma^2)h, 2\sigma^2\right), \quad i = 1, 2, \dots n$$

are independent identical distributions.Then, the above logarithmic quasi-likelihood function is a classical logarithm likelihood function, and we have

$$\begin{cases} \hat{\mu}_n = \dfrac{1}{nh}X_{t_n}^H + \dfrac{\beta_n}{2nh}\displaystyle\sum_{i=1}^{n}\left(X_{t_i}^H - X_{t_{i-1}}^H\right)^2 - \dfrac{\beta_n}{2n^2h}(X_{t_n}^H)^2, \\ \hat{\theta}_n = \dfrac{1}{2nh}\displaystyle\sum_{i=1}^{n}\left(X_{t_i}^H - X_{t_{i-1}}^H\right)^2 - \dfrac{1}{2n^2h}(X_{t_n}^H)^2. \end{cases} \tag{2.11}$$

The asymptotic behavior of the two estimators can be easily established. Thus, in the discussion later in this paper, unless otherwise stated, it is assumed that $H \neq \frac{1}{2}$.

3. Strong consistency of the estimator $\hat{\theta}_n$

In this section, we obtain the consistency of the estimator $\hat{\theta}_n$ in two cases. To obtain the strong consistency of the estimator $\hat{\theta}_n = \hat{\sigma}_n^2$, we need some lemmas; their proofs are given in Appendix 7.1. Using the strong law of large numbers for nonnegative random variables introduced by Etemadi [16], we can give the following strong laws of large numbers associated with fractional Brownian motion:

Lemma 3.1 *Let $B^H = \{B_t^H, t \geqslant 0\}$ be a fractional Brownian motion with Hurst index $0 < H < 1$ and let $t_i = ih, i = 0, 1, 2, \ldots, n$, such that the condition (C1) holds. Then, with probability one, we have*

$$G_n\left(B^H\right) := \frac{1}{nh^{2H}} \sum_{i=1}^{n} \left(B_{t_i}^H - B_{t_{i-1}}^H\right)^2 \longrightarrow 1 \quad (n \to \infty).$$

Proposition 3.2 *Let the conditions of Lemma 3.1 and the condition (C1) hold. Denote*

$$J_n\left(X^H\right) := \frac{1}{nh^{(2H)\wedge 1}} \sum_{i=1}^{n} \left(X_{t_i}^H - X_{t_{i-1}}^H\right)^2.$$

(i) *If $0 < H < \frac{1}{2}$, we have $J_n(X^H) \to \sigma^2$ almost surely, as n tends to infinity.*

(ii) *If $\frac{1}{2} < H < 1$ and the condition (C2) holds with $0 < \gamma < \frac{1}{4H-2}$, $J_n(X^H)$ converges to σ^2 almost surely, as n tends to infinity.*

(iii) *If $\frac{1}{2} < H < 1$ and the condition (C2) holds with $\gamma = \frac{1}{4H-2}$, $J_n(X^H)$ converges to $\sigma^2 + \frac{H^2}{4H-1}\sigma^4$ almost surely, as n tends to infinity.*

The proof of the proposition is given in Appendix 7.1. From the above proposition, we find that $J_n\left(X^H\right)$ is a strongly consistent estimator of σ^2 for $0 < H < \frac{1}{2}$. Moreover, for $\frac{1}{2} < H < 1$, if the conditions (C1) and (C2) hold with $0 < \gamma < \frac{1}{4H-2}$, $J_n\left(X^H\right)$ is a strongly consistent estimator of σ^2. For $\frac{1}{2} < H < 1$, if the conditions (C1) and (C2) hold with $\gamma = \frac{1}{4H-2}$, based on statement (iii), we can get the following estimator of $\theta = \sigma^2$:

$$\check{\theta}_n = \frac{4H-1}{2H^2} \left(\sqrt{1 + \frac{4H^2}{4H-1} J_n(X^H)} - 1 \right)$$

and the estimator is strongly consistent.

Corollary 3.3 *Let $\sigma > 0$ be known. If $0 < H < \frac{1}{2}$ and the condition (C1) holds, then*

$$\hat{H}_n = \frac{1}{2\log h} \left(\log \sum_{i=1}^{n} \left(X_{t_i}^H - X_{t_{i-1}}^H\right)^2 - \log(n\sigma^2) \right)$$

is a strongly consistent estimator of H.

Corollary 3.4 *Let $\sigma > 0$ be known and $\frac{1}{2} < H < 1$. If the conditions (C1) and (C2) hold with $\gamma = \frac{1}{4H-2}$, then*

$$\hat{H}_n = \frac{2(J_n\left(X^H\right) - \sigma^2) + \sqrt{4(J_n\left(X^H\right) - \sigma^2)^2 - \sigma^4(J_n\left(X^H\right) - \sigma^2)}}{\sigma^4}$$

is a strongly consistent estimator of H.

3.1 Case I: μ is known

Now, we consider the consistency of the estimator $\hat{\theta}_n$ given in (2.7) under the parameter μ being known. In this case, we have

$$\hat{\theta}_n = \frac{2n\hat{h}}{\sum\limits_{i=1}^{n}(\hat{t}_i)^2}\left(\sqrt{1+\Delta_n(H)}-1\right) \tag{3.1}$$

for every $n \geqslant 1$, where

$$\Delta_n(H) = \frac{1}{n^2\hat{h}^2}\sum_{i=1}^{n}(\hat{t}_i)^2 \cdot \sum_{i=1}^{n}\left(X_{t_i}^H - X_{t_{i-1}}^H - \mu h\right)^2. \tag{3.2}$$

Lemma 3.5 *Let μ and $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ be known and let the condition (C1) hold.*

(i) *For $0 < H < \frac{1}{2}$, we have $\lim\limits_{n\to 0}\Delta_n(H) = 0$ a.s.*
(ii) *For $\frac{1}{2} < H < 1$, if the condition (C2) holds with $0 < \gamma < \frac{1}{4H-2}$, $\lim\limits_{n\to 0}\Delta_n(H) = 0$ a.s.*

The proof of Lemma 3.5 is given in Appendix 7.1.

Theorem 3.6 *Let μ be known and the condition (C1) hold.*

(i) *If $0 < H < \frac{1}{2}$, the estimator $\hat{\theta}_n$ is strongly consistent.*
(ii) *If $\frac{1}{2} < H < 1$ and condition (C2) holds with $0 < \gamma < \frac{1}{4H-2}$, then the estimator $\hat{\theta}_n$ is strongly consistent.*

Proof Clearly, we have

$$\frac{h}{n\hat{h}}X_{t_n}^H \overset{a.s.}{\sim} Ch^{1-2H}\frac{1}{n}\left(\sigma B_{t_n} + \sigma B_{t_n}^H + \beta nh - \frac{1}{2}\sigma^2(nh)^{2H}\right) \xrightarrow{a.s.} 0 \quad (n\to\infty)$$

for $H \in (0, \frac{1}{2}]$ and

$$\frac{h}{n\hat{h}}X_{t_n}^H \overset{a.s.}{\sim} \frac{1}{n}\left(\sigma B_{t_n} + \sigma B_{t_n}^H + \beta nh - \frac{1}{2}\sigma^2(nh)^{2H}\right) \xrightarrow{a.s.} 0 \quad (n\to\infty)$$

for $H \in (\frac{1}{2}, 1)$, provided the conditions (C1) and (C2) hold with $0 < \gamma < \frac{1}{2H-1}$. It follows from (3.1), Lemma 3.5, the fact $\sqrt{1+x}-1 \sim \frac{1}{2}x$ $(x \to 0)$ and Proposition 3.2 that

$$\begin{aligned}
\hat{\theta}_n &\overset{a.s.}{\sim} \frac{n\hat{h}}{\sum_{i=1}^{n}(\hat{t}_i)^2}\cdot\Delta_n(H) = \frac{1}{n\hat{h}}\sum_{i=1}^{n}\left(X_{t_i}^H - X_{t_{i-1}}^H - \mu h\right)^2 \\
&= \frac{1}{n\hat{h}}\sum_{i=1}^{n}(X_{t_i}^H - X_{t_{i-1}}^H)^2 - 2\mu\frac{h}{n\hat{h}}\sum_{i=1}^{n}(X_{t_i}^H - X_{t_{i-1}}^H) + \mu^2\frac{h^2}{\hat{h}} \\
&= \frac{nh^{(2H)\wedge 1}}{n\hat{h}}J_n\left(X^H\right) - 2\mu\frac{h}{n\hat{h}}X_{t_n}^H + \mu^2\frac{h^2}{\hat{h}} \xrightarrow{a.s.} \sigma^2 \quad (n\to\infty)
\end{aligned} \tag{3.3}$$

for $0 < \gamma < \frac{1}{4H-2} < \frac{1}{2H-1}$. This completes the proof. □

3.2 Case II: μ is unknown

Here, we assume the parameters μ and σ are both unknown and obtain the strong consistency of the estimator $\hat{\theta}_n$ given by (2.10). In this case, when $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ we have that

$$\hat{\theta}_n = \frac{2n\hat{h}}{\rho_n}\left(\sqrt{1+\tilde{\Delta}_n(H)}-1\right) \tag{3.4}$$

for every $n \geqslant 1$, where $\beta_n = \frac{1}{2}(1+(nh)^{2H-1})$, $\rho_n = \alpha_n - 4n\beta_n^2 h^2$ with $\alpha_n = \sum_{i=1}^{n}(\hat{t}_i)^2$ and

$$\tilde{\Delta}_n(H) = \frac{\rho_n}{n^2\hat{h}^2}\sum_{i=1}^{n}\left(X_{t_i}^H - X_{t_{i-1}}^H - \frac{1}{n}X_{t_n}\right)^2. \tag{3.5}$$

Lemma 3.7 *Let condition (C1) hold.*

(i) *For* $0<H<\frac{1}{2}$ *, we have* $\lim_{n\to 0}\tilde{\Delta}_n(H)=0$ *a.s.*

(ii) *For* $\frac{1}{2}<H<1$ *, if condition (C2) holds with* $0<\gamma<\frac{1}{4H-2}$ *,* $\lim_{n\to 0}\tilde{\Delta}_n(H)=0$ *a.s.*

The proof of Lemma 3.7 is given in Appendix 7.1.

Theorem 3.8 *Let μ be unknown, and let condition (C1) hold.*

(i) *For* $0<H<\frac{1}{2}$ *, the estimator* $\hat{\theta}_n$ *is strongly consistent.*

(ii) *For* $\frac{1}{2}<H<1$ *, if condition (C2) holds with* $0<\gamma<\frac{1}{4H-2}$ *, then* $\hat{\theta}_n$ *is strongly consistent.*

Proof The conditions in statement (i) or statement (ii) imply that

$$\frac{1}{n\sqrt{\hat{h}}}X_{t_n}^H = \frac{1}{n\sqrt{\hat{h}}}\left(\sigma B_{t_n} + \sigma B_{t_n}^H + \beta nh - \frac{1}{2}\sigma^2(nh)^{2H}\right) \xrightarrow{a.s.} 0 \quad (n\to\infty).$$

Based on (3.4), the fact $\sqrt{1+x}-1 \sim \frac{1}{2}x$ $(x\to 0)$ and Proposition 3.2, we have

$$\begin{aligned}\hat{\theta}_n &\overset{a.s.}{\sim} \frac{1}{n\hat{h}}\sum_{i=1}^{n}\left(X_{t_i}^H - X_{t_{i-1}}^H - \frac{1}{n}X_{t_n}\right)^2 \\ &= \frac{1}{n\hat{h}}\sum_{i=1}^{n}(X_{t_i}^H - X_{t_{i-1}}^H)^2 - \frac{1}{n^2\hat{h}}(X_{t_n}^H)^2 \xrightarrow{a.s.} \sigma^2 \quad (n\to\infty)\end{aligned}$$

for all $H\in(0,\frac{1}{2})\cup(\frac{1}{2},1)$. This completes the proof. □

4. Asymptotic normality of the estimator $\hat{\theta}_n$

In this section, we consider the asymptotic distribution of $\hat{\theta}_n$.

Proposition 4.1 *Let the conditions in Lemma 3.1 hold.*

(i) *For* $0<H<\frac{3}{4}$ *we have*

$$\sqrt{n}\left(G_n(B^H)-1\right) \longrightarrow N\left(0, 2+\lambda_H\right) \quad (n\to\infty)$$

in distribution, where

$$\lambda_H = \sum_{n=1}^{\infty}\left((n+1)^{2H}+(n-1)^{2H}-2n^{2H}\right)^2.$$

(ii) *For* $H=\frac{3}{4}$ *, we have*

$$\sqrt{\frac{n}{\log n}}\left(G_n(B^H)-1\right) \longrightarrow N\left(0,\frac{9}{4}\right) \quad (n\to\infty)$$

in distribution.

(iii) *For* $\frac{3}{4}<H<1$ *, we have*

$$n^{2-2H}\left(G_n(B^H)-1\right) \longrightarrow \frac{2H^2(2H-1)}{4H-3}\mathscr{R}_H \quad (n\to\infty)$$

in L^2 *, where* $\mathscr{R}_H$ *denotes a Rosenblatt random variable with* $E(\mathscr{R}_H)^2=1$.

The lemma is an insignificant extension for some known results, and its proof will be omitted (see, for example, Theorem 5.4, Proposition 5.4, and Theorem 5.5 in Tudor [30]). In fact, $t_n = nh = T < \infty$. Such convergence has been studied and can be found in Breuer and Major [9], Dobrushin and Major [13], Giraitis and Surgailis [19], Nourdin [25], Nourdin and Reveillac [27] and Tudor [30]. Meanwhile, for more material on the Rosenblatt distribution and the related process, see Tudor [30].

4.1 Parameter μ is known

Here, we obtain the asymptotic distribution of $\hat{\theta}_n$, provided μ is known. Based on (2.7), Lemma 3.5, and the fact $\sqrt{1+x}-1=\frac{1}{2}x+O\left(x^2\right)\ (x\to 0)$, we get

$$\begin{aligned}\sqrt{n}\left(\hat{\theta}_n-\sigma^2\right) &\overset{a.s.}{=} \sqrt{n}\left(\frac{1}{n\hat{h}}\sum_{i=1}^{n}\left(X_{t_i}^H-X_{t_{i-1}}^H-\mu h\right)^2-\sigma^2\right)\\ &\quad+\frac{2n\hat{h}}{\sum_{i=1}^{n}(\hat{t}_i)^2}O(\Delta_n(H)^2)\cdot\sqrt{n}\\ &=\sqrt{n}\left(\frac{h^{(2H)\wedge 1}}{\hat{h}}J_n\left(X^H\right)-\sigma^2\right)-2\mu\frac{h}{\hat{h}\sqrt{n}}X_{t_n}^H+\mu^2\frac{h^2}{\hat{h}}\sqrt{n}\\ &\quad+\frac{2n\hat{h}}{\sum\limits_{i=1}^{n}(\hat{t}_i)^2}O(\Delta_n(H)^2)\cdot\sqrt{n}\end{aligned}\tag{4.1}$$

under the conditions of Lemma 3.5.

Lemma 4.2 *Let $J_n(X^H)$ be defined in Proposition 3.2.*

(i) *For $0<H<\frac{1}{4}$, we have*

$$\sqrt{n}\left(J_n\left(X^H\right)-\sigma^2\right)\longrightarrow\begin{cases}N\left(0,(2+\lambda_H)\sigma^4\right), & 0<\gamma<1-4H,\\ N\left(\sigma^2,(2+\lambda_H)\sigma^4\right), & \gamma=1-4H\end{cases}$$

in distribution, as n tends to infinity.

(ii) *For $\frac{3}{4}<H<1$, we have*

$$\sqrt{n}\left(J_n\left(X^H\right)-\sigma^2\right)\longrightarrow N\left(0,2\sigma^4\right)\quad(n\to\infty),$$

provided $0<\gamma<(4H-3)\wedge\frac{1}{8H-3}$. Moreover, when $\gamma=(4H-3)\wedge\frac{1}{8H-3}$, we have

$$\sqrt{n}\left(J_n\left(X^H\right)-\sigma^2\right)\longrightarrow\begin{cases}N\left(\sigma^2,2\sigma^4\right), & 4H-3<\frac{1}{8H-3},\\ N\left(\frac{H^2}{4H-1}\sigma^4\sigma^4,2\sigma^4\right), & 4H-3>\frac{1}{8H-3},\\ N\left(\sigma^2+\frac{H^2}{4H-1}\sigma^4,2\sigma^4\right), & 4H-3=\frac{1}{8H-3}\end{cases}$$

in distribution, as n tends to infinity.

