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On limit theorems under the Shilkret integral

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Abstract The Shilkret integral or idempotent expectation is a sublinear functional which is very close to being a sublinear expectation since it satisfies all the required properties but its domain is not a linear space. In this paper, we prove that it admits a law of large numbers which is structurally similar to Peng's LLN for sublinear expectations although significant differences exist. As regards the central limit theorem, the situation is radically different as the $\sqrt{n}$ normalization can lead to a trivial limit and other normalizations are possible for variables with a finite second moment or even bounded.

Keywords Law of large numbers, Possibility measure, Shilkret integral, Sublinear expectation

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1. Introduction

A lot of effort has been devoted to comparing the properties of nonlinear integrals to those of the Lebesgue integral. Often, those integrals were introduced due to theoretical developments (for instance, as early as 1925 Vitali [29] used what is now called the Choquet integral as a technical device in a proof). The practical motivation for using them came from tasks perceived to be outside traditional applications of integration. For example, De Campos and Bolaños [6, p. 23] write: '*Additivity does not seem suitable as a demandable property of set functions in many real situations, due to the lack of additivity in many facets of human reasoning*'.

A nice example comes from the application of the Choquet integral in multicriteria decision theory (see [10]). Unlike in probability theory, the initial space Ω does not contain experiment outcomes but criteria under which a number of alternatives are evaluated. Similarly, a function does not represent a random variable but the scores of an alternative under each criterion. Under this different semantics, additivity is unwarranted. If I were to hire an assistant to type my papers, using the criteria 'Math', 'TeX' and 'English', a person knowing only math and English would be an imperfect but interesting candidate (as she could learn TeX much faster than another candidate could learn math or English) while a person knowing only math or only English would be useless for the task. Thus the measure of {Math} and {English} is close to 0 but that of {Math, English} is much larger. That is due to a positive interaction between those two criteria

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explained by the time needed to master any of them. There is no such thing as interaction in the semantics of probability theory because experiment outcomes are mutually exclusive.

Shilkret [23] conceived an integration theory in which the sum operation is replaced by the maximum. Correspondingly, its measures are not additive but maxitive. The role of the Shilkret integral (under the name 'idempotent expectation') in probabilistic large deviation theory was analyzed by Puhalskii [22]. Some additional fields of application of this integral can be found in [15] (pattern recognition), [19] (image analysis) and [1] (decision support systems).

As will be recalled in Proposition 3.1, the Shilkret integral is subadditive and positively homogeneous (i.e., it is a sublinear functional) as well as monotone and constant-preserving. Therefore it satisfies the four defining properties of a sublinear expectation in the sense of Peng [21], a notion closely related to those of a coherent upper prevision [30] and a coherent risk measure [2]. However it is not translation invariant.

Consistently with the fact that those four properties imply translation invariance in a linear space [21, p. 4], the domain of the Shilkret integral is not linear: it is formed by non-negative functions only. While it could be extended to negative functions (e.g., by decomposing a function into its positive and negative parts), the failure of translation invariance underlines that it would necessarily lose some other of the sublinear expectation properties.

The Shilkret integral is, arguably, as close to being a sublinear expectation as a functional can be, but at the same time the lack of domain linearity means it is not a nonlinear expectation in Peng's sense [21], and the lack of translation invariance means it is not a Varadhan functional in Bell and Bryc's sense [4] or a monetary utility function in Delbaen's sense [7]. Although, it is a fuzzy integral in Mesiar's sense [17]. Thus it is in a very special situation which can cast some light on the position of sublinear expectations among more general functionals. In particular, we will study the behaviour of the Shilkret integral with respect to the law of large numbers and the central limit problem.

The structure of the paper is as follows. Section 2 details the preliminaries which are necessary to follow the paper. These include the basics about possibilistic measures and variables, their distributions and density functions. In Section 3, support results are presented. For instance, we will show that a sequence of variables is (in the terminology of possibility theory) product related and identically distributed if and only if it is i.i.d. in Peng's sense with respect to the Shilkret integral. Section 4 develops several forms of the law of large numbers. Since the assumptions in Peng's LLN for sublinear expectations are not satisfied (that is, the assumptions *on the functionals*, not the variables), the techniques rely on estimates of the level sets of the density function of the sample average instead of the ideas on the various proofs of LLNs for sublinear expectations.

In Section 5, we apply the law of a large numbers to obtain a convergence result for weak L^p quasinorms with respect to possibility measures. Then Section 6 proceeds to a comparison of the LLN with Peng's law. The most striking feature, from the perspective of sublinear expectations, is that the limit maximal distribution is not determined by the Shilkret integral itself but by another functional. Thus the LLN has a family resemblance but is not identical.

In Section 7, it is shown that the situation as regards the central limit theorem may be very different. We show by example that a non-normalized sum of 'i.i.d.' variables can converge to a non-trivial limit (correspondingly, if the sum is normalized by $\sqrt{n}$ it tends to 0). The limit satisfies the property that a sum of n 'i.i.d.' summands is identical in distribution to one summand alone. Note that the paper does not aim to prove a Central Limit Theorem with $\sqrt{n}$ normalization, but only to underline that other normalizations are possible under assumptions (finite variance or even boundedness) which imply the CLT for the ordinary expectation and sublinear expectations. Some final comments close the paper in Section 8.

2. Preliminaries

A *directed set* is a set I endowed with a preorder $\prec$ such that for any $i, j \in I$ there exists some $k \in I$ such that $i \prec k$ and $j \prec k$. A *net* is a generalization of the notion of a sequence, in which elements are indexed by a directed set instead of $\mathbb{N}$. Subnets are the analog of subsequences although some subtleties appear (a subnet of a sequence can fail to be a subsequence, a countable subset of a net can fail to define a subnet). Precisely, a net $\{x_j\}_{j\in J}$ is a *subnet* of a net $\{x_i\}_{i\in I}$ if each x_i can be written as $x_{h(j)}$ for a mapping $h : J \to I$ such that $j \prec j' \Rightarrow h(j) \prec h(j')$ and for each $i \in I$ there holds $i \prec h(j)$ for some $j \in J$.

A net $\{x_i\}_{i\in I} \subseteq \mathbb{R}$ is *convergent* to $x \in \mathbb{R}$ if, for each $\varepsilon > 0$, there exists $i \in I$ such that $i \prec j \Rightarrow |x_j - x| < \varepsilon$. Every subnet of a convergent net converges to the same limit. A net is said to have a property *eventually* if, for some $i \in I$, all x_j such that $i \prec j$ have that property. For example, convergence is the property that $|x_j - x| < \varepsilon$ eventually for each $\varepsilon > 0$.

A net of sets $\{A_i\}_{i\in I}$ is *increasing* (resp. *decreasing*) if $i \prec j$ implies $A_i \subseteq A_j$ (resp. $A_j \subseteq A_i$). Then we write $A_i \nearrow A$ and $A_i \searrow A$ if $A = \bigcup_{i\in I} A_i$ or $A = \bigcap_{i\in I} A_i$, respectively.

The closure of a set A will be denoted cl A. A function $f : \mathbb{R} \to \mathbb{R}$ on a topological space is called *upper semicontinuous* if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $f(x) \geqslant \sup_{|y-x|<\delta} f(y) - \varepsilon$. Equivalently, for each $t \in \mathbb{R}$ the set $\{f \geqslant t\} := \{x \mid f(x) \geqslant t\}$ is closed, or $f(x) \geqslant \limsup_n f(x_n)$ for each convergent sequence $x_n \to x$. An upper semicontinuous function always attains its supremum on any compact set.

A *quasinorm* is a mapping $\|\cdot\| : V \to \mathbb{R}$ on a real linear space V with the usual properties of a norm except that only a relaxed form of the triangle inequality is required. Namely, for all $x, y \in V$,

(i) $\|x\| = 0 \Leftrightarrow x = 0$,
(ii) For all $a \in \mathbb{R}$, $\|ax\| = |a| \cdot \|x\|$,
(iii) For some $c > 0$ (which does not depend on x, y), $\|x + y\| \leqslant c(\|x\| + \|y\|)$.

The L^p spaces, for $0 < p < 1$, are examples of quasinormed spaces.

