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Optimal stopping under model uncertainty in a general setting

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Abstract We consider the optimal stopping time problem under model uncertainty, $R(v) = \operatorname*{ess\,sup}_{\mathbb{P}\in\mathcal{P}} \operatorname*{ess\,sup}_{\tau\in\mathcal{S}_v} E^{\mathbb{P}}[Y(\tau)|\mathcal{F}_v]$, for every stopping time v, within the framework of families of random variables indexed by stopping times. This setting is more general than the classical setup of stochastic processes, notably allowing for general payoff processes that are not necessarily right-continuous. Under weaker integrability, with regularity assumptions for the reward family $Y = (Y(v), v \in S)$, the existence of an optimal stopping time is demonstrated. Sufficient conditions for the existence of an optimal model are then determined. For this purpose, we present a universal optional decomposition for the generalized Snell envelope family associated with Y. This decomposition is then employed to prove the existence of an optimal probability model and to study its properties.

Keywords Optimal stopping, Supermartingale, Uncertainty, American options

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1. Introduction

Optimal stopping problems are a class of decision-making problems that involve finding the best time to take a particular action. These problems arise in various contexts where a decision must be made based on uncertain information and limited resources, such as in investment and job search scenarios, launching a new brand, exercising an option, etc. All of these problems naturally lead to an optimal stopping problem. Formally, the main target is to choose a random stopping time that maximizes the expected reward induced by the problem's payoff process.

The classical solution to optimal stopping problems assumes that the reward's probability law is given to the decision maker. The problem is then formulated with respect to a unique probability measure. However, in numerous real-world scenarios, decision-makers contend with uncertainty regarding the true probabilistic *model*. Consequently, the probability law that

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generates future rewards may be partially or completely unknown, creating what is known as model *uncertainty* or *ambiguity*. This can arise due to various reasons, such as incomplete information, parameter estimations errors, or the presence of hidden variables. An example is the analysis of American options in incomplete financial markets, where we have to deal with multiple equivalent martingale measures, and it is uncertain which one underlies the market. In such scenarios, the decision maker relies on a set of plausible probability measures or models, each of which could potentially result in vastly different optimal stopping strategies.

In recent years, optimal stopping under ambiguity has attracted a lot of attention: [2, 4, 5, 10–12, 17, 19, 27, 31]. This report focuses on two types of optimal stopping problems: the worst-case and the best-case optimal stopping. The former—the so-called robust optimal stopping—maximizes the worst-case expected value: $\inf_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}[Y_\tau]$, via the choice of τ; see, e.g. [3–5, 10, 30, 31]. The latter, on the other hand, maximizes the best-case expected value: $\sup_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}[Y_\tau]$; see, e.g. [4–7, 15]. Here, $\mathcal{P}$ denotes the set of probability measures and Y represents the payoff process. All of the aforementioned studies assumed that $\mathcal{P}$ is dominated by a single probability measure, and Y is an $\mathbb{F}$-adapted RCLL (càdlàg) process. Several studies have focused on weakening the assumptions for the probability class $\mathcal{P}$. Therefore, in the literature, weak conditions related to mathematical finance have been discovered and improved to achieve greater generality; see e.g. [3, 15, 29, 30]. However, relatively few studies have focused on allowing for more general payoff processes. To the best of our knowledge, the most general result given in the literature is that of Krätschmer et al. [22]: A numerically implementable method for single stopping problems under uncertainty in drift and jump intensity was proposed. For general reward processes driven by multi-dimensional jump-diffusions, we also refer to Roger et al. [24].

In this study, we solve an optimal stopping problem under uncertainty with respect to a dominated probability class $\mathcal{P}$, where the reward is given by a family $Y = (Y(\tau), \tau \in \mathcal{S})$ of random variables indexed by stopping times. More precisely, $\mathcal{S}$ is used to collect all stopping times with respect to a general filtration $\mathbb{F}$ on Ω. The main result is Theorem 5.3, in which we construct an optimal pair $(\mathbb{P}^*, \tau^*) \in \mathcal{P} \times \mathcal{S}$ such that

$$R \triangleq \sup_{\mathbb{P}\in\mathcal{P}} \sup_{\tau\in\mathcal{S}} E^{\mathbb{P}}[Y(\tau)] = \sup_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}[Y(\tau^*)] = E^{\mathbb{P}^*}[Y(\tau^*)] \quad \text{a.s.} \tag{1.1}$$

The optimal stopping problem (1.1) features generality along three dimensions: (i) the setup of families of random variables indexed by stopping times is more general than the classical setup of processes; therefore, (1.1) allows for more general payoff processes that are not necessarily right-continuous. (ii) particularly, for the existence of optimal stopping times, the developed approach allows for simpler proofs by using only classical tools of Probability Theory, and (iii) it allows the problem to be solved under weaker assumptions than those used in previous literature. We solve problem (1.1) for a non-negative family of random variables $Y = (Y(\tau), \tau \in \mathcal{S})$. However, by applying a simple shift, all results apply to a general payoff family (see Lemma 5.1).

The technical set-up closely follows that of [20], in which the classical optimal stopping in the case of a reward given by a family of random variables indexed by stopping times was revisited. Herein, this framework is extended to account for model uncertainty (1.1). The key to this study is the generalized Snell envelope family of Y:

$$R(v) \triangleq \operatorname*{ess\,sup}_{\mathbb{P}\in\mathcal{P}} \operatorname*{ess\,sup}_{\tau\in\mathcal{S}_v} E^{\mathbb{P}}[Y(\tau)|\mathcal{F}_v], \quad v \in \mathcal{S}, \tag{1.2}$$

which is characterized as the smallest $\mathcal{P}$-supermartingale family that is greater than the reward family Y. Within this setting, a family of random variables indexed by stopping times is called a $\mathcal{P}$-supermartingale family if it is a $\mathbb{P}$-supermatingale family with respect to each measure $\mathbb{P} \in \mathcal{P}$ (please refer to Section 2 for the notations). The properties of this generalized Snell envelope family follow naturally from the extant theory of optimal stopping in [20]. In the classical literature, optimal stopping under uncertainty is typically formulated within the framework of processes. The reward is determined by a right-continuous, adapted process (Y_t). The value function family $\{R_v\}_{v\in\mathcal{S}}$ is defined as above, and gives rise to a non-negative, adapted process $\mathcal{R} = \{R_t, \mathcal{F}_t; 0 \leqslant t \leqslant T\}$. However, equation (1.2) cannot be regarded as the process $\mathcal{R}$ evaluated at the stopping time v. The standard approach involves an important step known as the *aggregation* step, which involves finding a RCLL modification $\mathcal{R}^0 = \{R^0_t, \mathcal{F}_t; 0 \leqslant t \leqslant T\}$ of the process $\mathcal{R}$, such that for each $v \in \mathcal{S}$, $R_v(\omega) = R^0_{v(\omega)}(\omega)$ for a.e. $\omega \in \Omega$. This process $\mathcal{R}^0$ is the generalized Snell envelope process for the payoff Y. Note that this step is nontrivial and relies on firm and sophisticated results based on the general theory of stochastic processes (see e.g. [19, 32, 33]). The framework of the present study overcomes this complexity. One advantage of the set-up employing families of random variables is that the aggregation step can be circumvented. Another benefit is the weaker integrability assumption required for the existence of an optimal stopping time. Specifically, we only assume that $R < \infty$, which is a weaker condition compared to the classical assumption made in the literature, namely, that (Y_t) is of class $(\mathcal{D})$. Moreover, optimal stopping times are characterized differently from the framework of processes. Given further assumptions on the reward family, we express the optimal stopping time as the essential infimum of a set of stopping times, instead of as the hitting time of processes, namely, $\tau^* = \text{ess inf}\{\tau \in \mathcal{S}, R(\tau) = Y(\tau) \text{ a.s.}\}$.

In order to prove the existence of an optimal model, we shall establish a "universal" optional decomposition for the Snell envelope family $\mathcal{R} = (R(v), v \in \mathcal{S})$ in the sense that it holds simultaneously for all $\mathbb{P} \in \mathcal{P}$. Specifically, under suitable conditions for the family $\mathcal{P}$, $\mathcal{R}$ is decomposed as the difference between a $\mathcal{P}$-martingale with RCLL paths and an optional RCLL increasing process. There are two results, which combined, prove our claim. In the first, in Theorem 5.1, we prove a universal optional decomposition theorem for $\mathcal{P}$-supermartingale families. To the best of our knowledge, such a decomposition has not previously been obtained. In Proposition 5.8, under the condition $\sup\limits_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}[(Y^*)^2] < \infty$ for the random variable $Y^* := \text{ess}\sup\limits_{\tau\in\mathcal{S}} Y(\tau)$, we show that $\mathcal{R}$ satisfies the integrability condition of Theorem 5.1; thus, it obtains the "universal" optional decomposition. Taken together, this leads to Theorem 5.3, the main result in Section 5: there exists an optimal model $\mathbb{P}^* \in \mathcal{P}$ such that (1.1) holds.

Notably, the developed framework sheds new light on the valuation of American knock-in barrier options. Barrier options of the knock-in type are financial derivatives that only become valuable if the underlying asset price reaches a predetermined barrier H, known as the "knocked-in" event. Once triggered, the option holder retains a standard American option with strike K, for instance, a put. As an example, an up-and-in barrier option requires the asset price to first rise and hit the upper barrier, triggering the knocked-in event. Thereafter, the option holder would benefit from falling asset prices, resulting in a higher payoff for the put option. Consider an underlying asset with the price process S in a market environment where the riskless rate of return is r. Let T be the expiration date for any option on the asset. Then, $(K - S)^+$ is the payoff of the put option when exercised at price S. In such a context, we are interested in the valuation problem for the American put option of the "up-and-in" barrier type, with the payoff

$$Y(t) = e^{-rt}(K - S(t))^{+}\mathbf{1}_{\{t \geqslant \tau_H\}}, \quad 0 \leqslant t \leqslant T, \tag{1.3}$$

where $\tau_H \triangleq \inf\{t \leqslant T : S(t) \leqslant H\}$ is the time where the option becomes "knocked-in" (see e.g., [1]). The payoff process is not right-continuous. Thus, the developed framework is particularly well-suited for analyzing this type of option. Now, assume that there is model uncertainty and the decision maker has to choose a stopping time to maximize the expected reward under any of the models in a given probability class $\mathcal{P}$; for instance, $\mathcal{P}$ generated by an incomplete financial market. Therefore, the hedging price $H(S)$ of the American contingent claim (1.3) can be computed as the optimal expected reward in the following optimal stopping problem under uncertainty:

$$H(S) \triangleq \sup_{\mathbb{P}\in\mathcal{P}} \sup_{\tau\in\mathcal{S}} E^{\mathbb{P}}[e^{-r\tau}(K - S(\tau))^{+}\mathbf{1}_{\{\tau \geqslant \tau_H\}}], \tag{1.4}$$

where $S = S_0$ is the initial value. The interest for the setup of families of random variables indexed by the stopping time has also been stressed by Kobylanski et al. [21], in the framework of optimal multiple stopping. Further applications are prospectively feasible, such as in the indifference valuation (seller's perspective; Carmona [9], Roger et al. [24]) of general optimally stopped reward processes under the multiple priors model.

The rest of the paper is organized as follows: Section 2 introduces the general setup, including the formulation of the problem under model ambiguity. In Section 3, we study the properties of the generalized Snell envelope family $(R(v), v \in \mathcal{S})$. We also give the necessary and sufficient conditions for the existence of an optimal pair $(\mathbb{P}^*, \tau^*) \in \mathcal{P} \times \mathcal{S}$. Section 4 is devoted to establishing the existence of an optimal stopping time. In Section 5, we find conditions for the family $\mathcal{P}$, under which there exists an optimal model for the problem.

2. Setting and problem description

Consider a decision maker (e.g., seller, buyer, firm) that needs to choose the best time to exercise a certain action in order to maximize the expected revenue. The decision must be made before a fixed predetermined time $T > 0$. Formally, we fix a filtered measurable base with a finite horizon $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{0\leqslant t\leqslant T})$. We assume that the filtration $\mathbb{F}$ satisfies the usual conditions of right continuity and augmentation by the null sets of $\mathcal{F} = \mathcal{F}_T$. By $\mathcal{S}$ we denote the class of $\mathbb{F}$-stopping times with values in $[0, T]$.

The decision maker chooses a stopping time τ in $\mathcal{S}$. When the decision is made to stop at τ, we assume that the decision maker receives the quantity $Y(\tau)$, where $Y(\tau)$ is a $\mathcal{F}_\tau$-measurable random variable. The problem is thus expressed in terms of families of random variables indexed by stopping times. The family $Y = (Y(\tau), \tau \in S)$ is called the reward (or payoff) family. Because the decision maker is uncertain about the probability with which the reward Y is generated, a class $\mathcal{P} = \{\mathbb{P} \,|\, \mathbb{P} \sim \mathbb{Q}\}$ of probability models $\mathbb{P}$ on $(\Omega, \mathcal{F})$ that share the same null sets with a base reference model $\mathbb{Q} \in \mathcal{P}$ is utilized. Thereafter, the decision maker has to choose a random stopping time that maximizes the expected reward under any of the models. In the present context, this yields to formulation of the following optimal stopping problem (at time 0 and at time $v \in \mathcal{S}$):

$$\begin{aligned} R &\triangleq \sup_{\mathbb{P}\in\mathcal{P}} \sup_{\tau\in\mathcal{S}} E^{\mathbb{P}}[Y(\tau)], \\ R(v) &\triangleq \operatorname*{ess\,sup}_{\mathbb{P}\in\mathcal{P}} \operatorname*{ess\,sup}_{\tau\in\mathcal{S}_v} E^{\mathbb{P}}[Y(\tau)|\mathcal{F}_v], \end{aligned} \tag{2.1}$$

where $\mathcal{S}_v := \{\tau \in \mathcal{S} \mid \tau \geqslant v \text{ a.s.}\}$ Therefore, solving the optimal stopping time problem (2.1) at time v, mainly consists to prove the existence of an optimal stopping time $\tau^*(v)$ and an optimal model $\mathbb{P}^*$, such that

$$R(v) = E^{\mathbb{P}^*}[Y(\tau^*(v))|\mathcal{F}_v] \quad \text{a.s.} \tag{2.2}$$

Hereinafter, we assume that the payoff is a non-negative family of random variables $Y = (Y(\tau), \tau \in \mathcal{S})$. This assumption simplifies the analysis, but all results can be extended to the general case by applying a simple translation argument. The details of this argument are provided by Lemma 5.1.

