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A class of quadratic reflected BSDEs with singular coefficients

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Abstract In this paper, we study the existence and uniqueness of the solution to a reflected backward stochastic differential equation (RBSDE) with the generator $g(t,y,z)=G_f^F(t,y,z)+f(y)|z|^2$, where $f(y)$ is a locally integrable function defined on an open interval D, and $G_f^F(t,y,z)$ is induced by f and a Lipschitz continuous function F. Both the solution Y_t and the obstacle L_t of this RBSDE take values in D. As applications, we provide a probabilistic interpretation of an obstacle problem for a quadratic PDE with a singular term, whose solution takes values in D, and study an optimal stopping problem for the payoff of American options under general utilities.

Keywords Backward stochastic differential equation, Comparison theorem, Quadratic growth, Viscosity solution, American option

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1. Introduction

El Karoui et al. [9] introduced the notion of reflected backward stochastic differential equations (RBSDEs) with Lipschitz continuous generators. Several works have been done to study RBSDEs whose generator $g(t,y,z)$ is continuous in (y,z) and has a quadratic growth in the variable z (quadratic RBSDEs). The existence and uniqueness of quadratic RBSDEs have been investigated by Kobylanski et al. [13] and Xu [17] for bounded terminal variables and obstacles, and by Bayraktar and Yao [5] for exponentially integrable terminal variables and obstacles. When the generator is differentiable, the existence and uniqueness of quadratic RBSDEs driven by a continuous martingale were obtained by Lionnet [14] for bounded terminal variables and upper-bounded obstacles.

In the literature, the generator $g(\cdot,y,z)$ of the quadratic RBSDE is usually assumed to be continuous on $\mathbf{R}\times\mathbf{R}^d$. This means that such an RBSDE may not be applicable to some problems described by systems involving singular coefficients, such as the obstacle problem involving the following semilinear PDE with a singular term:

$$\partial_t v+\frac{1}{2}\Delta v+h(v)|\nabla v|^2=0, \tag{1.1}$$

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where $h(\cdot)$ is a continuous function defined on $(0,\infty)$. The existence of a bounded Sobolev solution of an obstacle problem related to (1.1) was studied by Arcoya et al. [1] in the elliptic case. However, to the best of our knowledge, the viscosity solution of the obstacle problem for (1.1) has not been obtained in the literature. Motivated by this, we will study the existence and uniqueness of the RBSDE with generator:

$$g(t,y,z) = G^F(t,y,z) + f(y)|z|^2, \quad \text{where} \quad G^F(t,y,z) := \frac{F(t,u_f(y),u_f'(y)z)}{u_f'(y)}, \tag{1.2}$$

where $f(y)$ is a locally integrable function defined on an open interval D,

$$u_f(x) := \int_\alpha^x \exp\left(2\int_\alpha^y f(z)\mathrm{d}z\right)\mathrm{d}y, \quad x \in D,$$

and F is a Lipschitz continuous function. Both the solution Y_t and obstacle L_t of such RBSDE take values in D. Some typical examples of u_f and G^F are given in Example 2.1. When $f(y) = \frac{1}{y}$, some related BSDEs were studied by Bahlali and Tangpi [4]. Our study is inspired by Bahlali et al. [3] and can be seen as an extension of a study by Zheng et al. [19]. Our proof of is based on the transformation $u_f(y)$, an Itô-Krylov formula, and an existence and uniqueness theorem for an RBSDE whose solution and obstacle take values in an open interval (see Proposition 3.2). The transformation $u_f(y)$ and the Itô-Krylov formula are used to remove the quadratic term $f(y)|z|^2$, where $f(y)$ may be discontinuous. The transformation $u_f(y)$ has been applied to stochastic differential utility by Duffie and Epstein [8] and to quadratic BSDEs by [3] (see also [2, 4, 19]). In our situation, it is crucial to guarantee that the solution Y takes values in D. To this end, we establish an existence and uniqueness result for an RBSDE with a Lipschitz continuous generator, whose solution Y_t and obstacle L_t take values in an open interval. Some typical examples of such RBSDEs are also provided; in particular, we present an example showing that the quadratic BSDE with a bounded terminal variable may have no bounded solution, even when $D = \mathbf{R}$, $f(y)$ is a constant, and the corresponding RBSDE has a bounded solution (see Example 3.9).

As an application, we study an obstacle problem for a PDE that is more general than (1.1). By using the comparison theorem established in this study and the relation between the RBSDE with generator $G_f^F + f(y)|z|^2$ and the RBSDE with generator F, we provide a viscosity solution to this obstacle problem.

Interestingly, we observe that the transformation $u_f(y)$ can be viewed as a utility function induced by $f(y)$, which includes exponential utility, power utility, and logarithmic utility as special cases. Moreover, $-2f(y)$ represents the Arrow–Pratt risk-aversion coefficient of the utility function $u_f(y)$. Based on this observation, we apply the RBSDE with generator $G_f^F + f(y)|z|^2$ to study an optimal stopping problem for the payoff of American options under the utility function $u_f(y)$.

The remained of the paper is organized as follows. Section 2 presents some assumptions. Section 3 investigates the RBSDE with generator (1.2). Section 4 presents two applications. The Appendix contains auxiliary results.

2. Preliminaries

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space. Let $(B_t)_{t\geqslant 0}$ be a d-dimensional standard Brownian motion defined on this probability space, and let $(\mathcal{F}_t)_{t\geqslant 0}$ be the natural filtration generated by $(B_t)_{t\geqslant 0}$, augmented by the P-null sets of $\mathcal{F}$. Let $\mathcal{P}$ be the progressively measurable sigma-field on $[0,T]\times\Omega$. For $z\in\mathbf{R}^d$, let $|z|$ denote the Euclidean norm. Let $T>0$ and $p>1$ be given real numbers, and let $\mathcal{T}_{t,T}$ be the set of all stopping times τ satisfying $t\leqslant\tau\leqslant T$. Throughout, we assume that $D\subset\mathbf{R}$ is an open interval. We define the following spaces:

$L_{1,loc}(D)=\{f:\ f:D\to\mathbf{R}$, is measurable and locally integrable $\}$;

$L_D(\mathcal{F}_T)=\{\xi:\ \mathcal{F}_T$-measurable random variable whose range is included in $D\}$;

$L^r_D(\mathcal{F}_T)=\{\xi\in L_D(\mathcal{F}_T)\colon E\left[|\xi|^r\right]<\infty\},\ r\geqslant 1$;

$\mathcal{C}_D=\{(\psi_t)_{t\in[0,T]}$: continuous and $(\mathcal{F}_t)$ -adapted process whose range is included in $D\}$;

$\mathcal{S}^r_D=\{(\psi_t)_{t\in[0,T]}\in\mathcal{C}_D\colon E\left[\sup_{0\leqslant t\leqslant T}|\psi_t|^r\right]<\infty\},\ r\geqslant 1$;

$\mathcal{A}=\{(\psi_t)_{t\in[0,T]}$: increasing, continuous and $(\mathcal{F}_t)$-adapted $\mathbf{R}$-valued process with $\psi_0=0\}$;

$\mathcal{A}^r=\{(\psi_t)_{t\in[0,T]}\in\mathcal{A}\colon E[|\psi_T|^r]<\infty\},\ r>0$;

$H^r=\{(\psi_t)_{t\in[0,T]}$: $\mathbf{R}^d$ -valued, $(\mathcal{F}_t)$ -progressively measurable and $\int_0^T|\psi_t|^r dt<\infty\},\ r\geqslant 1$;

$\mathcal{H}^r=\{(\psi_t)_{t\in[0,T]}\in H^2\colon E[(\int_0^T|\psi_t|^2dt)^{\frac{r}{2}}]<\infty\},\ r\geqslant 1$;

$W^2_{1,loc}(D)=\{f\in L_{1,loc}(D)$: its generalized derivation f' and f'' both belong to $L_{1,loc}(D)\}$.

For convenience, when the range of a random variable (or a process) is $\mathbf{R}$ or is clear from context, we use the simplified notations: $L(\mathcal{F}_T), L^r(\mathcal{F}_T),\mathcal{C}$, and $\mathcal{S}^r$. Note that in this paper, all the equalities and inequalities for random variables are understood to hold in the almost sure sense.

Let $f\in L_{1,loc}(D)$ be given. For $\alpha\in D$, we define the transformation

$$u_f^\alpha(x):=\int_\alpha^x\exp\left(2\int_\alpha^y f(z)\mathrm{d}z\right)\mathrm{d}y,\quad x\in D.$$

We assume that $\alpha\in D$ is given and denote $u_f^\alpha(x)$ by $u_f(x)$. Let $V:=\{y:y=u_f(x),x\in D\}$. Some properties of $u_f(x)$ are provided in Lemma A.1 in the Appendix. Since $u_f(x)$ is continuous, strictly increasing and $u_f(\alpha)=0$, it follows that V is an open interval such that $0\in V$. In particular, when $f\geqslant 0$ and $D=(a,\infty)$, since $u_f(x)\geqslant u_0(x)=x-\alpha$ (by Lemma A.1(v)), we have $V=(b,\infty)$ for some $b\geqslant-\infty$.