The *Hausdorff metric* between non-empty compact sets of $\mathbb{R}$ is given by

$$d_H(K, L) = \max\{\max_{x\in K} \min_{y\in L} |x - y|, \max_{y\in L} \min_{x\in K} |x - y|\}.$$

In particular,

$$d_H([a, b], [c, d]) = \max\{|a - c|, |b - d|\}.$$

The smallest value $k \geqslant 0$ for which a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(y)| \leqslant k \cdot |x - y|$ (its *Lipschitz constant*) is denoted $\mathrm{Lip}(f)$. The set

$$\mathrm{BL}_1 = \{f : \mathbb{R} \to [-1, 1] \mid \mathrm{Lip}(f) \leqslant 1\}$$

is the closed unit ball of the space of all bounded Lipschitz functions with the norm $f \mapsto \max\{\|f\|_\infty, \mathrm{Lip}(f)\}$. We also write

$$\mathrm{BL}_1{}^+ = \{f \in \mathrm{BL}_1 \mid f \geqslant 0\}.$$

A *τ-algebra* in a space Ω is a subset $\mathcal{A}$ of the powerset $\mathcal{P}(\Omega)$ which contains the empty set and is closed under complementation and arbitrary unions (this is unrelated to the notion of a

τ-smooth or τ-additive measure; notice instead that it is both a σ-algebra and a topology). A *possibility measure* is a set function $\Pi : \mathcal{A} \to [0,1]$ such that, for every index set I,

(a) $\Pi(\emptyset) = 0$,
(b) $\Pi(\Omega) = 1$,
(c) $\Pi(\bigcup_{i\in I} A_i) = \sup_{i\in I} \Pi(A_i)$.

Puhalskii developed the connections between possibility theory and large deviations in probability theory with the following closely related concept. If Ω is endowed with a topology (maybe different from $\mathcal{A}$ itself), a set function $\mu : \mathcal{A} \to [0,1]$ is an *$\mathcal{F}$-idempotent probability* [22, Definition 1.1.1] if it satisfies the following properties:

(i) $\mu(\emptyset) = 0, \mu(\Omega) = 1$,
(ii) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$,
(iii) $\mu(\bigcup_{i\in I} A_i) = \sup_{i\in I} \mu(A_i)$ for any increasing net $\{A_i\}_{i\in I}$,
(iv) $\mu(\bigcap_{i\in I} A_i) = \inf_{i\in I} \mu(A_i)$ for any decreasing net $\{A_i\}_{i\in I}$ of closed sets.

Let Π be a possibility measure. If $\mathcal{A} = \mathcal{P}(\Omega)$ then $\Pi(A) = \sup_{\omega\in A} \Pi(\{\omega\})$ holds for every A, and Π is determined by the function $\pi : \omega \in \Omega \mapsto \pi(\omega) = \Pi(\{\omega\})$. The set-theoretical problems to extend a probability measure from a σ-algebra to $\mathcal{P}(\Omega)$ (i.e., Ulam's theorem) are not present for possibility measures, and a possibility measure always has a natural extension from $\mathcal{A}$ to $\mathcal{P}(\Omega)$. Accordingly, we assume $\mathcal{A} = \mathcal{P}(\Omega)$ from now on as this will make formulas simpler and more transparent.

A *possibilistic variable* is a mapping $X : \Omega \to \mathbb{R}$ such that $X^{-1}(\{x\}) \in \mathcal{A}$ for each $x \in \mathbb{R}$. Since Ω is endowed with a τ-algebra, it is unnecessary to define variables via preimages of Borel sets.

Under the assumption $\mathcal{A} = \mathcal{P}(\Omega)$, every $X : \Omega \to \mathbb{R}$ is a possibilistic variable and generates its *induced possibility measure* in $\mathcal{P}(\mathbb{R})$ given by

$$\begin{aligned} \Pi_X(A) &= \Pi(X \in A) := \Pi(\{\omega \in \Omega \mid X(\omega) \in A\}) \\ &= \sup_{x\in A} \Pi(X = x) = \sup_{x\in A} \pi_X(x) = \sup_{\omega \mid X(\omega)\in A} \pi(\omega). \end{aligned}$$

In possibility theory, π_X is usually called the *possibility distribution* of X. However, we will reserve the word *distribution* for Π_X, consistently with the ordinary usage in probability theory. It will be important to avoid conflicting uses of that term since also in Peng's approach [21] 'distribution' has a third different meaning. We will call π_X the *density function* of X. A possibilistic variable is also called a *fuzzy variable* [20] or *idempotent variable* [22].

Let $A \subseteq \mathbb{R}$. A possibilistic variable X is *uniformly distributed* with distribution u_A if $\pi_X = I_A$ (the *indicator function* of A). That is, u_A is a possibility measure in $\mathbb{R}$ defined by

$$u_A(B) = \begin{cases} 1, & \text{if } A \cap B \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

A uniformly distributed variable takes on values in A, all of them with possibility 1, whereas values in A^c have possibility 0. In other contexts, u_A receives other names; for instance, in game theory it is called a *unanimity game*. A uniform distribution $u_{[a,b]}$ is linked to the *maximal distribution*

$$f \mapsto \max_{x\in[a,b]} f(x) = \sup_{x\in\mathbb{R}} f(x) u_{[a,b]}(\{x\})$$

in Peng's sense of the term.

When more than one variable is considered, the notions above extend easily. For example, a tuple $(X_1, \dots, X_n) : \Omega \to \mathbb{R}^n$ has a joint distribution $\Pi_{(X_1,\dots,X_n)}$ and a joint density function $\pi_{(X_1,\dots,X_n)}$ so that

$$\Pi_{(X_1,\dots,X_n)}(A) = \sup_{(x_1,\dots,x_n)\in A} \pi_{(X_1,\dots,X_n)}(x_1, \dots, x_n).$$

Variables will be called *identically distributed* if they have the same distribution.

A binary operation $\top : [0,1] \times [0,1] \to [0,1]$ is called a *triangular norm* (usually shortened to *t-norm*) if it has the following properties:

(i) $\top$ is associative and commutative,
(ii) 1 is a neutral element,
(iii) $\top$ is non-decreasing as a bivariate function.

Familiar examples of t-norms are the product and the minimum. Every copula having Lipschitz constant 1 is a t-norm. A t-norm is called *Archimedean* if, for any $\varepsilon > 0$ and $a \in (0,1)$, some iterate $a\top \dots \top a$ is smaller than ε. Provided $\top$ is continuous, it is Archimedean if and only if $a\top a < a$ for each $a \in (0,1)$.

Let $\{X_n\}_n$ be a sequence of possibilistic variables. If, for each finite subset $\{i_1, \dots, i_k\}$ of distinct natural numbers,

$$\Pi\Big(\bigcap_{j=1}^{k} \{X_{i_j} \in A_{i_j}\}\Big) = \Pi(X_{i_1} \in A_{i_1})\top \dots \top\Pi(X_{i_k} \in A_{i_k}),$$

the sequence is called $\top$-*related*. By [27, Lemma 3.2], the identity

$$\pi_{(X_1,\dots,X_n)}(x_1, \dots, x_n) = \pi_{X_1}(x_1)\top \dots \top\pi_{X_n}(x_n), \quad n \in \mathbb{N}, \quad x_1, \dots, x_n \in \mathbb{R} \tag{1}$$

is a sufficient and necessary condition for $\top$-relatedness in the case of a continuous t-norm. If the triangular norm $\top$ is the product, we will call $\{X_n\}_n$ *product related*. In that case, (1) is formally analogous to the independence of continuous random variables via their density functions.

The space $\mathcal{F}_c(\mathbb{R})$ is formed by all $\pi : \mathbb{R} \to [0,1]$ which are upper semicontinuous, normalized and quasiconcave, and have bounded support. That happens if and only if each level set π_α, defined as

$$\pi_\alpha = \{x \in \mathbb{R} \mid \pi(x) \geqslant \alpha\}, \quad \alpha \in (0,1]$$

and

$$\pi_0 = \text{cl}\ \{x \in \mathbb{R} \mid \pi(x) > 0\},$$

is a non-empty compact interval for $\alpha \in [0,1]$.

Let $X \geqslant 0$ be a possibilistic variable. The *Shilkret integral* of X against a possibility measure Π is

$$\text{Sh}[X; \Pi] = \sup_{\omega\in\Omega} X(\omega)\pi(\omega).$$

One can also write this as

$$\text{Sh}[X; \Pi] = \sup_{x\in\mathbb{R}} x \cdot \pi_X(x), \tag{2}$$

which parallels the formula for the expectation of a continuous random variable. Clearly, the value of the integral might be infinite in general but that will not happen under the assumptions in this paper.

The Shilkret integral admits a verbatim extension to possibly negative functions, under which the integral of X always equals that of its positive part X_+.