The proof of Lemma 4.2 is given in Appendix 7.2.

Lemma 4.3 *Let the conditions $(C1)$ and $(C2)$ hold and denote*

$$\Xi_n(H):=\frac{n\hat{h}}{\sum_{i=1}^{n}(\hat{t}_i)^2}\Delta_n(H)^2\cdot\sqrt{n}$$

for all $0<H<1$ and $n\geqslant 1$, where $\Delta_n(H)$ is given in Lemma 3.5.

(i) *For* $0 < H < \frac{1}{2}$, *if* $0 < \gamma < 3 - 4H$, *we have* $\lim_{n\to 0} \Xi_n(H) = 0$ *a.s.*
(ii) *For* $\frac{1}{2} < H < 1$, *if* $0 < \gamma < \frac{1}{8H-3}$, *we have* $\lim_{n\to 0} \Xi_n(H) = 0$ *a.s.*

The proof of Lemma 4.3 is given in Appendix 7.2.

Theorem 4.4 *Given* $0 < H < \frac{1}{2}$, *let* μ *be known and let the conditions (C1) and (C2) hold.*

(i) *Let* $0 < H < \frac{1}{4}$. *We then have*

$$\sqrt{n}\left(\hat{\theta}_n - \sigma^2\right) \longrightarrow \begin{cases} N\left(0, (2+\lambda_H)\sigma^4\right), & 0 < \gamma < 1 - 4H, \\ N\left(\sigma^2, (2+\lambda_H)\sigma^4\right), & \gamma = 1 - 4H \end{cases}$$

in distribution, as n tends to infinity.
(ii) *Let* $\frac{3}{4} < H < 1$. *If* $0 < \gamma < (4H-3) \wedge \frac{1}{8H-3}$, *we then have*

$$\sqrt{n}\left(\hat{\theta}_n - \sigma^2\right) \longrightarrow N\left(0, 2\sigma^4\right)$$

in distribution, as n tends to infinity. If $\gamma = (4H-3) \wedge \frac{1}{8H-3}$, *we then have*

$$\sqrt{n}\left(\hat{\theta}_n - \sigma^2\right) \longrightarrow \begin{cases} N\left(\sigma^2, 2\sigma^4\right), & 4H - 3 < \frac{1}{8H-3}, \\ N\left(\frac{H^2}{4H-1}\sigma^4, 2\sigma^4\right), & 4H - 3 > \frac{1}{8H-3}, \\ N\left(\sigma^2 + \frac{H^2}{4H-1}\sigma^4, 2\sigma^4\right), & 4H - 3 = \frac{1}{8H-3} \end{cases}$$

in distribution, as n tends to infinity.

Proof Clearly, we have

$$\lim_{n\to\infty} \sqrt{n} h^{2-2H} = \begin{cases} 0, & 0 < \gamma < 3 - 4H, \\ 1, & \gamma = 3 - 4H, \\ +\infty, & \gamma > 3 - 4H \end{cases}$$

for $0 < H < \frac{1}{2}$, moreover, $\lim_{n\to\infty} n^{2H-\frac{1}{2}} h \to 0$ for $0 < H \leqslant \frac{1}{4}$. It follows from the facts,

$$\frac{1}{\sqrt{n}} h^{1-2H} B_{t_n} \longrightarrow 0, \quad \frac{1}{\sqrt{n}} h^{1-2H} B^H_{t_n} \longrightarrow 0 \quad (n \to \infty)$$

almost surely for all $0 < H < \frac{1}{2}$, that

$$\frac{h}{\hat{h}\sqrt{n}} X^H_{t_n} \overset{a.s.}{\sim} \frac{1}{\sqrt{n}} h^{1-2H} X^H_{t_n} = \frac{1}{\sqrt{n}} h^{1-2H} \left(\sigma B_{t_n} + \sigma B^H_{t_n} + \beta nh - \frac{1}{2}\sigma^2 (nh)^{2H}\right)$$
$$\longrightarrow \begin{cases} 0, & 0 < \gamma < 3 - 4H, \\ \beta, & \gamma = 3 - 4H, \\ +\infty, & \gamma > 3 - 4H \end{cases}$$

for all $0 < H < \frac{1}{4}$ almost surely. Combining (3.3), (4.1), Lemma 4.2, Lemma 4.3 and Slutsky's theorem, we obtain statement (i).

Now, we consider the case $\frac{3}{4} < H < 1$. Similar to statement (i), we have that

$$\lim_{n\to\infty} \sqrt{n} h = \begin{cases} 0, & 0 < \gamma < 1, \\ 1, & \gamma = 1, \\ +\infty, & \gamma > 1 \end{cases} \quad \text{and} \quad \lim_{n\to\infty} n^{2H-\frac{1}{2}} h^{2H} = \begin{cases} 0, & 0 < \gamma < \frac{1}{4H-1}, \\ 1, & \gamma = \frac{1}{4H-1}, \\ +\infty, & \gamma > \frac{1}{4H-1}. \end{cases} \tag{4.2}$$

Based on the convergence $\lim_{n\to\infty} \frac{1}{\sqrt{n}} B_{t_n} \overset{a.s.}{=} 0$ and $\lim_{n\to\infty} \frac{1}{\sqrt{n}} B^H_{t_n} \overset{a.s.}{=} 0$ for $0 < \gamma < \frac{1}{2H-1}$, we get that

$$\frac{h}{\hat{h}\sqrt{n}}X^H_{t_n} \overset{a.s.}{\sim} \frac{X^H_{t_n}}{\sqrt{n}} = \frac{1}{\sqrt{n}}\left(\sigma B_{t_n} + \sigma B^H_{t_n} + \beta nh - \frac{1}{2}\sigma^2(nh)^{2H}\right)$$
$$\overset{a.s.}{\longrightarrow} \begin{cases} 0, & 0<\gamma<\frac{1}{4H-1}, \\ -\frac{1}{2}\sigma^2, & \gamma=\frac{1}{4H-1}, \\ +\infty, & \gamma>\frac{1}{4H-1} \end{cases} \tag{4.3}$$

as n tends to infinity. Thus, statement (ii) follows from (3.3), (4.1), Lemma 4.2, Lemma 4.3 and Slutsky's theorem, since $\frac{1}{8H-3}<\frac{1}{4H-1}<1<\frac{1}{2H-1}$, provided $\frac{3}{4}<H<1$. □

At the end of this subsection, we consider the asymptotic distribution of $\hat{\theta}_n$ with $\frac{1}{4}<H\leqslant\frac{3}{4}$. We have

$$\sqrt{n}h^{1-2H} = \left(nh^{1+\gamma}\right)^{\frac{1-2H}{1+\gamma}} n^{\frac{4H-1+\gamma}{2(1+\gamma)}} \longrightarrow +\infty$$

with $\frac{1}{4}\leqslant H<1$ and $\gamma>0$, as n tends to infinity. However, similar to (4.1), we also have that

$$\begin{aligned} &\sqrt{n}\left(\hat{\theta}_n - \sigma^2 - h^{1-2H}\sigma^2\right) \\ &\overset{a.s.}{=} \sqrt{n}\left(\frac{1}{n\hat{h}}\sum_{i=1}^n\left(X^H_{t_i} - X^H_{t_{i-1}} - \mu h\right)^2 - \sigma^2 - h^{1-2H}\sigma^2\right) + 2\Xi_n(H) \\ &= \sqrt{n}\left(\frac{h^{(2H)\wedge 1}}{\hat{h}}J_n\left(X^H\right) - \sigma^2 - h^{1-2H}\sigma^2\right) \\ &\quad - 2\mu\frac{h}{\hat{h}\sqrt{n}}X^H_{t_n} + \mu^2\frac{h^2}{\hat{h}}\sqrt{n} + 2\Xi_n(H) \end{aligned} \tag{4.4}$$

for all $n\geqslant 1$. Similarly, we have

$$\sqrt{n}h^{2H-1} = \left(nh^{1+\gamma}\right)^{\frac{2H-1}{1+\gamma}} n^{\frac{3-4H+\gamma}{2(1+\gamma)}} \longrightarrow +\infty$$

with $H\leqslant\frac{3}{4}$ and $\gamma>0$, as n tends to infinity. Then we also have that

$$\begin{aligned} &\sqrt{n}\left(\hat{\theta}_n - \sigma^2 - h^{2H-1}\sigma^2\right) \\ &\overset{a.s.}{=} \sqrt{n}\left(\frac{1}{n\hat{h}}\sum_{i=1}^n\left(X^H_{t_i} - X^H_{t_{i-1}} - \mu h\right)^2 - \sigma^2 - h^{2H-1}\sigma^2\right) + 2\Xi_n(H) \\ &= \sqrt{n}\left(\frac{h^{(2H)\wedge 1}}{\hat{h}}J_n\left(X^H\right) - \sigma^2 - h^{2H-1}\sigma^2\right) \\ &\quad - 2\mu\frac{h}{\hat{h}\sqrt{n}}X^H_{t_n} + \mu^2\frac{h^2}{\hat{h}}\sqrt{n} + 2\Xi_n(H) \end{aligned} \tag{4.5}$$

for all $n\geqslant 1$.

Lemma 4.5 *Let the conditions in Proposition 3.2 hold.*

(i) *For $\frac{1}{2}<H<\frac{3}{4}$, we have*

$$\sqrt{n}\left(J_n\left(X^H\right) - \sigma^2 - h^{2H-1}\sigma^2\right) \longrightarrow \begin{cases} N\left(0, 2\sigma^4\right), & 0<\gamma<\frac{1}{8H-3}, \\ N\left(\frac{H^2}{4H-1}\sigma^4, 2\sigma^4\right), & \gamma=\frac{1}{8H-3} \end{cases}$$

in distribution, as n tends to infinity.

(ii) *For $H=\frac{3}{4}$, we have*

$$\sqrt{\frac{n}{\log n}}\left(J_n\left(X^{\frac{3}{4}}\right) - \sigma^2 - \sqrt{h}\sigma^2\right) \longrightarrow N\left(0, \frac{9}{4}\sigma^4\right)$$

in distribution, as n tends to infinity, for all $0 < \gamma \leqslant \frac{1}{3}$.

(iii) *For* $\frac{1}{4} < H < \frac{1}{2}$, *we have*

$$\sqrt{n}\left(J_n\left(X^H\right) - \sigma^2 - h^{1-2H}\sigma^2\right) \longrightarrow \begin{cases} N\left(0, (2+\lambda_H)\sigma^4\right), & 0 < \gamma < 3-4H, \\ N\left(\beta^2, (2+\lambda_H)\sigma^4\right), & \gamma = 3-4H \end{cases}$$

in distribution, as n tends to infinity.

The proof of Lemma 4.5 is given in Appendix 7.2.

Theorem 4.6 *Let* $\frac{1}{2} < H \leqslant \frac{3}{4}$ *and conditions (C1) and (C2) hold.*

(i) *If* $\frac{1}{2} < H < \frac{3}{4}$, *we then have*

$$\sqrt{n}\left(\hat{\theta}_n - \sigma^2 - h^{2H-1}\sigma^2\right) \longrightarrow \begin{cases} N\left(0, 2\sigma^4\right), & 0 < \gamma < \frac{1}{8H-3}, \\ N\left(\sigma^4 H^2, 2\sigma^4\right), & \gamma = \frac{1}{8H-3} \end{cases}$$

in distribution, as n tends to infinity.

(ii) *If* $H = \frac{3}{4}$, *we have*

$$\sqrt{\frac{n}{\log n}}\left(\hat{\theta}_n - \sigma^2 - \sqrt{h}\sigma^2\right) \longrightarrow N\left(0, \frac{9}{4}\sigma^4\right)$$

for all $0 < \gamma \leqslant \frac{1}{3}$ *in distribution, as n tends to infinity.*

(iii) *For* $\frac{1}{4} < H < \frac{1}{2}$, *we have*

$$\sqrt{n}\left(\hat{\theta}_n) - \sigma^2 - h^{1-2H}\sigma^2\right) \longrightarrow \begin{cases} N\left(0, (2+\lambda_H)\sigma^4\right), & 0 < \gamma < 3-4H, \\ N\left(\frac{1}{4}\sigma^4, (2+\lambda_H)\sigma^4\right), & \gamma = 3-4H \end{cases}$$

in distribution, as n tends to infinity.

Proof Statement (i) follows from (3.3), (4.2), (4.3), (4.5), Lemma 4.3, Lemma 4.5 and Slutsky's theorem.

Now, let now $H = \frac{3}{4}$. We have

$$\lim_{n\to\infty} E\left(\frac{B^H_{t_n}}{\sqrt{n\log n}}\right)^2 = \lim_{n\to\infty} \frac{(nh)^{\frac{3}{2}}}{n\log n} = \lim_{n\to\infty} \frac{1}{\log n}\sqrt{n}h^{\frac{3}{2}} = 0$$

provided $0 < \gamma \leqslant 2 = \frac{1}{2H-1}$, and

$$\lim_{n\to\infty} \frac{1}{\sqrt{\log n}}\sqrt{n}h = \begin{cases} 0, & 0 < \gamma \leqslant 1, \\ +\infty, & \gamma > 1, \end{cases}$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{\log n}} nh^{\frac{3}{2}} = \begin{cases} 0, & 0 < \gamma \leqslant \frac{1}{2}, \\ +\infty, & \gamma > \frac{1}{2}. \end{cases}$$

It follows that

$$\begin{aligned} \frac{h}{\hat{h}\sqrt{n\log n}} X^H_{t_n} &\overset{a.s.}{\sim} \frac{1}{\sqrt{n\log n}} X^H_{t_n} \\ &= \frac{\sigma}{\sqrt{n\log n}} B_{t_n} + \frac{\sigma}{\sqrt{n\log n}} B^H_{t_n} + \frac{\beta}{\sqrt{\log n}}\sqrt{n}h - \frac{\sigma^2}{2\sqrt{\log n}} nh^{\frac{3}{2}} \\ &\longrightarrow \begin{cases} 0, & 0 < \gamma \leqslant \frac{1}{2}, \\ -\infty, & \gamma > \frac{1}{2} \end{cases} \end{aligned}$$

in probability, as n tends to infinity. Combining (3.3), (4.5), Lemma 4.3, Lemma 4.5 and Slutsky's theorem, we obtain statement (ii).

Statement (iii) follows from (3.3), (4.4), Lemma 4.3, Lemma 4.5, and Slutsky's theorem. □

4.2 Parameter μ is unknown

In this subsection, we obtain the asymptotic distribution of $\hat{\theta}_n$, provided μ is unknown. Base on (2.10) and Lemma 3.7, similar to (4.1), we have

$$\begin{aligned}\sqrt{n}\left(\hat{\theta}_n-\sigma^2\right) &\overset{a.s.}{=} \sqrt{n}\left(\frac{1}{n\hat{h}}\sum_{i=1}^{n}\left(X^H_{t_i}-X^H_{t_{i-1}}-\frac{1}{n}X^H_{t_n}\right)^2-\sigma^2\right)\\ &\quad+\frac{n\hat{h}}{\rho_n}O\left(\tilde{\Delta}_n(H)\right)^2\sqrt{n}\\ &=\sqrt{n}\left(\frac{h^{(2H)\wedge 1}}{\hat{h}}J_n(X^H)-\sigma^2\right)-\frac{\sqrt{n}}{n^2\hat{h}}\left(X^H_{t_n}\right)^2+\frac{n\hat{h}}{\rho_n}O\left(\tilde{\Delta}_n(H)\right)^2\sqrt{n} \qquad (4.6)\end{aligned}$$

for all $H\in(0,\frac{1}{2})\cup(\frac{1}{2},1)$, where $\tilde{\Delta}_n(H)$ is given in Lemma 3.7. As a corollary of Lemma 3.7, the following lemma gives the estimation of the remainder term:

$$\tilde{\Xi}_n(H):=\frac{n\hat{h}}{\rho_n}\left(\tilde{\Delta}_n(H)\right)^2\sqrt{n}.$$

Lemma 4.7 *Let conditions (C1) and (C2) hold.*

(i) *For* $0<H\leqslant\frac{3}{8}$, *we have* $\lim_{n\to 0}\tilde{\Xi}_n(H)=0$ *a.s.*
(ii) *For* $\frac{3}{8}<H<\frac{1}{2}$, *if* $0<\gamma<\frac{3-4H}{8H-3}$, *we have* $\lim_{n\to 0}\tilde{\Xi}_n(H)=0$ *a.s.*
(iii) *For* $\frac{1}{2}<H<1$, *if* $0<\gamma<\frac{1}{8H-3}$, *we have* $\lim_{n\to 0}\tilde{\Xi}_n(H)=0$ *a.s.*

The proof of Lemma 4.7 is given in Appendix 7.2.