The following notations are used herein:

- $L^0(\mathcal{F}, \mathbb{R})$ is the algebra of equivalence classes of $\mathbb{R}$-valued random variables on Ω.
- $L^0_+(\mathcal{F}, \mathbb{R}) := \{\xi \in L^0(\mathcal{F}, \mathbb{R}) | \xi \geqslant 0\}$.
- $L^1(\Omega, \mathcal{F}, \mathbb{Q})$ is the set of real-valued optional processes ξ with $\sup\limits_{0 \leqslant t \leqslant T} E^{\mathbb{Q}}[|\xi_t|] < \infty$.
- $\mathcal{S}_{\sqsubseteq+} := \{\tau \in \mathcal{S} \mid \tau > \sqsubseteq \text{ a.s. on } \{\sqsubseteq < \mathcal{T}\} \text{ and } \tau = \sqsubseteq \text{ a.s. on } \{\sqsubseteq = \mathcal{T}\}\}$.
- For any sub-σ-field $\mathcal{G}$ of $\mathcal{F}$, for any probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$, let $L^2(\mathcal{G}, \mathbb{P})$ be the collection of real-valued $\mathcal{G}$-measurable random variables with the absolute value admitting a 2-moment under $\mathbb{P}$. This space is endowed with its usual norm.
- For a làdlàg process ϕ, ϕ_{t+} and ϕ_{t-} represent the right-hand and left-hand limit of ϕ at time t, respectively. $\Delta\phi_t := \phi_t - \phi_{t-}$ denotes the size of the left jump for ϕ at time t (with the convention $\phi_{0^-} = \phi_0$), and $\Delta_+\phi_t := \phi_{t+} - \phi_t$ represents the size of the right jump for ϕ at time t.
- For real-valued random variables X and X_n, $n \in \mathbb{N}$, " $X_n \uparrow X$ " indicates that " the sequence (X_n) is non-decreasing and converges to X a.s."

The following definition can be found in [18].

Definition 2.1 *A subset U of $L^1(\Omega, \mathcal{F}, \mathbb{Q})$ is said to be L^0-convex if $x\xi + (1-x)\zeta \in U$ for all $\xi, \zeta \in U$ and $x \in L^0_+(\mathcal{F}, \mathbb{R})$ such that $0 \leqslant x \leqslant 1$.*

3. First properties and necessary and sufficient conditions for optimality

In this subsection, we present some preliminary results for the value families R and R^+ when the reward is given by an admissible family of random variables indexed by stopping times.

Definition 3.1 *A family $Y = (Y(\tau), \tau \in \mathcal{S})$ is admissible if it satisfies the following conditions:*

(1) For all $\tau \in \mathcal{S}$, $Y(\tau)$ is a $\mathcal{F}_\tau$-measurable non-negative random variable.
(2) For all $\tau, \tau' \in \mathcal{S}$, $Y(\tau) = Y(\tau')$ a.s. on $\{\tau = \tau'\}$.

Moreover, if for all $\tau \in \mathcal{S}$, $Y(\tau)$ is square-integrable, the admissible family Y is considered square integrable.

Remark 3.1 *It is always possible to define an admissible family associated with a given process. More precisely, let $(Y_t)_{t\in[0,T]}$ be a non-negative progressive process. Define $Y(\tau) := Y_\tau$, for each $\tau \in \mathcal{S}$. The family $Y = (Y_\tau, \tau \in \mathcal{S})$ is clearly admissible.*

Now, let $(Y(\tau), \tau \in \mathcal{S})$ be an admissible reward family. For $v \in \mathcal{S}$, the value function at time v is defined by

$$R(v) := \operatorname*{ess\,sup}_{\mathbb{P}\in\mathcal{P}} \operatorname*{ess\,sup}_{\tau\in\mathcal{S}_v} E^{\mathbb{P}}[Y(\tau)|\mathcal{F}_v], \tag{3.1}$$

and the strict value function at time v is defined by

$$R^+(v) := \operatorname*{ess\,sup}_{\mathbb{P}\in\mathcal{P}} \operatorname*{ess\,sup}_{\tau\in\mathcal{S}_v{}^+} E^{\mathbb{P}}[Y(\tau)|\mathcal{F}_v]. \tag{3.2}$$

Based on these assumptions, we can define the density process $Z^{\mathbb{P}}_t \triangleq \frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}|_{\mathcal{F}_t}$ for any $t \in [0,T]$ and any $\mathbb{P}\in\mathcal{P}$. It is clear that $Z^{\mathbb{P}}$ is a $\mathbb{Q}$-martingale, $Z^{\mathbb{P}}_0 = 1$ and $Z^{\mathbb{P}}_t > 0$ a.s., for all $t \in [0,T]$. Hence, we define the set of $\mathbb{Q}$ -martingales

$$\mathcal{Z} = \Big\{ Z^{\mathbb{P}}_t = \frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}\Big|_{\mathcal{F}_t}, \quad 0 \leqslant t \leqslant T \quad \text{for} \quad \mathbb{P}\in\mathcal{P}\Big\}. \tag{3.3}$$

For technical reasons, discussed hereinafter, we assume that the set $\mathcal{Z}$ is L^0 -convex. Based on Bayes rule, this is re-expressed as

$$E^{\mathbb{P}}[Y(\tau)|\mathcal{F}_v] = E^{\mathbb{Q}}\Big[\frac{Z^{\mathbb{P}}_\tau}{Z^{\mathbb{P}}_v} Y(\tau)\Big|\mathcal{F}_v\Big] = \frac{1}{Z^{\mathbb{P}}_v} E^{\mathbb{Q}}[Z^{\mathbb{P}}_\tau Y(\tau)|\mathcal{F}_v], \quad \tau\in\mathcal{S}_v. \tag{3.4}$$

Therefore, $R(v)$ and $R^+(v)$ become

$$R(v) = \operatorname*{ess\,sup}_{Z\in\mathcal{Z}} \operatorname*{ess\,sup}_{\tau\in\mathcal{S}_v} \Gamma(v|\tau, Z), \tag{3.5}$$

$$R^+(v) = \operatorname*{ess\,sup}_{Z\in\mathcal{Z}} \operatorname*{ess\,sup}_{\tau\in\mathcal{S}_{v^+}} \Gamma(v|\tau, Z), \tag{3.6}$$

where $\Gamma(v|\tau,Z) := E^{\mathbb{P}}[Y(\tau)|\mathcal{F}_v] \triangleq \frac{1}{Z_v} E^{\mathbb{Q}}[Z_\tau Y(\tau)|\mathcal{F}_v]$. Because this random variable depends only on the restriction of the process Z to the stochastic interval $[v,\tau]$, we consider $\mathcal{Z}_{v,\tau}$ to be the restriction of $\mathcal{Z}$ to this interval.

Proposition 3.1 *(Admissibility of v and v^+) The families $R = (R(v), v\in\mathcal{S})$ and $R^+ = (R^+(v), v\in\mathcal{S})$, defined by (3.5) and (3.6), are admissible.*

Proof Let us prove the property for $R^+ = (R^+(v), v\in\mathcal{S})$. Because of the definition of the essential supremum (see Neveu [28]), one can see that for each $v\in\mathcal{S}$, $R^+(v)$ is an $\mathcal{F}_v$ -measurable random variable. Now, let $v, v'\in\mathcal{S}$ and set $A := \{v = v'\}$. For every $\tau\in\mathcal{S}_{v^+}$, put $\tau_A = \tau\mathbf{1}_A + T\mathbf{1}_{A^c}$. It is immediately clear that $\tau_A\in\mathcal{S}_{v^+}$. Because $A\in\mathcal{F}_v\cap\mathcal{F}_{v'}$, we get a.s. on A,

$$\begin{aligned}\Gamma(v|\tau,Z) &= \frac{1}{Z_v}E^{\mathbb{Q}}[Z_\tau Y(\tau)|\mathcal{F}_v] = \frac{1}{Z_v}E^{\mathbb{Q}}[Z_{\tau_A} Y(\tau_A)|\mathcal{F}_{v'}] = \Gamma(v'|\tau_A, Z)\\ &\leqslant \operatorname*{ess\,sup}_{Z\in\mathcal{Z}} \operatorname*{ess\,sup}_{\tau\in\mathcal{S}_{v'+}} \Gamma(v'|\tau, Z) = R^+(v').\end{aligned}$$

Then, taking the essential supremum over $Z\in\mathcal{Z}$ and $\tau\in\mathcal{S}_{v^+}$, we obtain $R^+(v)\leqslant R^+(v')$ a.s. Based on the symmetry of v and v', we obtain the converse inequality and the proof is complete. Similar arguments show that R is admissible. □

Proposition 3.2 *(Optimizing sequence for R and R^+) For any $v\in\mathcal{S}$, the family of random variables $\{\Gamma(v|\tau,Z)/\tau\in\mathcal{S}_v, Z\in\mathcal{Z}_{v,\tau}\}$ (resp. $\{\Gamma(v|\tau,Z)/\tau\in\mathcal{S}_{v^+}, Z\in\mathcal{Z}_{v,\tau}\}$) is closed under pairwise maximization. That is, there exists a sequence $\{(\tau_n, Z^n)\}_{n\in\mathbb{N}}$ with τ_n in $\mathcal{S}_v$ (resp. $\mathcal{S}_{v^+}$) and $Z^n\in\mathcal{Z}_{v,\tau_n}$ such that the sequence $\{\Gamma(v|\tau_n, Z^n)\}_{n\in\mathbb{N}}$ is increasing and such that*

$$R(v) \quad (\textit{resp. } R^+(v)) \quad = \lim_{n\to\infty}\Gamma(v|\tau_n, Z^n) \quad \text{a.s.}$$

Proof We prove the property for R, where the arguments are the same for R. Let $\tau_1,\tau_2\in\mathcal{S}_v$ and Z^1, $Z^2\in\mathcal{Z}$, and consider $A = \{\Gamma(v|\tau_2, Z^2)\geqslant\Gamma(v|\tau_1, Z^1)\}\in\mathcal{F}_v$. We also set

$$\begin{aligned}
&\tau = \tau_1 \mathbf{1}_{A^c} + \tau_2 \mathbf{1}_A,\\
&Z_t = Z^1_t \mathbb{Q}(A^c|\mathcal{F}_t) + Z^2_t \mathbb{Q}(A|\mathcal{F}_t), \quad 0 \leqslant t \leqslant T.
\end{aligned}$$

Then, τ is a stopping in $\mathcal{S}_v$, and because $\mathcal{Z}$ is L^0 -convex, we obtain $Z \in \mathcal{Z}$. Therefore,

$$\begin{aligned}
\Gamma(v|\tau, Z) &= E^{\mathbb{Q}}\Big[\frac{Z_\tau}{Z_v} Y(\tau)\Big|\mathcal{F}_v\Big]\\
&= E^{\mathbb{Q}}\Big[\frac{Z^1_{\tau_1}}{Z^1_v} Y(\tau_1)\Big|\mathcal{F}_v\Big]\mathbf{1}_{A^c} + E^{\mathbb{Q}}\Big[\frac{Z^2_{\tau_2}}{Z^2_v} Y(\tau_2)\Big|\mathcal{F}_v\Big]\mathbf{1}_A\\
&= \Gamma(v|\tau_1, Z^1)\mathbf{1}_{A^c} + \Gamma(v|\tau_2, Z^2)\mathbf{1}_A\\
&= \Gamma(v|\tau_1, Z^1) \vee \Gamma(v|\tau_2, Z^2).
\end{aligned}$$

Hence, the set $\{\Gamma(v|\tau, Z)/\tau \in \mathcal{S}_v, Z \in \mathcal{Z}_{v,\tau}\}$ is closed under pairwise maximization. The existence of an optimization sequence follows from the classical result for an essential supremum (Neveu (1975)). □

Definition 3.2 *An admissible family $Y = (Y(\tau), \tau \in \mathcal{S})$, such that $R < \infty$, is said to be a $\mathcal{P}$-supermartingale family (resp. a $\mathcal{P}$-martingale family) if for all $\tau, \tau' \in \mathcal{S}$, such that $\tau \geqslant \tau'$ a.s.,*

$$\begin{aligned}
&E^{\mathbb{P}}[Y(\tau)|\mathcal{F}_{\tau'}] \leqslant Y(\tau') \quad a.s.\ for\ each\ measure \quad \mathbb{P} \in \mathcal{P},\\
&(resp. \quad E^{\mathbb{P}}[Y(\tau)|\mathcal{F}_{\tau'}] = Y(\tau') \quad a.s.\ for\ all\ measures \quad \mathbb{P} \in \mathcal{P}).
\end{aligned}$$

The following proposition states that the value function R and the strict value function R^+ are both supermartingale families.