The *upper modal value* [26, p. 118] of a possibilistic variable X with respect to Π is

$$\mathbb{M}[X;\Pi] = \sup\{x \in \mathbb{R} \mid \pi_X(x_n) \to 1 \text{ for some sequence } x_n \to x\},$$

equivalently

$$\mathbb{M}[X;\Pi] = \sup\{x \in \mathbb{R} \mid \pi(\omega_n) \to 1 \text{ for some sequence } \{\omega_n\}_n \subseteq \Omega \mid X(\omega_n) \to x\}.$$

The *lower modal value* of X with respect to Π is

$$\begin{aligned}\mathcal{M}[X;\Pi] &= -\mathbb{M}[-X;\Pi] = \inf\{x \in \mathbb{R} \mid \pi_X(x_n) \to 1 \text{ for some sequence } x_n \to x\} \\ &= \inf\{x \in \mathbb{R} \mid \pi(\omega_n) \to 1 \text{ for some sequence } \{\omega_n\}_n \subseteq \Omega \mid X(\omega_n) \to x\}.\end{aligned}$$

Under the assumption that $\pi_X \in \mathcal{F}_c(\mathbb{R})$, by [27, Lemma 3.3],

$$[\mathcal{M}[X;\Pi], \mathbb{M}[X;\Pi]] = (\pi_X)_1. \tag{3}$$

We will say that a sequence of variables $\{X_n\}_n$ *converges in distribution to X under the Shilkret integral* if

$$\mathrm{Sh}[f(X_n);\Pi] \to \mathrm{Sh}[f(X);\tilde{\Pi}]$$

for each bounded continuous $f : \mathbb{R} \to [0,\infty)$. Here, X might be defined on a different sample space with a different possibility measure $\tilde{\Pi}$. There is no problem in assuming that each X_n is defined on a different possibility space but the *sample average* $\overline{X}_n = S_n/n$, where S_n denotes the *partial sum* $S_n = \sum_{i=1}^n X_i$, will not make sense unless all X_n are defined on the same space.

3. Support results

This section collects results which will be used in the sequel. For the reader's benefit, we start by collecting some basic properties of the Shilkret integral, all of which follow easily from the definitions.

Proposition 3.1 *Let Π, Π' be possibility measures. Let $X, Y \geqslant 0$ be possibilistic variables and $c \in [0,\infty)$. Then*

(a) $X \leqslant Y \Rightarrow \mathrm{Sh}[X;\Pi] \leqslant \mathrm{Sh}[Y;\Pi]$;
(b) $\mathrm{Sh}[c;\Pi] = c$;
(c) $\mathrm{Sh}[X+Y;\Pi] \leqslant \mathrm{Sh}[X;\Pi] + \mathrm{Sh}[Y;\Pi]$;
(d) $\mathrm{Sh}[cX;\Pi] = c \cdot \mathrm{Sh}[X;\Pi]$;
(e) $\Pi \leqslant \Pi' \Rightarrow \mathrm{Sh}[X;\Pi] \leqslant \mathrm{Sh}[X;\Pi']$;
(f) $\mathrm{Sh}[\max(X,Y);\Pi] = \max\{\mathrm{Sh}[X;\Pi], \mathrm{Sh}[Y;\Pi]\}$.

The following lemma is a reformulation of [27, Lemma 3.1].

Lemma 3.2 *Let $\top$ be a continuous Archimedean triangular norm. Let X be a possibilistic variable such that $\pi_X \in \mathcal{F}_c(\mathbb{R})$. Let $\{X_n\}_n$ be a sequence of $\top$-related possibilistic variables identically distributed as X. Then*

$$\pi_{\overline{X}_n}(x) = \sup_{n^{-1}(x_1+\ldots+x_n)=x} \pi_X(x_1)\top\ldots\top\pi_X(x_n)$$

for each $x \in \mathbb{R}$, and

$$d_H((\pi_{\overline{X}_n})_\alpha, (\pi_X)_1) \to 0$$

for each $\alpha \in (0,1]$*.*

We will also need the following bridge with $\mathcal{F}$-idempotent probabilities in order to benefit from some results in [22].

Lemma 3.3 *Let* Π *be a possibility measure in* $\mathbb{R}$*. If* $\pi \in \mathcal{F}_c(\mathbb{R})$ *then* Π *is an* $\mathcal{F}$*-idempotent probability.*

Proof The first three properties of an $\mathcal{F}$-idempotent probability follow from the definition of a possibility measure (in fact, they are equivalent to it [22, Lemma 1.1.4]). The fourth one will be established if we show that, whenever $\{A_i\}_{i\in I}$ is a net of closed sets with $A_i \searrow A$, one has $\sup_{x\in A_i} \pi(x) \to \sup_{x\in A} \pi(x)$. By the upper semicontinuity of π and the compactness of $A_i \cap \pi_0$, the supremum $\sup_{x\in A_i} \pi(x) = \sup_{x\in A_i\cap\pi_0} \pi(x)$ is attained at some point $x_i \in \pi_0$ for each $i \in I$. By the compactness, the net $\{x_i\}_{i\in I}$ has a convergent subnet whose limit x must be in A (since $\{A_i\}_{i\in I}$ is decreasing). By the upper semicontinuity of π at x, for any fixed $\varepsilon > 0$ there is some $\delta > 0$ such that

$$\pi(x) \geqslant \sup_{|y-x|<\delta} \pi(y) - \varepsilon.$$

But eventually $|x_i - x| < \delta$, whence

$$\sup_{y\in A_i} \pi(y) \geqslant \sup_{y\in A} \pi(y) \geqslant \pi(x) \geqslant \pi(x_i) - \varepsilon = \sup_{y\in A_i} \pi(y) - \varepsilon$$

eventually. That proves

$$\Pi(A_i) = \sup_{y\in A_i} \pi(y) \to \sup_{y\in A} \pi(y) = \Pi(A)$$

as wished. □

In order to compare our LLN to Peng's in Section 6, we will need to understand the relationship between the conditions on $\{X_n\}_n$ in both theorems. Although the Shilkret integral is not a sublinear expectation, it is immediate to adapt Peng's i.i.d. notion [21, Definitions 1.3.1 and 1.3.11] by just considering non-negative functions. Thus, the X_n will be called *Peng identically distributed* if

$$\mathrm{Sh}[f(X_i);\Pi] = \mathrm{Sh}[f(X_j);\Pi]$$

for every bounded Lipschitz function $f : \mathbb{R} \to [0,\infty)$ and $i, j \in \mathbb{N}$. They will be called *Peng independent* if

$$\mathrm{Sh}[f(X_1,\ldots,X_n,X_{n+1});\Pi] = \mathrm{Sh}[\mathrm{Sh}[f(x_1,\ldots,x_n,X_{n+1});\Pi]_{(x_1,\ldots,x_n)=(X_1,\ldots,X_n)};\Pi]$$

for every bounded Lipschitz function $f : \mathbb{R}^{n+1} \to [0,\infty)$ and all $n \in \mathbb{N}$.

Proposition 3.4 *Let* $\{X_n\}_n$ *be a sequence of possibilistic variables such that* $\pi_{X_n} \in \mathcal{F}_c(\mathbb{R})$ *for each* $n \in \mathbb{N}$*. The following conditions are equivalent.*

(1) X_n *are Peng independent and Peng identically distributed (Peng i.i.d.).*
(2) X_n *are product related and identically distributed.*

Proof For simplicity, we will prove the case of two variables X, Y. The general case is analogous.

Implication (1)⇒(2). If X, Y are Peng identically distributed, fix an arbitrary $x \in \mathbb{R}$ and let us show $\pi_X(x) = \pi_Y(x)$. Let f_m be a sequence of tent functions

$$f_m(y) = (1 - m|x-y|) \cdot I_{[x-m^{-1},x+m^{-1}]}.$$

By the assumption, for each $m \in \mathbb{N}$,

$$\sup_{y\in\mathbb{R}} f_m(y)\pi_X(y) = \sup_{y\in\mathbb{R}} f_m(y)\pi_Y(y). \tag{4}$$

Fix $\varepsilon > 0$. By the upper semicontinuity of π_X, there exists some $\delta > 0$ for which $\pi_X(x) \geqslant \sup_{|y-x|<\delta} \pi_X(y) - \varepsilon$. Accordingly, for m large enough,

$$\begin{aligned}\pi_X(x) &\geqslant \sup_{y\in[x-m^{-1},x+m^{-1}]} \pi_X(y) - \varepsilon \geqslant \sup_{y\in[x-m^{-1},x+m^{-1}]} f_m(y)\pi_X(y) - \varepsilon \\ &= \sup_{y\in\mathbb{R}} f_m(y)\pi_X(y) - \varepsilon \geqslant \pi_X(x) - \varepsilon,\end{aligned}$$

where the inequalities $I_{\{x\}} \leqslant f_m \leqslant 1$ have been used. By the arbitrariness of ε,

$$\sup_{y\in\mathbb{R}} f_m(y)\pi_X(y) \to \pi_X(x)$$

as $m \to \infty$ and, analogously,

$$\sup_{y\in\mathbb{R}} f_m(y)\pi_Y(y) \to \pi_Y(x),$$

whence (4) implies $\pi_X = \pi_Y$.

As regards independence, since

$$\mathrm{Sh}[f(X,Y)] = \sup_{x,y\in\mathbb{R}} f(x,y)\pi_{(X,Y)}(x,y)$$

and

$$\mathrm{Sh}[\mathrm{Sh}[f(x,Y);\Pi]_{x=X};\Pi] = \mathrm{Sh}[\sup_{y\in\mathbb{R}} f(X,y)\pi_Y(y);\Pi] = \sup_{x,y\in\mathbb{R}} f(x,y)\pi_X(x)\pi_Y(y),$$

a similar sandwich argument establishes

$$\pi_{(X,Y)}(x,y) = \pi_X(x)\pi_Y(y)$$

for all $x, y \in \mathbb{R}$, whence X, Y are product related.