Let $\delta\geqslant 0$, $\gamma\geqslant 0$, and $\kappa\geqslant 0$ be given constants. Define a function

$$F\left(\omega,t,y,z\right):\Omega\times[0,T]\times\mathbf{R}\times\mathbf{R}^d\longmapsto\mathbf{R},$$

such that F is measurable with respect to $\mathcal{P}\otimes\mathcal{B}(\mathbf{R})\otimes\mathcal{B}(\mathbf{R}^d)$. We always assume that F satisfies

Assumption (A) For each $t\in[0,T]$ and $(y,z),(\tilde{y},\tilde{z})\in\mathbf{R}\times\mathbf{R}^d$, we have

$$|F(t,0,0)|\leqslant\delta\ \text{ and }\ |F(t,y,z)-F(t,\tilde{y},\tilde{z})|\leqslant\gamma|y-\tilde{y}|+\kappa|z-\tilde{z}|.$$

We define the function $G_f^F(t,y,z)$ as follows:

$$G_f^F(t,y,z):=\frac{F(t,u_f(y),u_f'(y)z)}{u_f'(y)},\quad (t,y,z)\in[0,T]\times D\times\mathbf{R}^d.$$

Since F satisfies Assumption (A), it follows that G_f^F is continuous in (y,z) and when $\delta=\gamma=0$, $|G_f^F(t,y,z)|\leqslant\kappa|z|$. Moreover, if $f\geqslant 0$, then $u_f'(y)$ is nondecreasing, and by the definitions of G_f^F and u_f, we can check that for each $c\in D$ and $(t,y,z)\in[0,T]\times([c,\infty)\cap D)\times\mathbf{R}^d$,

$$\begin{aligned}|G_f^F(t,y,z)|&\leqslant\frac{\delta}{u_f'(y)}+\frac{\gamma|\int_\alpha^y u_f'(x)\mathrm{d}x|}{u_f'(y)}+\kappa|z|\\&\leqslant\frac{\delta}{u_f'(c)}+\frac{\gamma|\int_\alpha^c u_f'(x)\mathrm{d}x|}{u_f'(c)}+\gamma\left|\int_c^y\frac{u_f'(x)}{u_f'(y)}\mathrm{d}x\right|+\kappa|z|\\&\leqslant\frac{\delta}{u_f'(c)}+\frac{\gamma|u_f(c)|}{u_f'(c)}+\gamma|c|+\gamma|y|+\kappa|z|.\end{aligned}\tag{2.1}$$

Let $\beta \in \mathbf{R}$, and $\delta_1, \gamma_1,\ \kappa_1 \in \mathbf{R}$ such that $|\delta_1| \leqslant \delta, |\gamma_1| \leqslant \gamma$, and $|\kappa_1| \leqslant \kappa$. We list some typical examples of the functions u_f and G_f^F.

Example 2.1 (i) *Let* $F(t,a,b) = \kappa_1 b$ *(or* $F(t,a,b) = \kappa_1 |b|$ *). Then, for each* $f \in L_{1,loc}(D)$ *,*

$$G_f^F(t,y,z) = \kappa_1 z \text{ (or } G_f^F(t,y,z) = \kappa_1 |z|).$$

(ii) *Let* $D = \mathbf{R}$ *and* $f(y) = 0$*. Then*

$$u_f(y) = y - \alpha \quad \text{and} \quad G_f^F(t,y,z) = F(t, y-\alpha, z).$$

(iii) *Let* $D = (0,\infty)$, $f(y) = \frac{\beta}{y}, \beta \neq -\frac{1}{2}$, *and* $F(t,a,b) = \delta_1 + \gamma_1 a + \kappa_1 b$*. Then*

$$u_f(y) = \frac{\alpha}{1+2\beta}\left(\left(\frac{y}{\alpha}\right)^{1+2\beta} - 1\right) \quad \text{and} \quad G_f^F(t,y,z) = \frac{\alpha^{2\beta}(\delta_1 - \frac{\gamma_1 \alpha}{1+2\beta})}{y^{2\beta}} + \frac{\gamma_1}{1+2\beta} y + \kappa_1 z.$$

(iv) *Let* $D = (0,\infty)$, $f(y) = -\frac{1}{2y}$, *and* $F(t,a,b) = \delta_1 + \gamma_1 a + \kappa_1 b$*. Then*

$$u_f(y) = \alpha \ln\left(\frac{y}{\alpha}\right) \quad \text{and} \quad G_f^F(t,y,z) = \frac{\delta_1}{\alpha} y + \gamma_1 y \ln\left(\frac{y}{\alpha}\right) + \kappa_1 z.$$

(v) *Let* $D = \mathbf{R}$, $f(y) = \frac{\beta}{2}$, $\beta \neq 0$, *and* $F(t,a,b) = \delta_1 + \gamma_1 a + \kappa_1 b$*. Then*

$$u_f(y) = \frac{1}{\beta}(\exp(\beta(y-\alpha)) - 1) \quad \text{and} \quad G_f^F(t,y,z) = \frac{\delta_1 - \frac{\gamma_1}{\beta}}{\exp(\beta(y-\alpha))} + \frac{\gamma_1}{\beta} + \kappa_1 z.$$

In this paper, we study the following RBSDE $(G_f^F + f(y)|z|^2, \xi, L_t)$:

$$\begin{cases} Y_t = \xi + \displaystyle\int_t^T (G_f^F(s,Y_s,Z_s) + f(Y_s)|Z_s|^2)\mathrm{d}s + K_T - K_t - \int_t^T Z_s \mathrm{d}B_s, \quad t \in [0,T], \\ \forall t \in [0,T], \quad Y_t \geqslant L_t, \\ \displaystyle\int_0^T (Y_t - L_t)\mathrm{d}K_t = 0, \end{cases} \tag{2.2}$$

where T is the terminal time, ξ is the terminal variable, and L_t is the lower obstacle.

Definition 2.2 *A solution of the RBSDE* $(G_f^F(t,y,z) + f(y)|z|^2, \xi, L_t)$ *is a triple of processes* $(Y_t, Z_t, K_t) \in \mathcal{C}_D \times H_d^2 \times \mathcal{A}$, *which satisfies* $\int_0^T |G_f^F(s,Y_s,Z_s) + f(Y_s)|Z_s|^2)|\mathrm{d}s < \infty$ *and (2.2).*

When the RBSDE $(G_f^F(t,y,z) + f(y)|z|^2, \xi, L_t)$ is not restricted by L_t, it becomes the standard BSDE (g, ξ) :

$$Y_t = \xi + \int_t^T (G_f^F(s,Y_s,Z_s) + f(Y_s)|Z_s|^2)\mathrm{d}s - \int_t^T Z_s \mathrm{d}B_s, \quad t \in [0,T]. \tag{2.3}$$

Definition 2.3 *A solution of the BSDE* $(G_f^F(t,y,z) + f(y)|z|^2, \xi)$ *is a pair of processes* $(Y_t, Z_t) \in \mathcal{C}_D \times H_d^2$, *which satisfies* $\int_0^T |G_f^F(s,Y_s,Z_s) + f(Y_s)|Z_s|^2|\mathrm{d}s < \infty$ *and (2.3).*

3. Existence and uniqueness

In this section, we study the existence and uniqueness of the solution to the RBSDE $(G_f^F + f(y)|z|^2, \xi, L_t)$. First, we provide a necessary condition for the existence.

Proposition 3.1 *Let* (Y_t, Z_t, K_t) *be a solution to the RBSDE* $(f(y)|z|^2, \xi, L_t)$*. If there exists a constant* β *such that, for each* $x \in D$, $u_f(x) \geqslant \beta$ *, then we have* $u_f(\xi) \in L^1(\mathcal{F}_T)$.

Proof Applying Lemma A.2(ii) to $u_f(Y_t)$, and then by Lemma A.1(iii), we have

$$u_f(Y_t) = u_f(\xi) + \int_t^T u_f'(Y_s)\mathrm{d}K_s - \int_t^T u_f'(Y_s)Z_s\mathrm{d}B_s, \quad t \in [0,T].$$

Set

$$\tau_n = \inf\left\{t \geqslant 0, \int_0^t |u_f'(Y_s)|^2|Z_s|^2\mathrm{d}s \geqslant n\right\} \wedge T, \quad n \geqslant 1.$$

By the two equalities above and the fact that $\int_0^t u_f'(Y_s)\mathrm{d}K_s \geqslant 0$, we have $u_f(Y_0) \geqslant E[u_f(Y_{\tau_n})]$. Then by the continuity of Y_t and u_f, and Fatou's lemma, we have

$$\beta \leqslant E[u_f(\xi)] = E[\liminf_{n\to\infty} u_f(Y_{\tau_n})] \leqslant \liminf_{n\to\infty} E[u_f(Y_{\tau_n})] \leqslant u_f(Y_0),$$

which implies $u_f(\xi) \in L^1(\mathcal{F}_T)$. □

The following existence and uniqueness result plays an important role in this paper.

Proposition 3.2 *Let $e^{\gamma T}(\xi^+ + \delta T) \in L_D(\mathcal{F}_T)$ with $\xi \in L^p(\mathcal{F}_T)$, and $e^{\gamma t}(L_t^+ + \delta t) \in \mathcal{C}_D$ with $L_t \in \mathcal{S}^p$. Then the RBSDE (F, ξ, L_t) has a unique solution (y_t, z_t, k_t) such that $y_t \in \mathcal{S}_D^p$. Moreover, we have $(z_t, k_t) \in \mathcal{H}^p \times \mathcal{A}^p$.*

Proof According to Bouchard et al. [7, Theorem 3.1], the RBSDE (F, ξ, L_t) has a solution (y_t, z_t, k_t) such that $y_t \in \mathcal{S}^p$. If the RBSDE (F, ξ, L_t) has another solution $(\tilde{y}_t, \tilde{z}_t, \tilde{k}_t)$ such that $\tilde{y}_t \in \mathcal{S}^p$, then by [7, Proposition 2.1] and a localization argument, we have (y_t, z_t, k_t), $(\tilde{y}_t, \tilde{z}_t, \tilde{k}_t) \in \mathcal{S}^p \times \mathcal{H}^p \times \mathcal{A}^p$. By [7, Theorem 3.1] again, we obtain $(y_t, z_t, k_t) = (\tilde{y}_t, \tilde{z}_t, \tilde{k}_t)$, $\mathrm{d}t \times \mathrm{d}P$-*a.e.* In other words, the RBSDE (F, ξ, L_t) has a unique solution (y_t, z_t, k_t) such that $y_t \in \mathcal{S}^p$. Moreover, we have $(z_t, k_t) \in \mathcal{H}^p \times \mathcal{A}^p$.