Proposition 3.3 *The admissible families $R = (R(v), v \in \mathcal{S})$ and $R^+ = (R^+(v), v \in \mathcal{S})$ are $\mathcal{P}$-supermartingale families in the sense of the above definition. Moreover, the value family $R = (R(v), v \in \mathcal{S})$ is characterized as the Snell envelope family associated with $(Y(v), v \in \mathcal{S})$, which is the smallest $\mathcal{P}$-supermartingale family that is greater a.s. than $(Y(v), v \in \mathcal{S})$.*

Proof Let us prove the first property for R. Let $v, v' \in \mathcal{S}$ with $v \geqslant v'$ a.s. Based on Proposition 3, there exists an optimization sequence $\{(\tau_n, Z^n)\}_{n\in\mathbb{N}}$ with τ_n in $\mathcal{S}_v$ and $Z^n \in \mathcal{Z}_{v,\tau_n}$. Now, let $\mathbb{P} \in \mathcal{P}$, based on the monotone convergence theorem, we get

$$E^{\mathbb{P}}[R(v)|\mathcal{F}_{v'}] = \lim_{n\to\infty} E^{\mathbb{P}}[\Gamma(v|\tau_n, Z^n)|\mathcal{F}_{v'}] \quad \text{a.s.}$$

For each n, $\tau_n \in \mathcal{S}_{v'}$ and $Z^n \in \mathcal{Z}_{v,\tau_n}$, we have

$$E^{\mathbb{P}}[\Gamma(v|\tau_n, Z^n)|\mathcal{F}_{v'}] \leqslant R(v') \quad \text{a.s.}$$

Hence,

$$E^{\mathbb{P}}[R(v)|\mathcal{F}'_v] \leqslant R(v') \quad \text{a.s.},$$

which gives the $\mathcal{P}$-supermartingale property of R. The next task is to prove the second statement. It is clear that $(R(v), v \in \mathcal{S})$ is a $\mathcal{P}$-supermartingale family and for each $v \in \mathcal{S}$, $R(v) \geqslant Y(v)$ a.s. Let $(\bar{R}(v), v \in \mathcal{S})$ be another $\mathcal{P}$-supermartingale family such that for each $v \in \mathcal{S}$, $\bar{R}(v) \geqslant Y(v)$ a.s. Fix $v \in \mathcal{S}$. Given the $\mathcal{P}$-supermartingale property of $\bar{R}$, for every stopping time, we have $\tau \in \mathcal{S}_v$, and for every measure $\mathbb{P} \in \mathcal{P}$

$$\bar{R}(v) \geqslant E^{\mathbb{P}}[\bar{R}(\tau)|\mathcal{F}_v] \geqslant E^{\mathbb{P}}[Y(\tau)|\mathcal{F}_v] \quad \text{a.s.}$$

Taking the supremum over $\tau \in \mathcal{S}_v$ and $\mathbb{P} \in \mathcal{P}$, we obtain $\bar{R}(v) \geqslant R(v)$ a.s., and the proposition follows. □

Proposition 3.4 *For every $v \in \mathcal{S}$, $R(v) = Y(v) \vee R^+(v)$ a.s.*

Proof Let v be a stopping time in $\mathcal{S}$. Taking $\tau \in \mathcal{S}_v$, we first show that for every $Z \in \mathcal{Z}$,

$$\Gamma(v|\tau, Z) \leqslant Y(v) \vee R^+(v) \quad \text{a.s.} \tag{3.7}$$

We set $\bar{\tau} = \tau \mathbf{1}_{\{\tau > v\}} + T\mathbf{1}_{\{\tau = v\}}$. It is clear that $\bar{\tau}$ belongs to $\mathcal{S}_{v^+}$. Hence,

$$\Gamma(v|\tau, Z)\mathbf{1}_{\{\tau > v\}} = \Gamma(v|\bar{\tau}, Z)\mathbf{1}_{\{\tau > v\}} \leqslant R^+(v)\mathbf{1}_{\{\tau > v\}} \quad \text{a.s.} \tag{3.8}$$

Therefore,

$$\Gamma(v|\tau, Z) = Y(v)\mathbf{1}_{\{\tau = v\}} + \Gamma(v|\tau, Z)\mathbf{1}_{\{\tau > v\}} \leqslant Y(v)\mathbf{1}_{\{\tau = v\}} + R^+(v)\mathbf{1}_{\{\tau > v\}} \quad \text{a.s.}$$

By taking the essential supremum over $\tau \in \mathcal{S}_v$ and then taking the essential supremum over $Z \in \mathcal{Z}$, we get $R(v) \leqslant Y(v) \vee R^+(v)$ a.s. The other inequality follows immediately from the fact that $R(v) \geqslant R^+(v)$ a.s. and $R(v) \geqslant Y(v)$ a.s., which completes the proof. ☐

Proposition 3.5 *For any $v \in \mathcal{S}$, $\tau \in \mathcal{S}_v$ and $\mathbb{P} \in \mathcal{P}$, we have*

$$E^{\mathbb{P}}[R^+(\tau)|\mathcal{F}_v] = \operatorname*{ess\,sup}_{\sigma \in \mathcal{S}_{\tau^+}} E^{\mathbb{P}}[Y(\sigma)|\mathcal{F}_v] \quad \textit{a.s.} \tag{3.9}$$

In particular, $E^{\mathbb{P}}[R^+(\tau)] = \sup_{\sigma \in \mathcal{S}_{\tau^+}} E^{\mathbb{P}}[Y(\sigma)]$.

Proof Let $\mathbb{P} \in \mathcal{P}$. Denote by $Z = Z^{\mathbb{P}}$. From Proposition 3, there exists a sequence $\{(\tau_n, Z^n)\}_{n \in \mathbb{N}}$ with τ_n in $\mathcal{S}_{\tau^+}$ and $Z^n \in \mathcal{Z}_{\tau,\tau_n}$, such that

$$R^+(v) = \lim_{n \to \infty} \Gamma(\tau|\tau_n, Z^n) \quad \text{a.s.}$$

We can assume without loss of generality that $Z^n_u = Z_u$, $\forall u \in [v, \tau]$. Using Fatou's lemma, we get

$$\begin{aligned}
E^{\mathbb{P}}[R^+(\tau)|\mathcal{F}_v] &= E^{\mathbb{Q}}\Big[\frac{Z_\tau}{Z_v} R^+(\tau)\Big|\mathcal{F}_v\Big] \\
&= E^{\mathbb{Q}}\Big[\frac{Z_\tau}{Z_v} \lim_{n \to \infty} E^{\mathbb{Q}}\Big[\frac{Z^n_{\tau_n}}{Z^n_\tau} Y(\tau_n)\Big|\mathcal{F}_\tau\Big]\Big|\mathcal{F}_v\Big] \\
&= E^{\mathbb{Q}}\Big[\lim_{n \to \infty} E^{\mathbb{Q}}\Big[\frac{Z_\tau}{Z_v} \frac{Z^n_{\tau_n}}{Z^n_\tau} Y(\tau_n)\Big|\mathcal{F}_\tau\Big]\Big|\mathcal{F}_v\Big] \\
&\leqslant \lim_{n \to \infty} E^{\mathbb{Q}}\Big[\frac{Z^n_{\tau_n}}{Z^n_v} Y(\tau_n)\Big|\mathcal{F}_v\Big] \\
&= \lim_{n \to \infty} E^{\mathbb{P}}[Y(\tau_n)|\mathcal{F}_v] \\
&\leqslant \operatorname*{ess\,sup}_{\sigma \in \mathcal{S}_{\tau^+}} E^{\mathbb{P}}[Y(\sigma)|\mathcal{F}_v].
\end{aligned}$$

Now, given the $\mathcal{P}$-supermartinglae property of R^+, we have for all $\tau \in \mathcal{S}_v$ and all $\sigma \in \mathcal{S}_{\tau^+}$

$$E^{\mathbb{P}}[Y(\sigma)|\mathcal{F}_\tau] \leqslant R^+(\tau) \quad \text{a.s.}$$

Thus, for each $\sigma \in \mathcal{S}_{\tau^+}$, we have

$$E^{\mathbb{P}}[E^{\mathbb{P}}[Y(\sigma)|\mathcal{F}_\tau]|\mathcal{F}_v] = E^{\mathbb{P}}[Y(\sigma)|\mathcal{F}_v] \leqslant E^{\mathbb{P}}[R^+(\tau)|\mathcal{F}_v] \quad \text{a.s.}$$

By taking the essential supremum over $\sigma \in \mathcal{S}_{\tau^+}$ we derive the reverse inequality

$$\operatorname*{ess\,sup}_{\sigma \in \mathcal{S}_{\tau^+}} E^{\mathbb{P}}[Y(\sigma)|\mathcal{F}_v] \leqslant E^{\mathbb{P}}[R^+(\tau)|\mathcal{F}_v] \quad \text{a.s.}$$

The proof is complete. ☐

We now present a crucial property of regularity for the strict value function family, namely the $\mathcal{P}$-right continuity along stopping times in expectation. Let us first introduce the following definition.

Definition 3.3 *An admissible family $Y = (Y(\tau), \tau \in \mathcal{S})$ is said to be $\mathcal{P}$-right continuous in expectation ($\mathcal{P}$-RCE) if for every $\tau \in \mathcal{S}$ and for any sequence of stopping times $(\tau_n)_{n\in\mathbb{N}} \in \mathcal{S}$ such that $\tau_n \downarrow v$, one has $E^{\mathbb{P}}[Y(\tau)] = \lim_{n\to\infty} E^{\mathbb{P}}[Y(\tau_n)]$ with respect to each measure $\mathbb{P} \in \mathcal{P}$.*

Proposition 3.6 *($\mathcal{P}$-RCE property for R^+) Let $(Y(\tau), \tau \in \mathcal{S})$ be an admissible family. The associated strict value function family $R^+ = (R^+(\tau), \tau \in \mathcal{S})$ is $\mathcal{P}$-RCE.*

Proof It is clear that for each $\mathbb{P} \in \mathcal{P}$, the function $\tau \to E^{\mathbb{P}}[R^+(\tau)]$ is a non-increasing function of stopping times, because R^+ is a $\mathcal{P}$-supermartingal family. Assume that, contrary to our claim, that there exists a probability measure $\bar{\mathbb{P}} \in \mathcal{P}$ such that the family $(R^+(\tau), \tau \in \mathcal{S})$ is not RCE at $\tau \in \mathcal{S}$. Firstly, consider the case where $E^{\bar{\mathbb{P}}}[R^+(\tau)] < \infty$. Hence, there exists a constant $\epsilon > 0$ and a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}} \in \mathcal{S}$ such that $\tau_n \downarrow \tau$ and

$$\lim_{n\to\infty} \uparrow E^{\bar{\mathbb{P}}}[R^+(\tau_n)] + \epsilon \leqslant E^{\bar{\mathbb{P}}}[R^+(\tau)].$$

From Proposition 3.5, we have $E^{\bar{\mathbb{P}}}[R^+(\tau)] = \sup_{\sigma\in\mathcal{S}_{\tau^+}} E^{\bar{\mathbb{P}}}[Y(\sigma)]$. Thus, there exists $\tau' \in \mathcal{S}_{\tau^+}$ such that

$$\lim_{n\to\infty} \uparrow E^{\bar{\mathbb{P}}}[R^+(\tau_n)] + \frac{\epsilon}{2} \leqslant E^{\bar{\mathbb{P}}}[Y(\tau')]. \tag{3.10}$$

In order to find a contradiction, first suppose that $\tau < T$ a.s. In this case, $\tau' \in \mathcal{S}_{\tau^+}$ gives $\tau' > \tau$ a.s. We write $\{\tau' > \tau\} = \bigcup_{n\in\mathbb{N}} \uparrow \{\tau' > \tau_n\}$ and we get $E^{\bar{\mathbb{P}}}[Y(\tau')] = \lim_{n\to\infty} \uparrow E^{\bar{\mathbb{P}}}[Y(\tau')\mathbf{1}_{\tau'>\tau_n}]$. Thus, there exists n_0 such that

$$\lim_{n\to\infty} \uparrow E^{\bar{\mathbb{P}}}[R^+(\tau_n)] + \frac{\epsilon}{4} \leqslant E^{\bar{\mathbb{P}}}[Y(\tau')\mathbf{1}_{\tau'>\tau_{n_0}}].$$

Set $\bar{\tau} := \tau'\mathbf{1}_{\tau'>\tau_{n_0}} + T\mathbf{1}_{\tau'\leqslant\tau_{n_0}}$. One can see that $\bar{\tau} > \tau_{n_0}$ a.s. Therefore, the positivity of Y yields

$$E^{\bar{\mathbb{P}}}[R^+(\tau_{n_0})] + \frac{\epsilon}{4} \leqslant \lim_{n\to\infty} \uparrow E^{\bar{\mathbb{P}}}[R^+(\tau_n)] + \frac{\epsilon}{4} \leqslant E^{\bar{\mathbb{P}}}[Y(\bar{\tau})] \leqslant E^{\bar{\mathbb{P}}}[R^+(\tau_{n_0})], \tag{3.11}$$

which is impossible. To study the general case, take $\tau \in \mathcal{S}$. Because $\tau' \in \mathcal{S}_{\tau^+}$, $\tau' > \tau$ a.s. on $\{\tau < T\}$ and $\tau' = T$ a.s. on $\{\tau = T\}$. Then, one has

$$E^{\bar{\mathbb{P}}}[Y(\tau')] = E^{\bar{\mathbb{P}}}[Y(\tau')\mathbf{1}_{\tau<T}] + E^{\bar{\mathbb{P}}}[Y(T)\mathbf{1}_{\tau=T}] \quad \text{and}$$
$$E^{\bar{\mathbb{P}}}[Y(\tau')\mathbf{1}_{\tau<T}] = \lim_{n\to\infty} \uparrow E^{\bar{\mathbb{P}}}[Y(\tau')\mathbf{1}_{\{\tau<T\}\cap\{\tau'>\tau_n\}}].$$