Implication (2)⇒(1). This implication just follows from the definitions, even if $\pi_{X_n} \notin \mathcal{F}_c(\mathbb{R})$. □

Remark 3.1 *Proposition 3.4 does not hold if the density functions are not upper semicontinuous. Indeed, let A be a non-closed subset of $\mathbb{R}$. Let X, Y be such that $\pi_X = I_A$ and $\pi_Y = I_{\mathrm{cl}\,A}$. Then X, Y are not identically distributed (as $\pi_X(x) \neq \pi_Y(x)$ for $x \in \mathrm{cl}\,A \setminus A$) but they are Peng identically distributed, since*

$$\mathrm{Sh}[f(X);\Pi] = \sup_{x\in A} f(x) = \sup_{x\in \mathrm{cl}\,A} f(x) = \mathrm{Sh}[f(Y);\Pi]$$

for every continuous function f.

4. Laws of large numbers

This section is devoted to proving several forms of the law of large numbers for possibilistic variables, based on the Shilkret integral. Theorem 4.1.(a) below is formally similar to Peng's law of large numbers for sublinear expectations (a discussion of the differences follows in Section 6). Note the convergence for non-continuous test functions. Part (b) restates that result as a convergence in distribution under the Shilkret integral. Part (c) shows a convergence behaviour qualitatively different from the ordinary probabilistic law of large numbers. Finally, part (d) shows that convergence in part (a) is uniform over balls of bounded Lipschitz functions. Unlike part (c), this parallels the probabilistic situation in which there is an equivalence between weak convergence and Kantorovich–Rubinstein convergence.

Theorem 4.1 *Let $\top$ be a continuous Archimedean triangular norm. Let $\{X_n\}_n$ be a sequence of $\top$-related possibilistic variables identically distributed as X. If $\pi_X \in \mathcal{F}_c(\mathbb{R})$ then*

(a) $\mathrm{Sh}[f(\overline{X}_n); \Pi] \to \max_{x\in[\mathcal{M}[X;\Pi],\mathbb{M}[X;\Pi]]} f(x)$ *for each upper semicontinuous function* $f : \mathbb{R} \to [0, \infty)$.

(b) $\overline{X}_n \to Y$ *in distribution under the Shilkret integral for a uniform possibilistic variable Y with density function* $I_{[\mathcal{M}[X;\Pi],\mathbb{M}[X;\Pi]]}$.

(c) $\Pi(\overline{X}_n \in A) \to u_{[\mathcal{M}[X;\Pi],\mathbb{M}[X;\Pi]]}(A)$ *for every closed set A, and* $\Pi(\overline{X}_n \in B) = 1$ *for every set B intersecting* $[\mathcal{M}[X;\Pi], \mathbb{M}[X;\Pi]]$.

(d) $\sup_{f\in \mathrm{BL}_1{}^+} \left| \mathrm{Sh}[f(\overline{X}_n); \Pi] - \max_{x\in[\mathcal{M}[X;\Pi],\mathbb{M}[X;\Pi]]} f(x)\right| \to 0.$

Proof Observe

$$\mathrm{Sh}[f(\overline{X}_n); \Pi] = \mathrm{Sh}[f; \Pi_{\overline{X}_n}] = \sup_{x\in\mathbb{R}} f(x)\pi_{\overline{X}_n}(x). \tag{5}$$

Also notice

$$[\mathcal{M}[X;\Pi], \mathbb{M}[X;\Pi]] \subseteq (\pi_{\overline{X}_n})_1$$

since, for any $y \in [\mathcal{M}[X;\Pi], \mathbb{M}[X;\Pi]]$, by Lemma 3.2,

$$\pi_{\overline{X}_n}(y) \geqslant \pi_{X_1}(y)\top \ldots \top\pi_{X_n}(y) = 1\top \ldots \top 1 = 1.$$

The second to last identity uses the fact that $[\mathcal{M}[X;\Pi], \mathbb{M}[X;\Pi]] = (\pi_X)_1$ since $\pi_X \in \mathcal{F}_c(\mathbb{R})$.

Fix an arbitrary $\varepsilon > 0$. By Lemma 3.2,

$$(\pi_{\overline{X}_n})_\varepsilon \to [\mathcal{M}[X;\Pi], \mathbb{M}[X;\Pi]]$$

in the Hausdorff metric. For any $\alpha \in (\varepsilon, 1]$, we obtain

$$[\mathcal{M}[X;\Pi], \mathbb{M}[X;\Pi]] \subseteq (\pi_{\overline{X}_n})_1 \subseteq (\pi_{\overline{X}_n})_\alpha \subseteq (\pi_{\overline{X}_n})_\varepsilon \subseteq [\mathcal{M}[X;\Pi] - \varepsilon, \mathbb{M}[X;\Pi] + \varepsilon] \tag{6}$$

for all sufficiently large n. And for any $\alpha \in [0, \varepsilon]$, by Lemma 3.2,

$$\begin{aligned}
(\pi_{\overline{X}_n})_\alpha &= \{x \in \mathbb{R} \mid \sup_{n^{-1}(x_1+\ldots+x_n)=x} \pi_{X_1}(x_1)\top \ldots \top\pi_{X_n}(x_n) \geqslant \alpha\}\\
&\subseteq \{x \in \mathbb{R} \mid \exists x_1, \ldots, x_n \mid n^{-1}(x_1 + \ldots + x_n) = x, \pi_X(x_1) > 0, \ldots, \pi_X(x_n) > 0\}\\
&\subseteq n^{-1}((\pi_X)_0 + \ldots + (\pi_X)_0) = (\pi_X)_0
\end{aligned}$$

by the convexity of $(\pi_X)_0$. Accordingly, still for $\alpha \in [0, \varepsilon]$,

$$[\mathcal{M}[X;\Pi], \mathbb{M}[X;\Pi]] \subseteq (\pi_{\overline{X}_n})_1 \subseteq (\pi_{\overline{X}_n})_\alpha \subseteq (\pi_X)_0. \tag{7}$$

Let $U^{(\varepsilon)} : \mathbb{R} \to [0, 1]$ be given by

$$U^{(\varepsilon)}(x) = \varepsilon I_{[\min(\pi_X)_0-\varepsilon,\max(\pi_X)_0+\varepsilon]}(x) + (1 - \varepsilon)I_{[\mathcal{M}[X;\Pi]-\varepsilon,\mathbb{M}[X;\Pi]+\varepsilon]}(x).$$

This function is in $\mathcal{F}_c(\mathbb{R})$, as can be shown by checking

$$U^{(\varepsilon)}_\alpha = \begin{cases} [\min(\pi_X)_0 - \varepsilon, \max(\pi_X)_0 + \varepsilon], & \alpha \in [0, \varepsilon],\\ [\mathcal{M}[X;\Pi] - \varepsilon, \mathbb{M}[X;\Pi] + \varepsilon], & \alpha \in (\varepsilon, 1], \end{cases} \tag{8}$$

for all $\alpha \in [0, 1]$. From (6), (7), and (8),

$$I_{[[\mathcal{M}[X;\Pi],\mathbb{M}[X;\Pi]]} \leqslant \pi_{\overline{X}_n} \leqslant U^{(\varepsilon)} \tag{9}$$

for all sufficiently large n.

Now let us estimate $\limsup_n \mathrm{Sh}[f(\overline{X}_n);\Pi]$. Recalling (5),

$$\limsup_n \mathrm{Sh}[f(\overline{X}_n);\Pi]) = \limsup_n \sup_{x\in\mathbb{R}} f(x)\pi_{\overline{X}_n}(x) \leqslant \sup_{x\in\mathbb{R}} f(x)U^{(\varepsilon)}(x).$$

From the definition of $U^{(\varepsilon)}$,

$$\sup_{x\in\mathbb{R}} f(x)U^{(\varepsilon)}(x) \leqslant \varepsilon \cdot \sup_{x\in[\min(\pi_X)_0-\varepsilon,\max(\pi_X)_0+\varepsilon]} f(x) + (1-\varepsilon)\cdot \sup_{x\in[\mathcal{M}[X;\Pi]-\varepsilon,\mathbb{M}[X;\Pi]+\varepsilon]} f(x).$$

By the arbitrariness of ε,

$$\limsup_n \mathrm{Sh}[f(\overline{X}_n);\Pi]) \leqslant \inf_{\varepsilon>0}\left(\varepsilon \cdot \sup_{x\in[\min(\pi_X)_0-\varepsilon,\max(\pi_X)_0+\varepsilon]} f(x) + (1-\varepsilon)\cdot \sup_{x\in[\mathcal{M}[X;\Pi]-\varepsilon,\mathbb{M}[X;\Pi]+\varepsilon]} f(x)\right).$$

