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Optimal investment and consumption under logarithmic utility and uncertainty model

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Abstract We study a robust utility maximization problem in the case of incomplete market and logarithmic utility with general stochastic constraints. Our problem is equivalent to the maximization of nonlinear expected logarithmic utility. We characterize the optimal solution using quadratic backward stochastic differential equations.

Keywords Backward stochastic differential equations, g-expectation, g-martingale, Logarithmic utility, Robust utility

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1. Introduction

Utility maximization is an important problem in financial mathematics. Specifically, it is an optimal investment problem faced by an economic agent who has the potential to invest in a financial market over a fixed investment horizon T. The goal of this agent is to identify an optimal portfolio that allows him to maximize his "welfare" at time T. The founding work of Von Neumann and Morgenstern [1] made it possible to represent the preferences of the investor using a utility function U and a given probability measure $\mathbb{P}$ that reflect his views as follows:

$$\mathbb{E}_{\mathbb{P}}[U(X_T^{x,\pi,c})],$$

where $X_T^{x,\pi,c}$ is the investor wealth at time T starting from an initial wealth x and adopting an investment-consumption strategy (π, c). The investor's problem then requires solving the following optimization problem:

$$\sup_{\pi,c} \mathbb{E}_{\mathbb{P}}[U(X_T^{x,\pi,c})].$$

To solve this type of problem, there are two important approaches: the dual approach [9] and the backward stochastic differential equation (BSDE) approach [5, 7].

In reality, several scenarios are plausible, and it is difficult to precisely identify $\mathbb{P}$. Therefore, we must take into account this ambiguity in the model, also known as Knightian uncertainty. Knightian uncertainty studies have undergone enormous theoretical and practical developments.

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The work of Maccheroni, Marinacci, and Rustichini [2] led to a new representation of preferences in the presence of model uncertainty:

$$\inf_{Q\in\mathcal{Q}} \mathbb{E}_{\mathbb{P}}[U(X_T^{x,\pi,c})+\gamma(Q)],$$

where $\mathcal{Q}$ is the set of plausible scenarios and $\gamma(Q)$ is the penalty term. In other words, the investor will decide in the worst case. Thus, he will solve the following optimization problem, known as the robust utility maximization problem:

$$\sup_{\pi,c}\inf_{Q\in\mathcal{Q}} \mathbb{E}_{Q}[U(X_T^{x,\pi,c})+\gamma(Q)]. \tag{1.1}$$

Several developments have been implemented on this subject, either using the duality method [12–14] or stochastic control techniques based on BSDEs [3, 11]. Using a duality technique combined with a PDE approach to the dual problem, Hernández and Schied [10] successfully characterized the optimal value function using an HJB equation. By employing a stochastic control approach, Matoussi et al. [11] and Faidi et al. [3] studied the maximization part of the problem (1.1) in the case of entropic utility and general consistent time penalty, respectively. In such studies, quadratic BSDEs play a crucial role in describing the value function. The aim of this paper is to continue the study conducted by Faidi et al. [3]. Their findings allowed us to return to the study of the utility maximization problem under g-expectation, one of the fundamental examples of nonlinear expectation introduced by Peng [6]. We provide an explicit solution for investors with logarithmic utility in a constrained financial market that may be incomplete. As Cheridito et al. [8], the constraints that we impose on admissible strategies are not necessarily compact or closed. These constraints are weaker than those imposed by Jiang et al. [4] who studied a similar problem but under another class of nonlinear expectation named g^*-expectation. Based on the optimality principle of nonlinear g-martingales, we characterize the value function of our optimization problem using a quadratic BSDE. The results extend the findings of Cheridito et al. [7] in the context of robust utility as well as those of Jiang et al. [4] by allowing for intertemporal consumption. Unlike Jiang et al.'s proof, which is based on a measurable selection theorem, our proof is based on conditional analysis in $\mathbb{R}^n$, developed by Cheridito et al. [8].

The outline of the paper is as follows. Section 2 outlines the problem setting and the necessary notations. Section 3 describes the notion of g-expectation. In section 4, we specify the financial market and the set of admissible strategies. In section 5, we characterize the optimal investment-consumption strategy by using a BSDE. Finally, for illustration, we give an example corresponding to the case of entropic penalty.