Theorem 4.8 *Let μ be unknown and conditions (C1) and (C2) hold.*

(i) *Let* $0<H<\frac{1}{4}$. *We then have*

$$\sqrt{n}\left(\hat{\theta}_n-\sigma^2\right)\longrightarrow\begin{cases}N\left(0,(2+\lambda_H)\sigma^4\right), & 0<\gamma<1-4H,\\ N\left(\sigma^2,(2+\lambda_H)\sigma^4\right), & \gamma=1-4H\end{cases}$$

in distribution, as n tends to infinity.

(ii) *Let* $\frac{3}{4}<H<1$ *and* $0<\gamma<(4H-3)\wedge\frac{1}{8H-3}$; *we then have*

$$\sqrt{n}\left(\hat{\theta}_n-\sigma^2\right)\longrightarrow N\left(0,2\sigma^4\right)$$

in distribution, as n tends to infinity. If $\gamma=(4H-3)\wedge\frac{1}{8H-3}$, *we then have*

$$\sqrt{n}\left(\hat{\theta}_n-\sigma^2\right)\longrightarrow\begin{cases}N\left(\sigma^2,2\sigma^4\right), & 4H-3<\frac{1}{8H-3},\\ N\left(\frac{H^2}{4H-1}\sigma^4+\frac{1}{4}\sigma^4,2\sigma^4\right), & 4H-3>\frac{1}{8H-3},\\ N\left(\sigma^2+\frac{H^2}{4H-1}\sigma^4+\frac{1}{4}\sigma^4,2\sigma^4\right), & 4H-3=\frac{1}{8H-3}\end{cases}$$

in distribution, as n tends to infinity.

Proof Clearly, as n tends to infinity, we have $n^{2H-\frac{3}{4}}h^H\to 0$ for all $0<H\leqslant\frac{3}{8}$. Moreover, we also have

$$\lim_{n\to\infty} n^{\frac{1}{4}}h^{1-H} = \begin{cases} 0, & 0<\gamma<3-4H, \\ 1, & \gamma = 3-4H, \\ \infty, & \gamma > 3-4H \end{cases}$$

for $0 < H < \frac{1}{2}$. It follows that

$$\frac{1}{n^{\frac{3}{4}}h^H}X^H_{t_n} = \sigma\frac{B_{t_n}}{n^{\frac{3}{4}}h^H} + \sigma\frac{B^H_{t_n}}{n^{\frac{3}{4}}h^H} + \beta n^{\frac{1}{4}}h^{1-H} - \frac{1}{2}\sigma^2 n^{2H-\frac{3}{4}}h^H \longrightarrow \begin{cases} 0, & 0<\gamma<3-4H, \\ \beta, & \gamma=3-4H, \\ \infty, & \gamma>3-4H, \end{cases}$$

in probability for all $0 < H < \frac{1}{4}$, as n tends to infinity, since $3-4H < \frac{3-4H}{8H-3}$. Combining this with (3.3) (4.6), Lemma 4.2, Lemma 4.7 and Slutsky's theorem, we obtain statement (i).

Next, for $\frac{3}{4} < H < 1$, we have

$$\lim_{n\to\infty} n^{\frac{1}{4}}\sqrt{h} = \begin{cases} 0, & 0<\gamma<1, \\ 1, & \gamma=1, \\ \infty, & \gamma>1, \end{cases} \quad \text{and} \quad \lim_{n\to\infty} n^{2H-\frac{3}{4}}h^{2H-\frac{1}{2}} = \begin{cases} 0, & 0<\gamma<\frac{1}{8H-3}, \\ 1, & \gamma=\frac{1}{8H-3}, \\ \infty, & \gamma>\frac{1}{8H-3}. \end{cases} \tag{4.7}$$

It follows from the convergence

$$\lim_{n\to\infty} E\left(\frac{B^H_{t_n}}{n^{\frac{3}{4}}\sqrt{h}}\right)^2 = \lim_{n\to\infty} n^{2H-\frac{3}{2}}h^{2H-1} = \begin{cases} 0, & 0<\gamma<\frac{1}{4H-3}, \\ 1, & \gamma=\frac{1}{4H-3}, \\ +\infty, & \gamma>\frac{1}{4H-3}. \end{cases}$$

that

$$\begin{aligned} \frac{1}{n^{\frac{3}{4}}\sqrt{\hat{h}}}X^H_{t_n} &\overset{a.s.}{\sim} \frac{1}{n^{\frac{3}{4}}\sqrt{h}}X^H_{t_n} = \sigma\frac{B_{t_n}}{n^{\frac{3}{4}}\sqrt{h}} + \sigma\frac{B^H_{t_n}}{n^{\frac{3}{4}}\sqrt{h}} + \mu n^{\frac{1}{4}}\sqrt{h} - \frac{1}{2}\sigma^2 n^{2H-\frac{3}{4}}h^{2H-\frac{1}{2}} \\ &\xrightarrow{a.s.} \begin{cases} 0, & 0<\gamma<\frac{1}{8H-3}, \\ -\frac{1}{2}\sigma^2, & \gamma=\frac{1}{8H-3}, \\ \infty, & \gamma>\frac{1}{8H-3}, \end{cases} \end{aligned}$$

as n tends to infinity. Combining this with (3.3) (4.6), Lemma 4.2, Lemma 4.7 and Slutsky's theorem, we obtain statement (ii). □

Theorem 4.9 *Given $\frac{1}{4} < H \leqslant \frac{3}{4}$, let μ be unknown and conditions (C1) and (C2) hold.*

(i) *For $\frac{1}{2} < H < \frac{3}{4}$, we have*

$$\sqrt{n}\left(\hat{\theta}_n - \sigma^2 - h^{2H-1}\sigma^2\right) \longrightarrow \begin{cases} N\left(0, 2\sigma^4\right), & 0<\gamma<\frac{1}{8H-3}, \\ N\left(\frac{H^2}{4H-1}\sigma^4 + \frac{1}{4}\sigma^2, 2\sigma^4\right), & \gamma=\frac{1}{8H-3} \end{cases}$$

in distribution, as n tends to infinity.

(ii) *For $H = \frac{3}{4}$, we have*

$$\sqrt{\frac{n}{\log n}}\left(\hat{\theta}_n - \sigma^2 - \sqrt{h}\sigma^2\right) \longrightarrow N\left(0, \frac{9}{4}\sigma^4\right)$$

for all $0 < \gamma \leqslant \frac{1}{3}$ in distribution, as n tends to infinity.

(iii) *For $\frac{1}{4} < H < \frac{1}{2}$, we have*

$$\sqrt{n}\left(\hat{\theta}_n - \sigma^2 - h^{1-2H}\sigma^2\right) \longrightarrow N\left(0, (2+\lambda_H)\sigma^4\right)$$

in distribution, as n tends to infinity.

Proof Let $\frac{1}{2} < H < \frac{3}{4}$. It follows from (4.7) that

$$\begin{aligned}\frac{1}{n^{\frac{3}{4}}\sqrt{h}}X^H_{t_n} &= \sigma\frac{B_{t_n}}{n^{\frac{3}{4}}\sqrt{h}} + \sigma\frac{B^H_{t_n}}{n^{\frac{3}{4}}\sqrt{h}} + \frac{\beta t_n - \frac{1}{2}\sigma^2(t_n)^{2H}}{n^{\frac{3}{4}}\sqrt{h}}\\ &= \sigma\frac{B_{t_n}}{n^{\frac{3}{4}}\sqrt{h}} + \sigma\frac{B^H_{t_n}}{n^{\frac{3}{4}}\sqrt{h}} + \beta n^{\frac{1}{4}}\sqrt{h} - \frac{1}{2}\sigma^2 n^{2H-\frac{3}{4}}h^{2H-\frac{1}{2}}\\ &\longrightarrow \begin{cases}0, & 0<\gamma<3-4H,\\ \beta, & \gamma = 3-4H,\\ \infty, & \gamma > 3-4H,\end{cases}\end{aligned}$$

in probability for all $\frac{1}{2} < H < \frac{3}{4}$, as n tends to infinity since $1 < 8H - 3 < 3$. Thus, statement (i) follows from Lemma 4.5, Lemma 4.7, (3.3), Slutsky's theorem, and the decomposition

$$\begin{aligned}&\sqrt{n}\left(\hat{\theta}_n - \sigma^2 - h^{2H-1}\sigma^2\right)\\ &\overset{a.s.}{=} \sqrt{n}\left(\frac{1}{n\hat{h}}\sum_{i=1}^n\left(X^H_{t_i} - X^H_{t_{i-1}} - \frac{1}{n}X_{t_n}\right)^2 - \sigma^2 - h^{2H-1}\sigma^2\right)\\ &\quad + \frac{2n\hat{h}}{\rho_n}O\left(\tilde{\Delta}_n(H)^2\right)\cdot\sqrt{n}\\ &= \sqrt{n}\left(\frac{h}{\hat{h}}J_n(X^H) - \sigma^2 - h^{2H-1}\sigma^2\right) - \frac{\sqrt{n}}{n^2\hat{h}}\left(X^H_{t_n}\right)^2 + \frac{2n\hat{h}}{\rho_n}O\left(\tilde{\Delta}_n(H)^2\right)\cdot\sqrt{n}. \qquad (4.8)\end{aligned}$$

Now, let $H = \frac{3}{4}$. Clearly, the following convergence holds:

$$\begin{aligned}&\lim_{n\to\infty} E\left(\frac{B^H_{t_n}}{n^{\frac{3}{4}}\sqrt{h}\sqrt[4]{\log n}}\right)^2 = \lim_{n\to\infty}\frac{(nh)^{\frac{3}{2}}}{n^{\frac{3}{2}}h\sqrt{\log n}} = \lim_{n\to\infty}\frac{1}{\sqrt{\log n}}\sqrt{h} = 0,\\ &\lim_{n\to\infty}\frac{1}{\sqrt[4]{\log n}}n^{\frac{1}{4}}\sqrt{h} = \begin{cases}0, & 0<\gamma\leqslant 1,\\ +\infty, & \gamma>1,\end{cases}\\ &\lim_{n\to\infty}\frac{1}{\sqrt[4]{\log n}}n^{2H-\frac{3}{4}}h^{2H-\frac{1}{2}} = \lim_{n\to\infty}\frac{1}{\sqrt[4]{\log n}}n^{\frac{3}{4}}h = \begin{cases}0, & 0<\gamma\leqslant\frac{1}{3},\\ +\infty, & \gamma>\frac{1}{3}.\end{cases}\end{aligned}$$

It follows that

$$\begin{aligned}\frac{1}{n^{\frac{3}{4}}\sqrt{\hat{h}}\sqrt[4]{\log n}}X^H_{t_n} &\overset{a.s.}{\sim} \frac{1}{n^{\frac{3}{4}}\sqrt{h}\sqrt[4]{\log n}}X^H_{t_n}\\ &= \frac{1}{n^{\frac{3}{4}}\sqrt{h}\sqrt[4]{\log n}}\left(\sigma B_{t_n} + \sigma B^H_{t_n} + \beta nh - \frac{1}{2}\sigma^2(nh)^{2H}\right)\\ &\longrightarrow\begin{cases}0, & 0<\gamma\leqslant\frac{1}{3},\\ -\infty, & \gamma>\frac{1}{3}\end{cases}\end{aligned}$$

in probability, as n tends to infinity. Combining this with (3.3), (4.8), Lemma 4.5, Lemma 4.7 and Slutsky's theorem, we obtain statement (ii).

Finally, for $\frac{1}{4} < H < \frac{1}{2}$, we have

$$\lim_{n\to\infty} n^{2H-\frac{3}{4}}h^H = \begin{cases}0, & 0<\gamma<\frac{3-4H}{8H-3},\\ 1, & \gamma=\frac{3-4H}{8H-3},\\ \infty, & \gamma>\frac{3-4H}{8H-3}\end{cases}$$

for $\frac{3}{8} < H < \frac{1}{2}$. Moreover, we also have $3 - 4H < \frac{3-4H}{8H-3}$. Similarly, combining this with (3.3), Lemma 4.5, Lemma 4.7 and Slutsky's theorem, we obtain (iii). □

Remark 4.10 *At the end of this section, we will discuss the exact convergence rate of $\hat{\theta}_n$, that is, the rate at which sequence $\hat{\theta}_n - \sigma^2$ converges to zero almost surely. Taking the known μ as an example, based on the decomposition (4.1) and (7.7), we get*

$$\begin{aligned}\hat{\theta}_n - \sigma^2 &\overset{a.s.}{=} \frac{h^{(2H)\wedge 1}}{\hat{h}}\left(J_n\left(X^H\right) - \sigma^2\right)\\ &\quad + \frac{h^{(2H)\wedge 1} - \hat{h}}{h^{(2H)\wedge 1}}\sigma^2 - 2\mu\frac{h}{\hat{h}n}X^H_{t_n} + \mu^2\frac{h^2}{\hat{h}} + \frac{2n\hat{h}}{\sum_{i=1}^{n}(\hat{t}_i)^2}O(\Delta_n(H)^2)\\ &= \frac{h^{(2H)\wedge 1}}{\hat{h}}\sigma^2\sum_{j=1}^{5}R_n(H;j)\end{aligned}$$

for all $0 < H < 1$ and $n \geqslant 1$, where

$$\begin{aligned}R_n(H;1) &= \frac{1}{nh^{(2H)\wedge 1}}\sum_{i=1}^{n}\left(B_{t_i} - B_{t_{i-1}}\right)^2 + \frac{1}{nh^{(2H)\wedge 1}}\sum_{i=1}^{n}\left(B^H_{t_i} - B^H_{t_{i-1}}\right)^2 - 1,\\ R_n(H;2) &= \frac{2}{nh^{(2H)\wedge 1}}\sum_{i=1}^{n}\left(B_{t_i} - B_{t_{i-1}}\right)\left(B^H_{t_i} - B^H_{t_{i-1}}\right),\\ R_n(H;3) &= \frac{1}{\sigma^2 nh^{(2H)\wedge 1}}\sum_{i=1}^{n}\left[f\left(t_i\right) - f\left(t_{i-1}\right)\right]^2,\\ R_n(H;4) &= \frac{2}{\sigma nh^{(2H)\wedge 1}}\sum_{i=1}^{n}\left[f\left(t_i\right) - f\left(t_{i-1}\right)\right]\left((B^H_{t_i} - B^H_{t_{i-1}}) + (B_{t_i} - B_{t_{i-1}})\right),\\ R_n(H;5) &= \frac{h^{(2H)\wedge 1} - \hat{h}}{h^{(2H)\wedge 1}}\sigma^2 - 2\mu\frac{h}{\hat{h}n}X^H_{t_n} + \mu^2\frac{h^2}{\hat{h}} + \frac{2n\hat{h}}{\sum_{i=1}^{n}(\hat{t}_i)^2}O(\Delta_n(H)^2)\end{aligned}$$

for all $0 < H < 1$ and $n \geqslant 1$. We can easily obtain the exact convergence rates of $R_n(H;3)$, $R_n(H;4)$, and $R_n(H;5)$. However, the exact convergence rate of $R_n(H;1)$ is still quite difficult. This, along with some related issues, will be addressed in a forthcoming paper.