Consider only the values of ε such that $\varepsilon = 1/k$ for some $k\in\mathbb{N}$ (doing so preserves the direction of the inequality). Then

$$\sup_{x\in[\min(\pi_X)_0-\varepsilon,\max(\pi_X)_0+\varepsilon]} f(x) \leqslant \sup_{x\in[\min(\pi_X)_0-1,\max(\pi_X)_0+1]} f(x) =: M < \infty$$

since an upper semicontinuous function attains its supremum on a compact set. That bounds the term with the ε factor; for the one with the $(1-\varepsilon)$ factor, simply estimate

$$(1-\varepsilon)\cdot \sup_{x\in[\mathcal{M}[X;\Pi]-\varepsilon,\mathbb{M}[X;\Pi]+\varepsilon]} f(x) \leqslant \sup_{x\in[\mathcal{M}[X;\Pi]-\varepsilon,\mathbb{M}[X;\Pi]+\varepsilon]} f(x).$$

For each $k\in\mathbb{N}$, the latter supremum is attained at some $x_k \in [\mathcal{M}[X;\Pi]-\varepsilon,\mathbb{M}[X;\Pi]+\varepsilon]$. Since the sequence $\{x_k\}_k$ is contained in the compact set $[\mathcal{M}[X;\Pi]-1,\mathbb{M}[X;\Pi]+1]$, it has a convergent subsequence

$$x_{k'} \to x_0 \in \bigcap_{k\in\mathbb{N}} [\mathcal{M}[X;\Pi]-k^{-1},\mathbb{M}[X;\Pi]+k^{-1}] = [\mathcal{M}[X;\Pi],\mathbb{M}[X;\Pi]].$$

By the upper semicontinuity of f,

$$\sup_{x\in[\mathcal{M}[X;\Pi],\mathbb{M}[X;\Pi]} f(x) \geqslant f(x_0) \geqslant \limsup_k f(x_{k'}) \geqslant \inf_k f(x_k).$$

Therefore

$$\limsup_n \mathrm{Sh}[f(\overline{X}_n);\Pi]) \leqslant \inf_{k\in\mathbb{N}} \left(k^{-1}\cdot M + f(x_k)\right) \leqslant \sup_{x\in[\mathcal{M}[X;\Pi],\mathbb{M}[X;\Pi]} f(x). \tag{10}$$

Estimating with (5) and (9), we easily obtain

$$\liminf_n \mathrm{Sh}[f(\overline{X}_n);\Pi]) \geqslant \sup_{x\in[\mathcal{M}[X;\Pi],\mathbb{M}[X;\Pi]]} f(x),$$

which together with (10) implies

$$\mathrm{Sh}[f(\overline{X}_n);\Pi] \to \sup_{x\in[\mathcal{M}[X;\Pi],\mathbb{M}[X;\Pi]]} f(x).$$

To prove part (b), just notice that, denoting by $\widetilde{\Pi}$ the possibility measure in the space where Y is defined,

$$\mathrm{Sh}[f(Y);\widetilde{\Pi}] = \mathrm{Sh}[f;\widetilde{\Pi}_Y] = \sup_{x\in\mathbb{R}} f(x) I_{[\mathcal{M}[X;\Pi],\mathbb{M}[X;\Pi]]}(x) = \sup_{x\in[\mathcal{M}[X;\Pi],\mathbb{M}[X;\Pi]]} f(x).$$

Thus $\mathrm{Sh}[f(\overline{X}_n);\Pi] \to \mathrm{Sh}[f(Y);\widetilde{\Pi}]$ for all upper semicontinuous non-negative f, in particular $\overline{X}_n \to Y$ in distribution under the Shilkret integral.

To prove part (c), apply part (a) with $f = I_A$, since the indicator function of a closed set is upper semicontinuous. Then the sequence is

$$\mathrm{Sh}[I_A(\overline{X}_n);\Pi] = \Pi(\overline{X}_n \in A)$$

and its limit is

$$\max_{x\in[\mathcal{M}[X;\Pi],\mathbb{M}[X;\Pi]]} I_A(x) = \begin{cases} 1, & x \in [\mathcal{M}[X;\Pi],\mathbb{M}[X;\Pi]] \cap A, \\ 0, & \text{otherwise}, \end{cases}$$

which equals $u_{[\mathcal{M}[X;\Pi],\mathbb{M}[X;\Pi]]}(A)$.

For a general set B intersecting $[\mathcal{M}[X;\Pi],\mathbb{M}[X;\Pi]]$ at some point x, one has by Lemma 3.2

$$\Pi(\overline{X}_n \in B) \geqslant \Pi(\overline{X}_n \in \{x\}) \geqslant \pi_{X_1}(x)\top\ldots\top\pi_{X_n}(x) = 1\top\ldots\top 1 = 1.$$

As regards part (d), it follows from (a). Indeed, by results of Puhalskii [22, Theorem 1.9.25 and equivalence (1)$\Longleftrightarrow$(2) in Theorem 1.9.2], (d) is equivalent to the statement in (a) but restricted to continuous bounded nonnegative functions. Those theorems apply because, by Lemma 3.3, $\Pi_{\overline{X}_n}$ and $u_{[\mathcal{M}[X;\Pi],\mathbb{M}[X;\Pi]]}$ are $\mathcal{F}$-idempotent probabilities. □

5. Convergence of weak L^p quasinorms

In this section, we will apply Theorem 4.1 to obtain convergence of closely related functionals which can be described in terms of generalized weak moments (in probabilistic language) or generalized weak L^p quasinorms (in analytic language).

Recall that the *weak moment of order* $p \in (0,\infty)$ of a random variable X in a probability space is the quantity

$$\mathbb{W}_p[X] = \sup_{t>0} t^p \cdot P(|X| > t).$$

Finiteness of $\mathbb{W}_1[X]$ is equivalent to the weak law of large numbers for i.i.d. sequences.

In its turn, the *weak L^p quasinorm* of X is

$$\|X\|_{p,w} = \sup_{t>0} t \cdot P(|X| > t)^{1/p} = \mathbb{W}_p[X]^{1/p}.$$

The space of all X for which $\|X\|_{p,w} < \infty$ is the *weak L^p space* (of the underlying probability space), which is a quasinormed space.

In the case of possibilistic variables, we naturally define

$$\mathbb{W}_p[X;\Pi] = \sup_{t>0} t^p \cdot \Pi(|X| > t)$$

and

$$\|X\|_{p,w,\Pi} = \sup_{t>0} t \cdot \Pi(|X| > t)^{1/p}.$$

Notice $\mathbb{W}_p[X]$ is usually not called a 'weak' moment in possibility theory, see, e.g., [22]. It is easy to obtain the following basic result.

Proposition 5.1 *Let $p \in (0,\infty)$. Let X be a possibilistic variable and Π a possibility measure. Then*

(a) $\Pi^{1/p}$ *is a possibility measure,*

(b) $\pi_X^{1/p}$ *is the density function for X with respect to $\Pi^{1/p}$,*

(c) $\|X\|_{p,w,\Pi} = \|X\|_{1,w,\Pi^{1/p}} = \mathrm{Sh}[|X|;\Pi^{1/p}]$,

(d) $\mathbb{W}_p[X;\Pi] = \mathbb{W}_1[X;\Pi^{1/p}] = \mathrm{Sh}[|X|;\Pi^{1/p}]^{1/p}$.

The functional $\|\cdot\|_{p,w,\Pi}$ defines a quasinorm in the *weak L^p space* $L_{p,w}(\Pi)$ of all variables for which the value of the functional is finite.

Proposition 5.2 *Let $p \in (0,\infty)$. Let Π be a possibility measure. Then, for any $X, Y \in L_{p,w}(\Pi)$,*

$$\|X+Y\|_{p,w,\Pi} \leqslant 2\max\{\|X\|_{p,w,\Pi}, \|Y\|_{p,w,\Pi}\}.$$

Moreover, $\|\cdot\|_{p,w,\Pi}$ is a quasinorm in $L_{p,w}(\Pi)$.

Proof From Proposition 5.1.(c) and the properties of the Shilkret integral (Proposition 3.1),

$$\begin{aligned}\|X+Y\|_{p,w,\Pi} &= \mathrm{Sh}[|X+Y|;\Pi^{1/p}] \leqslant \mathrm{Sh}[|X|+|Y|;\Pi^{1/p}] \\ &\leqslant \mathrm{Sh}[2\max(|X|,|Y|);\Pi^{1/p}] = 2\mathrm{Sh}[\max(|X|,|Y|);\Pi^{1/p}] \\ &= 2\max\{\mathrm{Sh}[|X|;\Pi^{1/p}], \mathrm{Sh}[|Y|;\Pi^{1/p}]\} = 2\max\{\|X\|_{p,w,\Pi}, \|Y\|_{p,w,\Pi}\}.\end{aligned}$$

The proof of the properties of a quasinorm is easy, taking into account that the relaxed triangle inequality

$$\|X+Y\|_{p,w,\Pi} \leqslant 2(\|X\|_{p,w,\Pi} + \|Y\|_{p,w,\Pi})$$

follows from the former inequality. □

Remark 5.1 T*he constant 2 in Proposition 5.2 is optimal, as one easily checks by taking $X = Y = 1$.*

Before proceeding to the main result in this section, we apply a recent metrization theorem of Mitrea.