We now consider the range of y_t. Using a classic linearization method, we have

$$y_t = \xi + \int_t^T (F(s,0,0) + a_s y_s + b_s z_s)\mathrm{d}s + k_T - k_t - \int_t^T z_s\mathrm{d}B_s, \quad t \in [0,T], \tag{3.1}$$

where

$$a_s = \frac{F(s, y_s, z_s) - F(s, 0, z_s)}{y_s} 1_{\{|y_s|>0\}} \quad \text{and} \quad b_s = \frac{(F(s,0,z_s) - F(s,0,0))z_s}{|z_s|^2} 1_{\{|z_s|>0\}}.$$

Clearly, $|a_s| \leqslant \gamma, |b_s| \leqslant \kappa$. Let Q be a probability measure such that

$$\frac{\mathrm{d}Q}{\mathrm{d}P} = \exp\left\{\int_0^T b_s\mathrm{d}B_s - \frac{1}{2}\int_0^T |b_s|^2\mathrm{d}s\right\}.$$

Then $X_t := \exp\{\int_0^t b_s\mathrm{d}B_s - \frac{1}{2}\int_0^t |b_s|^2\mathrm{d}s\}$ is a solution to the the following linear SDE:

$$X_t = 1 + \int_0^t b_s X_s \mathrm{d}B_s, \quad t \in [0,T],$$

such that for each $r > 1$, $X_t \in \mathcal{S}^r$ (see [15, Theorem 4.4, page 61 and Lemma 2.3, page 93]). By Girsanov's theorem, $\bar{B}_t = B_t - \int_0^t b_s ds$ is a standard Brownian motion under Q. Then, (3.1) can be rewritten as

$$y_t = \xi + \int_t^T (F(s,0,0) + a_s y_s)\mathrm{d}s + k_T - k_t - \int_t^T z_s\mathrm{d}\bar{B}_s, \quad t \in [0,T].$$

By Itô's formula, we have

$$e^{\int_0^t a_r\mathrm{d}r}y_t=e^{\int_0^T a_r\mathrm{d}r}\xi+\int_t^T F(s,0,0)e^{\int_0^s a_r\mathrm{d}r}\mathrm{d}s+\int_t^T e^{\int_0^s a_r\mathrm{d}r}\mathrm{d}k_s-\int_t^T e^{\int_0^s a_r\mathrm{d}r}z_s\mathrm{d}\bar{B}_s,\quad t\in[0,T]. \quad (3.2)$$

Since $|a_s|\leqslant\gamma$ and for each $r>1$, $X_t\in\mathcal{S}^r$, by Hölder's inequality, we have, for $1<q<p$,

$$E_Q\left[\sup_{t\in[0,T]}|e^{\int_0^t a_r\mathrm{d}r}y_t|^q\right]\leqslant e^{\gamma Tq}E\left[X_T\sup_{t\in[0,T]}|y_t|^q\right]\leqslant e^{\gamma Tq}E\left[|X_T|^{\frac{p}{p-q}}\right]^{\frac{p-q}{p}}E\left[\sup_{t\in[0,T]}|y_t|^p\right]^{\frac{q}{p}}<\infty.$$

Similarly, we can also get $e^{\int_0^t a_r\mathrm{d}r}z_t\in\mathcal{H}^q,\int_0^t e^{\int_0^s a_r\mathrm{d}r}\mathrm{d}k_s\in\mathcal{A}^q$ and $e^{\int_0^t a_r\mathrm{d}r}L_t\in\mathcal{S}^q$ under probability measure Q. Thus, $(e^{\int_0^t a_r\mathrm{d}r}y_t,e^{\int_0^t a_r\mathrm{d}r}z_t,\int_0^t e^{\int_0^s a_r\mathrm{d}r}\mathrm{d}k_s)\in\mathcal{S}^q\times\mathcal{H}^q\times\mathcal{A}^q$ under probability measure Q is a solution to the RBSDE $(F(t,0,0)e^{\int_0^t a_r\mathrm{d}r},e^{\int_0^T a_r\mathrm{d}r}\xi,e^{\int_0^t a_r\mathrm{d}r}L_t)$. Then, by the proof of [9, Proposition 2.3] and (3.2), we have

$$e^{\int_0^t a_r\mathrm{d}r}y_t=\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}}E_Q\left[\int_t^\tau F(s,0,0)e^{\int_0^s a_r\mathrm{d}r}\mathrm{d}s+e^{\int_0^\tau a_r\mathrm{d}r}\eta_\tau|\mathcal{F}_t\right],\quad\forall t\in[0,T],$$

where $\eta_\tau:=L_\tau 1_{\{\tau<T\}}+\xi 1_{\{\tau=T\}}$. Thus, we have

$$y_t=\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}}E_Q\left[\int_t^\tau F(s,0,0)e^{\int_t^s a_r\mathrm{d}r}\mathrm{d}s+e^{\int_t^\tau a_r\mathrm{d}r}\eta_\tau|\mathcal{F}_t\right],\quad\forall t\in[0,T]. \quad (3.3)$$

Since $|F(s,0,0)|\leqslant\delta$ and $|a_s|\leqslant\gamma$, we have

$$\begin{aligned}L_t\leqslant y_t&\leqslant E_Q[\sup_{\tau\in\mathcal{T}_{t,T}}e^{\gamma(\tau-t)}(\delta(\tau-t)+\eta_\tau^+)|\mathcal{F}_t]\\&\leqslant E_Q[\sup_{\tau\in\mathcal{T}_{0,T}}e^{\gamma\tau}(\delta\tau+L_\tau^+)\vee e^{\gamma T}(\delta T+\xi^+)|\mathcal{F}_t],\quad\forall t\in[0,T].\end{aligned} \quad (3.4)$$

We can prove that, for each $\eta\in L^1(\mathcal{F}_T)$ and $c\in\mathbf{R}$ such that $\eta<c$, we have $E[\eta|\mathcal{F}_t]<c$ for each $t\in[0,T]$. Thus, (3.4) leads to $y_t\in\mathcal{S}_D$. The proof is complete. □

We have the following existence and uniqueness result for the RBSDE $(G_f^F+f(y)|z|^2,\xi,L_t)$.

Theorem 3.3 *Let $e^{\gamma T}(u_f(\xi)\vee 0+\delta T)\in L_V(\mathcal{F}_T)$ with $u_f(\xi)\in L^p(\mathcal{F}_T)$, and $e^{\gamma t}(u_f(L_t)\vee 0+\delta t)\in\mathcal{C}_V$ with $u_f(L_t)\in\mathcal{S}^p$. Then the RBSDE $(G_f^F+f(y)|z|^2,\xi,L_t)$ has a unique solution (Y_t,Z_t,K_t) such that $u_f(Y_t)\in\mathcal{S}^p$. Moreover, we have*

(i) $(Y_t,Z_t,K_t)\in\mathcal{S}^p\times\mathcal{H}^p\times\mathcal{A}^p$, *when f is integrable on D;*

(ii) $(Y_t,Z_t,K_t)\in\mathcal{S}^p\times\mathcal{H}^{2p}\times\mathcal{A}^p$, *when there exist constants* $c\in D$ and $\beta>0$ *such that $L_t\geqslant c,\mathrm{d}t\times\mathrm{d}P$-a.e. and $f\geqslant\beta,a.e.$*

Proof By Proposition 3.2, the RBSDE $(F,u_f(\xi),u_f(L_t))$ has a unique solution (y_t,z_t,k_t) such that $y_t\in\mathcal{S}_V^p$. Moreover, we have $(z_t,k_t)\in\mathcal{H}^p\times\mathcal{A}^p$. In view of Lemma A.1(iv), we can apply Lemma A.2(ii) to $u_f^{-1}(y_t)$, and then, by setting

$$(Y_t,Z_t,K_t):=\left(u_f^{-1}(y_t),\frac{z_t}{u_f'(u_f^{-1}(y_t))},\int_0^t\frac{1}{u_f'(u_f^{-1}(y_s))}\mathrm{d}k_s\right),\quad t\in[0,T], \quad (3.5)$$

and using Lemma A.1(iii), we get that (Y_t,Z_t,K_t) is a solution to the RBSDE $(G_f^F+f(y)|z|^2,\xi,L)$ such that $u_f(Y_t)\in\mathcal{S}^p$. We now prove the uniqueness. Since $u_f(x)\in W_{1,loc}^2(D)$ (see Lemma A.1(ii)), for a solution $(\bar{Y}_t,\bar{Z}_t,\bar{K}_t)$ of RBSDE $(G_f^F+f(y)|z|^2,\xi,L)$ such that $u_f(\bar{Y}_t)\in\mathcal{S}^p$, applying Lemma A.2(ii) to $u_f(\bar{Y}_t)$, and then by Lemma A.1(iii), we have

$$u_f(\bar{Y}_t)=u_f(\xi)+\int_t^T F(s,u_f(\bar{Y}_t),u_f'(\bar{Y}_s)\bar{Z}_s)\mathrm{d}s+\int_t^T u_f'(\bar{Y}_s)\mathrm{d}\bar{K}_s-\int_t^T u_f'(\bar{Y}_s)\bar{Z}_s\mathrm{d}B_s,\quad t\in[0,T],$$

which means that $(u_f(\bar{Y}_t), u_f'(\bar{Y}_t)\bar{Z}_t, \int_0^t u_f'(\bar{Y}_s)\mathrm{d}\bar{K}_s)$ is a solution to the RBSDE $(F, u_f(\xi), u_f(L_t))$ such that $u_f(\bar{Y}_t) \in \mathcal{S}^p$. Moreover, from (3.5), it follows that $(u_f(Y_t), u_f'(Y_t)\bar{Z}_t, \int_0^t u_f'(Y_s)\mathrm{d}\bar{K}_s)$ is a unique solution to the RBSDE $(F, u_f(\xi), u_f(L_t))$ such that $u_f(Y_t) \in \mathcal{S}^p$. Then, by Lemma A.1(ii), we obtain the uniqueness.