From this and (3.10), there exists n_0 such that

$$\lim_{n\to\infty} \uparrow E^{\bar{\mathbb{P}}}[R^+(\tau_n)] + \frac{\epsilon}{4} \leqslant E^{\bar{\mathbb{P}}}[Y(\tau')\mathbf{1}_{\tau'>\tau_{n_0}\cap\{\tau<T\}}] + E^{\bar{\mathbb{P}}}[Y(T)\mathbf{1}_{\tau=T}].$$

Consider $\bar{\tau} := \tau'\mathbf{1}_{\{\tau'>\tau_{n_0}\}\cap\{\tau<T\}} + T\mathbf{1}_{\{\tau'\leqslant\tau_{n_0}\}\cap\{\tau<T\}} + T\mathbf{1}_{\{\tau=T\}}$. One can check that $\bar{\tau} \in \mathcal{S}_{\tau^+_{n_0}}$, then

$$E^{\bar{\mathbb{P}}}[Y(\tau')\mathbf{1}_{\{\tau'>\tau_{n_0}\}\cap\{\tau<T\}}] + E^{\bar{\mathbb{P}}}[Y(T)\mathbf{1}_{\{\tau=T\}}] \leqslant +E^{\bar{\mathbb{P}}}[Y(\bar{\tau})] \leqslant E^{\bar{\mathbb{P}}}[R^+(\tau_{n_0})],$$

we derive again (3.11), which gives a contradiction. □

The following definition plays a fundamental role in solving the optimal stopping problem (2.1).

Definition 3.4 *An admissible family* $(Y(\tau), \tau \in \mathcal{S})$ *is said to be* $\mathcal{P}$*-right (resp.* $\mathcal{P}$*-left) upper-semicontinuous in expectation along stopping times if for all* $\mathbb{P} \in \mathcal{P}$ *, for all* $\tau \in \mathcal{S}$ *and for all sequences of stopping times* $(\tau_n)_n$ *such that* $\tau_n \downarrow \tau$ *(resp.* $\tau_n \uparrow \tau$ *), we have*

$$E^{\mathbb{P}}[Y(\tau)] \geqslant \lim_{n\to\infty} \sup E^{\mathbb{P}}[Y(\tau_n)].$$

Remark 3.2 *Let* $(Y(\tau), \tau \in \mathcal{S})$ *be a* $\mathcal{P}$ *-right (resp.* $\mathcal{P}$ *-left) USCE admissible family. For each* $v \in \mathcal{S}$ *and* $B \in \mathcal{F}_v$ *, the admissible family* $(Y(\tau)\mathbf{1}_B, \tau \in \mathcal{S}_v)$ *is also a* $\mathcal{P}$ *-right (resp.* $\mathcal{P}$ *-left) USCE admissible family. Indeed, fix* $\mathbb{P} \in \mathcal{P}$ *and* $v \in \mathcal{S}$*. Let* $(\tau_n)_n$ *be a sequence of stopping times such that* $\tau_n \downarrow \tau$*. For each* n *, set* $\bar{\tau}_n := \tau_n \mathbf{1}_B + T\mathbf{1}_{B^c}$ *and* $\bar{\tau} := \tau\mathbf{1}_B + T\mathbf{1}_{B^c}$*. It is clear that* $\bar{\tau}_n \downarrow \tau$*. Therefore,* $E^{\mathbb{P}}[Y(\bar{\tau})] \geqslant \lim_{n\to\infty} \sup E^{\mathbb{P}}[Y(\bar{\tau}_n)]$*. Hence,* $E^{\mathbb{P}}[Y(\tau)\mathbf{1}_B] \geqslant \lim_{n\to\infty} \sup E^{\mathbb{P}}[Y(\tau_n)\mathbf{1}_B]$.

We give now the necessary and sufficient conditions for optimality.

Theorem 3.1 *Let* $\mathbb{P}^* \in \mathcal{P}$ *and* $\tau^* \in \mathcal{S}$ *be such that* $E^{\mathbb{P}^*}[Y(\tau^*)] < \infty$*. The stopping time* τ^* *and the probability measure* $\mathbb{P}^*$ *re optimal in (3.1), i.e.,*

$$E^{\mathbb{P}^*}[Y(\tau^*)] = R(0) = \sup_{\mathbb{P}\in\mathcal{P}} \sup_{\tau\in\mathcal{S}} E^{\mathbb{P}}[Y(\tau)] \tag{3.12}$$

holds, if and only if

(1) $R(\tau^*) = Y(\tau^*)$ *a.s.,*

(2) *The* $\mathcal{P}$*-supermartingale family* $(R(\tau^* \wedge \sigma), \sigma \in \mathcal{S})$ *is a* $\mathbb{P}^*$ *-martingale family. That is, for all* $\theta, \theta' \in \mathcal{S}$ *such that* $\theta, \theta' \leqslant \tau^*$ *a.s., we have*

$$E^{\mathbb{P}^*}[R(\theta')|\mathcal{F}_\theta] = R(\theta) \quad a.s. \text{ on } \quad \{\theta \leqslant \theta'\}.$$

Proof Suppose τ^* and $\mathbb{P}^*$ are optimal, i.e., (3.12) holds. From Proposition 3.5, we have

$$\begin{aligned} E^{\mathbb{P}^*}[R(\tau^*)] &= \sup_{\sigma\in\mathcal{S}^*_\tau} E^{\mathbb{P}^*}[Y(\sigma)] \leqslant \sup_{\mathbb{P}\in\mathcal{P}} \sup_{\sigma\in\mathcal{S}^*_\tau} E^{\mathbb{P}}[Y(\sigma)] \\ &\leqslant \sup_{\mathbb{P}\in\mathcal{P}} \sup_{\sigma\in\mathcal{S}} E^{\mathbb{P}}[Y(\sigma)] = E^{\mathbb{P}^*}[Y(\tau^*)] \\ &\leqslant E^{\mathbb{P}^*}[R(\tau^*)] \quad \text{a.s.} \end{aligned}$$

Because R dominates Y and $E^{\mathbb{P}^*}[Y(\tau^*)] < \infty$, we get $R(\tau^*) = Y(\tau^*)$ $\mathbb{P}^*$-a.s. and the first assertion follows. Let us prove the second assertion. For this, take $\sigma \in \mathcal{S}$ and note that

$$E^{\mathbb{P}^*}[Y(\tau^*)] = \sup_{\mathbb{P}\in\mathcal{P}} \sup_{\tau\in\mathcal{S}} E^{\mathbb{P}}[Y(\tau)] = \sup_{\mathbb{P}\in\mathcal{P}} \sup_{\tau\in\mathcal{S}_{\sigma\wedge\tau^*}} E^{\mathbb{P}}[Y(\tau)]. \tag{3.13}$$

Moreover, from Proposition 3.5, we have

$$\sup_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}[R(\sigma \wedge \tau^*)] = \sup_{\mathbb{P}\in\mathcal{P}} \sup_{\tau\in\mathcal{S}_{\sigma\wedge\tau^*}} E^{\mathbb{P}}[Y(\tau)] = E^{\mathbb{P}^*}[Y(\tau^*)].$$

From this and the supermartingale property of R, we get

$$E^{\mathbb{P}}[R(\sigma \wedge \tau^*)] \leqslant \sup_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}[R(\sigma \wedge \tau^*)] = E^{\mathbb{P}^*}[Y(\tau^*)] \leqslant E^{\mathbb{P}^*}[R(\tau^*)] \leqslant E^{\mathbb{P}^*}[R(\tau^* \wedge \sigma)].$$

It follows that, $E^{\mathbb{P}}[R(\sigma \wedge \tau^*)] = E^{\mathbb{P}^*}[R(\tau^*)]$, which gives the second assertion. Conversely, from (1) and (2) we have

$$E^{\mathbb{P}^*}[R(\tau^*)] = E^{\mathbb{P}^*}[R(0)] = R(0) = E^{\mathbb{P}^*}[Y(\tau^*)] = \sup_{\mathbb{P}\in\mathcal{P}} \sup_{\tau\in\mathcal{S}} E^{\mathbb{P}}[Y(\tau)].$$

The proof is thus complete. □

4. Existence of Optimal Stopping Times

In this section, we establish the existence of optimal stopping times using an approximation method introduced by Maingueneau [26] (see also El Karoui [16]). We begin by constructing a family of *approximatly optimal* stopping times. Fix $v \in \mathcal{S}$. For $\alpha \in (0,1)$, a stopping time τ_α^* is said to be $(1-\alpha)$-optimal for $R(v)$ if it satisfies

$$\alpha R(v) \leqslant \operatorname*{ess\,sup}_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}[Y(\tau_\alpha^*)|\mathcal{F}_v]. \tag{4.1}$$

Passing to the limit later when $\alpha \uparrow 1$ implies the existence of an optimal stopping time. For $\alpha \in (0,1)$, we now define the following $\mathcal{F}_v$-measurable random variable

$$U^\alpha(v) := \operatorname{ess\,inf}\{\tau \in \mathcal{S}_v, \alpha R(\tau) \leqslant Y(\tau) \quad \text{a.s.}\}. \tag{4.2}$$

Remark 4.1 (1) *The random variable $U^\alpha(v)$ is a stopping time, and one has $U^\alpha(v) \geqslant v$ a.s. Indeed, by seting $\mathbb{L}_v^\alpha := \{\tau \in \mathcal{S}_v, \alpha R(\tau) \leqslant Y(\tau) \quad \text{a.s.}\}$, τ_1, $\tau_2 \in \mathbb{L}_v^\alpha$, and $\bar{\tau} := \tau_1 \vee \tau_2 \in \mathcal{S}_v$, we get*

$$\begin{aligned}\alpha R(\bar{\tau}) &= \alpha R(\tau_1)\mathbf{1}_{\{\tau_1 \geqslant \tau_2\}} + \alpha R(\tau_2)\mathbf{1}_{\{\tau_1<\tau_2\}}\\ &\leqslant Y(\tau_1)\mathbf{1}_{\{\tau_1 \geqslant \tau_2\}} + Y(\tau_2)\mathbf{1}_{\{\tau_1<\tau_2\}} \quad a.s.\\ &= Y(\bar{\tau}) \quad a.s.\end{aligned}$$

Thus, $\mathbb{L}_v^\alpha$ is stable through pairwise minimization. From [28], there exists a sequence of stopping times $(\tau_n)_n \in \mathbb{L}_v^\alpha$ such that $\tau_n \downarrow U^\alpha(v)$. Therefore, $U^\alpha(v)$ is a stopping time and $U^\alpha(v) \geqslant v$ a.s.

(2) *Let $\alpha \in (0,1)$. Let $v \in \mathcal{S}$ such that $\alpha R(v) \leqslant Y(v)$ a.s. We have $U^\alpha(v) = v$ a.s. Indeed, by setting $\bar{A} := \{\alpha R(v) \leqslant Y(v)\}$ and $\bar{v} := v\mathbf{1}_{\bar{A}} + T\mathbf{1}_{\bar{A}^c}$, we clearly obtain $\bar{v} \in \mathbb{L}_v^\alpha$. Based on the definition of $U^\alpha(v)$, we have $U^\alpha(v) \leqslant \bar{v}$ a.s. Therefore, $U^\alpha(v)\mathbf{1}_{\bar{A}} \leqslant \bar{v}\mathbf{1}_{\bar{A}} = v\mathbf{1}_{\bar{A}}$ a.s., and hence $U^\alpha(v) = v$ a.s.*

The following proposition holds.