Proposition 5.3 *Let Π be a possibility measure. Then the topology of $L_{p,w}(\Pi)$ is generated by a uniformly equivalent metric d satisfying*

$$\frac{1}{4}\cdot\|X-Y\|_{p,w,\Pi} \leqslant d(X,Y) \leqslant \|X-Y\|_{p,w,\Pi}.$$

Proof By [18, Theorem 3.26.(12), p. 148], the topology generated in a set S by a mapping $\rho : S\times S \to [0,\infty)$ which satisfies

(i) $\rho(x,y) \leqslant C_1 \cdot \max\{\rho(x,z), \rho(z,y)\}$,

(ii) $\rho(x,y) \leqslant C_0 \cdot \rho(y,x)$,

(iii) $\rho(x,y) = 0 \Leftrightarrow x = y$,

for all $x, y, z \in S$, where $C_0 \geqslant 1$ and $C_1 > 1$, admits a metric d such that

$$C_1^{-2}\cdot\rho(x,y) \leqslant d(x,y)^{\log_2 C_1} \leqslant \max\{1, C_0\}\cdot\rho(x,y).$$

By Proposition 5.2, in $L_{p,w}(\Pi)$ we can take $\rho(X,Y) = \|X-Y\|_{p,w,\Pi}$ with $C_0 = 1$ and $C_1 = 2$, whence the conclusion follows. □

As an application of Theorem 4.1, we obtain the following law of large numbers for weak L^p quasinorms.

Theorem 5.4 *Let $\top$ be a continuous Archimedean triangular norm. Let $\{X_n\}_n$ be a sequence of $\top$-related possibilistic variables identically distributed as X. If $\pi_X \in \mathcal{F}_c(\mathbb{R})$ then*

(a) $\|f(\overline{X}_n)\|_{p,w,\Pi} \to \max_{x\in[\mathcal{M}[X;\Pi],\mathbb{M}[X;\Pi]]} |f(x)|$ *for each function $f:\mathbb{R}\to\mathbb{R}$ whose positive part f_+ and negative part f_- are upper semicontinuous (in particular, every continuous function),*

(b) $\sup_{f\in \mathrm{BL_1}^+} \left| \|f(\overline{X}_n)\|_{p,w,\Pi} - \max_{x\in[\mathcal{M}[X;\Pi],\mathbb{M}[X;\Pi]]} |f(x)| \right| \to 0.$

Proof Since $|f| = f_+ + f_-$, the function $|f|:\mathbb{R}\to[0,\infty)$ is upper semicontinuous. Moreover, since $\pi_X \in \mathcal{F}_c(\mathbb{R})$, also $\pi_X^{1/p} \in \mathcal{F}_c(\mathbb{R})$. By Proposition 5.1.(b), $\pi_X^{1/p}$ is the density function of X with respect to $\Pi^{1/p}$. There follows that X is bounded except for a set of $\Pi^{1/p}$-possibility 0, whence $X \in L_{p,w}(\Pi)$. Applying Theorem 4.1.(a) to $|f|$, X, and $\Pi^{1/p}$, by Proposition 5.1.(c)

$$\|f(\overline{X}_n)\|_{p,w,\Pi} = \mathrm{Sh}[|f|(\overline{X}_n);\Pi^{1/p}] \to \max_{x\in[\mathcal{M}[X;\Pi^{1/p}],\mathbb{M}[X;\Pi^{1/p}]]} |f|(x).$$

To obtain part (a), there only remains to show $\mathcal{M}[X;\Pi^{1/p}] = \mathcal{M}[X;\Pi]$ and $\mathbb{M}[X;\Pi^{1/p}] = \mathbb{M}[X;\Pi]$. But a direct inspection of the definitions of $\mathcal{M}$ and $\mathbb{M}$ reveals that those identities hold since a sequence converges to 1 if and only if the $(1/p)$-th power of its terms converges to 1.

The proof of part (b) follows the same idea. □

6. Comparison to Peng's law of large numbers

We will recall now Peng's LLN for sublinear functionals which generalize the ordinary expectation [21, Theorem 2.4.1, p. 34]. It is very interesting to contrast it with the statement of Theorem 4.1.

Theorem 6.1 (Peng) *Let $\mathcal{H}$ be a linear space of random variables such that*

(i) *The constant function c is in $\mathcal{H}$ for each $c\in\mathbb{R}$,*
(ii) *$X\in\mathcal{H}$ implies $|X|\in\mathcal{H}$,*
(iii) *$f(X_1,\ldots,X_n)\in\mathcal{H}$ for each $n\in\mathbb{N}$ and each bounded Lipschitz function $f:\mathbb{R}^n\to\mathbb{R}$.*

Let $\mathbb{E}:\mathcal{H}\to\mathbb{R}$ be a sublinear expectation, namely

(a) *$X\leqslant Y$ implies $\mathbb{E}[X]\leqslant\mathbb{E}[Y]$,*
(b) *$\mathbb{E}[c]=c$ for all $c\in\mathbb{R}$,*
(c) *$\mathbb{E}[X+Y]\leqslant\mathbb{E}[X]+\mathbb{E}[Y]$ for all $X,Y\in\mathcal{H}$,*
(d) *$\mathbb{E}[cX]=c\mathbb{E}[X]$ for all $X\in\mathcal{H}$ and $c\geqslant 0$.*

Let $\{X_n\}_n$ be a sequence of random variables which are independent and identically distributed (with respect to $\mathbb{E}$) in the sense of Peng ([21, Definitions 1.3.1 and 1.3.11], see Section 2) and such that $\mathbb{E}[(|X_1|-t)_+]\to 0$ as $t\to\infty$. Then

$$\mathbb{E}[f(\overline{X}_n)] \to \max_{x\in[-\mathbb{E}[-X],\mathbb{E}[X]]} f(x)$$

for each bounded Lipschitz function $f:\mathbb{R}\to\mathbb{R}$.

Consider the idea of taking $\mathbb{E}=\mathrm{Sh}[\cdot;\Pi]$, although the Shilkret integral is not a sublinear expectation. By Proposition 3.4, when the triangular norm $\top$ is taken to be the product, a sequence $\{X_n\}_n$ under Theorem 4.1 also satisfies the i.i.d. assumptions of Theorem 6.1. Besides, $\pi_X \in \mathcal{F}_c(\mathbb{R})$ implies $(|X_1| - t)_+ = 0$ for all $t > \max\{|\min(\pi_X)_0|, |\max(\pi_X)_0|\}$, except for a set of possibility 0, whence $\mathbb{E}[(|X_1| - t)_+] \to 0$. Finally, $\mathbb{E}$ satisfies all the assumptions in Theorem 6.1, except that its domain is not a linear space. Notice we regard (b) as being satisfied on the grounds that it holds for every constant function in the domain.

Strictly speaking, the conclusion of Theorem 6.1 cannot be true for the Shilkret integral since $-\mathbb{E}[-X]$ is not defined, but it is easy to rewrite the 'limit interval' in it so as not to involve $-X$. A sublinear expectation can be represented as the supremum of a family of additive functionals [21, Theorem 1.2.1]. Letting $\mathcal{E}$ be the infimum of that family, $[-\mathbb{E}[-X], \mathbb{E}[X]]$ is rewritten as $[\mathcal{E}[X], \mathbb{E}[X]]$. Thus the appearance of $-X$ in the statement is just a superficial difficulty.

Let us call $[\mathcal{E}[X], \mathbb{E}[X]]$ and $[\mathcal{M}[X;\Pi], \mathbb{M}[X;\Pi]]$ in Theorems 6.1 and 4.1 the *limit intervals*. The formal similarity between Theorem 4.1.(a) and Peng's law is apparent. The qualitative difference is that Peng's has $\mathbb{E}$ both in the expectation functional and in the limit interval, while in Theorem 4.1.(a) the converging sequence is analogous but the limit interval depends on the upper and lower modal values instead of the Shilkret integral itself.

The Shilkret integral is very close to satisfying the assumptions in Theorem 6.1, and it is not so surprising that a formally similar convergence holds. But remarkably, the match between the expectation functional and the limit interval is broken. Even so, convergence can be rewritten so that both the sequence and its limit involve Shilkret integrals in the form $\mathrm{Sh}[f(\overline{X}_n)] \to \mathrm{Sh}[f(Y)]$ (Theorem 4.1.(b)).

It is also remarkable that $\mathcal{M}[X;\Pi] = -\mathbb{M}[-X;\Pi]$, i.e., the limit intervals $[-\mathbb{M}[-X;\Pi], \mathbb{M}[X;\Pi]]$ and $[-\mathbb{E}[-X], \mathbb{E}[X]]$ are analogous in structure. Actually, $\mathbb{M}$ *is* a sublinear expectation [28, Proposition 3.3]. That raises the following open questions.