Proof of (i): Since f is integrable on D, by [3, Lemma A.1(j)], there exist two positive constants c_1 and c_2 such that, for each $x, y \in D$, we have $c_1|x-y| \leqslant |u_f(x) - u_f(y)| \leqslant c_2|x-y|$. Then, by (3.5), Lemma A.1(ii), and the fact that $(y_t, z_t, k_t) \in \mathcal{S}^p \times \mathcal{H}^p \times \mathcal{A}^p$, we get (i).

Proof of (ii): Since $f > 0, a.e.$, by Lemma A.1(v), we have

$$u_f(Y_t) \geqslant u_0(Y_t) = Y_t - \alpha \geqslant L_t - \alpha,$$

which together with $L_t \geqslant c, \mathrm{d}t \times \mathrm{d}P\text{-}a.e.$, implies $Y_t \in \mathcal{S}^p$ and $Y_t \geqslant c, \mathrm{d}t \times \mathrm{d}P\text{-}a.e.$

For $n \geqslant 1$, we define the following stopping time

$$\sigma_n = \inf\left\{t \geqslant 0, \int_0^t f(Y_s)|Z_s|^2\mathrm{d}s \geqslant n\right\} \wedge T.$$

Since $Y_t \geqslant c,\ \mathrm{d}t \times \mathrm{d}P\text{-}a.e.$, by the assumption $f \geqslant \beta, a.e.$, (2.1), (2.2), and the fact that $K_s \in \mathcal{A}$, for any stopping time $\tau \leqslant \sigma_1$, we have

$$\begin{aligned}\beta \int_\tau^{\sigma_n} |Z_s|^2\mathrm{d}s &\leqslant \int_\tau^{\sigma_n} f(Y_s)|Z_s|^2\mathrm{d}s \\ &= Y_\tau - Y_{\sigma_n} - \int_\tau^{\sigma_n} G_f^F(s, Y_s, Z_s)\mathrm{d}s - (K_{\sigma_n} - K_\tau) + \int_\tau^{\sigma_n} Z_s\mathrm{d}B_s \\ &\leqslant |Y_\tau| + |Y_{\sigma_n}| + \int_\tau^{\sigma_n} C(1 + |Y_s| + |Z_s|)\mathrm{d}s + \int_\tau^{\sigma_n} Z_s\mathrm{d}B_s. \end{aligned} \tag{3.6}$$

where C is a constant depending only on $c, u_f'(c), u_f(c), \delta, \gamma,$ and κ. By Jensen's inequality, (3.6), and BDG inequality, we have

$$\begin{aligned}\beta^p \left(E\left[\left(\int_\tau^{\sigma_n} |Z_s|^2\mathrm{d}s\right)^{\frac{p}{2}}\right]\right)^2 &\leqslant E\left[\left(\beta\int_\tau^{\sigma_n} |Z_s|^2\mathrm{d}s\right)^p\right] \\ &\leqslant E\left[\left(\int_\tau^{\sigma_n} f(Y_s)|Z_s|^2\mathrm{d}s\right)^p\right] \\ &\leqslant C_1\left(1 + E\left[\left(\int_\tau^{\sigma_n} |Z_s|^2\mathrm{d}s\right)^{\frac{p}{2}}\right] + E\left(\left|\int_\tau^{\sigma_n} Z_s\mathrm{d}B_s\right|^p\right)\right) \\ &\leqslant C_2\left(1 + E\left[\left(\int_\tau^{\sigma_n} |Z_s|^2\mathrm{d}s\right)^{\frac{p}{2}}\right]\right), \end{aligned} \tag{3.7}$$

where C_1 and C_2 are two positive constants depending only on $E[\sup_{t\in[0,T]} |Y_t|^p]$, $T, C,$ and p. Then by solving the quadratic inequality (3.7) with $E[(\int_\tau^{\sigma_n} |Z_s|^2\mathrm{d}s)^{\frac{p}{2}}]$ as the unknown variable, we get $E[(\int_\tau^{\sigma_n} |Z_s|^2\mathrm{d}s)^{\frac{p}{2}}] < C_3$, where $C_3 > 0$ is a constant depending only on C_2 and β. By plugging this inequality into (3.7), we get $E\left[\left(\int_\tau^{\sigma_n} |Z_s|^2\mathrm{d}s\right)^p\right] < C_4$, where $C_4 > 0$ is a constant depending only on $C_2, C_3,$ and β. Fatou's Lemma then gives

$$E\left[\left(\int_0^T |Z_s|^2\mathrm{d}s\right)^p\right] < \infty. \tag{3.8}$$

Since $f > 0, a.e.$, by (2.1) and (2.2), we have

$$
\begin{aligned}
0 \leqslant K_T &\leqslant Y_0 - \xi - \int_0^T G_f^F(s, Y_s, Z_s)\mathrm{d}s + \int_0^T Z_s \mathrm{d}B_s \\
&\leqslant |Y_0| + |\xi| + \int_0^T C(1 + |Y_s| + |Z_s|)\mathrm{d}s + \left| \int_0^T Z_s \mathrm{d}B_s \right|.
\end{aligned}
$$

where C is the constant in (3.6). Then by the fact that $Y_t \in \mathcal{S}^p$, BDG inequality, and (3.8), we get $K_t \in \mathcal{A}^p$. Thus, (ii) holds. The proof is complete. □

From Theorem 3.3, we get the following Corollary 3.4 directly.

Corollary 3.4 *Let $V = (b, \infty)$ with $b \geqslant -\infty$. Let $u_f(\xi) \in L^p(\mathcal{F}_T)$ and $u_f(L_t) \in \mathcal{S}^p$. Then the RBSDE $(G_f^F + f(y)|z|^2, \xi, L_t)$ has a unique solution (Y_t, Z_t, K_t) such that $u_f(Y_t) \in \mathcal{S}^p$.*

Corollary 3.5 *Let $\delta = \gamma = 0$ in Assumption (A). Let $u_f(\xi) \in L^p(\mathcal{F}_T)$ and $u_f(L_t) \in \mathcal{S}^p$. Then the RBSDE $(G_f^F + f(y)|z|^2, \xi, L_t)$ has a unique solution (Y_t, Z_t, K_t) such that $u_f(Y_t) \in \mathcal{S}^p$. In particular, for each $t \in [0, T]$, we have*

$$
u_f(Y_t) = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} E_Q[u_f(\eta_\tau)|\mathcal{F}_t], \tag{3.9}
$$

where $\eta_\tau := L_\tau 1_{\{\tau < T\}} + \xi 1_{\{\tau = T\}}$ and Q is a probability measure equivalent to P.

Proof Since $\delta = \gamma = 0$ and $0 \in V$, by Theorem 3.3, we get that the RBSDE $(G_f^F + f(y)|z|^2, \xi, L_t)$ has a unique solution (Y_t, Z_t, K_t) such that $u_f(Y_t) \in \mathcal{S}^p$. By (3.3) and (3.5), we deduce that, for each $t \in [0, T]$,

$$
u_f(Y_t) = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} E_Q\left[\int_t^\tau F(s, 0, 0)e^{\int_t^s a_r \mathrm{d}r}\mathrm{d}s + e^{\int_t^\tau a_r \mathrm{d}r} u_f(\eta_\tau)|\mathcal{F}_t\right],
$$

where $\eta_\tau := L_\tau 1_{\{\tau < T\}} + \xi 1_{\{\tau = T\}}$, Q is a probability measure equivalent to P, and a_t is a process such that $|a_t| \leqslant \gamma$. Since $\delta = \gamma = 0$, we have $F(s, 0, 0) = 0$ and $a_s = 0$. The proof is complete. □

Corollary 3.6 *Let $\delta = \gamma = 0$ in Assumption (A) and $u_f(\xi) \in L^p(\mathcal{F}_T)$. Then the BSDE $(G_f^F + f(y)|z|^2, \xi)$ has a unique solution (Y_t, Z_t) such that $u_f(Y_t) \in \mathcal{S}^p$. In particular, for each $t \in [0, T]$, we have*

$$
u_f(Y_t) = E_Q[u_f(\xi)|\mathcal{F}_t], \tag{3.10}
$$

where Q is a probability measure equivalent to P.

Proof By [6, Theorem 4.2 and Lemma 3.1], we deduce that the BSDE $(F, u_f(\xi))$ has a unique solution (y_t, z_t) such that $y_t \in \mathcal{S}^p$. Since $\delta = \gamma = 0$, from (3.2), we get that

$$
y_t = E_Q\left[u_f(\xi)|\mathcal{F}_t\right],
$$

where Q is a probability measure equivalent to P. This implies that $y_t \in \mathcal{S}_V^p$. Then applying Lemma A.2(ii) to $u_f^{-1}(y_t)$, and by a similar argument as in the proof of Theorem 3.3, we deduce that the BSDE $(G_f^F + f(y)|z|^2, \xi)$ has a unique solution (Y_t, Z_t) such that $u_f(Y_t) = y_t$. The proof is complete. □

Remark 3.7 *According to the definition of F, Corollary 3.6 is an extension of [19, Proposition 3.3], which studied the BSDE $(K|z| + f(y)|z|^2, \xi)$, where K is a constant. In Corollary 3.5 and 3.6, if we further assume that $\kappa = 0$ in Assumption (A), then the probability measure Q in (3.9) and (3.10) are both the probability measure P.*

Example 3.8 *We show some typical cases of Corollary 3.4.*

(i) *For* $f(y) = \frac{\beta}{y}$ *with* $\beta > -\frac{1}{2}$, *if* $\xi^{1+2\beta} \in L^p(\mathcal{F}_T)$ *and* $L_t^{1+2\beta} \in \mathcal{S}^p$, *then the RBSDE* $(G_f^F + f(y)|z|^2, \xi, L_t)$ *has a unique solution* (Y_t, Z_t, K_t) *such that* $Y_t^{1+2\beta} \in \mathcal{S}^p$.