5. Asymptotic behavior of estimator $\hat{\mu}_n$

In this section, we consider the strong consistency and asymptotic distribution of $\hat{\mu}_n$.

Theorem 5.1 *Let $H \in (0,1)$ and condition (C1) hold. If $\sigma > 0$ is known, the estimator given by (2.7)*

$$\hat{\mu}_n = \frac{1}{nh}\sum_{i=1}^{n}\left(X^H_{t_i} - X^H_{t_{i-1}} + \frac{1}{2}\theta\hat{t}_i\right)$$

is strongly consistent, and

$$(nh)^{(1-H)\wedge\frac{1}{2}}\left(\hat{\mu}_n - \mu\right) \longrightarrow N\left(0, \sigma^2\right) \tag{5.1}$$

in distribution, as n tends to infinity.

Proof Based on the laws of the iterated logarithm of Brownian motion and fractional Brownian motion, we have

$$\frac{B_t}{t^\alpha}, \frac{B_t^H}{t^\beta} \longrightarrow 0 \quad (t \to \infty)$$

a.s. for all $\alpha > \frac{1}{2}$ and $\beta > H$. It follows that

$$\hat{\mu}_n - \mu = \frac{\sigma\left(B_{t_n} + B_{t_n}^H\right)}{nh} \xrightarrow{a.s.} 0$$

and

$$(nh)^{(1-H)\wedge\frac{1}{2}}\left(\hat{\mu}_n - \mu\right) = \frac{\sigma\left(B_{t_n} + B_{t_n}^H\right)}{(nh)^{1-(1-H)\wedge\frac{1}{2}}} \longrightarrow N(0, \sigma^2)$$

in distribution, as n tends to infinity, by Slutsky's theorem. □

Lemma 5.2 *Given $\frac{1}{2} < H < 1$. Let $\sigma > 0$ be unknown and conditions $(C1)$ and $(C2)$ hold. Then, the estimator $\hat{\theta}_n$ given by* (2.10) *satisfies*

$$(nh)^{2H-1}\left(\hat{\theta}_n - \sigma^2\right) \xrightarrow{a.s.} 0 \tag{5.2}$$

as $n \to \infty$, provided one of the following conditions holds:

(i) $\frac{3}{4} < H < 1$ *and* $0 < \gamma < (4H-3) \wedge \frac{1}{8H-3}$.
(ii) $\frac{1}{2} < H \leqslant \frac{3}{4}$ *and* $0 < \gamma < \frac{1}{8H-3}$.

Proof When $\frac{3}{4} < H < 1$, based on (i) in Theorem 4.8 have that

$$\begin{aligned}(nh)^{2H-1}\left(\hat{\theta}_n - \sigma^2\right) &= n^{2H-\frac{3}{2}}h^{2H-1} \cdot \sqrt{n}\left(\hat{\theta}_n - \sigma^2\right) \\ &= \left(nh^{1+\gamma}\right)^{\frac{2H-1}{1+\gamma}} n^{-\frac{1-(4H-3)\gamma}{2(1+\gamma)}} \cdot \sqrt{n}\left(\hat{\theta}_n - \sigma^2\right) \xrightarrow{a.s.} 0,\end{aligned}$$

as $n \to \infty$, provided $0 < \gamma < (4H-3) \wedge \frac{1}{8H-3}$.

When $\frac{1}{2} < H < \frac{3}{4}$, based on (i) in Theorem 4.9 have that

$$\begin{aligned}(nh)^{2H-1}\left(\hat{\theta}_n - \sigma^2\right) &= (nh)^{2H-1}\left(\hat{\theta}_n - \sigma^2 - h^{2H-1}\sigma^2\right) + n^{2H-1}h^{4H-2}\sigma^2 \\ &= n^{2H-\frac{3}{2}}h^{2H-1} \cdot \sqrt{n}\left(\hat{\theta}_n - \sigma^2 - h^{2H-1}\sigma^2\right) + n^{2H-1}h^{4H-2}\sigma^2 \\ &\xrightarrow{a.s.} 0\end{aligned}$$

as $n \to \infty$, provided $0 < \gamma < \frac{1}{8H-3}$.

When $H = \frac{3}{4}$, based on (ii) in Theorem 4.9, we have that

$$\begin{aligned}(nh)^{2H-1}\left(\hat{\theta}_n - \sigma^2\right) &= \sqrt{h\log n} \cdot \sqrt{\frac{n}{\log n}}\left(\hat{\theta}_n - \sigma^2 - h^{2H-1}\sigma^2\right) + \sqrt{n}h\sigma^2 \\ &\xrightarrow{a.s.} 0\end{aligned}$$

as $n \to \infty$, provided $0 < \gamma < \frac{1}{3}$. □

Theorem 5.3 *Let conditions (C1) and (C2) hold, and let $\sigma > 0$ be unknown.*

(i) *If $0 < H < \frac{1}{2}$, the estimator $\hat{\mu}_n$ given by* (2.10) *is strongly consistent, and when $0 < \gamma < 3 - 4H$, we have*

$$\sqrt{nh}\left(\hat{\mu}_n - \mu\right) \xrightarrow{d} N\left(0, (1 + 1_{\{H=\frac{1}{2}\}}\right) \quad (n \to \infty).$$

(ii) *If* $\frac{3}{4} < H < 1$ *and* $0 < \gamma < (4H-3) \wedge \frac{1}{8H-3}$, *the estimator* $\hat{\mu}_n$ *given by* (2.10) *is strongly consistent and*

$$(nh)^{1-H}\left(\hat{\mu}_n - \mu\right) \xrightarrow{d} N\left(0, \sigma^2\right) \quad (n \to \infty).$$

(iii) *If* $\frac{1}{2} < H \leqslant \frac{3}{4}$ *and* $0 < \gamma < \frac{2H-1}{1-H} \wedge \frac{1}{8H-3}$, *the estimator* $\hat{\mu}_n$ *given by* (2.10) *is strongly consistent and*

$$(nh)^{1-H}\left(\hat{\mu}_n - \mu\right) \xrightarrow{d} N\left(0, \sigma^2\right) \quad (n \to \infty).$$

Proof For (i), base on Theorem 3.8 and the fact

$$\beta_n = \frac{1}{2}(1 + (nh)^{2H-1}) \longrightarrow \frac{1}{2} \quad (n \to \infty)$$

we have that

$$\begin{aligned}\hat{\mu}_n &= \frac{1}{nh}X^H_{t_n} + \beta_n\hat{\theta}_n = \frac{\sigma}{nh}\left(B_{t_n} + B^H_{t_n}\right) + \frac{1}{nh}\left(\beta t_n - \frac{1}{2}\sigma^2(t_n)^{2H}\right) + \beta_n\hat{\theta}_n \\ &= \frac{\sigma}{nh}\left(B_{t_n} + B^H_{t_n}\right) + \beta - \frac{1}{2}\sigma^2(nh)^{2H-1} + \beta_n\hat{\theta}_n \longrightarrow \beta + \frac{1}{2}\sigma^2 \stackrel{a.s.}{=} \mu \end{aligned} \tag{5.3}$$

as $n \to \infty$. It follows from Theorem 4.8 and the fact that

$$\frac{1}{\sqrt{nh}}B_{t_n} \sim N(0,1), \quad \frac{1}{\sqrt{nh}}B^H_{t_n} \xrightarrow{a.s.} 0 \quad (n \to \infty)$$

with $0 < H < \frac{1}{2}$ that

$$\begin{aligned}(nh)^{(1-H)\wedge\frac{1}{2}}(\hat{\mu}_n - \mu) &= \sqrt{nh}(\hat{\mu}_n - \mu) \\ &= \frac{\sigma}{\sqrt{nh}}B_{t_n} + \frac{\sigma}{\sqrt{nh}}B^H_{t_n} + \beta_n\sqrt{nh}\left(\hat{\theta}_n - \sigma^2\right) \longrightarrow N\left(0, \sigma^2\right)\end{aligned}$$

in distribution, as n tends to infinity, provided $0 < \gamma < 3 - 4H$. This completes the proof of $0 < H \leqslant \frac{1}{4}$.

When $\frac{1}{4} < H \leqslant \frac{1}{2}$,

$$\begin{aligned}(nh)^{\frac{1}{2}}(\hat{\mu}_n - \mu) &= \frac{\sigma}{\sqrt{nh}}B_{t_n} + \frac{\sigma}{\sqrt{nh}}B^H_{t_n} + \beta_n\sqrt{nh}\left(\hat{\theta}_n - \sigma^2\right) + \frac{1}{2}(nh)^{2H-\frac{1}{2}}\left(\hat{\theta}_n - \sigma^2\right) \\ &\longrightarrow N(0, (1 + 1_{\{H=\frac{1}{2}\}}a^2)\end{aligned}$$

For (ii), based on Theorem 4.8 and Lemma 5.2, we have that

$$\begin{aligned}\hat{\mu}_n &= \frac{1}{nh}X^H_{t_n} + \beta_n\hat{\theta}_n = \frac{\sigma}{nh}\left(B_{t_n} + B^H_{t_n}\right) + \frac{1}{nh}\left(\beta t_n - \frac{1}{2}\sigma^2(t_n)^{2H}\right) + \beta_n\hat{\theta}_n \\ &= \mu + \frac{\sigma}{nh}\left(B_{t_n} + B^H_{t_n}\right) + \frac{1}{2}\left(\hat{\theta}_n - \sigma^2\right) + \frac{1}{2}(nh)^{2H-1}\left(\hat{\theta}_n - \sigma^2\right) \xrightarrow{a.s.} \mu\end{aligned} \tag{5.4}$$

as $n \to \infty$. Combining this with Theorem 4.8 and

$$\frac{1}{(nh)^H}B_{t_n} \xrightarrow{a.s.} 0 \quad (n \to \infty), \quad \frac{1}{(nh)^H}B^H_{t_n} \sim N(0,1)$$

with $\frac{1}{2} < H < 1$, we get that

$$\begin{aligned}(nh)^{1-H}(\hat{\mu}_n - \mu) &= \sigma\frac{B_{t_n}}{(nh)^H} + \sigma\frac{B^H_{t_n}}{(nh)^H} + \frac{1}{2}(nh)^{1-H}\left(\hat{\theta}_n - \sigma^2\right) + \frac{1}{2}(nh)^H\left(\hat{\theta}_n - \sigma^2\right) \\ &\longrightarrow N(0, \sigma^2)\end{aligned}$$

in distribution, as n tends to infinity. Similarly, we can obtain (iii). □

6. Simulation

Here, we present a comprehensive evaluation of the proposed parameter estimation method for the mixed fractional Brownian motion model, based on both **numerical simulations** and **empirical analysis**.

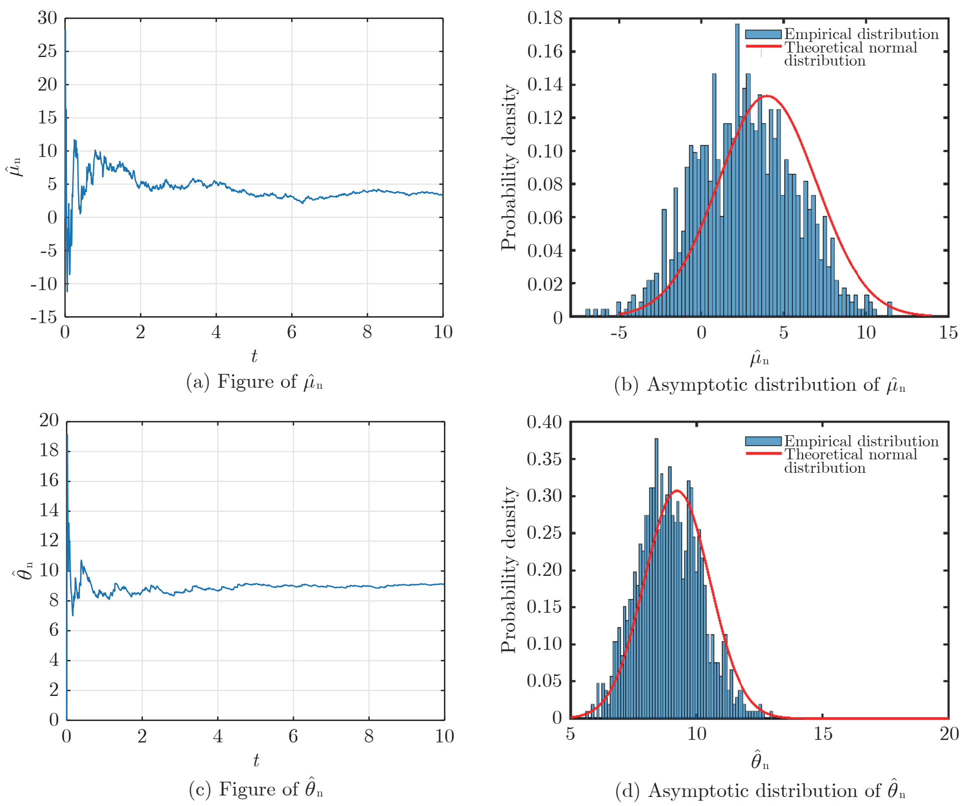
6.1 Numerical simulations

First, we should clarify that for all figures presented below, the sample size is set to $n = 1000$, the time step is chosen as $h = \frac{0.01}{\sqrt{n}}$, and the set values are $\mu = 4$ and $\sigma^2 = 9$. In the analysis of the asymptotic distribution, the number of trials-that is, the number of simulated sample paths is also set to $num\_trials = 1000$. To ensure notational uniformity, we reiterate that $\theta = \sigma^2$ throughout the subsequent discussion. To assess the effectiveness and robustness of the proposed estimation method, we design two primary experimental scenarios:

(1) **Case with partially known parameters**

• In the case where parameter μ is known, we estimate parameter σ and analyze its estimation path and asymptotic distribution. The figures present the estimation paths and asymptotic distribution for $\hat{\theta}_n$ under the condition that $H = 0.4$ (Figure 1(c)-Figure 1(d)) and $H = 0.6$ (Figure 2(c)-Figure 2(d)).

• In the case where parameter σ is known, we estimate parameter μ and examine its estimation path and asymptotic distribution. Similarly, Figures present the estimation paths and



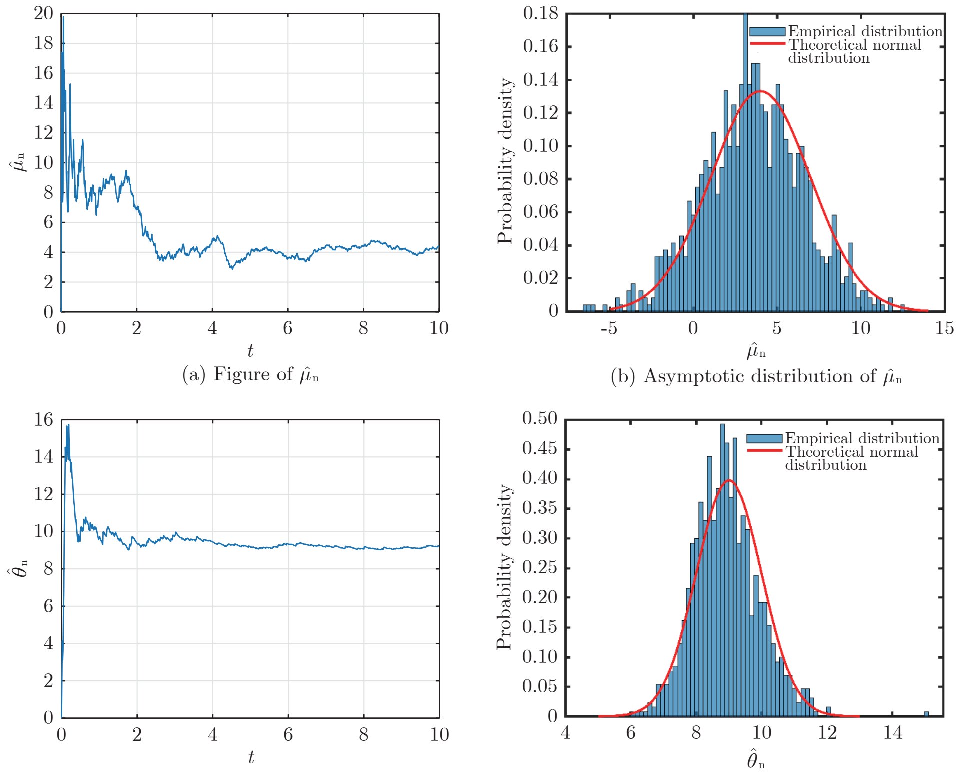
(a) Figure of $\hat{\mu}_n$

(b) Asymptotic distribution of $\hat{\mu}_n$

(c) Figure of $\hat{\theta}_n$

(d) Asymptotic distribution of $\hat{\theta}_n$

Figure 1 Asymptotic behavior of estimators $\hat{\mu}_n$ and $\hat{\theta}_n$ when one parameter is known under $H = 0.4$



(a) Figure of $\hat{\mu}_n$ (b) Asymptotic distribution of $\hat{\mu}_n$

(c) Figure of $\hat{\theta}_n$ (d) Asymptotic distribution of $\hat{\theta}_n$

Figure 2 Asymptotic behavior of estimators $\hat{\mu}_n$ and $\hat{\theta}_n$ when one parameter is known under $H = 0.6$

asymptotic distribution of $\hat{\mu}_n$ when $H = 0.4$ (Figure 1(a)-Figure 1(b)) and $H = 0.6$ (Figure 2(a)-Figure 2(b)).