(1) Does Theorem 4.1 admit an abstract version valid for all functionals satisfying a list of axioms?

(2) Is it possible to weaken the axiomatic assumptions in Peng's law in a way that covers Theorem 4.1 or unifies Theorem 6.1 with a hypothetical LLN as in Question 1?

(3) Under what conditions is the limit interval determined by a sublinear expectation?

(4) If the LLN for a functional Φ has a limit interval determined by another functional Ψ, how is Ψ derived from Φ ? (Trivially, $\Psi = \Phi$ in Peng's LLN.)

Question 3 does not have a general positive answer, see [28, Proposition 5.10] where Φ is the possibilistic mean value and $\Psi = (\mathcal{M} + \mathbb{M})/2$ which fails property (c) of a sublinear expectation.

The convergence in Theorem 4.1.(c) is stronger than its probabilistic counterpart (Theorem 6.1 applied to the usual expectation, equivalent to the weak law of large numbers) which is the unilateral bound

$$\limsup_n P(\overline{X}_n \in A) \leqslant I_A(E[X]), \quad \text{for all closed } A$$

in view of the portmanteau lemma [5, Theorem 2.1, p. 16]. That is due to achieving convergence for all upper semicontinuous functions in Theorem 4.1.(a). The full convergence just cannot hold in the probabilistic case as it would imply $P(\overline{X}_n = E[X]) \to 1$.

An analog of Theorem 4.1.(d) holds in the framework of Theorem 6.1 (at least under some additional conditions) due to results of Song [25, Theorem 5.1]. Precisely, it does under conditions (E5) and (H) in that paper, and a finite $(1+\varepsilon)$-th moment (see also [25, Remark 5.2] for the case $\varepsilon \in (0,1)$). While it can be worked out that those three assumptions are verified if the density function is in $\mathcal{F}_c(\mathbb{R})$, Song's theorem does not apply since the Shilkret integral is not a sublinear expectation.

7. On the central limit theorem

The law of large numbers for the Shilkret integral is simultaneously similar and different from Peng's law of large numbers for sublinear expectations. However, the situation for the central limit theorem is radically different. Let us show by example that a non-trivial limit can be achieved with other normalizations than $\sqrt{n}$, while $\sqrt{n}$ may yield trivial limits.

Recall that a *G-normally distributed* variable in Peng's framework is identified by the property that $aX + bY$ is identically distributed as $\sqrt{a^2 + b^2}X$ for any Peng independent copies X, Y and $a, b \geqslant 0$. Namely, the appropriate generalization of centered normal distributions for the CLT is achieved via the property of *strict 2-stability* (satisfied only by centered normal distributions in the probabilistic case).

Proposition 7.1 *There exists a bounded possibilistic variable X such that, for any Peng independent (equivalently, product related) sequence $\{X_n\}_n$ identically distributed as X, and any bounded continuous function $f : \mathbb{R} \to [0, \infty)$,*

$$\mathrm{Sh}[f(S_n); \Pi] \to \mathrm{Sh}[f(Y); \tilde{\Pi}]$$

for a non-degenerate variable Y having a finite Shilkret integral, whereas

$$\mathrm{Sh}[f(S_n/\sqrt{n}); \Pi] \to f(0),$$

i.e., $S_n/\sqrt{n} \to 0$ in distribution under the Shilkret integral.

Proof Let π_X be the function $x \mapsto (1 - x)I_{[0,1]}$. Since $\pi_X \in \mathcal{F}_c(\mathbb{R})$, indeed Peng independence and product relatedness are equivalent by Proposition 3.4.

From Lemma 3.2, writing S_n as the average of $nX_1, \ldots, nX_n$,

$$\begin{aligned}
\pi_{S_n}(x) &= \sup_{n^{-1}(x_1+\ldots+x_n)=x} \pi_{nX}(x_1) \ldots \pi_{nX}(x_n) \\
&= \sup_{n^{-1}(x_1+\ldots+x_n)=x} \pi_X(x_1/n) \ldots \pi_X(x_n/n) \\
&= \sup\{(1 - x_1/n) \ldots (1 - x_n/n) \mid x_i \in [0, 1], n^{-1}(x_1 + \ldots + x_n) = x\}.
\end{aligned}$$

With a Lagrange multiplier, the supremum is found to be attained at $x_1 = \ldots = x_n = x$, whence

$$\pi_{S_n}(x) = (1 - x/n)^n \cdot I_{[0,1]}(x/n) = (1 - x/n)^n \cdot I_{[0,n]}(x).$$

The sequence $(1 - x/n)^n$ is increasing, for $n > |x|$, and converges to e^{-x}.

Let Y have density $\pi_Y(x) = e^{-x}I_{[0,\infty)}(x)$. Its Shilkret integral is $\sup_{x\geqslant 0} xe^{-x}$, indeed finite (its value is e^{-1}). Now

$$\mathrm{Sh}[f(S_n); \Pi] = \mathrm{Sh}[f; \Pi_{S_n}] = \sup_{x\in\mathbb{R}} f(x)(1 - x/n)^n I_{[0,n]}(x)$$

while

$$\mathrm{Sh}[f(Y); \tilde{\Pi}] = \sup_{x\in\mathbb{R}} f(x)e^{-x}.$$

Since, for $n \geqslant 2$,

$$f(x)(1 - x/n)^n I_{[0,n]}(x) \nearrow f(x)e^{-x},$$

we have

$$\begin{aligned}
\mathrm{Sh}[f(Y); \tilde{\Pi}] &= \sup_{x\in\mathbb{R}} \sup_n f(x)(1 - x/n)^n I_{[0,n]}(x) = \sup_n \sup_{x\in\mathbb{R}} f(x)(1 - x/n)^n I_{[0,n]}(x) \\
&= \lim_n \sup_{x\in\mathbb{R}} f(x)(1 - x/n)^n I_{[0,n]}(x) = \lim_n \mathrm{Sh}[f(S_n); \Pi].
\end{aligned}$$

Since f, π_{S_n}, π_Y are bounded, all Shilkret integrals are finite.

On the other hand, one analogously calculates

$$\pi_{S_n/\sqrt{n}}(x) = (1 - x/\sqrt{n})^n \cdot I_{[0,\sqrt{n}]}(x),$$

which converges to 0 at rate $e^{-\sqrt{n}}$ except at $x = 0$ where it has the value 1. One can check then, using the continuity at 0,

$$\mathrm{Sh}[f(S_n/\sqrt{n})] \to f(0).$$

$\square$

While $S_n/\sqrt{n}$ *does* converge to a G-normally distributed variable, it is a *trivial* one, and the sequence S_n with no normalization converges to a non-trivial distribution.

Notice Y has a finite second moment $\mathrm{Sh}[Y^2; \tilde{\Pi}] = 4e^{-2}$ (and X is even bounded), whence a 'finite variance' condition is not equivalent anymore to being in the domain of attraction of a 2-stable distribution.

Considering the more general definition of *strict α-stability* for $\alpha > 0$, i.e., substituting $(a^\alpha + b^\alpha)^{1/\alpha}$ for $\sqrt{a^2 + b^2}$ in the definition of a G-normal distribution, is still not enough to accomodate this example. Making $\alpha \to \infty$ yields the condition

$$aX + bY \text{ is identically distributed as } \max\{a, b\} \cdot X,$$

which is impossible for non-trivial probability distributions and can appropriately be called *strict ∞-stability* ('max-stability' already refers to a different notion in probability theory). In the definition, X and Y are taken to be product related (or $\top$-related if a more general triangular norm were considered). Although Proposition 3.4 has not been established for unbounded distributions, recall that the implication 'product related and identically distributed $\Rightarrow$ Peng i.i.d. with respect to the Shilkret integral' follows directly from the definitions. Hence the following proposition includes as well the variant of ∞-stability in whose definition Peng independence replaces product relatedness.

Proposition 7.2 *Let Y be a possibilistic variable with density function $\pi_Y(x) = e^{-x} I_{[0,\infty)}(x)$. Then its distribution is strictly ∞-stable.*

Proof Let X be identically distributed as Y and product related. We need to prove $\Pi_{aX+bY} = \Pi_{\max\{a,b\}X}$ where, by symmetry, the case $a \geqslant b$ is enough.