Proposition 4.1 *Assume that $R < \infty$ and that the reward family $(Y(\tau), \tau \in \mathcal{S})$ is $\mathcal{P}$-right USCE. Then, for each $v \in \mathcal{S}$ and $\alpha \in (0,1)$, the stopping time $U^\alpha(v)$, defined by (4.2), satisfies*

$$\alpha R(U^\alpha(v)) \leqslant Y(U^\alpha(v)) \quad \text{a.s.} \tag{4.3}$$

Proof Let $v \in \mathcal{S}$ and $A \in \mathcal{F}_{U^\alpha(v)}$. Recall that, based on Proposition 3.4, for the set $B_\alpha := \{R(U^\alpha(v)) > Y(U^\alpha(v))\}$, we have $R(U^\alpha(v)) = R^+(U^\alpha(v))$ a.s. Using the $\mathcal{P}$-RCE property of R^+, it is easily seen that the family $(R^+(\tau)\mathbf{1}_{B_\alpha}, \tau \in \mathcal{S}_{U^\alpha(v)})$ is also $\mathcal{P}$-RCE. From Remark 1, there exists a minimizing sequence $(\tau_n)_n \in \mathbb{L}_v^\alpha$ such that $\tau_n \downarrow U^\alpha(v)$. Hence, for each $\mathbb{P} \in \mathcal{P}$, the following equality holds:

$$\alpha E^{\mathbb{P}}[R^+(U^\alpha(v))\mathbf{1}_{B_\alpha \cap A}] = \alpha \lim_{n\to\infty} E^{\mathbb{P}}[R^+(\tau_n)\mathbf{1}_{B_\alpha \cap A}].$$

Because $(\tau_n)_n \in \mathbb{L}_v^\alpha$ and $R^+ \leqslant R$, we have for each n; $\alpha R^+(\tau_n) \leqslant \alpha R(\tau_n) \leqslant Y(\tau_n)$ a.s. Therefore,

$$\alpha E^{\mathbb{P}}[R^+(U^\alpha(v))\mathbf{1}_{B_\alpha \cap A}] \leqslant \alpha \limsup_{n\to\infty} E^{\mathbb{P}}[Y(\tau_n)\mathbf{1}_{B_\alpha \cap A}]. \tag{4.4}$$

Hence, from the positivity of Y and Eq. (4.4), we get

$$\begin{aligned}\alpha E^{\mathbb{P}}[R(U^\alpha(v))\mathbf{1}_A] &= \alpha E^{\mathbb{P}}[R^+(U^\alpha(v))\mathbf{1}_{B_\alpha \cap A}] + \alpha E^{\mathbb{P}}[Y(U^\alpha(v))\mathbf{1}_{B_\alpha^c \cap A}]\\ &\leqslant \limsup_{n\to\infty} E^{\mathbb{P}}[Y(\tau_n)\mathbf{1}_{B_\alpha \cap A}] + \alpha E^{\mathbb{P}}[Y(U^\alpha(v))\mathbf{1}_{B_\alpha^c \cap A}]\\ &\leqslant \limsup_{n\to\infty} E^{\mathbb{P}}[Y(\tilde{\tau}_n)\mathbf{1}_A],\end{aligned}$$

where $\tilde{\tau}_n := \tau_n \mathbf{1}_{B_\alpha \cap A} + U^\alpha(v)\mathbf{1}_{B_\alpha^c \cap A} + U^\alpha(v)\mathbf{1}_{A^c}$. The sequence $(\tilde{\tau}_n)_n$ is a non-increasing sequence of stopping times that tends to $U^\alpha(v)$ as $n \to \infty$. Thus, using the $\mathcal{P}$-right USCE assumption for the reward family Y and Remark 3.2, we obtain

$$\alpha E^{\mathbb{P}}[R(U^\alpha(v))\mathbf{1}_A] \leqslant E^{\mathbb{P}}[Y(U^\alpha(v))\mathbf{1}_A],$$

for each $A \in \mathcal{F}_{U^\alpha(v)}$ and each $\mathbb{P} \in \mathcal{P}$. Consequently, $\alpha R(U^\alpha(v)) \leqslant Y(U^\alpha(v))$ a.s. and the proof is ended. □

Let $v \in \mathcal{S}$ and $\alpha \in (0,1)$. In the sequel, we show that $U^\alpha(v)$ defined by (4.2) is a $(1-\alpha)$-optimal stopping time for $R(v)$.

Theorem 4.1 *Assume that the reward $(Y(\tau), \tau \in \mathcal{S})$ is $\mathcal{P}$-right USCE and $R < \infty$. Let $v \in \mathcal{S}$. For each $\alpha \in (0,1)$, the stopping time $U^\alpha(v)$ is a $(1-\alpha)$-optimal stopping time for $R(v)$.*

Remark 4.2 *Note that the integrability condition $R < \infty$ is a weaker condition compared to the classical assumption made in the literature, namely, that $\{Y(\tau), \tau \in \mathcal{S}\}$ is of class $(\mathcal{D})$.*

Proof We have to prove inequality (4.8) for $U^\alpha(v)$. This is done via the following steps.

Step 1 We show that for each $\alpha \in (0,1)$ and each $v \in S$, we have

$$R(v) = \operatorname*{ess\,sup}_{\mathbb{P} \in \mathcal{P}} E^{\mathbb{P}}[R(U^\alpha(v))|\mathcal{F}_v] \quad \text{a.s.} \tag{4.5}$$

From the $\mathcal{P}$-supermartingale property of $(R(v), v \in \mathcal{S})$, we clearly have the property: for each $\mathbb{P} \in \mathcal{P}$, $E^{\mathbb{P}}[R(U^\alpha(v))|\mathcal{F}_v] \leqslant R(v)$ a.s., because $U^\alpha(v)$ is a stopping time greater than or equal to v. By taking the essential supremum over $\mathbb{P} \in \mathcal{P}$, we get $\operatorname*{ess\,sup}_{\mathbb{P} \in \mathcal{P}} E^{\mathbb{P}}[R(U^\alpha(v))|\mathcal{F}_v] \leqslant R(v)$ a.s. To prove the reverse inequality, consider for each $v \in \mathcal{S}$, the random variable $\bar{S}(v) = \operatorname*{ess\,sup}_{\mathbb{P} \in \mathcal{P}} E^{\mathbb{P}}[R(U^\alpha(v))|\mathcal{F}_v]$.

Step 1.1 We claim that the family $(\bar{S}(v), v \in \mathcal{S})$ is a $\mathcal{P}$-supermartingale family, i.e., for any $\mathbb{P} \in \mathcal{P}$ and $\tau, \tau' \in \mathcal{S}$ such that $\tau \geqslant \tau'$ a.s., we have

$$E^{\mathbb{P}}[\bar{S}(\tau)|\mathcal{F}_{\tau'}] \leqslant \bar{S}(\tau') \quad \text{a.s.} \tag{4.6}$$

Indeed, fix $\tau \in \mathcal{S}$, $Z \in \mathcal{Z}$. Let $Z^1, Z^2 \in \mathcal{Z}$, and set

$$\begin{aligned} A &:= \Big\{ E^{\mathbb{Q}}\Big[\frac{Z^1_{U^\alpha(\tau)}}{Z^1_\tau} R(U^\alpha(\tau))\Big|\mathcal{F}_\tau\Big] \leqslant E^{\mathbb{Q}}\Big[\frac{Z^2_{U^\alpha(\tau)}}{Z^2_\tau} R(U^\alpha(\tau))\Big|\mathcal{F}_\tau\Big]\Big\} \in \mathcal{F}_\tau; \\ \bar{Z}_t &:= Z^1_t \mathbb{Q}(A^c|\mathcal{F}_t) + Z^2_t \mathbb{Q}(A|\mathcal{F}_t), \quad 0 \leqslant t \leqslant T. \end{aligned}$$

Because $\mathcal{Z}$ is L^0 -convex, we have $\bar{Z} \in \mathcal{Z}$. Therefore,

$$\begin{aligned} & E^{\mathbb{Q}}\Big[\frac{\bar{Z}_{U^\alpha(\tau)}}{\bar{Z}_\tau} R(U^\alpha(\tau))\Big|\mathcal{F}_\tau\Big] \\ &= E^{\mathbb{Q}}\Big[\frac{Z^1_{U^\alpha(\tau)}}{Z^1_\tau} R(U^\alpha(\tau))\Big|\mathcal{F}_\tau\Big]\mathbf{1}_{A^c} + E^{\mathbb{Q}}\Big[\frac{Z^2_{U^\alpha(\tau)}}{Z^2_\tau} R(U^\alpha(\tau))\Big|\mathcal{F}_\tau\Big]\mathbf{1}_A \\ &= E^{\mathbb{Q}}\Big[\frac{Z^1_{U^\alpha(\tau)}}{Z^1_\tau} R(U^\alpha(\tau))\Big|\mathcal{F}_\tau\Big] \vee E^{\mathbb{Q}}\Big[\frac{Z^2_{U^\alpha(\tau)}}{Z^2_\tau} R(U^\alpha(\tau))\Big|\mathcal{F}_\tau\Big]. \end{aligned}$$

Hence, the family of random variables $(E^{\mathbb{Q}}\big[\frac{Z_{U^\alpha(\tau)}}{Z_\tau} R(U^\alpha(\tau))|\mathcal{F}_\tau\big], Z \in \mathcal{Z})$ is closed under pairwise maximization. Similarly, as in the proof of Proposition 3.9, without any loss of generality, there exists a sequence $(Z^n)_{n \in \mathbb{N}} \in \mathcal{Z}_{\tau, U^\alpha(\tau)}$ such that $Z^n_u = Z_u$, $\forall u \in [v, \tau]$ and

$$\bar{S}(\tau) = \lim_{n\to\infty} \uparrow E^{\mathbb{Q}}\Big[\frac{Z^n_{U^\alpha(\tau)}}{Z^n_\tau} R(U^\alpha(\tau))\Big|\mathcal{F}_\tau\Big] \quad \text{a.s.}$$

It follows that for any $\tau' \in \mathcal{S}_\tau$ a.s., we have

$$\begin{aligned}
E^{\mathbb{P}}[\bar{S}(\tau)|\mathcal{F}_{\tau'}] &= E^{\mathbb{Q}}\Big[\frac{Z_\tau}{Z_{\tau'}}\bar{S}(\tau)\Big|\mathcal{F}_{\tau'}\Big] \\
&= E^{\mathbb{Q}}\Big[\frac{Z_\tau}{Z_{\tau'}} \lim_{n\to\infty} \uparrow E^{\mathbb{Q}}\Big[\frac{Z^n_{U^\alpha(\tau)}}{Z^n_\tau} R(U^\alpha(\tau))\Big|\mathcal{F}_\tau\Big]\Big|\mathcal{F}_{\tau'}\Big] \\
&\leqslant \lim_{n\to\infty} E^{\mathbb{Q}}\Big[\frac{Z_\tau}{Z_{\tau'}}\frac{Z^n_{U^\alpha(\tau)}}{Z^n_\tau} R(U^\alpha(\tau))\Big|\mathcal{F}_{\tau'}\Big] \\
&\leqslant \lim_{n\to\infty} E^{\mathbb{Q}}\Big[\frac{Z^n_{U^\alpha(\tau)}}{Z^n_{\tau'}} R(U^\alpha(\tau))\Big|\mathcal{F}_{\tau'}\Big] \\
&\leqslant \operatorname*{ess\,sup}_{M\in\mathcal{Z}} E^{\mathbb{Q}}\Big[\frac{M_{U^\alpha(\tau)}}{M_{\tau'}} R(U^\alpha(\tau))\Big|\mathcal{F}_{\tau'}\Big] \\
&= \operatorname*{ess\,sup}_{M\in\mathcal{Z}} E^{\mathbb{Q}}\Big[\frac{M_{U^\alpha(\tau')}}{M_{\tau'}} E^{\mathbb{Q}}\Big[\frac{M_{U^\alpha(\tau)}}{M_{U^\alpha(\tau')}} R(U^\alpha(\tau))\Big|\mathcal{F}_{U^\alpha(\tau')}\Big]\Big|\mathcal{F}_{\tau'}\Big].
\end{aligned} \tag{4.7}$$

Because $\tau \geqslant \tau'$, it is clear that $U^\alpha(\tau) \geqslant U^\alpha(\tau')$. The $\mathcal{P}$-supermartingale property of R together with Eq. (4.7) yield

$$E^{\mathbb{P}}[\bar{S}(\tau)|\mathcal{F}_{\tau'}] \leqslant \operatorname*{ess\,sup}_{M\in\mathcal{Z}} E^{\mathbb{Q}}\Big[\frac{M_{U^\alpha(\tau')}}{M_{\tau'}} R(U^\alpha(\tau'))\Big|\mathcal{F}_{\tau'}\Big] = \bar{S}(\tau') \quad \text{a.s.},$$

which is our claim in (4.6).

Step 1.2 For a fixed $\alpha \in (0,1)$, consider the $\mathcal{P}$-supermartingale family $(\alpha R(v) + (1-\alpha)\bar{S}(v), v \in \mathcal{S})$. We will show that $\alpha R(v) + (1-\alpha)\bar{S}(v) \geqslant Y(v)$. Fix $\tau \in \mathcal{S}_v$. Set $\bar{A} := \{\alpha R(v) \leqslant Y(v)\} \in \mathcal{F}_v$. From 2 of Remark 4.1, we have $U^\alpha(v) = v$ a.s. on $\bar{A}$. Hence, $\bar{S}(v) = \operatorname*{ess\,sup}_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}[R(U^\alpha(v))|\mathcal{F}_v] = \operatorname*{ess\,sup}_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}[R(v)|\mathcal{F}_v] = R(v)$ a.s. on $\bar{A}$. Therefore,

$$\alpha R(v) + (1-\alpha)\bar{S}(v) = R(v) \geqslant Y(v) \quad \text{a.s. on } \bar{A}.$$

Moreover, for $\bar{A}^c = \{\alpha R(v) > Y(v)\}$ based on the positivity of $\bar{S}$, we get

$$\alpha R(v) + (1-\alpha)\bar{S}(v) \geqslant \alpha R(v) \geqslant Y(v) \quad \text{a.s. on } \bar{A}^c,$$

and Step 1.2 is proved.

Now, because $(R(v), v \in \mathcal{S})$ is the smallest $\mathcal{P}$-supermartingale family that dominates $(Y(v), v \in \mathcal{S})$, we get $\alpha R(v) + (1-\alpha)\bar{S}(v) \geqslant R(v)$ a.s. Moreover, because $R(v) < \infty$ a.s. and $\lambda < 1$, we conclude that $\bar{S}(v) \geqslant R(v)$ a.s. The proof of Step 1 is complete.