(2) **Case with completely unknown parameters**

• In this scenario, where both μ and σ are unknown, we estimate both parameters simultaneously and analyze their estimation paths and asymptotic distributions. The figures present the estimation paths and asymptotic distribution of $\hat{\mu}_n$ and $\hat{\theta}_n$ when $H = 0.4$ (Figure 3) and $H = 0.6$ (Figure 4).

These simulations are conducted using Monte Carlo methods in MATLAB, with the generated samples following the noise structure and path characteristics specified by the mixed fractional Black-Scholes model.

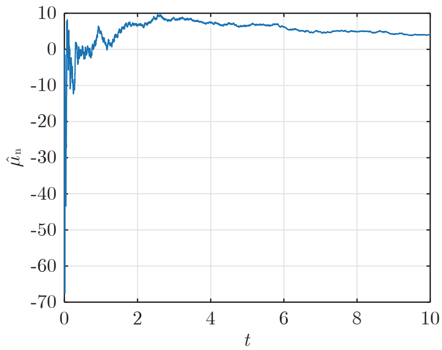
In addition, to investigate the asymptotic behavior of the proposed estimators for different sample sizes, we consider three sample sizes: $n = 1000$, 2000, and 3000. We carry out a comparison of theoretical variance with empirical variance, as well as the corresponding errors, is carried out. The specific experimental design is outlined as follows:

• Table 1: Theoretical variance, empirical variance, and their errors for parameter μ when $\theta = 9$ is known.

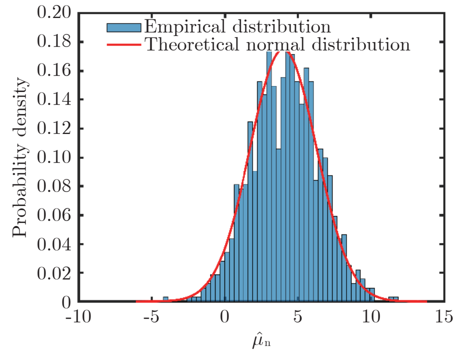
• Table 2: Theoretical variance, empirical variance, and their errors for parameter θ when $\mu = 4$ is known.

• Table 3: Joint analysis of the variance estimates and errors for both parameters when μ and θ are unknown.

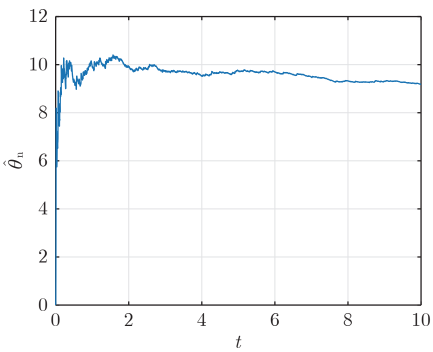
The results show that as the sample size increases, the discrepancy between theoretical and



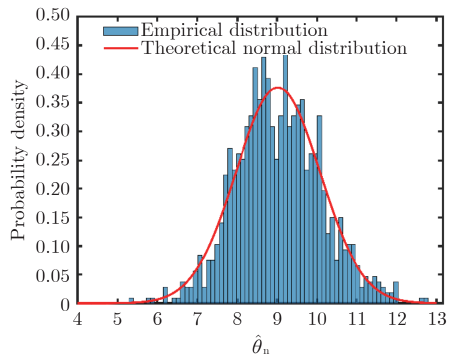
(a) Figure of $\hat{\mu}_n$



(b) Asymptotic distribution of $\hat{\mu}_n$

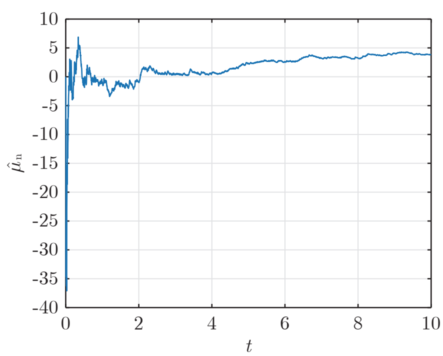


(c) Figure of $\hat{\theta}_n$

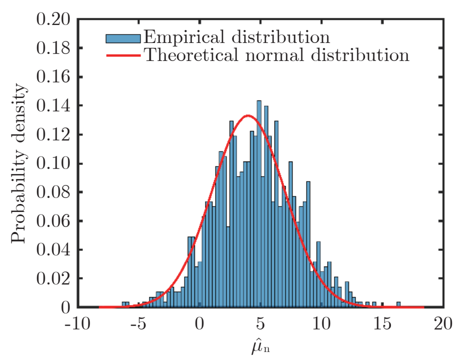


(d) Asymptotic distribution of $\hat{\theta}_n$

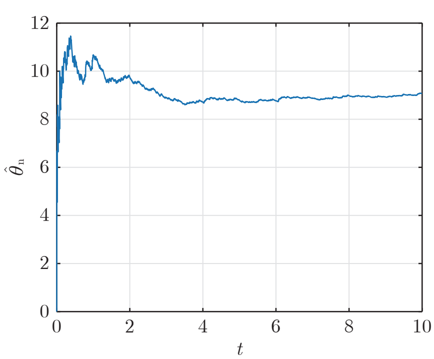
Figure 3 Asymptotic behavior of estimators $\hat{\mu}_n$ and $\hat{\theta}_n$ when both parameters are unknown under $H = 0.4$



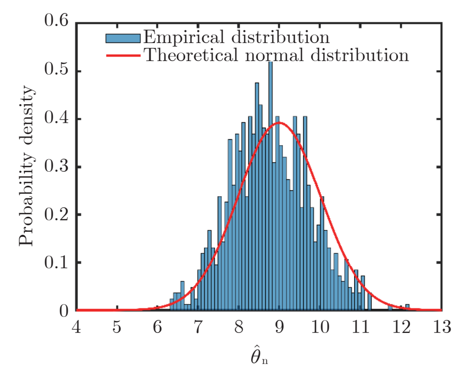
(a) Figure of $\hat{\mu}_n$



(b) Asymptotic distribution of $\hat{\mu}_n$



(c) Figure of $\hat{\theta}_n$



(d) Asymptotic distribution of $\hat{\theta}_n$

Figure 4 Asymptotic behavior of estimators $\hat{\mu}_n$ and $\hat{\theta}_n$ when both parameters are known under $H = 0.6$

Table 1 Comparison of theoretical and empirical variances of $\hat{\mu}_n$ under known $\theta = 9$ and Various Hurst Indices

H	$n = 1000$			$n = 2000$			$n = 3000$		
	Theoretical	Empirical	Abs. Error	Theoretical	Empirical	Abs. Error	Theoretical	Empirical	Abs. Error
0.2	9.000000	8.770946	0.229054	9.000000	8.916261	0.083739	9.000000	8.908694	0.091306
0.4	9.000000	9.348421	0.348421	9.000000	9.062997	0.062997	9.000000	8.963374	0.036626
0.6	9.000000	9.270003	0.270003	9.000000	9.116029	0.116029	9.000000	8.944521	0.055479
0.75	9.000000	9.386669	0.386669	9.000000	8.926289	0.073711	9.000000	8.972941	0.027059
0.8	9.000000	8.881655	0.118345	9.000000	9.080995	0.080995	9.000000	8.934139	0.065861

Table 2 Comparison of Theoretical and Empirical Variances of $\hat{\theta}_n$ under known $\mu = 4$ and Various Hurst Indices

H	$n = 1000$			$n = 2000$			$n = 3000$		
	Theoretical	Empirical	Abs. Error	Theoretical	Empirical	Abs. Error	Theoretical	Empirical	Abs. Error
0.2	0.20041	0.19950	0.00091	0.100210	0.090650	0.009560	0.066800	0.067570	0.000770
0.4	0.16829	0.18314	0.01485	0.084143	0.081240	0.002903	0.056095	0.059520	0.003425
0.6	0.17510	0.31716	0.14206	0.126380	0.087612	0.087612	0.058414	0.088453	0.030039
0.75	1.57470	1.25890	0.31574	0.719330	0.692630	0.026695	0.508230	0.486390	0.021847
0.8	1.33020	0.24190	1.08830	0.775520	0.146890	0.628630	0.565070	0.102580	0.462490

Table 3 Comparison of Theoretical and Empirical Variances of $\hat{\theta}_n$ and $\hat{\mu}_n$ under Various Hurst Indices and Sample Sizes

H	Estimator	$n = 1000$			$n = 2000$			$n = 3000$		
		Theoretical	Empirical	Abs. Error	Theoretical	Empirical	Abs. Error	Theoretical	Empirical	Abs. Error
0.2	$\hat{\theta}_n$	0.20041001	0.17950421	0.02090580	0.10020501	0.08074054	0.00956	0.06680	0.06757	0.00077
	$\hat{\mu}_n$	0.15412187	0.06279792	0.0913295	0.01843240	0.02171838	0.01946447	0.06680334	0.05460320	0.01220013
0.4	$\hat{\theta}_n$	0.084143	0.078907	0.01485	0.005236	0.079468	0.004675	0.056095	0.052597	0.003498
	$\hat{\mu}_n$	2.846050	2.210418	0.635631	2.012461	1.570847	0.441614	1.643168	1.311402	0.331766
0.6	$\hat{\theta}_n$	0.162000	0.169141	0.007141	0.081000	0.077786	0.003214	0.05841	0.08845	0.03004
	$\hat{\mu}_n$	3.582965	6.731692	3.148727	2.715379	4.955863	2.240484	0.210170	0.489641	0.279472
0.75	$\hat{\theta}_n$	1.122	1.3004	0.17836	0.83225	0.69426	0.13798	0.78326	0.69741	0.085841
	$\hat{\mu}_n$	4.0249	4.1408	0.11588	2.846	2.8633	0.017266	2.3238	2.4213	0.097489
0.8	$\hat{\theta}_n$	0.104205	0.081000	0.023205	0.149289	0.162000	0.012711	0.064852	0.054000	0.010852
	$\hat{\mu}_n$	4.307408	2.832263	1.475146	3.769689	2.444855	1.324833	3.208035	2.308841	0.899194

empirical variances diminishes, and the estimation errors decrease significantly, further evidence of the asymptotic consistency and convergence properties of the proposed estimation method as the sample size grows.

Furthermore, our preliminary analysis of a broad set of publicly listed companies indicates that the Hurst exponents estimated via the R/S method are typically within the range of $H \in [0.5, 0.75]$. For clarity and representativeness, we focus on $H = 0.6$ as a typical case and present the estimation paths and asymptotic distributions under both experimental settings, which have broader applicability. In addition, we include various plots for $H = 0.6$ when $H < \frac{1}{2}$, offering supplementary visual illustrations and analyses.

6.2 Empirical analysis

To further evaluate the performance of the model and estimation method in a real-world market setting, we conduct an empirical analysis using data from a representative stock in the Chinese A-share market, Hailan Home (stock code: 600398). This analysis is based on the mixed fractional Brownian motion Black-Scholes model, with parameter estimation carried out using our proposed approach.

The historical daily closing price data for this stock are obtained from the Tushare Pro platform, covering the period from December 28, 2001, to April 7, 2025. We perform data cleaning and preprocessing using Python.

Once the data are prepared, we first use the R/S method to estimate the Hurst exponent of the stock's return series, yielding a value of $H = 0.6275$, suggesting the presence of long-memory effects in the time series. Following this, we estimate the key model parameters $\hat{\mu}_n$ and $\hat{\theta}_n$ are estimated using the quasi-likelihood estimation framework proposed in this paper in Figure 5.

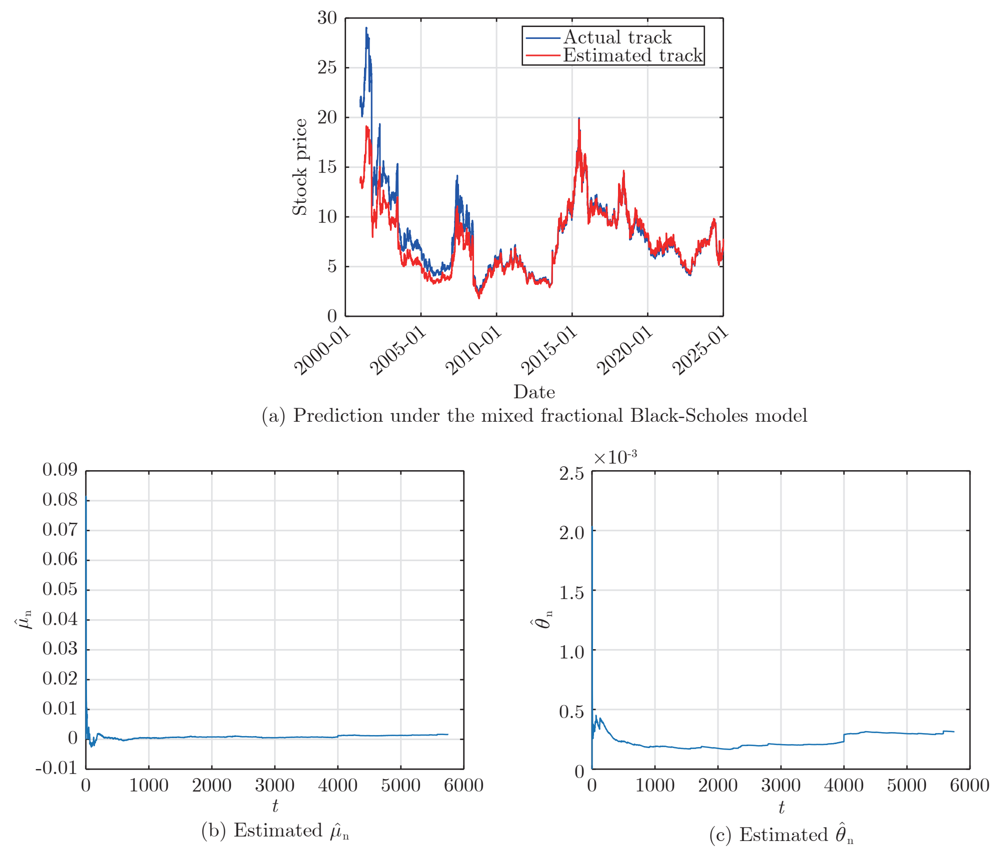
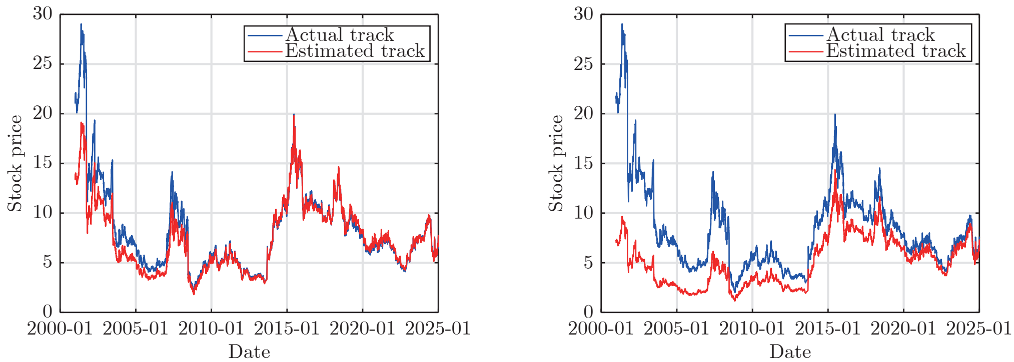


Figure 5 Parameter estimation and simulation results for stock 600398

To facilitate a more intuitive assessment of the model's fit, we generate simulated price trajectories based on the estimated parameters using MATLAB and compare these with the actual observed closing prices. The results indicate that the model captures the overall price dynamics well, further validating the applicability and effectiveness of the proposed estimation method for real financial data.