By the product relatedness of X and Y, analogously to Lemma 3.2,

$$\pi_{aX+bY}(z) = \sup_{ax+by=z} \pi_X(x) \cdot \pi_Y(y) = \sup_{ax+by=z} e^{-(x+y)} I_{[0,\infty)}(x) I_{[0,\infty)}(y)$$

which can be written as

$$\sup_{x,y \geqslant 0 | ax+by=z} e^{-(x+y)}$$

with the convention $\sup \emptyset = 0$. On the other hand,

$$\pi_{\max\{a,b\}\cdot X}(z) = \pi_{aX}(z) = \pi_X(z/a) = e^{-z/a} I_{[0,\infty)}(z/a) = e^{-z/a} I_{[0,\infty)}(z).$$

Fix an arbitrary $z \in \mathbb{R}$. Clearly $\pi_{aX+bY}(z) = 0 = \pi_{aX}(z)$ if $z < 0$, so assume $z \geqslant 0$. To prove $\pi_{aX+bY}(z) \geqslant \pi_{aX}(z)$, just take $x = z/a, y = 0$. For the converse, notice $ax + by = z$ implies $y = (z - ax)/b$. Then

$$x + y = x + \frac{z - ax}{b} \geqslant x + \frac{z - ax}{a} = z/a,$$

whence

$$sup_{x,y\geqslant 0|ax+by=z}e^{-(x+y)} \leqslant e^{-z/a},$$

which completes the proof. □

In particular, the sum $Y_1+\ldots+Y_n$ of Peng independent copies of Y is identically distributed as Y_1 alone. Incidentally, π_Y coincides with the (probabilistic) density function of an exponential random variable with mean 1.

It is cautious to ask whether the example could hide that a sequence of variables $\{X_n\}_n$ as specified in Proposition 7.1 does not exist. Indeed, void conclusions can follow from the postulated properties of a class of objects if we omit to check that the class is non-empty. Let us dispel that doubt.

For the reader's benefit, we translate into our terminology a particular case of the possibilistic Daniell–Kolmogorov theorem of Janssen *et al.* [13, Corollary 29].

Proposition 7.3 *Let T be a nonempty set and K a compact space. Let $\{\Pi_S \mid \emptyset \neq S \subseteq T$, S finite$\}$ be a consistent family of possibility measures, i.e., each Π_S is a possibility measure on $\mathcal{P}(K^S)$ and $S \subsetneq S'$ implies*

$$\pi_S(x) = \sup_{\mathrm{Pr}_S(y)=x} \pi_{S'}(y),$$

where Pr_S is the natural projection from $K^{S'}$ onto K^S. Assume moreover that each Π_S satisfies

$$\pi_S(x) = \inf_{x\in G \text{ open}} \Pi_S(G).$$

Then there exists some possibility space $(\Omega^*, \mathcal{P}(\Omega^*), \Pi^*)$ and variables $X_t : \Omega^* \to K, t \in T$, for which the joint distribution of $(X_t)_{t\in S}$ is Π_S for every non-empty finite $S \subseteq T$.

Proof Corollary 29 in [13] is used under its condition (C_3), for which the complete chain L is taken to be $[0,1]$. Moreover, $\mathfrak{T}_t$ is the topology of K for each t, the greatest possibilistic extension of Π_S is Π_S itself since it is assumed to be defined on $\mathcal{P}(K^S)$, and the required outer regularity condition in that corollary means exactly that the possibility of each singleton can be outer approximated by open sets.

The possibility measure which is obtained from that result need not be defined on all of $\mathcal{P}(\Omega^*)$ but on a τ-algebra $\mathcal{A}^*$. Since we claimed we would assume $\mathcal{A}^* = \mathcal{P}(\Omega^*)$ in the paper, the proposition is stated that way. An extension from $\mathcal{A}^*$ to $\mathcal{P}(\Omega^*)$ is always possible by defining

$$\pi^*(\omega^*) = \Pi^*(\bigcap_{\omega^*\in A\in\mathcal{A}^*} A)$$

and then setting

$$\Pi^*(A) = \sup_{\omega^*\in A} \pi^*(\omega^*)$$

for each $A \subseteq \Omega^*$. □

That result lets us solve our question with more than sufficient generality.

Corollary 7.4 *Let $\top$ be a continuous triangular norm and let X be a bounded possibilistic variable whose density function is upper semicontinuous. Then there exists a sequence $\{X_n\}_n$ in some possibility space which is $\top$-related and identically distributed as X.*

Proof Let $(\Omega, \mathcal{P}(\Omega), \Pi)$ be the domain of X. Take $T = \mathbb{N}$ and let K be a compact subset of $\mathbb{R}$ containing the range of X. Let Π_S be the possibility distribution determined by the definition of $\top$-relatedness, i.e.

$$\pi_S(x_1, \ldots, x_{\mathrm{card}(S)}) = \pi_X(x_1)\top \ldots \top\pi_X(x_{\mathrm{card}(S)}).$$

Consistency is satisfied. That stems from [27, Lemma 3.2] mentioned at the end of Section 2, because both Π_S and $\Pi_{S'}$ are consistent with $\Pi_{\{1,\ldots,n\}}$ for large enough n.

Since π_X is upper semicontinuous, each π_S is upper semicontinuous as well. Thus for each $x \in K^S$ and $k \in \mathbb{N}$ there exists an open neighbourhood G_k of x such that

$$\pi_S(x) \geqslant \sup_{y\in G_k} \pi_S(y) - k^{-1} = \Pi_S(G_k) - k^{-1} \geqslant \inf_{x\in G \text{ open}} \pi_S(G) - k^{-1} \geqslant \pi_S(x) - k^{-1}.$$

By the arbitrariness of k, we have $\pi_S(x) = \inf_{x\in G \text{ open}} \pi_S(G)$.

By Proposition 7.3, there exist variables X_n such that each finite subset S has joint distribution π_S, which by construction means $\{X_n\}_n$ are $\top$-related (taking $S = \{i_1, \ldots, i_k\}$ in the definition) and identically distributed as X (taking S to be a singleton). □

In particular, the variable X with density function $(1 - x)I_{[0,1]}$ used in Proposition 7.1 satisfies those assumptions, therefore by Lemma 3.4 it actually admits a Peng independent and identically distributed sequence.

8. Concluding remarks

(1) As shown in this paper, the statement of the law of large numbers for the Shilkret integral is similar to that of sublinear expectations but it involves two different functionals. It looks like the differences as regards the weak convergence of suitably normalized sums to a non-degenerate limit, could potentially be vast. A study of the central limit problem is left for further research.

It is interesting to point out that the situation in the Large Deviation Principle could also diverge from those two, namely there could be no difference! Zapata [31] has recently developed some abstract results on large deviations which do not require a probabilistic framework. As an application, he obtained a Gärtner–Ellis theorem for sublinear expectations [31, Theorem 5.4]. In his result, the domain of the sublinear expectations is the set of all bounded measurable *non-negative* functions. That means it covers simultaneously the Shilkret integral and sublinear expectations on the linear space of bounded measurable functions.

(2) It follows from Proposition 3.4 that Peng independence is symmetric in this context, a departure from the situation in sublinear expectation spaces. Indeed, there are possibilistic variables which are Peng independent with respect to the Shilkret integral but do not belong to the three cases of symmetric Peng independence specified by Hu and Li's trichotomy theorem [12]. Corollary 7.4 provides many such examples.

(3) The assumption in Proposition 7.3 that the possibility of each singleton is outer approximated by open sets is absent in the probabilistic Daniell–Kolmogorov theorem because all probability distributions in $\mathbb{R}^S$ are regular for finite S.

(4) While translation invariance is a limited form of linearity that has very often been required for generalized expectations, let us mention that some recent work in risk measures studies the alternative property of excess invariance [24], which is enjoyed by the Shilkret integral.

Excess invariance stems from the position that translation invariance is not always desirable for financial regulators evaluating risk. In some situations, it is argued that if a financial operation does not incur in loss the regulator should have nothing to object. When designing capital tests for financial institutions, a regulator's concern may be to protect liability holders from those who will enjoy the eventual profit of risky actions while having a limited exposure to losses; if there is no loss, liability holders do not need protection. From that perspective, it is claimed that in some situations a risk measure for regulatory purposes should not depend on both the negative and the positive part of a function but only on the part that represents losses. A detailed discussion of this kind of argument can be found in [16]; see also, e.g., [8, 9] for additional critical discussion of translation invariance (cash additivity, in the language of risk measures).

The Shilkret integral has that alternative property since its definition extends verbatim to general functions but the integral of a function equals then that of its positive part, irrespective of the negative part. Accordingly, the mapping $X \mapsto \mathrm{Sh}[-X; \Pi]$ is a shortfall risk measure in the sense of Staum [24]; the change of sign is due to the fact that losses are described as negative profit in the theory of risk measures.

(5) This paper brings closer the work on possibilistic variables and that on sublinear expectations. For instance, product relatedness and identical distribution (in the sense of possibility theory) are shown to be close to Peng's notion of i.i.d., only with respect to a 'sublinear expectation' which is not defined on a linear space. Thus readers with a background on sublinear expectations can give those notions a more familiar interpretation.

In that regard, one may ask whether the Shilkret integral can be replaced by some sublinear expectation in order to achieve the same equivalence, which would make the theory of possibilistic variables fully compatible with that of sublinear expectations.

(6) Concerning Proposition 7.2, it may be observed that the literature of sublinear expectations has considered α-stable distributions but only for the exponents $\alpha \in (1, 2]$ which are possible in traditional Probability Theory with a finite expectation, and $\alpha = 1$ corresponding to maximal distributions (uniform possibility distributions over a compact interval, in our language), see [3, 11, 14, 21].

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