(ii) *For* $f(y) = -\frac{1}{2y}$, *if* $\ln(\xi) \in L^p(\mathcal{F}_T)$ *and* $\ln(L_t) \in \mathcal{S}^p$, *then the RBSDE* $(G_f^F + f(y)|z|^2, \xi, L_t)$ *has a unique solution* (Y_t, Z_t, K_t) *such that* $\ln(Y_t) \in \mathcal{S}^p$.

(iii) *For* $f(y) = \frac{\beta}{2}$ *with* $\beta > 0$, *if* $\exp(\beta\xi) \in L^p(\mathcal{F}_T)$ *and* $\exp(\beta L_t) \in \mathcal{S}^p$, *then the RBSDE* $(G_f^F + f(y)|z|^2, \xi, L_t)$ *has a unique solution* (Y_t, Z_t, K_t) *such that* $\exp(\beta Y_t) \in \mathcal{S}^p$.

(iv) *For* $f(y) = \frac{\beta}{2}1_{\{y\geqslant 0\}}$ *with* $\beta > 0$, *if* $\xi^-, \exp(\beta\xi^+) \in L^p(\mathcal{F}_T)$ *and* $L_t^-, \exp(\beta L_t^+) \in \mathcal{S}^p$, *then the RBSDE* $(G_f^F + f(y)|z|^2, \xi, L_t)$ *has a unique solution* (Y_t, Z_t, K_t) *such that* $Y_t^-, u_f(\beta Y_t^+) \in \mathcal{S}^p$.

The following Example 3.9 shows that the quadratic BSDE with a bounded terminal variable may have no bounded solution, even when $D = \mathbf{R}$, $f(y)$ is a constant, and the corresponding RBSDE has a bounded solution.

Example 3.9 *We consider the Example 2.1(v) with* $\beta = 1, \delta_1 = 0, e^{\gamma_1 T} > 2$ *and* $\alpha = 0$. *Corollary 3.4 shows that when* $u_f(L_t) \in \mathcal{S}^p$ *and* $L_T \leqslant \ln(\frac{1}{2})$, *the RBSDE* $(G_f^F + \frac{|z|^2}{2}, \ln(\frac{1}{2}), L_t)$ *has a unique solution* (Y_t, Z_t, K_t) *such that* $u_f(Y_t) \in \mathcal{S}^p$. *Moreover, if* L_t *is bounded, by (3.4) and (3.5), we deduce that* Y_t *is bounded. However, we will see that the BSDE* $(G_f^F + \frac{|z|^2}{2}, \ln(\frac{1}{2}))$ *has no solution* (Y_t, Z_t) *such that* $u_f(Y_t) \in \mathcal{S}^p$. *In fact, by [6, Theorem 4.2 and Lemma 3.1], we deduce that the BSDE* $(F, -\frac{1}{2})$ *has a unique solution* (y_t, z_t) *such that* $y_t \in \mathcal{S}^p$. *Since* $u_f(x) = \exp(x) - 1$ *and* $V = (-1, \infty)$, *from (3.2), we get that*

$$y_0 = E_Q\left[-\frac{1}{2}e^{\int_0^T \gamma_1 \mathrm{d}r}\right] = -\frac{1}{2}e^{\gamma_1 T} \notin V, \tag{3.11}$$

where Q *is a probability measure equivalent to* P. *If we assume that the BSDE* $(G_f^F + \frac{|z|^2}{2}, \ln(\frac{1}{2}))$ *has a solution* (Y_t, Z_t) *such that* $u_f(Y_t) \in \mathcal{S}^p$, *then by applying Itô's formula to* $u_f(Y_t)$, *we get that* $(u_f(Y_t), u_f'(Y_t)Z_t)$ *is a solution to the BSDE* $(F, -\frac{1}{2})$. *This together with the uniqueness of the solution to the BSDE* $(F, -\frac{1}{2})$ *gives* $y_t = u_f(Y_t) \in V$, *which contradicts (3.11).*

We now establish some comparison results for the RBSDE $(G_f^F + f(y)|z|^2, \xi)$.

Proposition 3.10 *Let* F_1 *and* F_2 *satisfy Assumption (A) such that* $F_1(\cdot) \geqslant F_2(\cdot)$. *Let* $\xi_1, \xi_1 \in L_D(\mathcal{F}_T)$ *such that* $\xi_1 \geqslant \xi_2 \geqslant L_T$ *and* $u_f(\xi_1), u_f(\xi_2) \in L^p(\mathcal{F}_T)$. *Let* (Y_t^i, Z_t^i, K_t^i) *be the solution of the RBSDE* $(G_f^{F_i} + f(y)|z|^2, \xi_i, L)$ *satisfying* $u_f(Y_t^i) \in \mathcal{S}^p, i = 1, 2$. *Then for each* $t \in [0, T]$, $Y_t^1 \geqslant Y_t^2$.

Proof By the proof of Theorem 3.3, we can get that the RBSDE $(F_i, u_f(\xi_i), u_f(L_t))$ has a unique solution (y_t^i, z_t^i, k_t^i) such that $y_t^i = u_f(Y_t^i)$, $i = 1, 2$. By Lemma A.1(ii) and [9, Theorem 4.1], we get that for each $t \in [0, T]$, $u_f(Y_t^1) \geqslant u_f(Y_t^2)$, which gives $Y_t^1 \geqslant Y_t^2$. □

We assume two semimartingales:

$$Y_t^1 = Y_T^1 + \int_t^T h_1(s)\mathrm{d}s + A_T^1 - A_t^1 - \int_t^T Z_s^1 \mathrm{d}B_s, \quad t \in [0, T]$$

and

$$Y_t^2 = Y_T^2 + \int_t^T h_2(s)\mathrm{d}s - A_T^2 + A_t^2 - \int_t^T Z_s^2 \mathrm{d}B_s, \quad t \in [0, T],$$

where $Y_t^i \in \mathcal{C}_D$, $A_t^i \in \mathcal{A}$, $Z_t^i \in H^2$ and $h_i(t)$ is a progressively measurable process satisfying $\int_0^T |h_i(t)|\mathrm{d}t < \infty, i = 1, 2.$

Proposition 3.11 *Let* $u_f'(Y_t^i)Z_t^i \in \mathcal{H}^p$ *and* $u_f'(Y_t^i)h_i(t) - \frac{1}{2}u_f''(Y_t^i)|Z_t^i|^2 \in \mathcal{H}^p$, $i = 1, 2$. *Let* (Y_t, Z_t) *be the solution of the BSDE* $(G_f^F + f(y)|z|^2, \xi)$ *satisfying* $u_f(Y_t) \in \mathcal{S}^p$.

(i) *Assume that* $\xi \leqslant Y_T^1$ *and* $G_f^F(t, Y_t^1, Z_t^1) + f(Y_t^1)|Z_t^1|^2 \leqslant h_1(t)$, $\mathrm{d}t \times \mathrm{d}P$*-a.e. Then for each* $t \in [0, T]$, $Y_t \leqslant Y_t^1$. *Moreover, if* $Y_t = Y_t^1$ *for some* $t \in [0, T]$ *, then* $\xi = Y_T^1$, $A_T^1 = A_t^1$, and $G_f^F(s, Y_s^1, Z_s^1) + f(Y_s^1)|Z_s^1|^2 = h_1(s)$, $\mathrm{d}t \times \mathrm{d}P$*-a.e., on* $[t, T] \times \Omega$;

(ii) *Assume that* $\xi \geqslant Y_T^2$ *and* $G_f^F(t, Y_t^2, Z_t^2) + f(Y_t^2)|Z_t^2|^2 \geqslant h_2(t)$, $\mathrm{d}t \times \mathrm{d}P$*-a.e. Then for each* $t \in [0, T]$, $Y_t \geqslant Y_t^2$. *Moreover, if* $Y_t = Y_t^2$ *for some* $t \in [0, T]$, *then* $\xi = Y_T^2$, $A_T^2 = A_t^2$, *and* $G_f^F(s, Y_s^2, Z_s^2) + f(Y_s^2)|Z_s^2|^2 = h_2(s)$, $\mathrm{d}t \times \mathrm{d}P$*-a.e.*, on $[t, T] \times \Omega$.

Proof Proof of (i): Applying Lemma A.2(ii) to $u_f(Y_t)$, we get that $(u_f(Y_t), u_f'(Y_t)Z_t)$ is a solution to the BSDE $(F, u_f(\xi))$. Since $u_f(Y_t) \in \mathcal{S}^p$, by [6, Lemma 3.1], we have $u_f'(Y_t)Z_t \in \mathcal{H}^p$. Applying Lemma A.2(ii) to $u_f(Y_t^1)$, we have

$$\begin{aligned} u_f(Y_t^1) = u_f(Y_T^1) &+ \int_t^T (u_f'(Y_s^1)h_1(s) - \frac{1}{2}u_f''(Y_s^1)|Z_s^1|^2)\mathrm{d}s \\ &+ \int_t^T u_f'(Y_s^1)\mathrm{d}A_s^1 - \int_t^T u_f'(Y_t^1)Z_s^1\mathrm{d}B_s, \quad t \in [0, T]. \end{aligned}$$

Since for each $t \in [0, T]$,

$$\begin{aligned} F(t, u_f(Y_t^1), u_f'(Y_t^1)Z_t^1) &= G_f^F(t, Y_t^1, Z_t^1)u_f'(Y_t^1), \quad \text{(by the definition of } G_f^F) \\ &\leqslant (h_1(t) - f(Y_t^1)|Z_t^1|^2)u_f'(Y_t^1) \\ &= u_f'(Y_t^1)h_1(t) - \frac{1}{2}u_f''(Y_t^1)|Z_t^1|^2, \quad \text{(by Lemma A.1(iii))} \end{aligned}$$

it follows from the proof of [10, Theorem 2.2], and Lemma A.1(ii) that (i) holds.