Furthermore, we simulate the stock price trajectories using both the proposed mixed fractional Brownian motion-driven model proposed in this paper and the classical Black-Scholes model. Figure 6 presents the comparative results are presented in the figure below. As shown, the proposed model exhibits a superior fit to the actual price dynamics, especially in capturing the volatility clustering and long-memory characteristics of the price process. These findings further demonstrate the advantages and applicability of our model for financial data modeling and empirical analysis.



(a) Mixed fractional Black-Scholes model (b) Classical Black-Scholes model

Figure 6 Comparative prediction of stock 600398 under the mixed fractional and classical Black-Scholes models

7. Appendix

7.1 Proofs of lemmas in section 3

In this appendix, we provide proofs of the lemmas in section 3. In 1983, Etemadi [16] introduced a strong law of large numbers for nonnegative random variables as follows:

Lemma 7.1(Etemadi [16]). *Let $\{\xi_n\}$ be a sequence of nonnegative random variables with finite second moments and $S_n = \sum_{i=1}^{n} \xi_i$, such that*

(i) *the sequence $\{\omega_n = E\xi_n\}$ satisfies $\left(\sum_{i=1}^{n} \omega_i\right)^{-1} \omega_n \to 0$ and $\sum_{i=1}^{n} \omega_i \to \infty$ as $n \to \infty$.*

(ii) *the following series converges:*

$$\sum_{n=1}^{\infty} \frac{1}{(ES_n)^2} \sum_{i=1}^{n} \mathrm{Cov}^+(\xi_n, \xi_i).$$

Then, as $n \to \infty$, $\frac{S_n}{ES_n} \to 1$ almost surely.

Lemma 7.2 *For all $0 < r' < s' < r < s$ and $0 < H < 1$ we have*

$$\left|E\left[(B_s^H - B_r^H)(B_{s'}^H - B_{r'}^H)\right]\right| \leqslant \frac{(s-r)(s'-r')}{(r-s')^{2-2H}}. \tag{7.1}$$

Proof When $0 < H < \frac{1}{2}$, the lemma is obtained in the proof of Lemma 3.3 in Yan et al. [31]. Now, we assume that $\frac{1}{2} < H < 1$ and define the function $x \mapsto F_{r,s}(x)$ on $[r', s']$ by

$$F_{r,s}(x) = (r-x)^{2H} - (s-x)^{2H}$$

for $0 < r' < s' < r < s$. Thanks to the mean value theorem, we see that there are $\xi \in (r', s')$ and $\eta \in (r, s)$, such that

$$\begin{aligned} 2E\left[(B_s^H - B_r^H)(B_{s'}^H - B_{r'}^H)\right] &= F_{r,s}(s') - F_{r,s}(r') \\ &= 2H(s'-r')\left[(s-\xi)^{2H-1} - (r-\xi)^{2H-1}\right] \\ &= 2H(2H-1)(s'-r')(s-r)\left(\eta-\xi\right)^{2H-2} \geqslant 0 \end{aligned}$$

which gives

$$E\left[(B_s^H - B_r^H)(B_{s'}^H - B_{r'}^H)\right] \leqslant \frac{(s'-r')(s-r)}{(r-s')^{2-2H}}. \tag{7.2}$$

This completes the proof. $\square$

Proof of Lemma 3.1 When $H=\frac{1}{2}$, the lemma follows from the strong law of large numbers. Now let $H\in(0,\frac{1}{2})\cup(\frac{1}{2},1)$. Then, the sequence $\{E(B^H_{t_n}-B^H_{t_{n-1}})^2\}$ satisfies the condition (i) in Lemma 7.1. Meanwhile, by the fact

$$E\left[(B^H_{t_n}-B^H_{t_{n-1}})^2(B^H_{t_i}-B^H_{t_{i-1}})^2\right]=h^{4H}+2\left(E(B^H_{t_n}-B^H_{t_{n-1}})(B^H_{t_i}-B^H_{t_{i-1}})\right)^2$$

with $n>i$ and Lemma 7.2, we see that

$$\sum_{i=1}^{n}\left|\mathrm{Cov}\left((B^H_{t_n}-B^H_{t_{n-1}})^2,(B^H_{t_i}-B^H_{t_{i-1}})^2\right)\right|=\sum_{i=1}^{n}\left|E\left[(B^H_{t_n}-B^H_{t_{n-1}})^2(B^H_{t_i}-B^H_{t_{i-1}})^2\right]-h^{4H}\right|$$
$$=2\sum_{i=1}^{n}\left(E(B^H_{t_n}-B^H_{t_{n-1}})(B^H_{t_i}-B^H_{t_{i-1}})\right)^2$$
$$=2h^{4H}+2\left(E(B^H_{t_n}-B^H_{t_{n-1}})(B^H_{t_{n-1}}-B^H_{t_{n-2}})\right)^2+2\sum_{i=1}^{n-2}\left(E(B^H_{t_n}-B^H_{t_{n-1}})(B^H_{t_i}-B^H_{t_{i-1}})\right)^2$$
$$\leqslant 2h^{4H}+(2-2^{2H})^2h^{4h}+2h^{4H}\sum_{i=1}^{n-2}\frac{1}{(n-1-i)^{4-4H}}$$
$$=h^{4H}\left(2+(2-2^{2H})^2+2\sum_{j=1}^{n-2}\frac{1}{j^{4-4H}}\right)$$
$$\leqslant\begin{cases}Ch^{4H}, & 0<H<\frac{3}{4},\\ Ch^{3}(1+\log n), & H=\frac{3}{4},\\ Ch^{4H}n^{4H-3}, & \frac{3}{4}<H<1\end{cases}$$

for all $n\geqslant 1$. It follows that

$$\sum_{n=1}^{\infty}\frac{1}{\left\{\sum_{i=1}^{n}E(B^H_{t_i}-B^H_{t_{i-1}})^2\right\}^2}\sum_{i=1}^{n}\left|\mathrm{Cov}\left((B^H_{t_n}-B^H_{t_{n-1}})^2,(B^H_{t_i}-B^H_{t_{i-1}})^2\right)\right|<\infty$$

for all $H\in(0,\frac{1}{2})\cup(\frac{1}{2},1)$. Thus, condition (ii) in Lemma 7.1 holds, and the lemma follows. □

Lemma 7.3 *Let $H\in(0,\frac{1}{2})\cup(\frac{1}{2},1)$ and denote*

$$K_n(H):=\sum_{i=1}^{n}(i^{2H}-(i-1)^{2H})^2.$$

for $n\geqslant 1$. Then, the limit $\lim_{n\to\infty}\left(K_n(H)-n^{4H-1}\right)$ is finite and nonzero for $0<H\leqslant\frac{1}{4}$ and $K_n(H)-n^{4H-1}=O\left(n^{4H-1}\right)$ for all $H\in(\frac{1}{4},1)$, as n tends to infinity.

Proof Clearly, we have

$$\begin{aligned}K_n(H)-n^{4H-1}&=\sum_{i=1}^{n}i^{4H}\left(1-\left(1-\frac{1}{i}\right)^{2H}\right)^2-n^{4H-1}\\&=\sum_{i=1}^{n}i^{4H}\left(2Hi^{-1}+O\left(i^{-2}\right)\right)^2-n^{4H-1}\\&=4H^2\sum_{i=1}^{n}i^{4H-2}-n^{4H-1}+4H^2\sum_{i=1}^{n}i^{4H-3}+\sum_{i=1}^{n}i^{4H-1}O\left(i^{-3}\right)\\&=4H^2n^{4H-1}\left(\sum_{i=1}^{n}\left(\frac{i}{n}\right)^{4H-2}\cdot\frac{1}{n}-1\right)+4H^2n^{4H-2}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^{4H-3}\cdot\frac{1}{n}\\&\quad+\sum_{i=1}^{n}\left(\frac{i}{n}\right)^{4H-1}O\left(i^{-3}\right)=O\left(n^{4H-1}\right)\end{aligned}$$

for all $H \in (\frac{1}{4}, 1)$, as n tends to infinity, since

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^{4H-2} \cdot \frac{1}{n} = \int_0^1 x^{4H-2} \mathrm{d}x = \frac{1}{4H-1}.$$

This completes the proof. □

Lemma 7.4 *Let $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ and $f(t) = \beta t - \frac{1}{2}\sigma^2 t^{2H}$. Denote*

$$\Psi_H(n) := \frac{1}{nh^{(2H)\wedge 1}} \sum_{i=1}^{n} \left[f(t_i) - f(t_{i-1})\right]^2,$$

where $t_i = ih$ with conditions (C1) and (C2). Then, $\lim_{n\to\infty} \Psi_H(n) = 0$ for $0 < H < \frac{1}{2}$, and

$$\lim_{n\to\infty} \Psi_H(n) = \begin{cases} 0, & 0 < \gamma < \frac{1}{4H-2}, \\ \frac{H^2}{4H-1}\sigma^4, & \gamma = \frac{1}{4H-2}, \\ +\infty, & \gamma > \frac{1}{4H-2}, \end{cases}$$

for all $\frac{1}{2} < H < 1$.

Proof From the proof of Lemma 7.3, it follows that, as n tends to infinity

$$\begin{aligned} \Psi_H(n) &= \beta^2 h^{2-2H} - \beta\sigma^2 h n^{2H-1} + \frac{1}{4}\sigma^4 \sum_{i=1}^{n} \left((t_i)^{2H} - (t_{i-1})^{2H}\right)^2 \\ &= \beta^2 h^{2-2H} - \beta\sigma^2 h n^{2H-1} + \frac{1}{4}\sigma^4 h^{4H} \frac{1}{n} K_n(H) \longrightarrow 0 \end{aligned}$$

for all $0 < H < \frac{1}{2}$. Moreover, when $\frac{1}{2} < H < 1$, we have

$$\lim_{n\to\infty} h^{2H} n^{2H-1} = \begin{cases} 0, & 0 < \gamma < \frac{1}{2H-1}, \\ 1, & \gamma = \frac{1}{2H-1}, \\ +\infty, & \gamma > \frac{1}{2H-1}, \end{cases}$$

and

$$\lim_{n\to\infty} h^{4H-1} n^{4H-2} = \begin{cases} 0, & 0 < \gamma < \frac{1}{4H-2}, \\ 1, & \gamma = \frac{1}{4H-2}, \\ +\infty, & \gamma > \frac{1}{4H-2}, \end{cases}$$

which imply

$$\Psi_H(n) = \beta^2 h - \beta\sigma^2 h^{2H} n^{2H-1} + \sigma^4 \frac{1}{4n} h^{4H-1} K_n(H) \longrightarrow \begin{cases} 0, & 0 < \gamma < \frac{1}{4H-2}, \\ \frac{H^2}{4H-1}\sigma^4, & \gamma = \frac{1}{4H-2}, \\ +\infty, & \gamma > \frac{1}{4H-2}, \end{cases}$$

for all as n tends to infinity. □

Proof of Proposition 3.2 Recall that

$$X_t^H = \sigma\left(B_t + B_t^H\right) + f(t), \quad t \geqslant 0,$$

where $f(t) = \beta t - \frac{1}{2}\sigma^2 t^{2H}$ with $\beta = \mu - \frac{1}{2}\sigma^2$. It follows that

$$\begin{aligned} J_n\left(X^H\right) = &\frac{\sigma^2}{nh^{(2H)\wedge 1}} \sum_{i=1}^{n} \left(B_{t_i} - B_{t_{i-1}}\right)^2 + \frac{\sigma^2}{nh^{(2H)\wedge 1}} \sum_{i=1}^{n} \left(B_{t_i}^H - B_{t_{i-1}}^H\right)^2 \\ &+ \frac{2\sigma^2}{nh^{(2H)\wedge 1}} \sum_{i=1}^{n} \left(B_{t_i} - B_{t_{i-1}}\right)\left(B_{t_i}^H - B_{t_{i-1}}^H\right) + \frac{1}{nh^{(2H)\wedge 1}} \sum_{i=1}^{n} \left[f(t_i) - f(t_{i-1})\right]^2 \\ &+ \frac{2\sigma}{nh^{(2H)\wedge 1}} \sum_{i=1}^{n} \left[f(t_i) - f(t_{i-1})\right]\left((B_{t_i}^H - B_{t_{i-1}}^H) + (B_{t_i} - B_{t_{i-1}})\right) \end{aligned} \tag{7.3}$$

for all $0<H<1$ and $n\geqslant 1$.

When $0<H<\frac{1}{2}$, based on Cauchy's inequality, Lemma 7.4 and Lemma 3.1 we can show that

$$\begin{aligned}&\frac{1}{n^2h^{4H}}\left(\sum_{i=1}^{n}\left(B_{t_i}-B_{t_{i-1}}\right)\left(B_{t_i}^H-B_{t_{i-1}}^H\right)\right)^2\\ \leqslant\ & h^{1-2H}\cdot\frac{1}{nh}\sum_{i=1}^{n}\left(B_{t_i}-B_{t_{i-1}}\right)^2\cdot\frac{1}{nh^{2H}}\sum_{i=1}^{n}\left(B_{t_i}^H-B_{t_{i-1}}^H\right)^2\xrightarrow{a.s.}0,\\ &\frac{1}{nh^{2H}}\sum_{i=1}^{n}\left|f\left(t_i\right)-f\left(t_{i-1}\right)\right|\left|\left(B_{t_i}^H-B_{t_{i-1}}^H\right)+\left(B_{t_i}-B_{t_{i-1}}\right)\right|\xrightarrow{a.s.}0\end{aligned}$$

as n tends to infinity. It follows from (7.3), Lemma 7.4 and Lemma 3.1 that

$$J_n\left(X^H\right)\xrightarrow{a.s.}\sigma^2 \tag{7.4}$$

for all $0<H<\frac{1}{2}$, as n tends to infinity. This shows that (i) holds. Similarly, we can obtain (ii) by using (7.3), Lemma 7.4, and Lemma 3.1. □