Step 2 We now show that

$$\alpha R(v) \leqslant \operatorname*{ess\,sup}_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}[Y(U^\alpha(v))|\mathcal{F}_v] \quad \text{a.s.} \tag{4.8}$$

Based on Proposition 4.1 and Step 1, we have

$$\alpha R(v) = \operatorname*{ess\,sup}_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}[\alpha R(U^\alpha(v))|\mathcal{F}_v] \leqslant \operatorname*{ess\,sup}_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}[Y(U^\alpha(v))|\mathcal{F}_v] \quad \text{a.s.}$$

Thereupon, $U^\alpha(v)$ is $(1-\alpha)$ -optimal for $R(v)$, and the proof is complete. □

Let $v \in \mathcal{S}$ and $\lambda_1, \lambda_2 \in (0,1)$ such that $\lambda_1 \leqslant \lambda_2$. Clearly $U^{\lambda_1}(v) \leqslant U^{\lambda_2}(v)$ a.s. Accordingly, the map $\alpha \to U^\alpha(v)$ is non-decreasing for $(0,1)$. Thus, we define the stopping time

$$U^*(v) := \lim_{\lambda\uparrow 1} U^\alpha(v) \quad \text{a.s.} \tag{4.9}$$

Note that $U^*(v) \geqslant v$ a.s. Inspired by classical literature, the stopping time $U^*(v)$ appears to be a good candidate as the optimal stopping time for $R(v)$. Under further assumptions regarding the reward family, we state the main existence result of an optimal stopping time in the following theorem.

Theorem 4.2 *(Existence of an optimal stopping time) Assume that the reward family* $(Y(v), v \in \mathcal{S})$ *is* $\mathcal{P}$*-USCE , and that* $R < \infty$*. Then for every* $v \in \mathcal{S}$ *, the stopping time* $U^*(v)$ *(defined by (4.9)) is an optimal stopping time for* $R(v)$ *, that is,*

$$R(v) = \operatorname*{ess\,sup}_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}[Y(U^*(v))|\mathcal{F}_v]. \tag{4.10}$$

Additionally, $U^*(v) = \mathcal{U}(v) := \text{ess inf}\{\tau \in \mathcal{S}_v, R(\tau) = Y(\tau) \quad \text{a.s.}\}$ *a.s.*

Proof Let $v \in \mathcal{S}$. By characterizing $(R(v), v \in \mathcal{S})$ as the smallest $\mathcal{P}$-supermartingale family that dominates $(Y(v), v \in \mathcal{S})$, for every $\mathbb{P} \in \mathcal{P}$, we have

$$E^{\mathbb{P}}[Y(U^*(v))|\mathcal{F}_v] \leqslant E^{\mathbb{P}}[R(U^*(v))|\mathcal{F}_v] \leqslant R(v) \quad \text{a.s.},$$

which yields $\operatorname*{ess\,sup}_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}[Y(U^*(v))|\mathcal{F}_v] \leqslant R(v)$ a.s. We now demonstrate the other inequality

$$\operatorname*{ess\,sup}_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}[Y(U^*(v))|\mathcal{F}_v] \geqslant R(v). \tag{4.11}$$

Suppose by contradiction that (4.11) does not hold. That is, for all $\epsilon > 0$, we have

$$\bar{\mathbb{P}}\big(\{R(v) - \epsilon \geqslant E^{\bar{\mathbb{P}}}[Y(U^*(v))|\mathcal{F}_v]\}\big) > 0,$$

for some $\bar{\mathbb{P}} \in \mathcal{P}$. Because $\mathbb{P} \sim \mathbb{Q}$, for all $\mathbb{P} \in \mathcal{P}$, it follows that

$$\mathbb{P}\big(\{R(v) - \epsilon \geqslant E^{\mathbb{P}}[Y(U^*(v))|\mathcal{F}_v]\}\big) > 0,$$

for every $\mathbb{P} \in \mathcal{P}$. Set $C := \{R(v) - \epsilon \geqslant E^{\mathbb{P}}[Y(U^*(v))|\mathcal{F}_v]\}$. For all $A \in \mathcal{F}_v$, we have

$$\begin{aligned}
E^{\mathbb{P}}[Y(U^*(v))\mathbf{1}_A] &= E^{\mathbb{P}}[E^{\mathbb{P}}[Y(U^*(v))|\mathcal{F}_v]\mathbf{1}_A] \\
&= E^{\mathbb{P}}\Big[E^{\mathbb{P}}[Y(U^*(v))|\mathcal{F}_v]\mathbf{1}_{A\cap C}\Big] + E^{\mathbb{P}}[Y(U^*(v))\mathbf{1}_{A\cap C^c}] \\
&\leqslant E^{\mathbb{P}}[(R(v) - \epsilon)\mathbf{1}_{A\cap C}] + E^{\mathbb{P}}[R(U^*(v))\mathbf{1}_{A\cap C^c}] \quad \text{a.s.} \\
&\leqslant E^{\mathbb{P}}[(R(v) - \epsilon)\mathbf{1}_{A\cap C}] + E^{\mathbb{P}}[R(v)\mathbf{1}_{A\cap C^c}] \quad \text{a.s.} \\
&\leqslant E^{\mathbb{P}}[(R(v) - \epsilon\mathbf{1}_C)\mathbf{1}_A] \quad \text{a.s.}
\end{aligned}$$

Therefore, $E^{\mathbb{P}}[Y(U^*(v))|\mathcal{F}_v] \leqslant R(v) - \epsilon\mathbf{1}_C$ a.s. Based on the integrability condition $R < \infty$, Proposition 4.3 and Remark 3.2, we have for each $A \in \mathcal{F}_v$

$$\begin{aligned}
E^{\mathbb{P}}[\lim_{\alpha\uparrow 1} E^{\mathbb{P}}[R(U^\alpha(v))|\mathcal{F}_v]\mathbf{1}_A] &\leqslant E^{\mathbb{P}}\Big[\lim_{\alpha\uparrow 1} E^{\mathbb{P}}\Big[\frac{1}{\alpha}Y(U^\alpha(v))\mathbf{1}_A|\mathcal{F}_v\Big]\Big] \\
&\leqslant \lim_{\alpha\uparrow 1} E^{\mathbb{P}}\Big[\Big(\frac{1}{\alpha}Y(U^\alpha(v))\mathbf{1}_A\Big)\Big] \\
&\leqslant E^{\mathbb{P}}[Y(U^*(v))\mathbf{1}_A].
\end{aligned}$$

Hence,

$$\lim_{\alpha\uparrow 1} E^{\mathbb{P}}[R(U^\alpha(v))|\mathcal{F}_v] \leqslant E^{\mathbb{P}}[Y(U^*(v))|\mathcal{F}_v] \leqslant R(v) - \epsilon\mathbf{1}_C, \quad \forall \mathbb{P} \in \mathcal{P}.$$

Then, there exists $\bar{\alpha} \in (0, 1)$ such that for each $\mathbb{P} \in \mathcal{P}$,

$$E^{\mathbb{P}}[R(U^{\bar{\alpha}}(v))|\mathcal{F}_v] \leqslant E^{\mathbb{P}}[Y(U^*(v))|\mathcal{F}_v] \leqslant R(v) - \epsilon \mathbf{1}_C.$$

Taking the essential supremum over $\mathbb{P} \in \mathcal{P}$ and using Eq. (4.5), we get

$$R(v) \leqslant R(v) - \epsilon \mathbf{1}_C,$$

which gives the desired contradiction, and hence Eq. (4.10) follows. The final task is to show that $\mathcal{U}(v) = U^*(v)$ a.s. Because the map $\alpha \to U^\alpha(v)$ is nondecreasing on $(0,1)$, for each $\alpha \in (0,1)$, we have $U^\alpha(v) \leqslant U^*(v)$ a.s. Based on Proposition 4.1 and the $\mathcal{P}$-supermartingale property of $(R(v), v \in \mathcal{S})$, we obtain

$$\alpha E^{\mathbb{P}}[R(U^*(v))] \leqslant \alpha E^{\mathbb{P}}[R(U^\alpha(v))] \leqslant E^{\mathbb{P}}[Y(U^\alpha(v))] \quad \text{a.s.,} \quad \forall \alpha \in (0,1), \quad \forall \mathbb{P} \in \mathcal{P}.$$

Passing to the limit $\alpha \uparrow 1$, and using the fact that $(Y(v), v \in \mathcal{S})$ is $\mathcal{P}$-USCE, we get

$$E^{\mathbb{P}}[R(U^*(v))] \leqslant \liminf_{\alpha \uparrow 1} E^{\mathbb{P}}[R(U^\alpha(v))] \leqslant \lim_{\alpha \uparrow 1} E^{\mathbb{P}}[Y(U^\alpha(v))] \leqslant E^{\mathbb{P}}[Y(U^*(v))] \quad \text{a.s.,}$$

for every $\mathbb{P} \in \mathcal{P}$. This combined with the fact that the family $(R(v), v \in \mathcal{S})$ dominates Y, leads to $R(U^*(v)) = Y(U^*(v))$ a.s. It follows that based on the definition of $\mathcal{U}(v)$, we have $U^*(v) \geqslant \mathcal{U}(v)$. Let us demonstrate the other inequality. Observe that for $\alpha = 1$, we have $U^1(v) = \mathcal{U}(v)$ a.s. Thus, for all $\alpha \leqslant 1$, $U^\alpha(v) \leqslant U^1(v) = \mathcal{U}(v)$ a.s. Taking the limit $\alpha \uparrow 1$, we obtain $U^*(v) \leqslant \mathcal{U}(v)$ a.s., which ends the proof. □

5. Existence of optimal models

We now consider the conditions under which, for the family $\mathcal{P}$, there exists an optimal probability model for our problem. To this end, under suitable conditions for the family $\mathcal{P}$, we establish a "universal" optional decomposition for our Snell envelope family $\mathcal{R} = (R(v), v \in \mathcal{S})$ in the sense that it holds simultaneously for all $\mathbb{P} \in \mathcal{P}$. Specifically, we decompose $\mathcal{R}$ as the difference between a $\mathcal{P}$-martingale with RCLL paths and an optional RCLL increasing process.

Recall some definitions and results. Here and subsequently,

- $\mathbf{H}^2$ (resp. $\mathbf{H}^{loc}$) denotes the set of all zero-mean, square-integrable $\mathbb{Q}$-martingales (resp. local $\mathbb{Q}$-martingales). We define seminorms on $\mathbf{H}^2$ using the formula: $\|M\|_t := \sqrt{E^{\mathbb{Q}}[M_t^2]}$ for $M \in \mathbf{H}^2$. The space $\{\mathbf{H}^2, \|.\|_t\}$ is a complete separable space (see [23, Proposition 4.1.]).
- $\mathbf{X}^2_{lr}$ denotes the collection of all optional processes X such that X_τ lies in $L^2(\mathcal{F}_\tau, \mathbb{Q})$ for all $\tau \in \mathcal{S}$, and such that X admits right and left limits.
- For every $\mathbb{Q}$-local martingale H, $\mathcal{E}(H)$ denotes the Doléans-Dade exponential of H, that is, the solution of the stochastic differential equation: $\mathrm{d}U_t = U_{t-}\mathrm{d}H_t$, with $U_0 = 1$.

Definition 5.1 *A subset $\mathcal{H}$ of $\mathbf{H}^2$ is called a subspace of $\mathbf{H}^2$ if it satisfies the following three conditions: (i) $X, Y \in \mathcal{H}$ then $X + Y \in \mathcal{H}$; (ii) If $X \in \mathcal{H}$ and ψ is predictable, then $\int \psi \mathrm{d}X \in \mathcal{H}$; (iii) $\mathcal{H}$ is closed in $\{\mathbf{H}^2, \|.\|_t\}$.*

We have the following fundamental "Kunita-Watanabe decomposition" (see [23, Proposition 4.2.].

Proposition 5.1 *If $\mathcal{H}$ is a subspace of $\mathbf{H}^2$, then any element W of $\mathbf{H}^2$ can be decomposed uniquely as $W = W' + W''$, where $W' \in \mathcal{H}$, and $W'' \in \mathcal{H}^\perp$.*

Note that if two probability measures $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, then there is a $\mathbb{Q}$-local martingale H, such that $\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}\Big|_{\mathcal{F}_t} = \mathcal{E}(H)_t$ (see e.g., [25, Proposition 5.8.]). We denote by $\mathcal{H}$ the set

$$\mathcal{H} \triangleq \{H \in \mathbf{H}^{loc} / \ \mathcal{E}(H) = Z, \quad \text{for some } Z \in \mathcal{Z}\}. \tag{5.1}$$

We can now state the following universal optional decomposition that holds simultaneously for all $\mathbb{P} \in \mathcal{P}$. The proof is adapted from Kunita-Watanabe's work on square-integrable martingales (see [23, 33]).