Proof of (ii): The proof is similar to (i), so it is not given explicitly. □

Remark 3.12 *If* $f = 0$, *then Proposition 3.10 becomes a comparison theorem for RBSDEs with Lipschitz continuous generators, which was studied by [9, Theorem 4.1]. If* $G_f^F = 0$ *, then Proposition 3.11 becomes the comparison theorem for the BSDE* $(f(y)|z|^2, \xi)$ *, which was studied by [3, Proposition 3.2], [2, Proposition 2.3], and [19, Proposition 4.1, 4.3].*

4. Applications

4.1 An obstacle problem for PDEs with singular coefficients

In this subsection, we study the following obstacle problem for a quadratic PDE:

$$\begin{cases} \min\{v(t,x) - h(t,x), -\partial_t v(t,x) - \mathcal{L}v(t,x) - \widetilde{G}(t, v(t,x), \sigma^*\nabla_x v(t,x))\} = 0, \ (t,x) \in [0,T) \times \mathbf{R}^d; \\ v(T,x) = \psi(x), \quad x \in \mathbf{R}^d, \end{cases} \tag{4.1}$$

where $\widetilde{G}(t, y, z) = G_f^F(t, y, z) + f(y)|z|^2$ with $\delta = \gamma = 0$, $\psi(x) : \mathbf{R}^d \mapsto D$, $h(t,x) : [0,T] \times \mathbf{R}^d \mapsto D$, and $\mathcal{L}$ is the infinitesimal generator of the solution $X_s^{t,x}$ of the SDE:

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})\mathrm{d}r + \int_t^s \sigma(r, X_r^{t,x})\mathrm{d}B_r, \quad x \in \mathbf{R}^d, \quad s \in [t, T],$$

where $b : [0,T] \times \mathbf{R}^d \mapsto \mathbf{R}^d$, $\sigma : [0,T] \times \mathbf{R}^d \mapsto \mathbf{R}^{d\times d}$. The operator $\mathcal{L}$ is given by

$$\mathcal{L} := \frac{1}{2}\sum_{i,j=1}^{d} (\sigma\sigma^*)_{i,j}(s,x)\frac{\partial^2}{\partial_{x_i}\partial_{x_j}} + \sum_{i=1}^{d} b_i(s,x)\frac{\partial}{\partial_{x_i}}.$$

Definition 4.1 *A function $v(t,x) \in C_D([0,T] \times \mathbf{R}^d)$ [1] is called a viscosity subsolution of (4.1), if $v(T,\cdot) \leqslant \psi(\cdot)$ and for each $(t,x,\phi) \in [0,T] \times \mathbf{R}^d \times C_D^{1,2}([0,T] \times \mathbf{R}^d)$ such that $\phi(t,x) = v(t,x)$ and (t,x) is a local minimum point of $\phi - v$, we have*

$$\min\{\phi(t,x) - h(t,x), -\partial_t\phi(t,x) - \mathcal{L}\phi(t,x) - \widetilde{G}(t,\phi(t,x),\sigma^*\nabla_x\phi(t,x))\} \leqslant 0.$$

A function $v(t,x) \in C_D([0,T] \times \mathbf{R}^d)$ is called a viscosity supersolution of (4.1), if $v(T,\cdot) \geqslant \psi(\cdot)$ and for each $(t,x,\phi) \in [0,T] \times \mathbf{R}^d \times C_D^{1,2}([0,T] \times \mathbf{R}^d)$ such that $\phi(t,x) = v(t,x)$ and (t,x) is a local maximum point of $\phi - v$, we have

$$\min\{\phi(t,x) - h(t,x), -\partial_t\phi(t,x) - \mathcal{L}\phi(t,x) - \widetilde{G}(t,\phi(t,x),\sigma^*\nabla_x\phi(t,x))\} \geqslant 0.$$

A function $v(t,x) \in C_D([0,T] \times \mathbf{R}^d)$ is called a viscosity solution of (4.1), if it is a viscosity subsolution and a viscosity supersolution of (4.1).

Assumption (B) (i) f is continuous.
(ii) $\psi(\cdot)$ is continuous such that $\psi(\cdot) \geqslant h(T,\cdot)$, and $u_f(\psi(\cdot))$ has polynomial growth.
(iii) $u_f(h(\cdot,\cdot))$ is continuous and $u_f(h(t,\cdot))$ has polynomial growth (uniformly in t).
(iv) $b(t,\cdot)$ and $\sigma(t,\cdot)$ are both Lipschitz continuous with linear growth (uniformly in t).

Let Assumption (B) hold. For $t \in [0,T)$, let $(\mathcal{F}_s^t)_{t\leqslant s\leqslant T}$ be the natural filtration generated by $(B_s - B_t)_{s\geqslant t}$, augmented by the P-null sets of $\mathcal{F}$. Then by [15, Theorem 4.4, page 61 and Lemma 2.3, page 93]), for $(t,x) \in [0,T) \times \mathbf{R}^d$, we have $u_f(\psi(X_T^{t,x})) \in L^2(\mathcal{F}_T^t)$ and $u_f(h(s,X_s^{t,x})) \in \mathcal{S}^2$ ($(\mathcal{F}_s^t)$-progressively measurable). It follows from Corollary 3.5 that the following Markovian RBSDE:

$$\begin{cases} Y_s^{t,x} = \psi(X_T^{t,x}) + \displaystyle\int_s^T \widetilde{G}(t,Y_r^{t,x},Z_r^{t,x})\mathrm{d}r + K_T^{t,x} - K_s^{t,x} - \int_s^T Z_r^{t,x}\mathrm{d}B_r, \quad s \in [t,T], \\ Y_s^{t,x} \geqslant h(s,X_s^{t,x}), \quad s \in [t,T], \\ \displaystyle\int_t^T (Y_s^{t,x} - h(s,X_s^{t,x}))\mathrm{d}K_s^{t,x} = 0, \end{cases} \tag{4.2}$$

has a unique $(\mathcal{F}_s^t)$-progressively measurable solution $(Y_s^{t,x}, Z_s^{t,x}, K_s^{t,x})$ such that $u_f(Y_s^{t,x}) \in \mathcal{S}^2$. In addition, we have $u_f'(Y_s^{t,x})Z_s^{t,x} \in \mathcal{H}^2$. Set $v(t,x) := Y_t^{t,x}$. We have the following proposition:

Proposition 4.2 *$v(t,x)$ is a viscosity solution of the obstacle problem (4.1).*

Proof By (3.5), for each $(t,x) \in [0,T) \times \mathbf{R}^d$, the RBSDE $(F, u_f(\psi(X_T^{t,x})), u_f(h(s,X_s^{t,x})))$ admits a unique $(\mathcal{F}_s^t)$-progressively measurable solution $(y_s^{t,x}, z_s^{t,x}, k_s^{t,x})$ such that $y_s^{t,x} = u_f(Y_s^{t,x})$. Thus, we have $u_f(v(t,x)) = u_f(Y_t^{t,x}) = y_t^{t,x}$. It follows from [9, Lemma 8.4] and Lemma A.1(ii) that $v(t,x)$ is continuous in (t,x). By (6.4.9) in [18], we further get that, for $s \in [t,T]$,

$$Y_s^{t,x} = v(s,X_s^{t,x}). \tag{4.3}$$

[1] $v(t,x) \in C_D([0,T] \times \mathbf{R}^d)$ means that $v(t,x) \in C([0,T] \times \mathbf{R}^d)$ takes values in D.

Step 1 We will show that $v(t,x)$ is a viscosity supersolution of (4.1). Suppose that $v(t,x)$ is not a viscosity supersolution of (4.1). This means that there exists $(t,x,\phi)\in[0,T]\times\mathbf{R}^d\times C_D^{1,2}([0,T]\times\mathbf{R}^d)$ satisfying the condition that $\phi(t,x)=v(t,x)$ and that (t,x) is a local maximum point of $\phi-v$, such that

$$\frac{\partial\phi}{\partial t}(t,x)+\mathcal{L}\phi(t,x)+\widetilde{G}(t,\phi(t,x),\sigma^*\nabla_x\phi(t,x))>0.$$

By continuity, there exist $\beta\in(0,T-t]$, $c>0$, and $C>0$ such that for each $s\in[t,t+\beta]$ and $y\in[x-c,x+c]$, we have

$$v(s,y)\geqslant\phi(s,y)\quad\text{and}\quad\frac{\partial\phi}{\partial t}(s,y)+\mathcal{L}\phi(s,y)+\widetilde{G}(s,\phi(s,y),\sigma^*\nabla_x\phi(s,y))\geqslant C. \tag{4.4}$$

We define a stopping time $\tau=\inf\{s\geqslant t;|X_s^{t,x}-x|\geqslant c\}\wedge(t+\beta)$. Thus, $t<\tau\leqslant t+\beta$, and the ranges of $v(s\wedge\tau,X_{s\wedge\tau}^{t,x})_{s\in[t,t+\beta]}$, $h(s\wedge\tau,X_{s\wedge\tau}^{t,x})_{s\in[t,t+\beta]}$ and $\phi(s\wedge\tau,X_{s\wedge\tau}^{t,x})_{s\in[t,t+\beta]}$ are all included in a closed subset of D. By (4.2) and (4.3), we get that the following RBSDE:

$$\bar{Y}_s=v(\tau,X_\tau^{t,x})+\int_s^{t+\beta}1_{\{r\leqslant\tau\}}\widetilde{G}(r,\bar{Y}_r,\bar{Z}_r)\mathrm{d}r+\bar{K}_\tau^{t,x}-\bar{K}_s^{t,x}-\int_s^{t+\beta}\bar{Z}_r\mathrm{d}B_r,\quad s\in[t,t+\beta], \tag{4.5}$$

has a solution $(\bar{Y}_s,\bar{Z}_s,\bar{K}_s)=(Y_{s\wedge\tau}^{t,x},1_{\{s\leqslant\tau\}}Z_s^{t,x},K_{s\wedge\tau}^{t,x})\in\mathcal{S}_D^\infty\times\mathcal{H}^2\times\mathcal{A}$, where $\bar{Z}_s\in\mathcal{H}^2$ is due to $u_f'(Y_s^{t,x})Z_s^{t,x}\in\mathcal{H}^2$ and $Y_{s\wedge\tau}^{t,x}\in\mathcal{S}_D^\infty$. By Corollary 3.6, the BSDE $(\widetilde{G},v(\tau,X_\tau^{t,x}))$ with terminal time τ has a unique solution $(\hat{Y}_s,\hat{Z}_s)$ on $[t,\tau]$ such that $\hat{Y}_s\in\mathcal{S}_D^\infty$. Since $(\bar{Y}_s,\bar{Z}_s,\bar{K}_s)\in\mathcal{S}_D^\infty\times\mathcal{H}^2\times\mathcal{A}$ and F has a linear growth, we have $u_f'(\bar{Y}_t)\bar{Z}_t\in\mathcal{H}^2$ and

$$u_f'(\bar{Y}_t)\widetilde{G}(t,\bar{Y}_t,\bar{Z}_t)-\frac{1}{2}u_f''(\bar{Y}_t)|\bar{Z}_t|^2=F(t,u_f(\bar{Y}_t),u_f'(\bar{Y}_t)\bar{Z}_t)\in\mathcal{H}^2.$$

Then by (4.5) and Proposition 3.11(i), we get $\bar{Y}_t\geqslant\hat{Y}_t$.