Proof of Lemma 3.5 Clearly, we have that

$$\frac{1}{n}\sum_{i=1}^{n}(\hat{t}_i)^2=h^2+2h^{2H+1}n^{2H-1}+\frac{1}{n}h^{4H}K_n(H)\sim h^2 \tag{7.5}$$

for all $0<H<\frac{1}{2}$ by Lemma 7.3. It follows from Proposition 3.2 and convergence

$$h^{3-2H}\frac{1}{n}X_{t_n}^H=h^{3-2H}\frac{1}{n}\left(\sigma B_{t_n}+\sigma B_{t_n}^H\right)+\beta h^{4-2H}-\frac{1}{2}\sigma^2h^3n^{2H-1}\xrightarrow{a.s.}0\quad(n\to\infty)$$

that

$$\Delta_n(H)\overset{a.s.}{\sim}C\frac{h^{2-4H}}{n}\left(\sum_{i=1}^{n}(X_{t_i}^H-X_{t_{i-1}}^H)^2-2\mu hX_{t_n}^H+\mu^2h^2n\right)\xrightarrow{a.s.}0\quad(n\to\infty)$$

for all $H\in(0,\frac{1}{2})$. This gives (i). When $\frac{1}{2}<H<1$, Lemma 7.3 implies that

$$\frac{1}{n}\sum_{i=1}^{n}(\hat{t}_i)^2=h^{4H}O\left(n^{4H-2}\right)\left(O(nh)^{2-4H}+O(nh)^{1-2H}+1\right)=h^{4H}O\left(n^{4H-2}\right)\quad(n\to\infty) \tag{7.6}$$

and

$$\begin{aligned}h^{4H-1}n^{4H-3}X_{t_n}^H=&h^{4H-1}n^{4H-3}\left(\sigma B_{t_n}+\sigma B_{t_n}^H\right)+\beta h^{4H}n^{4H-2}-\frac{1}{2}\sigma^2h^{6H-1}n^{6H-3}\\ =&O\left(n^{-\frac{2-\gamma(4H-2)}{1+\gamma}}\right)\cdot\frac{1}{nh}\left(\sigma B_{t_n}+\sigma B_{t_n}^H\right)+\beta O\left(n^{-\frac{2-\gamma(4H-2)}{1+\gamma}}\right)\\ &-\frac{1}{2}\sigma^2O\left(n^{-\frac{2-\gamma(6H-3)}{1+\gamma}}\right)\xrightarrow{a.s.}0\quad(n\to\infty)\end{aligned}$$

for $0<\gamma<\frac{2}{6H-3}<\frac{2}{4H-2}$. It follows from Proposition 3.2 that

$$\begin{aligned}\Delta_n(H)&\overset{a.s.}{\sim}Ch^{4H-1}n^{4H-3}\left(\sum_{i=1}^{n}(X_{t_i}^H-X_{t_{i-1}}^H)^2-2\mu hX_{t_n}^H+\mu^2h^2n\right)\\ &=Ch^{4H-1}n^{4H-2}J_n(X^H)-2\mu h^{4H-1}n^{4H-3}X_{t_n}^H+\mu^2h^{4H}n^{4H-2}\xrightarrow{a.s.}0\quad(n\to\infty)\end{aligned}$$

for all $\frac{1}{2}<H<1$ and $0<\gamma<\frac{1}{4H-2}$ since $\frac{1}{4H-2}<\frac{2}{6H-3}<\frac{2}{4H-2}$. This gives (ii). □

Proof of Lemma 3.7 Let $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. Then, we have

$$\rho_n = \alpha_n - 4n\beta_n^2 h^2 = \sum_{i=1}^{n} (\hat{t}_i)^2 - 4nh^2 \cdot \left(\frac{1}{2} + \frac{(nh)^{2H-1}}{2}\right)^2$$

$$= \sum_{i=1}^{n} ({t_i}^{2H} - t_{i-1}^{2H})^2 - h^{4H} n^{4H-1} = h^{4H}\left(K_n(H) - n^{4H-1}\right) = h^{4H} O\left(n^{4H-1}\right) \quad (n \to \infty)$$

by Lemma 7.3. It follows that

$$\tilde{\Delta}_n(H) = \frac{\rho_n}{n^2 \hat{h}^2} \sum_{i=1}^{n} \left(X_{t_i}^H - X_{t_{i-1}}^H - \frac{1}{n} X_{t_n}\right)^2 \overset{a.s.}{\sim} C n^{4H-3} \sum_{i=1}^{n} \left(X_{t_i}^H - X_{t_{i-1}}^H - \frac{1}{n} X_{t_n}\right)^2$$

$$= C h^{2H} n^{4H-2} J_n(X^H) - C n^{4H-4} (X_{t_n})^2 \xrightarrow{a.s.} 0 \quad (n \to \infty)$$

for all $0 < H < \frac{1}{2}$, based on the fact that

$$n^{2H-2} X_{t_n} = n^{2H-2}\left(\sigma B_{t_n} + \sigma B_{t_n}^H\right) + \beta h n^{2H-1} - \frac{1}{2}\sigma^2 h^{2H} n^{4H-2} \xrightarrow{a.s.} 0 \quad (n \to \infty).$$

This obtains (i).

Similarly, when $\frac{1}{2} < H < 1$, we have

$$h^{2H-1} n^{2H-2} X_{t_n} = h^{2H} n^{2H-1} \left(\frac{1}{nh}\left(\sigma B_{t_n} + \sigma B_{t_n}^H\right) + \beta\right) - \frac{1}{2}\sigma^2 h^{4H-1} n^{4H-2}$$

$$= O\left(n^{-\frac{1-(2H-1)\gamma}{1+\gamma}}\right)\left(\frac{1}{nh}\left(\sigma B_{t_n} + \sigma B_{t_n}^H\right) + \beta\right) - \frac{1}{2}\sigma^2 O\left(n^{-\frac{1-(4H-2)\gamma}{1+\gamma}}\right) \xrightarrow{a.s.} 0$$

for all $0 < \gamma < \frac{1}{4H-2}$, as n tends to infinity. It follows that

$$\tilde{\Delta}_n(H) \overset{a.s.}{\sim} C h^{4H-2} n^{4H-3} \sum_{i=1}^{n} \left(X_{t_i}^H - X_{t_{i-1}}^H - \frac{1}{n} X_{t_n}\right)^2$$

$$= C h^{4H-1} n^{4H-2} J_n(X^H) - C h^{4H-2} n^{4H-4} (X_{t_n})^2 \xrightarrow{a.s.} 0 \quad (n \to \infty)$$

for all $\frac{1}{2} < H < 1$ and $0 < \gamma < \frac{1}{4H-2}$. This obtains (ii) and the lemma follows. □

7.2 Proof of lemmas in section 4

Proof of Lemma 4.2. Based on (7.3), we find that

$$\begin{aligned}
&\sqrt{n}\left(J_n\left(X^H\right) - \sigma^2\right) \\
&= \sqrt{n}\left(\frac{\sigma^2}{nh^{(2H)\wedge 1}} \sum_{i=1}^{n} \left(B_{t_i} - B_{t_{i-1}}\right)^2 + \frac{\sigma^2}{nh^{(2H)\wedge 1}} \sum_{i=1}^{n} \left(B_{t_i}^H - B_{t_{i-1}}^H\right)^2 - \sigma^2\right) \\
&\quad + \frac{2\sigma^2}{\sqrt{n} h^{(2H)\wedge 1}} \sum_{i=1}^{n} \left(B_{t_i} - B_{t_{i-1}}\right)\left(B_{t_i}^H - B_{t_{i-1}}^H\right) \\
&\quad + \frac{1}{\sqrt{n} h^{(2H)\wedge 1}} \sum_{i=1}^{n} \left[f\left(t_i\right) - f\left(t_{i-1}\right)\right]^2 \\
&\quad + \frac{2\sigma}{\sqrt{n} h^{(2H)\wedge 1}} \sum_{i=1}^{n} \left[f\left(t_i\right) - f\left(t_{i-1}\right)\right]\left((B_{t_i}^H - B_{t_{i-1}}^H) + (B_{t_i} - B_{t_{i-1}})\right)
\end{aligned} \tag{7.7}$$

for all $0 < H < 1$ and $n \geqslant 1$.

(i) Let $0 < H < \frac{1}{2}$. Clearly,

$$\lim_{n\to\infty} \frac{1}{nh^{4H}} E\left(\sum_{i=1}^{n} \left(B_{t_i} - B_{t_{i-1}}\right)\left(B^H_{t_i} - B^H_{t_{i-1}}\right)\right)^2 = \lim_{n\to\infty} h^{1-2H} = 0 \tag{7.8}$$

and Lemma 7.4 imply that

$$\begin{aligned}&\lim_{n\to\infty} \frac{1}{nh^{4H}} E\left(\sum_{i=1}^{n} \left[f(t_i) - f(t_{i-1})\right]\left(B_{t_i} - B_{t_{i-1}}\right)\right)^2 \\ &= \lim_{n\to\infty} \frac{h^{1-2H}}{nh^{2H}} \sum_{i=1}^{n} \left[f(t_i) - f(t_{i-1})\right]^2 = 0.\end{aligned} \tag{7.9}$$

We now need to estimate the term

$$\begin{aligned}\Lambda_n(f,H) &:= \frac{1}{nh^{4H}} E\left(\sum_{i=1}^{n} \left[f(t_i) - f(t_{i-1})\right]\left(B^H_{t_i} - B^H_{t_{i-1}}\right)\right)^2 \\ &= \frac{1}{nh^{2H}} \sum_{i=1}^{n} \left[f(t_i) - f(t_{i-1})\right]^2 \\ &\quad + \frac{1}{nh^{2H}} \sum_{1\leqslant i<j\leqslant n} \left[f(t_i) - f(t_{i-1})\right]\left[f(t_j) - f(t_{j-1})\right] R(i,j)\end{aligned} \tag{7.10}$$

for all $H \in (0, \frac{1}{2})$, where $R(i,j) = (j-i+1)^{2H} + (j-i-1)^{2H} - 2(j-i)^{2H}$. Based on the mean value theorem, we get that

$$\begin{aligned}|\Lambda_n(f,H;1)| :&= \lim_{n\to\infty} \frac{1}{nh^{2H}} |f(t_1)| \left|\sum_{j=2}^{n} [f(t_j) - f(t_{j-1})] R(1,j)\right| \\ &= \frac{1}{2}\sigma^2 \lim_{n\to\infty} \frac{h}{n} \sum_{j=2}^{n} \left|\beta - \sigma^2 H \xi_j^{2H-1}\right| \left|j^{2H} + (j-2)^{2H} - 2(j-1)^{2H}\right| \\ &= \frac{1}{2}\sigma^2 \lim_{n\to\infty} h \left|\beta - \sigma^2 H \xi_n^{2H-1}\right| \left|n^{2H} + (n-2)^{2H} - 2(n-1)^{2H}\right| = 0\end{aligned}$$

for some $\xi_j \in (t_{j-1}, t_j)$, $j = 2, 3, \ldots, n$ and

$$\begin{aligned}|\Lambda_n(f,H;2)| :&= \lim_{n\to\infty} \frac{1}{nh^{2H}} \left|\sum_{2\leqslant i<j} [f(t_i) - f(t_{i-1})][f(t_j) - f(t_{j-1})] R(i,j)\right| \\ &\leqslant \lim_{n\to\infty} \frac{h^{2-2H}}{n} \sum_{2\leqslant i<j} \left|\beta - \sigma^2 H \xi_i^{2H-1}\right| \left|\beta - \sigma^2 H \xi_j^{2H-1}\right| R(i,j) \\ &= \frac{1}{2}\sigma^2 \lim_{n\to\infty} h^{2-2H} \sum_{i=2}^{n-1} \left|\beta - \sigma^2 H \xi_i^{2H-1}\right| R(i,n) \\ &\leqslant \frac{1}{2}\sigma^2 \lim_{n\to\infty} h^{2-2H} \sum_{i=2}^{n-1} \left(|\beta| + \sigma^2 H((i-1)h)^{2H-1}\right) R(i,n) \\ &\leqslant \frac{1}{2}\sigma^2 \lim_{n\to\infty} h \sum_{i=2}^{n-1} \left(|\beta| h^{1-2H} + \sigma^2\right) R(i,n) = 0\end{aligned}$$

for all $0 < H < \frac{1}{2}$ and some $\xi_i \in (t_{i-1}, t_i)$ since the series

$$\sum_{n=1}^{\infty} \left((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}\right)$$

converges. It follows that $\lim_{n\to\infty} \Lambda_n(f, H) = 0$ for all $0 < H < \frac{1}{2}$.

$$\sqrt{n}h^{1-2H}\, G_n(B) = \left(nh^{1+\gamma}\right)^{\frac{1-2H}{1+\gamma}}\, n^{-\frac{1-4H+\gamma}{1+\gamma}} \cdot G_n(B^H) \longrightarrow \begin{cases} 0, & 0 < \gamma < 1-4H, \\ 1, & \gamma = 1-4H, \\ +\infty, & \gamma > 1-4H, \end{cases}$$

almost surely, as n tends to infinity.

Combining this with (7.7), (7.8), (7.9), Propostion 4.1, and Slutsky's theorem, we get that

$$\sqrt{n}\left(J_n\left(X^H\right) - \sigma^2\right) \longrightarrow N\left(0, (2+\lambda_H)\sigma^4\right) \tag{7.11}$$

for all $0 < H < \frac{1}{4}$ in distribution, as n tends to infinity.

(ii) Let $\frac{3}{4} < H < 1$. Based on (7.7), we find that

$$\begin{aligned}\sqrt{n}\left(J_n\left(X^H\right) - \sigma^2\right) &= \sigma^2\sqrt{n}\left(G_n(B) - 1\right) + \sigma^2\sqrt{n}h^{2H-1}G_n(B^H) \\ &\quad + \frac{2\sigma^2}{\sqrt{n}h}\sum_{i=1}^{n}\left(B_{t_i} - B_{t_{i-1}}\right)\left(B^H_{t_i} - B^H_{t_{i-1}}\right) + \frac{1}{\sqrt{n}h}\sum_{i=1}^{n}\left[f(t_i) - f(t_{i-1})\right]^2 \\ &\quad + \frac{2\sigma}{\sqrt{n}h}\sum_{i=1}^{n}\left[f(t_i) - f(t_{i-1})\right]\left(B_{t_i} - B_{t_{i-1}}\right) \\ &\quad + \frac{2\sigma}{\sqrt{n}h}\sum_{i=1}^{n}\left[f(t_i) - f(t_{i-1})\right]\left(B^H_{t_i} - B^H_{t_{i-1}}\right)\end{aligned} \tag{7.12}$$

for all $n \geqslant 1$. First, the strong law of large numbers (Lemma 3.1) implies that

$$\sqrt{n}h^{2H-1}G_n(B^H) = (nh^{1+\gamma})^{\frac{2H-1}{1+\gamma}} n^{-\frac{4H-3-\gamma}{1+\gamma}} \cdot G_n(B^H) \longrightarrow \begin{cases} 0, & 0 < \gamma < 4H-3, \\ 1, & \gamma = 4H-3, \\ +\infty, & \gamma > 4H-3 \end{cases} \tag{7.13}$$

almost surely, as n tends to infinity. Next, we have

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{nh^2}E\left(\sum_{i=1}^{n}\left(B_{t_i} - B_{t_{i-1}}\right)\left(B^H_{t_i} - B^H_{t_{i-1}}\right)\right)^2 \\ &= \lim_{n\to\infty}\frac{1}{nh^2}\sum_{i=1}^{n}E\left(B_{t_i} - B_{t_{i-1}}\right)^2\left(B^H_{t_i} - B^H_{t_{i-1}}\right)^2 \\ &= \lim_{n\to\infty} h^{2H-1} = 0,\end{aligned} \tag{7.14}$$

and by Lemma 7.4, we get

$$\lim_{n\to\infty}\frac{1}{nh^2}E\left(\sum_{i=1}^{n}\left[f(t_i) - f(t_{i-1})\right]\left(B_{t_i} - B_{t_{i-1}}\right)\right)^2 = \lim_{n\to\infty}\frac{1}{nh}\sum_{i=1}^{n}\left[f(t_i) - f(t_{i-1})\right]^2 = 0 \tag{7.15}$$

for all $\frac{1}{2} < H < 1$ and $0 < \gamma \leqslant \frac{1}{4H-2}$. Moreover, from the proof of Lemma 7.4, we can obtain