Theorem 5.1 *(Optional decomposition) Suppose that $\mathcal{H}$ is a subspace of $\mathbf{H}^2$ in the sense of Definition 5.1. Let S be a $\mathcal{P}$-supermartingale family S, such that $(S(\tau), \tau \in \mathcal{S})$ is uniformly integrable w.r.t. the reference probability $\mathbb{Q}$. Then, there exists $X \in \mathbf{X}^2_{lr}$ such that $S(\tau) = X_\tau$ a.s. for all $\tau \in \mathcal{S}$. If in addition, $\operatorname{ess\,sup}_{\tau \in \mathcal{S}} S(\tau) \in L^2(\mathcal{F}, \mathbb{Q})$, then there exists C, an optional non-decreasing process starting at 0, along with a RCLL $\mathcal{P}$-martingale $M \in \mathbf{H}^2$, such that a.s.*

$$S(\tau) = X_\tau = X_0 + M_\tau - C_\tau, \quad \textit{for all } \tau \in \mathcal{S}. \tag{5.2}$$

Proof The existence of the process $X \in \mathbf{X}^2_{lr}$ such that $S(\tau) = X_\tau$ a.s. for all $\tau \in \mathcal{S}$ follows from [8, Theorem 3.1]. Using Theorem 3.1 in [8], we deduce the existence of an increasing predictable process $A^{\mathbb{Q}}$ such that $A^{\mathbb{Q}}_0 = 0$ and $A^{\mathbb{Q}}_T \in L^2(\mathcal{F}, \mathbb{Q})$, and of a $\mathbb{Q}$-martingale $\tilde{M}^{\mathbb{Q}} \in \mathbf{H}^2$ such that a.s.

$$S(\tau) = X_\tau = X_0 + \tilde{M}^{\mathbb{Q}}_\tau - A^{\mathbb{Q}}_\tau, \quad \forall \tau \in \mathcal{S}.$$

Because $\mathcal{H}$ is a subspace of $\mathbf{H}^2$, the martingale $\tilde{M}^{\mathbb{Q}}$ admits the Kunita-Watanabe decomposition: $M^{\mathbb{Q}} = K + M$, where $K \in \mathcal{H}$ and $M \in \mathcal{H}^\perp$, i.e., $\langle H, M\rangle = 0, \forall H \in \mathcal{H}$. Based on [33, Remark 2.16], the set $\mathcal{Z}$ of (3.3) is also closed in the set of all square-intergable $\mathbb{Q}$-martingales Z with mean 1, under the semi-norms $\|.\|_t$ defined by: $\|Z\|_t := E^{\mathbb{Q}}[(Z_t - 1)^2]^{\frac{1}{2}}$. Consequently, $\langle \mathcal{E}(H), M\rangle = 0$ for all $H \in \mathcal{H}$, and for each $\mathbb{P} \in \mathcal{P}$, $Z^{\mathbb{P}} M$ is a $\mathbb{Q}$-martingale. This suggests that M is a $\mathcal{P}$-martingale, which we identify with the $\mathcal{P}$-martingale in our decomposition. Therefore, we have a.s.

$$S(\tau) = X_\tau = X_0 + M_\tau - (A^{\mathbb{Q}}_\tau - K_\tau), \quad \forall \tau \in \mathcal{S}.$$

The proof is completed by showing that the optional process $C := A^{\mathbb{Q}} - K$ is non-decreasing. Note that the square-integrable martingale $K \in \mathcal{H}$ admits a decomposition of the form

$$K = K^c + K^d,$$

in which K^c is the continuous martingale part and K^d is the purely discontinuous martingale part, such that $\langle K^c, K^d\rangle = 0$ (see e.g., [14]). If we prove that K is a purely discontinuous martingale that has only negative jumps, the assertion holds. Indeed, because S is a $\mathcal{P}$-supermartingale family, X is a $\mathcal{P}$-supermartingale; thus considering Girsanov Theorem, the process $\langle H, K + M\rangle - A^{\mathbb{Q}}$ is non-increasing for all $H \in \mathcal{H}$. It follows that

$$A^{\mathbb{Q}} - \langle H, K\rangle \text{ is an increasing process}, \forall H \in \mathcal{H}. \tag{5.3}$$

Because $K \in \mathbf{H}^2$, the process $\langle K^c\rangle$ is integrable, and based on Lebesgue decomposition theorem, the measure $dA^{\mathbb{Q}}_t$ admits a decomposition of the form

$$\mathrm{d}A^{\mathbb{Q}}_t = \psi_t \mathrm{d}\langle K^c\rangle_t + \mathrm{d}D_t, \tag{5.4}$$

in which ψ is a positive predictable process in $L^1([0,\infty) \times \ \Omega, \mathrm{d}\langle K^c\rangle \mathrm{d}Q)$, and D is an integrable predictable increasing process such that, $\mathbb{Q}$-almost surely, $\mathrm{d}D_t \perp \mathrm{d}\langle K^c\rangle_t$. By fixing $m \in \mathbb{Z}$, we have the following version of (5.4)

$$\mathrm{d}A^{\mathbb{Q}}_t = \psi_t \mathbf{1}_{\{\psi_t \leqslant m\}} \mathrm{d}\langle K^c\rangle_t + \mathrm{d}D^m_t, \tag{5.5}$$

where, $\mathbb{Q}$-almost surely, the measure $\mathrm{d}D^m_t$ is singular w.r.t. $\mathbf{1}_{\{\psi_t \leqslant m\}} \mathrm{d}\langle K^c\rangle_t$. Thus, on $\{\psi \leqslant m\}$

we have

$$A_t^{\mathbb{Q}} = \int_0^t \psi_s \mathbf{1}_{\{\psi_s \leqslant m\}} \mathrm{d}\langle K^c \rangle_s. \tag{5.6}$$

Because $\mathcal{H}$ is a subspace of $\mathbf{H}^2$, we have $(\int_0^t (1+\psi_s)\mathbf{1}_{\{\psi_s \leqslant m\}} dK_s^c)_{t\in[0,T]} \in \mathcal{H}$. Therefore, for the event $\{\psi \leqslant m\}$, (5.3) shows that the process

$$A^{\mathbb{Q}} - \left\langle \int (1+\psi_s)\mathbf{1}_{\{\psi_s \leqslant m\}} \mathrm{d}K_s^c, K \right\rangle = A^{\mathbb{Q}} - \int (1+\psi_s)\mathbf{1}_{\{\psi_s \leqslant m\}} \mathrm{d}\langle K^c \rangle_s \text{ is increasing.}$$

Based on (5.6), we conclude that $\mathbb{Q}$-almost surely, the process

$$\langle K^c \rangle_t = \int_0^t (1+\psi_s)\mathbf{1}_{\{\psi_s \leqslant m\}} \mathrm{d}\langle K^c \rangle_s - \int_0^t \psi_s \mathbf{1}_{\{\psi_s \leqslant m\}} \mathrm{d}\langle K^c \rangle_s$$

is non-increasing on $\{\psi \leqslant m\}$, $\forall m \in \mathbb{Z}$. Consequently, $\langle K^c \rangle_\infty = 0$; thus, $K^c = 0$ and $K = K^d$. We now proceed to show that K has only negative jumps. The square-integrable martingale K can be decomposed further with respect to the sign of its jumps: $K = K^+ + K^-$, where K^+ (resp. K^-) is the compensated integral of $\mathbf{1}_{\{\Delta K > 0\}}$ (resp. $\mathbf{1}_{\{\Delta K < 0\}}$) with respect to K. The processes K^+ and K^- are both square-integrable martingales (see [14, Ch. VIII, p. 357]). We proceed analogously to the proof of $K^c = 0$, and we show that $K^+ = 0$. We conclude that K is a purely discontinuous martingale with negative jumps; hence, the process $C_t = A_t^{\mathbb{Q}} - K_t$ is non-decreasing, which is the desired conclusion. □

The following proposition shows that our Snell envelope family $\mathcal{R} = (R(v), v \in \mathcal{S})$ satisfies the integrability condition of Theorem 5.1; thus, it admits the “universal” optional decomposition.

Proposition 5.2 *If $\mathcal{H}$ is a subspace of $\mathbf{H}^2$ in the sense of Definition 5.1, and if*

$$\sup_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}[(\operatorname*{ess\,sup}_{\tau\in\mathcal{S}} Y(\tau))^2] < \infty, \tag{5.7}$$

then the Snell envelope family $\mathcal{R} = (R(v), v \in \mathcal{S})$ associated with $(Y(\tau), \tau \in \mathcal{S})$ admits a “universal” optional decomposition; i.e., there exists $X \in \mathbf{X}_{lr}^2$ such that $R(\tau) = X_\tau$ a.s. Moreover, there exists C an optional non-decreasing process starting at 0, along with a RCLL $\mathcal{P}$-martingale $M \in \mathbf{H}^2$, such that a.s.

$$R(\tau) = X_\tau = X_0 + M_\tau - C_\tau, \quad \textit{for all } \tau \in \mathcal{S}. \tag{5.8}$$

Proof We now show that $\operatorname*{ess\,sup}_{\tau\in\mathcal{S}} R(\tau) \in L^2(\mathcal{F}, \mathbb{Q})$. In fact, condition (5.7) provides even more scope. Here, we prove that $E^{\mathbb{P}}[(\operatorname*{ess\,sup}_{\tau\in\mathcal{S}} R(\tau))^2] < \infty$, for any probability measure $\mathbb{P} \in \mathcal{P}$. Indeed, let $\bar{Y}$ denote the random variable $\operatorname*{ess\,sup}_{\sigma\in\mathcal{S}} Y(\sigma)$. Note that we have a.s.

$$R(\tau) = \operatorname*{ess\,sup}_{\mathbb{P}\in\mathcal{P}} \operatorname*{ess\,sup}_{\sigma\in\mathcal{S}_\tau} E^{\mathbb{P}}[Y(\sigma)|\mathcal{F}_\tau] \leqslant \operatorname*{ess\,sup}_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}[\bar{Y}|\mathcal{F}_\tau] =: G(\tau), \quad \forall \tau \in \mathcal{S}.$$

We first show that the family $\mathcal{G} = (G(\tau), \tau \in \mathcal{S})$ is a square-integrable $\mathcal{P}$-supermartingale family, with $G(T) = \bar{Y}$. It is easy to check that $\{E^{\mathbb{P}}[\bar{Y}|\mathcal{F}_\tau], \mathbb{P} \in \mathcal{P}\}$, for each $\tau \in \mathcal{S}$, and $\{E^{\mathbb{P}}[\bar{Y}^2|\mathcal{F}_v]/v \in \mathcal{S}, \mathbb{P} \in \mathcal{P}\}$ is closed under pairwise maximization. Fix an arbitrary $\bar{\mathbb{P}} \in \mathcal{P}$. There exists a sequence $\{\mathbb{P}_n\}_n$ of probability measures in $\mathcal{P}$, such that $G(\tau) = \lim_{n\to\infty} \uparrow E^{\mathbb{P}_n}[\bar{Y}|\mathcal{F}_\tau]$ a.s. Without loss of generality, we can take $\mathbb{P}_n = \bar{P}$ on $\mathcal{F}_\tau$, $\forall n$. Hence, by monotone convergence we get, for each $\tau, v \in \mathcal{S}$ s.t. $T \geqslant \tau \geqslant v \geqslant 0$,

$$\begin{aligned} E^{\bar{\mathbb{P}}}[G(\tau)|\mathcal{F}_v] &= E^{\bar{\mathbb{P}}}\Big[\lim_{n\to\infty}\uparrow E^{\mathbb{P}_n}[\bar{Y}|\mathcal{F}_\tau]\Big|\mathcal{F}_v\Big] = \lim_{n\to\infty}\uparrow E^{\bar{\mathbb{P}}}\Big[E^{\mathbb{P}_n}[\bar{Y}|\mathcal{F}_\tau]\Big|\mathcal{F}_v\Big] \\ &= \lim_{n\to\infty}\uparrow E^{\mathbb{P}_n}\Big[E^{\mathbb{P}_n}[\bar{Y}|\mathcal{F}_\tau]\Big|\mathcal{F}_v\Big] = \lim_{n\to\infty}\uparrow E^{\mathbb{P}_n}[\bar{Y}|\mathcal{F}_v] \\ &\leqslant \operatorname*{ess\,sup}_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}[\bar{Y}|\mathcal{F}_v] = G(v) \quad \text{a.s.,} \end{aligned} \tag{5.9}$$

and thus the "supermartingale" property of $\mathcal{G}$ holds. In the same manner, we obtain the square-integrable property;

$$\begin{aligned} E^{\bar{\mathbb{P}}}[G(\tau)^2] &= E^{\bar{\mathbb{P}}}\Big[\lim_{n\to\infty}(E^{\mathbb{P}_n}[\bar{Y}|\mathcal{F}_\tau])^2\Big] \leqslant E^{\bar{\mathbb{P}}}\Big[\liminf_{n\to\infty} E^{\mathbb{P}_n}[(\bar{Y}^2)|\mathcal{F}_\tau]\Big] \\ &\leqslant \liminf_{n\to\infty} E^{\bar{\mathbb{P}}}\Big[E^{\mathbb{P}_n}[(\bar{Y})^2|\mathcal{F}_\tau]\Big] = \liminf_{n\to\infty} E^{\mathbb{P}_n}\Big[E^{\mathbb{P}_n}[(\bar{Y})^2|\mathcal{F}_\tau]\Big] \\ &\leqslant \sup_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}[(\bar{Y})^2] < \infty, \end{aligned} \tag{5.10}$$

using Fatou's lemma, Jensen's inequality, and the condition (5.7). Next, we consider the following $\bar{\mathbb{P}}$-supermartingale family

$$H(\tau) := G(\tau) - E^{\bar{\mathbb{P}}}[G(T)|\mathcal{F}_\tau], \quad \tau \in \mathcal{S}. \tag{5.11}$$