Applying Itô's formula to $\phi(s,X_s^{t,x})$ for $s\in[t,\tau]$, we get that the following BSDE:

$$\tilde{Y}_s=\phi(\tau,X_\tau^{t,x})+\int_s^{t+\beta}-1_{\{r\leqslant\tau\}}\left(\frac{\partial\phi}{\partial t}(r,X_r^{t,x})+\mathcal{L}\phi(r,X_r^{t,x})\right)\mathrm{d}r-\int_s^{t+\beta}\tilde{Z}_r\mathrm{d}B_r,\quad s\in[t,t+\beta], \tag{4.6}$$

has a solution $(\tilde{Y}_s,\tilde{Z}_s)=(\phi(s\wedge\tau,X_{s\wedge\tau}^{t,x}),1_{\{s\leqslant\tau\}}\sigma^*\nabla_x\phi(s,X_s^{t,x}))\in\mathcal{S}_D^\infty\times\mathcal{H}^\infty$. By (4.4), we have $v(\tau,X_\tau^{t,x})\geqslant\phi(\tau,X_\tau^{t,x})$ and for each $r\in[t,\tau]$,

$$\widetilde{G}(r,\phi(r,X_r^{t,x}),\sigma^*\nabla_x\phi(r,X_r^{t,x}))\geqslant-\frac{\partial\phi}{\partial t}(r,X_r^{t,x})-\mathcal{L}\phi(r,X_r^{t,x})+C. \tag{4.7}$$

Then, since $(\tilde{Y}_s,\tilde{Z}_s)\in\mathcal{S}_D^\infty\times\mathcal{H}^\infty$, by (4.6), (4.7) and Proposition 3.11(ii) (strict comparison), we get $\hat{Y}_t>\tilde{Y}_t$. Since $\bar{Y}_t\geqslant\hat{Y}_t$, it follows that $v(t,x)=\bar{Y}_t>\tilde{Y}_t=\phi(t,x)$, which contradicts the condition that $v(t,x)=\phi(t,x)$. Thus, $v(t,x)$ is a viscosity supersolution of (4.1).

Step 2 We will show that $v(t,x)$ is a viscosity subsolution of (4.1). This proof is similar to Step 1. Suppose that $v(t,x)$ is not a viscosity subsolution of (4.1). This means that there exists $(t,x,\phi)\in[0,T]\times\mathbf{R}^d\times C_D^{1,2}([0,T]\times\mathbf{R}^d)$ satisfying the condition that $\phi(t,x)=v(t,x)$ and that (t,x) is a local minimum point of $\phi-v$, such that $v(t,x)>h(t,x)$ and

$$\frac{\partial\phi}{\partial t}(t,x)+\mathcal{L}\phi(t,x)+\widetilde{G}(t,\phi(t,x),\sigma^*\nabla_x\phi(t,x))<0.$$

By continuity, there exist $\beta\in(0,T-t]$, $c>0$, and $C>0$ such that for each $s\in[t,t+\beta]$ and $y\in[x-c,x+c]$, we have $v(s,y)\leqslant\phi(s,y)$,

$$v(s,y)\geqslant h(s,y)+C,\quad \text{and}\quad \frac{\partial \phi}{\partial t}(s,y)+\mathcal{L}\phi(s,y)+\widetilde{G}(s,\phi(s,y),\sigma^*\nabla_x\phi(s,y))\leqslant -C. \tag{4.8}$$

We define a stopping time $\tau=\inf\{s\geqslant t; |X_s^{t,x}-x|\geqslant c\}\wedge(t+\beta)$, then $t<\tau\leqslant t+\beta$, and the ranges of $v(s\wedge\tau,X_{s\wedge\tau}^{t,x})_{s\in[t,t+\beta]}$, $h(s\wedge\tau,X_{s\wedge\tau}^{t,x})_{s\in[t,t+\beta]}$ and $\phi(s\wedge\tau,X_{s\wedge\tau}^{t,x})_{s\in[t,t+\beta]}$ are all included in a closed subset of D. By (4.3) and (4.8), we have, for each $s\in[t,\tau]$,

$$Y_s^{t,x}=v(s,X_s^{t,x})\geqslant h(s,X_s^{t,x})+C,$$

which implies that for each $s\in[t,\tau]$, $\mathrm{d}K_s^{t,x}=0$. Then by (4.2) and (4.3), we get that the following BSDE:

$$\bar{Y}_s=v(\tau,X_\tau^{t,x})+\int_s^{t+\beta}1_{\{r\leqslant\tau\}}\widetilde{G}(r,\bar{Y}_r,\bar{Z}_r)\mathrm{d}r-\int_s^{t+\beta}\bar{Z}_r\mathrm{d}B_r,\quad s\in[t,t+\beta], \tag{4.9}$$

has a solution $(\bar{Y}_s,\bar{Z}_s)=(Y_{s\wedge\tau}^{t,x},1_{\{s\leqslant\tau\}}Z_s^{t,x})\in\mathcal{S}_D^\infty\times H^2$. Applying Itô's formula to $\phi(s,X_s^{t,x})$ for $s\in[t,\tau]$, we get that the following BSDE:

$$\tilde{Y}_s=\phi(\tau,X_\tau^{t,x})+\int_s^{t+\beta}-1_{\{r\leqslant\tau\}}\left(\frac{\partial\phi}{\partial t}(r,X_r^{t,x})+\mathcal{L}\phi(r,X_r^{t,x})\right)\mathrm{d}r-\int_s^{t+\beta}\tilde{Z}_r\mathrm{d}B_r,\quad s\in[t,t+\beta], \tag{4.10}$$

has a solution $(\tilde{Y}_s,\tilde{Z}_s)=(\phi(s\wedge\tau,X_{s\wedge\tau}^{t,x}),1_{\{s\leqslant\tau\}}\sigma^*\nabla_x\phi(s,X_s^{t,x}))\in\mathcal{S}_D^\infty\times\mathcal{H}^\infty$. By (4.8), we have $v(\tau,X_\tau^{t,x})\leqslant\phi(\tau,X_\tau^{t,x})$ and for each $r\in[t,\tau]$,

$$-\frac{\partial\phi}{\partial t}(r,X_r^{t,x})-\mathcal{L}\phi(r,X_r^{t,x})\geqslant\widetilde{G}(r,v(r,X_r^{t,x}),\sigma^*\nabla_x\phi(r,X_r^{t,x})+C.$$

Then, since $(\tilde{Y}_s,\tilde{Z}_s)\in\mathcal{S}_D^\infty\times\mathcal{H}^\infty$, it follows from that (4.9), (4.10) and Proposition 3.11(i) that $v(t,x)=\bar{Y}_t<\tilde{Y}_t=\phi(t,x)$, which contradicts the condition that $v(t,x)=\phi(t,x)$. Thus, $v(t,x)$ is a viscosity subsolution of (4.1).

The proof is complete. □

4.2 An optimal stopping problem for the payoff of American options

From the definition of u_f, we observe that u_f is a utility function (concave and strictly increasing), when f is nonpositive. More specially, from Example 2.1, we get that

u_f is an exponential utility, when $f(y)=-\beta,\ \beta>0$;

u_f is a power utility, when $f(y)=-\frac{\beta}{y},\ \beta>0,\ \beta\neq\frac{1}{2}$;

u_f is a logarithmic utility, when $f(y)=-\frac{1}{2y}$.

From Lemma A.1(iii), we observe that $-2f(x)$ is the Arrow-Pratt risk aversion coefficient of the utility function $u_f(x)$ (see [12, Definition 2.45 and Example 2.46]).