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}h}\sum_{i=1}^{n}\left[f(t_i) - f(t_{i-1})\right]^2 = \begin{cases} 0, & 0 < \gamma < \frac{1}{8H-3}, \\ \frac{H^2}{4H-1}\sigma^4, & \gamma = \frac{1}{8H-3}, \\ +\infty, & \gamma > \frac{1}{8H-3} \end{cases} \tag{7.16}$$

for all $\frac{1}{2} < H < 1$. Similar to the proof of (7.10), we have

$$\begin{aligned}|\tilde{\Lambda}_n(f,H;1)| :&= \lim_{n\to\infty} \frac{1}{nh^{2-2H}} f(t_1) \sum_{j=2}^{n} [f(t_j)-f(t_{j-1})]\, R(1,j) \\ &\leqslant \sigma^2 H \lim_{n\to\infty} h^{4H-1}\cdot \frac{1}{n}\sum_{j=2}^{n} j^{2H-1} R(1,j) = \sigma^2 H \lim_{n\to\infty} h^{4H-1} n^{2H-1} R(1,n) \\ &= \sigma^2 H \lim_{n\to\infty} h^{4H-1} n^{2H-1}\left(n^{2H}+(n-2)^{2H}-2(n-1)^{2H}\right) \\ &= C\lim_{n\to\infty} h^{4H-1} n^{4H-3} = 0, \end{aligned} \tag{7.17}$$

provided $\frac{3}{4} < H < 1$ and $0 < \gamma < \frac{2}{4H-3}$. Moreover

$$\begin{aligned}|\tilde{\Lambda}_n(f,H;2)| :&\leqslant \lim_{n\to\infty} \frac{1}{nh^{2-2H}} |\sum_{i<j} [f(t_i)-f(t_{i-1})][f(t_j)-f(t_{j-1})] R(i,j)| \\ &\leqslant C \lim_{n\to\infty} h^{6H-2} \frac{1}{n} \sum_{i<j} (ij)^{2H-1} R(i,j) = C \lim_{n\to\infty} h^{6H-2} n^{2H-1} \sum_{i=1}^{n-1} i^{2H-1} R(i,n) \\ &\leqslant C \lim_{n\to\infty} h^{6H-2} n^{4H-2} \sum_{i=1}^{n-1} \left((n-i+1)^{2H}+(n-i-1)^{2H}-2(n-i)^{2H}\right) \\ &= C \lim_{n\to\infty} h^{6H-2} n^{4H-2} \sum_{k=1}^{n-1} \left((k+1)^{2H}+(k-1)^{2H}-2k^{2H}\right) \\ &\leqslant C \lim_{n\to\infty} h^{6H-2} n^{4H-2} \sum_{k=1}^{n-1} k^{2H-2} = C \lim_{n\to\infty} h^{6H-2} n^{6H-3} = 0 \end{aligned}$$

for all $\frac{1}{2} < H < 1$, provided $0 < \gamma < \frac{1}{6H-3}$. It follows that

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{nh^2} E\left(\sum_{i=1}^{n} [f(t_i)-f(t_{i-1})]\left(B^H_{t_i}-B^H_{t_{i-1}}\right)\right)^2 \\ =& \lim_{n\to\infty} \frac{1}{nh^{2-2H}} \sum_{i=1}^{n} [f(t_i)-f(t_{i-1})]^2 \\ &+ \lim_{n\to\infty} \frac{1}{nh^{2-2H}} \sum_{i<j} [f(t_i)-f(t_{i-1})]\,[f(t_j)-f(t_{j-1})]\, R(i,j) \\ \leqslant& \lim_{n\to\infty} \frac{1}{nh^{2-2H}} \sum_{i=1}^{n} [f(t_i)-f(t_{i-1})]^2 + |\tilde{\Lambda}_n(f,H;1)| + |\tilde{\Lambda}_n(f,H;2)| = 0 \end{aligned} \tag{7.18}$$

for all $0 < \gamma < \frac{1}{6H-3}$. Combining these with (7.12), (7.13), (7.14), (7.15), (7.16), (7.18) Proposition 4.1 and Slutsky's theorem, we obtain

$$\sqrt{n}\left(J_n\left(X^H\right)-\sigma^2\right) \longrightarrow N\left(0,2\sigma^4\right) \quad (n\to\infty) \tag{7.19}$$

in distribution, provided $0 < \gamma < (4H-3)\wedge \frac{1}{8H-3}$. Moreover, when $\gamma = (4H-3)\wedge\frac{1}{8H-3}$, we have

$$\sqrt{n}\left(J_n\left(X^H\right)-\sigma^2\right) \longrightarrow \begin{cases} N\left(\sigma^2, 2\sigma^4\right), & 4H-3 < \frac{1}{8H-3}, \\ N\left(\frac{H^2}{4H-1}\sigma^4, 2\sigma^4\right), & 4H-3 > \frac{1}{8H-3}, \\ N\left(\sigma^2+\frac{H^2}{4H-1}\sigma^4, 2\sigma^4\right), & 4H-3 = \frac{1}{8H-3} \end{cases}$$

in distribution, as n tends to infinity. □

Proof of Lemma 4.3 For $0 < H \leqslant \frac{1}{2}$, based on the proof of Lemma 3.5, we have

$$\begin{aligned}\Xi_n(H) &= \frac{n\hat{h}}{\sum_{i=1}^{n}(\hat{t}_i)^2}\Delta_n(H)^2 \cdot \sqrt{n} = \frac{\sqrt{n}}{n^3\hat{h}^3}\sum_{i=1}^{n}(\hat{t}_i)^2\left(\sum_{i=1}^{n}\left(X_{t_i}^H - X_{t_{i-1}}^H - \mu h\right)^2\right)^2 \\ &\overset{a.s.}{\sim} Cn^{-\frac{3}{2}}h^{2-6H}\left(nh^{2H}J_n(X^H) - 2\mu h X_{t_n}^H + \mu^2 h^2 n\right)^2 \\ &= C\left(n^{\frac{1}{4}}h^{1-H}J_n(X^H) - 2\mu n^{-\frac{3}{4}}h^{2-3H}X_{t_n}^H + \mu^2 n^{\frac{1}{4}}h^{3-3H}n\right)^2 \xrightarrow{a.s.} 0,\end{aligned} \tag{7.20}$$

as n tends to infinity, if $0 < \gamma < 3 - 4H$ and (i) is proved.

Similarly, for $\frac{1}{2} < H < 1$, using Lemma 3.5 again, we have

$$\begin{aligned}\Xi_n(H) &\overset{a.s.}{\sim} C\sqrt{n}h^{4H-3}n^{4H-4}\left(nhJ_n(X^H) - 2\mu h X_{t_n}^H + \mu^2h^2n\right)^2 \\ &= C\left(n^{2H-\frac{3}{4}}h^{2H-\frac{1}{2}}J_n(X^H) - 2\mu n^{2H-\frac{7}{4}}h^{2H-\frac{1}{2}}X_{t_n}^H + \mu^2 n^{2H-\frac{3}{4}}h^{2H+\frac{1}{2}}\right)^2 \xrightarrow{a.s.} 0,\end{aligned} \tag{7.21}$$

as n tends to infinity, if $0 < \gamma < \frac{1}{8H-3}$, we can get (ii). □

Lemma 7.5 *Let $\frac{1}{2} < H \leqslant \frac{3}{4}$ and conditions (C1) and (C2) hold. Then, we have*

$$\sqrt{n}\left(J_n(M^H) - 1 - h^{2H-1}\right) \longrightarrow N(0,2)$$

in distribution, as n tends to infinity, where $M^H = B + B^H$ and

$$J_n(M^H) = \frac{1}{nh}\sum_{i=1}^{n}\left((B_{t_i} - B_{t_{i-1}}) + (B_{t_i}^H - B_{t_{i-1}}^H)\right)^2.$$

Proof Based on law of large numbers (Lemma 3.1), we find that

$$J_n(M^H) \xrightarrow{a.s.} 1 \quad (n \to \infty)$$

and based on Proposition 4.1 and (7.14),

$$\begin{aligned}&\sqrt{n}\left(J_n(M^H) - 1 - h^{2H-1}\right) \\ &= \frac{1}{\sqrt{n}h}\sum_{i=1}^{n}\left\{\left((B_{t_i} - B_{t_{i-1}}) + (B_{t_i}^H - B_{t_{i-1}}^H)\right)^2 - h - h^{2H}\right\} \\ &= \sqrt{n}\left(G_n(B) - 1\right) + \sqrt{n}h^{2H-1}\left(G_n(B^H) - 1\right) \\ &\quad + \frac{2}{\sqrt{n}h}\sum_{i=1}^{n}(B_{t_i} - B_{t_{i-1}})(B_{t_i}^H - B_{t_{i-1}}^H) \\ &\longrightarrow N(0,2)\end{aligned} \tag{7.22}$$

in distribution, as n tends to infinity. □

Lemma 7.6 *Let $\frac{1}{4} < H \leqslant \frac{1}{2}$ and conditions (C1) and (C2) hold. Then, we have*

$$\sqrt{n}\left(J_n(M^H) - 1 - h^{1-2H}\right) \longrightarrow N(0, 2 + \lambda_H)$$

in distribution, as n tends to infinity, where $M^H = B + B^H$ and

$$J_n(M^H) = \frac{1}{nh^{2H}}\sum_{i=1}^{n}\left((B_{t_i} - B_{t_{i-1}}) + (B_{t_i}^H - B_{t_{i-1}}^H)\right)^2.$$

Proof Based on the strong law of large numbers (Lemma 3.1), we find that

$$J_n(M^H) \xrightarrow{a.s.} 1 \quad (n \to \infty)$$

and based on Proposition 4.1 and (7.8),

$$
\begin{aligned}
&\sqrt{n}\left(J_n(M^H)-1-h^{1-2H}\right)\\
&=\frac{1}{\sqrt{n}h^{2H}}\sum_{i=1}^{n}\left\{\left((B_{t_i}-B_{t_{i-1}})+(B^H_{t_i}-B^H_{t_{i-1}})\right)^2-h-h^{2H}\right\}\\
&=\sqrt{n}\left(G_n(B^H)-1\right)+\sqrt{n}h^{1-2H}\left(G_n(B)-1\right)\\
&\quad+\frac{2}{\sqrt{n}h^{2H}}\sum_{i=1}^{n}(B_{t_i}-B_{t_{i-1}})(B^H_{t_i}-B^H_{t_{i-1}})\\
&\longrightarrow N(0,2+\lambda_H)
\end{aligned}
\tag{7.23}
$$

in distribution, as n tends to infinity. □

Proof of Lemma 4.5 Let $\frac{1}{2}<H<\frac{3}{4}$. Based on (7.7), Lemma 7.5, (7.14), (7.15), (7.16), (7.18), and Slutsky's theorem, we find that

$$
\begin{aligned}
&\sqrt{n}\left(J_n\left(X^H\right)-\sigma^2-h^{2H-1}\sigma^2\right)\\
&=\sigma^2\sqrt{n}\left(J_n(M^H)-1-h^{2H-1}\right)+\frac{1}{\sqrt{n}h}\sum_{i=1}^{n}\left[f\left(t_i\right)-f\left(t_{i-1}\right)\right]^2\\
&\quad+\frac{2\sigma}{\sqrt{n}h}\sum_{i=1}^{n}\left[f\left(t_i\right)-f\left(t_{i-1}\right)\right]\left(M^H_{t_i}-M^H_{t_{i-1}}\right)\\
&\longrightarrow\begin{cases}N\left(0,2\sigma^4\right), & 0<\gamma<\frac{1}{8H-3},\\ N\left(\frac{H^2}{4H-1}\sigma^4,2\sigma^4\right), & \gamma=\frac{1}{8H-3}\end{cases}
\end{aligned}
\tag{7.24}
$$

in distribution, as n tends to infinity.

Let $H=\frac{3}{4}$. Based on (7.23) and Proposition 4.1, it follows that

$$
\begin{aligned}
\sqrt{\frac{n}{\log n}}\left(J_n(M^{3/4})-1-\sqrt{h}\right)&=\sqrt{\frac{n}{\log n}}\left(G_n(B)-1\right)+\sqrt{\frac{n}{\log n}}\left(G_n(B^{3/4})-1\right)\\
&\quad+2\frac{1}{\sqrt{n\log n}h}\sum_{i=1}^{n}(B_{t_i}-B_{t_{i-1}})(B^H_{t_i}-B^H_{t_{i-1}})\longrightarrow N\left(0,\frac{9}{4}\right)
\end{aligned}
$$

in distribution, as n tends to infinity. Combining this with (7.7), Lemma 7.6, (7.14), (7.15), (7.16), (7.18), and Slutsky's theorem, we get that

$$
\begin{aligned}
&\sqrt{\frac{n}{\log n}}\left(J_n\left(X^{3/4}\right)-\sigma^2-\sqrt{h}\sigma^2\right)\\
&=\sigma^2\sqrt{\frac{n}{\log n}}\left(J_n(M^{3/4})-1-\sqrt{h}\right)+\frac{1}{\sqrt{n\log n}h}\sum_{i=1}^{n}\left[f\left(t_i\right)-f\left(t_{i-1}\right)\right]^2\\
&\quad+\frac{2\sigma}{\sqrt{n\log n}h}\sum_{i=1}^{n}\left[f\left(t_i\right)-f\left(t_{i-1}\right)\right]\left(M^{3/4}_{t_i}-M^{3/4}_{t_{i-1}}\right)\longrightarrow N\left(0,\frac{9}{4}\sigma^4\right)
\end{aligned}
\tag{7.25}
$$

for all $0<\gamma\leqslant\frac{1}{3}$ in distribution, as n tends to infinity.

When $\frac{1}{4}<H<\frac{1}{2}$, based on (7.7), Lemma 7.6, (7.8), (7.9), (7.10) and Slutsky's theorem, we find that

$$
\begin{aligned}
&\sqrt{n}\left(J_n\left(X^H\right)-\sigma^2-h^{2H-1}\sigma^2\right)\\
&=\sigma^2\sqrt{n}\left(J_n(M^H)-1-h^{1-2H}\right)+\frac{1}{\sqrt{n}h^{2H}}\sum_{i=1}^{n}\left[f\left(t_i\right)-f\left(t_{i-1}\right)\right]^2\\
&\quad+\frac{2\sigma}{\sqrt{n}h^{2H}}\sum_{i=1}^{n}\left[f\left(t_i\right)-f\left(t_{i-1}\right)\right]\left(M^H_{t_i}-M^H_{t_{i-1}}\right)\\
&\longrightarrow\begin{cases}N\left(0,(2+\lambda_H)\sigma^4\right), & 0<\gamma<3-4H,\\ N\left(\beta^2,(2+\lambda_H)\sigma^4\right), & \gamma=3-4H\end{cases}
\end{aligned}
\tag{7.26}
$$

in distribution, as n tends to infinity. □

Proof of Lemma 4.7 Let $H\in(0,\frac{1}{2})\cup(\frac{1}{2},1)$. Based on the proof of Lemma 3.7, we have that

$$
\begin{aligned}
\tilde{\Xi}_n(H)&=\frac{n\hat{h}}{\rho_n}\left(\tilde{\Delta}_n(H)\right)^2\sqrt{n}\\
&=\frac{\rho_n}{n^3\hat{h}^3}\left(\sum_{i=1}^{n}\left(X^H_{t_i}-X^H_{t_{i-1}}-\frac{1}{n}X_{t_n}\right)^2\right)^2\sqrt{n}\\
&\overset{a.s.}{\sim}C\frac{\sqrt{n}}{n^{4-4H}h^{2H}}\left(nh^{2H}J_n(X^H)-\frac{1}{n}(X_{t_n})^2\right)^2\\
&=C\left(h^Hn^{2H-\frac{3}{4}}J_n(X^H)-h^{-H}n^{2H-\frac{11}{4}}(X_{t_n})^2\right)^2\xrightarrow{a.s.}0
\end{aligned}
$$

for all $\frac{3}{8}\leqslant H<\frac{1}{2}$ and $0<\gamma<\frac{3-4H}{8H-3}$. When $0<H\leqslant\frac{3}{8}$, the above convergence is also true, and statements (i) and (ii) follow. Similarly, we also have

$$
\begin{aligned}
\tilde{\Xi}_n(H)&=\frac{\rho_n}{n^3\hat{h}^3}\left(\sum_{i=1}^{n}\left(X^H_{t_i}-X^H_{t_{i-1}}-\frac{1}{n}X_{t_n}\right)^2\right)^2\sqrt{n}\\
&\overset{a.s.}{\sim}Cn^{4H-\frac{7}{2}}h^{4H-3}\left(nhJ_n(X^H)-\frac{1}{n}(X_{t_n})^2\right)^2\\
&=C\left(n^{2H-\frac{3}{4}}h^{2H-\frac{1}{2}}J_n(X^H)-h^{2H-\frac{3}{2}}n^{2H-\frac{11}{4}}(X_{t_n})^2\right)^2\xrightarrow{a.s.}0
\end{aligned}
$$

for all $\frac{1}{2}<H<1$ and $0<\gamma<\frac{1}{8H-3}$. This gives (iii), and the lemma follows. □

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