Note that for each $\tau \in \mathcal{S}$, $0 \leqslant H(\tau) \leqslant G(\tau)$. Thus, we have

$$H(\tau)^2 \leqslant G(\tau)^2 \leqslant \liminf_{n\to\infty} E^{\mathbb{P}_n}[(\bar{Y})^2|\mathcal{F}_\tau] \leqslant \operatorname*{ess\,sup}_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}[(\bar{Y})^2|\mathcal{F}_\tau],$$

and

$$\operatorname*{ess\,sup}_{\tau\in\mathcal{S}} H(\tau)^2 \leqslant \operatorname*{ess\,sup}_{\mathbb{P}\in\mathcal{P}} \operatorname*{ess\,sup}_{\tau\in\mathcal{S}} E^{\mathbb{P}}[(\bar{Y})^2|\mathcal{F}_\tau] = \lim_{n\to\infty}\uparrow E^{\mathbb{Q}_n}[(\bar{Y})^2|\mathcal{F}_{\tau_n}] \quad \text{a.s.,}$$

where $\{(\tau_n, \mathbb{Q}_n)\}_{n\in\mathbb{N}}$ is a conveniently chosen sequence in $\mathcal{S}\times\mathcal{P}$ such that $\mathbb{Q}_n = \bar{\mathbb{P}}$ on $\mathcal{F}_{\tau_n}$, $\forall n \in \mathbb{N}$. We take the expectation with respect to $\bar{\mathbb{P}}$ and apply the monotone convergence theorem to obtain

$$\begin{aligned} E^{\bar{\mathbb{P}}}[\operatorname*{ess\,sup}_{\tau\in\mathcal{S}} H(\tau)^2] &\leqslant \lim_{n\to\infty}\uparrow E^{\bar{\mathbb{P}}}\Big[E^{\mathbb{Q}_n}[(\bar{Y})^2|\mathcal{F}_{\tau_n}]\Big] \leqslant \lim_{n\to\infty}\uparrow E^{\mathbb{Q}_n}[(\bar{Y})^2] \\ &\leqslant \sup_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}[(\bar{Y})^2] < \infty. \end{aligned}$$

Hence, the system $(H(\tau), \tau \in \mathcal{S})$ is a square-integrable $\bar{\mathbb{P}}$-supermartingale family with the property: $\operatorname*{ess\,sup}_{\tau\in\mathcal{S}} H(\tau) \in L^2(\mathcal{F}, \bar{\mathbb{P}})$. We then apply Theorem 3.1 in [8] to deduce the existence of a process $Z \in \mathbf{X}^2_{lr}$, an increasing predictable process B such that $B_0 = 0$ and $B_T \in L^2(\mathcal{F}, \bar{\mathbb{P}})$, and of a $\bar{\mathbb{P}}$-martingale $N \in \mathbf{H}^2$ such that a.s.

$$H(\tau) = Z_\tau = Z_0 + N_\tau - B_\tau, \quad \forall \tau \in \mathcal{S}. \tag{5.12}$$

We then check that $H(\tau) = G(\tau) - E^{\bar{\mathbb{P}}}[\bar{Y}|\mathcal{F}_\tau] = E^{\bar{\mathbb{P}}}[B_T|\mathcal{F}_\tau] - B_\tau$, for each $\tau \in \mathcal{S}$ a.s., and hence

$$G(\tau) \leqslant E^{\bar{\mathbb{P}}}[\bar{Y} + B_T|\mathcal{F}_\tau], \quad \forall \tau \in \mathcal{S} \text{ a.s.}$$

Note that the process $(E^{\bar{\mathbb{P}}}[\bar{Y} + B_T|\mathcal{F}_t])_{t\in[0,T]}$ is a RCLL square-integrable $\bar{\mathbb{P}}$-martingale because $B_T \in L^2(\mathcal{F}, \bar{\mathbb{P}})$ and (5.7) hold. Therefore, based on the Burkholder–Davis–Gundy inequalities, we deduce that

$$E^{\bar{\mathbb{P}}}\Big[(\operatorname*{ess\,sup}_{\tau\in\mathcal{S}} R(\tau))^2\Big] \leqslant E^{\bar{\mathbb{P}}}\Big[(\operatorname*{ess\,sup}_{\tau\in\mathcal{S}} E^{\bar{\mathbb{P}}}[\bar{Y}+B_T|\mathcal{F}_\tau])^2\Big]$$
$$= E^{\bar{\mathbb{P}}}\Big[(\sup_{0\leqslant t\leqslant T} E^{\bar{\mathbb{P}}}[\bar{Y}+B_T|\mathcal{F}_t])^2\Big] < \infty,$$

for every $\mathbb{P}\in\mathcal{P}$. Hence, we can use Theorem 5.1, which provides the desired decomposition. The proof is complete. □

Before proving the existence of an optimal model $\mathbb{P}^*\in\mathcal{P}$, we first recall the following Theorem by Delbaen and Protter [13], which provides necessary and sufficient conditions for the existence of an optimal model P^*.

Theorem 5.2 *Suppose that the non-decreasing process C in the optional decomposition of (5.8) is predictable. Then, we have $E^{\mathbb{P}^*}\Big[Y(U^*(v))\Big|\mathcal{F}_v\Big] = R(v)$ a.s. for any $v\in\mathcal{S}$ and $\mathbb{P}^*\in\mathcal{P}$, if and only if C is "flat" away from the set $\mathcal{H}(\omega)=\{v\in\mathcal{S}, R(v)(\omega)=Y(v)(\omega)\}$.*

Note that in [13], Delbaen and Protter show that the process C for the optional decomposition is predictable if all the martingales in the set $\mathcal{Z}$ of (3.3) have continuous paths. The following is an example where $\mathcal{Z}$ fulfils this condition.

Example 5.1 *Let X be the Ito process solving the equation:*

$$\mathrm{d}X_t = b_t\mathrm{d}W_t,$$

where W is an $\mathbb{F}$-Brownian motion. $\mathrm{d}H_t = H_t\mathrm{d}X_t$ *, with $H_0=1$. We define the set $\mathcal{Z}$ as follows:*

$$\mathcal{Z} \triangleq \left\{ Z = \mathcal{E}\left(\int_0^\cdot b_t\mathrm{d}W_t\right), \quad \textit{for some } b\in\mathcal{BMO}\right\}. \tag{5.13}$$

We are now in a position to state the main result of this section. The objective is to characterize an optimal probability model using the above decomposition (5.8). This is the developed Theorem 5.3, stated below.

Theorem 5.3 *Assume that all the martingales in the set $\mathcal{Z}$ of (3.3) have continuous paths, and the set $\mathcal{H}$ is a subspace of $\mathbf{H}^2$, wherein the reward family $(Y(v), v\in\mathcal{S})$ is $\mathcal{P}$-USCE, and that equation (5.7) holds. Then, there exists a probability measure $\mathbb{P}^*\in\mathcal{P}$ such that, for every $v\in\mathcal{S}$,*

$$E^{\mathbb{P}^*}\Big[Y(U^*(v))\Big|\mathcal{F}_v\Big] = R(v) = \operatorname*{ess\,sup}_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}\Big[Y(U^*(v))\Big|\mathcal{F}_v\Big] \quad \text{a.s.} \tag{5.14}$$

Moreover, any model $\mathbb{P}\in\mathcal{P}$ is then optimal.

Proof Let $\mathbb{P}\in\mathcal{P}$ and $v\in\mathcal{S}$. The conditions of Theorem 4.2 and Theorem 5.2 are fulfilled, thus

$$E^{\mathbb{P}}[R(U^*(v))|\mathcal{F}_v] = E^{\mathbb{P}}[Y(U^*(v))|\mathcal{F}_v] = R(v) - E^{\mathbb{P}}[C_{U^*(v)} - C_v|\mathcal{F}_v], \quad \text{a.s.}$$

Taking the essential supremum with respect to $\mathbb{P}$ yields

$$\operatorname*{ess\,inf}_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}\Big[C_{U^*(v)} - C_v\Big|\mathcal{F}_v\Big] = \operatorname*{ess\,inf}_{Z\in\mathcal{Z}} E^{\mathbb{Q}}\Big[\frac{Z_{U^*(v)}}{Z_v}(C_{U^*(v)} - C_v)\Big|\mathcal{F}_v\Big] = 0, \quad \text{a.s.} \tag{5.15}$$

The set $\{E^{\mathbb{Q}}[\frac{Z_{U^*(v)}}{Z_v}(C(U^*(v)) - C_v)|\mathcal{F}_v], Z\in\mathcal{Z}\}$ is closed under pairwise maximization. Thus, once more, the fundamental property of the essential infimum/supremum guarantees that there is a sequence $(Z^n)_{n\in\mathbb{N}}\subseteq\mathcal{Z}$ such that a.s.

$$\operatorname*{ess\,inf}_{\mathbb{P}\in\mathcal{P}} E^{\mathbb{P}}[C_{U^*(v)} - C_v|\mathcal{F}_v] = \lim_{n\to\infty}\downarrow E^{\mathbb{Q}}\Big[\frac{Z^n_{U^*(v)}}{Z^n_v}(C_{U^*(v)} - C_v)\Big|\mathcal{F}_v\Big] = 0.$$

Using Fatou's lemma, we get

$$\begin{aligned}0 &\leqslant E^{\mathbb{Q}}\Big[\lim_{n\to\infty}\downarrow \frac{Z^n_{U^*(v)}}{Z^n_v}\cdot(C_{U^*(v)}-C_v)\Big|\mathcal{F}_v\Big]\\ &\leqslant \lim_{n\to\infty}\downarrow E^{\mathbb{Q}}\Big[\frac{Z^n_{U^*(v)}}{Z^n_v}(C_{U^*(v)}-C_v)\Big|\mathcal{F}_v\Big]=0.\end{aligned}$$

Thus,

$$E^{\mathbb{Q}}\Big[\lim_{n\to\infty}\downarrow \frac{Z^n_{U^*(v)}}{Z^n_v}\cdot(C_{U^*(v)}-C_v)\Big|\mathcal{F}_v\Big]\equiv 0. \tag{5.16}$$

Now, because $\mathcal{H}$ is closed in $\mathbf{H}^2$, it follows from [33, Lemma 2.22.] that for all $\tau,\sigma\in\mathcal{S}$,

$$\operatorname*{ess\,inf}_{Z\in\mathcal{Z}}\frac{Z_\sigma}{Z_\tau}>0 \quad \text{a.s.} \tag{5.17}$$

Therefore, $\lim_{n\to\infty}\downarrow\frac{Z^n_{U^*(v)}}{Z^n_v}>0$, and based on (5.16), we deduce that $C_{U^*(v)}=C_v$ a.s. because the process C is increasing. Based on equation (5.8), this leads immediately to

$$E^{\mathbb{P}}[R(v)]=X_0+E^{\mathbb{P}}[M_v]-E^{\mathbb{P}}[C_v]=X_0+E^{\mathbb{P}}[M_{U^*(v)}]-E^{\mathbb{P}}[C_{U^*(v)}]=E^{\mathbb{P}}[R(U^*(v))],$$

from which we conclude that $E^{\mathbb{P}}[Y(U^*(v))|\mathcal{F}_v]=E^{\mathbb{P}}[R(U^*(v))|\mathcal{F}_v]=R(v)$, a.s. Hence, $\mathbb{P}\equiv\mathbb{P}^*$ is an optimal model, and given the arbitrariness of $\mathbb{P}$, any model is then optimal. □

Lemma 5.1 *Set* $Y=(Y(\tau), then \tau\in\mathcal{S})$ *is an admissible family that is not necessarily non-negative, and supposedly satisfies the following integrability condition:*

$$\sup_{\mathbb{P}\in\mathcal{P}}E^{\mathbb{P}}[\operatorname*{ess\,sup}_{\tau\in\mathcal{S}}(Y(\tau))^-]<\infty, \tag{5.18}$$

where for $x\in\mathbb{R}$, $x^-:=-\min(x,0)$*. For each* $v\in\mathcal{S}$ *, the value function at time* v *is given by*

$$R(v):=\operatorname*{ess\,sup}_{\mathbb{P}\in\mathcal{P}}\operatorname*{ess\,sup}_{\tau\in\mathcal{S}_v}E^{\mathbb{P}}[Y(\tau)|\mathcal{F}_v]. \tag{5.19}$$

Then, the optimal stopping problem associated with Y *can be reduced to an equivalent problem with a non-negative reward process.*

Proof Define, for each $v\in\mathcal{S}$

$$X(v):=\operatorname*{ess\,sup}_{\mathbb{P}\in\mathcal{P}}E^{\mathbb{P}}[\operatorname*{ess\,sup}_{\tau\in\mathcal{S}}(Y(\tau))^-|\mathcal{F}_v], \tag{5.20}$$

and set $\bar{Y}(v):=Y(v)+X(v)$. Similarly to (5.9) and (5.10), we can show that $X=(X(v),v\in\mathcal{S})$ is a $\mathcal{P}$-supermartingale family. Hence, the new reward family $\bar{Y}$ is non-negative, and the associated new value function $\bar{R}$ satisfies

$$\begin{aligned}\bar{R}(v)&=\operatorname*{ess\,sup}_{\mathbb{P}\in\mathcal{P}}\operatorname*{ess\,sup}_{\tau\in\mathcal{S}_v}E^{\mathbb{P}}[\bar{Y}(\tau)|\mathcal{F}_v]\\ &=R(v)+\operatorname*{ess\,sup}_{\mathbb{P}\in\mathcal{P}}\operatorname*{ess\,sup}_{\tau\in\mathcal{S}_v}E^{\mathbb{P}}[X(\tau)|\mathcal{F}_v]\\ &\leqslant R(v)+X(v)\quad\text{a.s.}\end{aligned}$$

Conversely, we also have

$$\bar{R}(v)-R(v)=\operatorname*{ess\,sup}_{\mathbb{P}\in\mathcal{P}}\operatorname*{ess\,sup}_{\tau\in\mathcal{S}_v}E^{\mathbb{P}}[X(\tau)|\mathcal{F}_v]\geqslant\operatorname*{ess\,sup}_{\mathbb{P}\in\mathcal{P}}E^{\mathbb{P}}[X(v)|\mathcal{F}_v]=X(v).$$

Therefore, $\bar{R}(v)=R(v)+X(v)$ a.s., and the optimal stopping problem associated with the reward Y can be thus solved by translation. □

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