The holder of an American option has the right to exercise the option at any stopping time $\tau\in\mathcal{T}_{0,T}$ according to the expected utility of its payoff. Let f be nonpositive and let the payoff of the American option be described by $\eta_t:=L_t1_{\{t<T\}}+\xi1_{\{t=T\}}$ with $u_f(\eta_t)\in\mathcal{S}^p$. For $\tau\in\mathcal{T}_{0,T}$, the conditional expected utility of η_τ is given by $E[u_f(\eta_\tau)|\mathcal{F}_t]$. Then, by Corollary 3.5 and Remark 3.7, the RBSDE $(f(y)|z|^2,\xi,L_t)$ has a unique solution (y_t,z_t,k_t) such that $u_f(y_t)$ is the maximal conditional expected utility of the payoff η_t, i.e.,

$$u_f(y_t)=\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}}E[u_f(\eta_\tau)|\mathcal{F}_t],\quad t\in[0,T]. \tag{4.11}$$

In particular, we have $y_0=u_f^{-1}(\operatorname{ess\,sup}_{\tau\in\mathcal{T}_{0,T}}E[u_f(\eta_\tau)])$, which is the certainty equivalent value of the payoff of the American option. For $t\in[0,T]$, set $\sigma_t^*:=\inf\{s\geqslant t,y_s=\eta_s\}\wedge T$. Since

$\mathrm{d}k_s = 0$ on $[t, \sigma_t^*]$, by Lemma A.1(ii) and (4.11), we get that σ_t^* is an optimal stopping time such that

$$E[u_f(\eta_{\sigma_t^*})|\mathcal{F}_t] = \operatorname{ess}\sup_{\tau\in\mathcal{T}_{s,T}} E[u_f(\eta_\tau)|\mathcal{F}_t].$$

More generally, we consider the notion of F-evaluation $\mathcal{E}^F_{s,t}[\cdot]$, which is a nonlinear evaluation introduced by [16, Definition 3.1]. For $\sigma \in \mathcal{T}_{0,T}$, let $(\hat{y}_t, \hat{z}_t)$ be the solution to the BSDE $(F, u_f(\eta_\sigma))$ with terminal time σ such that $\hat{y}_t \in \mathcal{S}^p$. Then, we denote the F-evaluation of the utility of η_σ at time t by

$$\mathcal{E}^F_{t,\sigma}[u_f(\eta_\sigma)] := \hat{y}_t, \quad t \in [0,T].$$

By [11, Theorem 5.9], the RBSDE $(F, u_f(\xi), u_f(L_t))$ has a unique solution $(\tilde{y}_t, \tilde{z}_t, \tilde{k}_t)$ such that

$$\tilde{y}_t = \operatorname{ess}\sup_{\tau\in\mathcal{T}_{t,\tau}} \mathcal{E}^F_{t,\tau}[u_f(\eta_\tau)], \quad t \in [0,T].$$

Let us further assume that $e^{\gamma t}(u_f(\eta_t) \vee 0) + \delta t) \in \mathcal{C}_V$ with $u_f(\eta_t) \in \mathcal{S}^p$. By Theorem 3.3 and (3.5), the RBSDE $(G^F_f + f(y)|z|^2, \xi, L_t)$ has a unique solution (y_t, z_t, k_t) such that

$$u_f(y_t) = \tilde{y}_t = \operatorname{ess}\sup_{\tau\in\mathcal{T}_{t,\tau}} \mathcal{E}^F_{t,T}[u_f(\eta_\tau)], \quad t \in [0,T]. \tag{4.12}$$

In particular, we have $y_0 = u_f^{-1}(\operatorname{ess}\sup_{\tau\in\mathcal{T}_{0,T}} \mathcal{E}^F_{0,T}[u_f(\eta_\tau)])$, which is the certainty equivalent value of the payoff of American option under the nonlinear evaluation $\mathcal{E}^F$. For $t \in [0,T]$, set $\sigma_t^* := \inf\{s \geqslant t, y_s = \eta_s\} \wedge T$. Since $\mathrm{d}k_s = 0$ on $[t, \sigma_t^*]$, by Lemma A.1(ii) and (4.12), we get that σ_t^* is an optimal stopping time such that

$$\mathcal{E}^F_{t,\sigma_t^*}[u_f(\eta_{\sigma_t^*})] = \operatorname{ess}\sup_{\tau\in\mathcal{T}_{t,\tau}} \mathcal{E}^F_{t,\tau}[u_f(\eta_\tau)].$$

Appendix A

Let $f \in L_{1,loc}(D)$ be given. Given $\alpha \in D$, we define the following:

$$u_f(x) := \int_\alpha^x \exp\left(2\int_\alpha^y f(z)\mathrm{d}z\right)\mathrm{d}y, \quad x \in D.$$

The following properties for $u_f(x)$ come from [3, Lemma A.1] and [19, Lemma 2.1].

Lemma A.1 *The following properties of $u_f(x)$ hold:*

(i) $u_f(x) \in W^2_{1,loc}(D)$ *, in particular,* $u_f(x) \in C^1(D)$*;*
(ii) $u_f(x)$ *is strictly increasing;*
(iii) $u_f''(x) - 2f(x)u_f'(x) = 0$*, a.e. on* D*;*
(iv) $u_f^{-1}(x) \in W^2_{1,loc}(V)$ *, in particular,* $u_f^{-1}(x) \in C^1(V)$ *and is strictly increasing;*
(v) *If* $l \in L_{1,loc}(D)$ *and* $l(x) \leqslant f(x), a.e.$ *, then for every* $x \in D$*,* $u_l(x) \leqslant u_f(x)$*.*

To deal with discontinuous generators, an Itô-Krylov formula was established in [3, Theorem 2.1]. To treat our situation conveniently, we give in Lemma A.2 this Itô-Krylov formula for semimartingales with values in D, using a similar argument.

Lemma A.2 *Let* $Y_t = Y_T + \int_t^T h(s)\mathrm{d}s + A_T - A_t - \int_t^T Z_s\mathrm{d}B_s, t \in [0,T]$*, where* $Y_t \in \mathcal{C}_D$*,* $A_t \in \mathcal{A}$*,* $Z_t \in H^2$ *and* $h(t)$ *is a progressively measurable process satisfying* $\int_0^T |h(t)|\mathrm{d}t < \infty$*. Then the following two statements hold:*

(i) *(Krylov's estimate) Let* $\{O_n\}_{n\geqslant 1}$ *be a sequence of closed intervals such that for each* $n \geqslant 1$, $O_n \subset O_{n+1}$, *and* $Y_0 \in O_1$, $\cup_{n\geqslant 1} O_n = D$. *Let* $\tau_n^1 = \inf\{t \geqslant 0, Y_t \notin O_n\} \wedge T$, $\tau_n^2 = \inf\{t \geqslant 0, (A_t + \int_0^t |h(s)|\mathrm{d}s + \int_0^t |Z_s|^2 \mathrm{d}s) > n\} \wedge T$ *and* $\tau_n = \tau_n^1 \wedge \tau_n^2$. *Then for each nonnegative* $\psi \in L_{1,loc}(D)$, *we have*

$$E\left[\int_0^{T\wedge\tau_n} \psi(Y_t)|Z_t|^2 \mathrm{d}t\right] \leqslant (2n + 2\lambda(O_n)) \int_{O_n} \psi(x)\mathrm{d}x.$$

(ii) *(Itô-Krylov formula) For each* $u \in W^2_{1,loc}(D)$, *we have*

$$u(Y_t) = u(Y_0) + \int_0^t u'(Y_s)\mathrm{d}Y_s + \frac{1}{2}\int_0^t u''(Y_s)|Z_s|^2 \mathrm{d}s, \quad t \in [0,T].$$

Proof It is clear that there exists a sequence of closed intervals $\{O_n\}_{n\geqslant 1}$ such that for any $n \geqslant 1$, $O_n \subset O_{n+1}$, and $Y_0 \in O_1$, $\cup_{n\geqslant 1} O_n = D$. Let $\tau_n^1 = \inf\{t \geqslant 0, Y_t \notin O_n\} \wedge T$, $\tau_n^2 = \inf\{t \geqslant 0, (A_t + \int_0^t |h(s)|\mathrm{d}s + \int_0^t |Z_s|^2 \mathrm{d}s) > n\} \wedge T$ and $\tau_n = \tau_n^1 \wedge \tau_n^2$.

Proof (i): Using Tanaka's formula, for $a \in D$, we have

$$\begin{aligned}(Y_{t\wedge\tau_n} - a)^- = (Y_0 - a)^- &+ \int_0^{t\wedge\tau_n} 1_{\{Y_s \leqslant a\}} h(s)\mathrm{d}s + \int_0^{t\wedge\tau_n} 1_{\{Y_s\leqslant a\}}\mathrm{d}A_s \\ &- \int_0^{t\wedge\tau_n} 1_{\{Y_s\leqslant a\}} Z_s \mathrm{d}B_s + \frac{1}{2} L^a_{t\wedge\tau_n}(Y), \quad t \in [0,T].\end{aligned}$$

Since

$$|(Y_{t\wedge\tau_n} - a)^{-}(Y_0 - a)^-| \leqslant |Y_{t\wedge\tau_n} - Y_0| \leqslant \lambda(O_n),$$

we deduce that for each $a \in D$ and $t \in [0,T]$,

$$E(L^a_{t\wedge\tau_n}(Y)) \leqslant 2n + 2\lambda(O_n).$$

It follows from occupation times formula that for each nonnegative $\psi \in L_{1,loc}(D)$,

$$\begin{aligned}E\left[\int_0^{T\wedge\tau_n} \psi(Y_t)|Z_t|^2 \mathrm{d}t\right] &= E\left[\int_0^{T\wedge\tau_n} \psi(Y_t)\mathrm{d}\left\langle Y\right\rangle, Y\left\langle t \right\rangle\right] \\ &= \int_{O_n} \psi(a) E(L^a_{T\wedge\tau_n}(Y)\mathrm{d}a \\ &\leqslant (2n + 2\lambda(O_n)) \int_{O_n} \psi(a)\mathrm{d}a.\end{aligned}$$

Proof of (ii): For any $u \in W^2_{1,loc}(D)$, using a convolution method, we can find a sequence $\{u_m\}_{m\geqslant 1}$ in $C^2(D)$ such that

(1) u_m converges uniformly to u in O_n;

(2) u'_m converges uniformly to u' in O_n;

(3) u''_m converges to u' in $L^1(O_n)$.

Using Itô's formula, we have

$$u_m(Y_{t\wedge\tau_n}) = u_m(Y_0) + \int_0^{t\wedge\tau_n} u'_m(Y_s)\mathrm{d}Y_s + \frac{1}{2}\int_0^{t\wedge\tau_n} u''_m(Y_s)|Z_s|^2 \mathrm{d}s, \quad t \in [0,T].$$

Then, by (i) and the same arguments as in the proof of [3, Theorem 2.1], we get (ii). □

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