2. Formulation Problem

We consider a filtered probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ over a finite horizon time T, where the filtration $\mathbb{F}=(\mathcal{F}_t^W)_{t\in[0,T]}$ is generated by standard d-dimensional Brownian motion $W=(W^1,\dots,W^d)$. To present our problem rigorously, we introduce some process spaces.

- For $(n,k)\in\mathbb{N}\times\mathbb{N}, \mathcal{P}^{n\times k}$ is the space of all predictable processes with values in $\mathbb{R}^{n\times k}$. $\mathcal{P}^{1\times 1}$ will be denoted as simply $\mathcal{P}$.
- For $p\in\mathbb{N}, \mathscr{H}_T^p(\mathbb{R}^m)$ is the set of all $\mathbb{R}^m$-valued stochastic processes Z, which are predictable with respect to $\mathbb{F}$ and satisfy $\mathbb{E}_{\mathbb{P}}\left[\left(\int_0^T |Z_t|^2\,\mathrm{d}t\right)^{\frac{p}{2}}\right]<\infty$.

- $L^{\exp}$ is the space of all $\mathcal{F}_T$-measurable random variable ξ satisfying

$$\mathbb{E}_{\mathbb{P}}\left[\exp\left(\gamma|\xi|\right)\right] < \infty, \quad \text{for all } \gamma > 0.$$

- $D_0^{\exp}$ is the space of all progressively measurable processes $y = (y_t)_{0\leqslant t\leqslant T}$ with

$$\mathbb{E}_{\mathbb{P}}\left[\exp\left(\gamma \operatorname*{ess\,sup}_{0\leqslant t\leqslant T}|y_t|\right)\right] < \infty, \quad \text{for all } \gamma > 0.$$

- $D_1^{\exp}$ denotes the space of all progressively measurable processes $y = (y_t)_{0\leqslant t\leqslant T}$ such that,

$$\mathbb{E}_{\mathbb{P}}\left[\exp\left(\gamma\int_0^T |y_s|\,\mathrm{d}s\right)\right] < \infty, \quad \text{for all } \gamma > 0.$$

According to exponential martingale representation results, it is well known that for every measure $Q \ll \mathbb{P}$ on $\mathcal{F}_T$ there is a predictable process $(\eta_t)_{t\in[0,T]}$ such that $\mathbb{E}_{\mathbb{P}}[\int_0^T \parallel \eta_t \parallel^2 \mathrm{d}t] < +\infty$ Q-a.s and the density process of Q with respect to $\mathbb{P}$ is an RCLL martingale $Z^Q = (Z_t^Q)_{t\in[0,T]}$ given by

$$\forall t \in [0,T], \quad Z_t^Q = \mathcal{E}\left(\int_0^t \eta_u \mathrm{d}W_u\right) \quad Q\text{-a.s.},$$

where $\mathcal{E}(M)_t = \exp\left(M_t - \frac{1}{2}\langle M\rangle_t\right)$ denotes the stochastic exponential of a continuous local martingale M. We introduce a consistent time penalty as follows:

$$\gamma_t(Q) = \mathbb{E}_Q\left[\int_t^T h\left(\eta_s\right)\mathrm{d}s \mid \mathcal{F}_t\right],$$

where $h : \mathbb{R}^d \to [0, +\infty[$ is a convex function such that $h(0) \equiv 0$. We also assume that there are four positive constants $\kappa_1, \kappa_2, \kappa_3$, and κ_4 satisfying

$$\kappa_1\|x\|^2 - \kappa_2 \leqslant h(x) \leqslant \kappa_3\|x\|^2 + \kappa_4.$$

Our optimization problem is written as follows:

$$\sup_{(\pi,c)\in\mathfrak{A}} \inf_{Q^\eta\in\mathcal{Q}} \mathbb{E}_{Q^\eta}\left[\bar{\alpha}U(X_T^{x,\pi,c}) + \alpha\int_0^T e^{-\int_0^s \delta_u \mathrm{d}u}u(c_s)\mathrm{d}s + \beta\int_0^T e^{-\int_0^s \delta_u \mathrm{d}u}h\left(\eta_s\right)\mathrm{d}s\right],$$

where $\mathcal{Q}$ is the space of all probability measures Q^η on $(\Omega, \mathcal{F})$ such that $Q^\eta = \mathcal{E}\left(\int_0^T \eta_u \mathrm{d}W_u\right)\cdot\mathbb{P}$ on $\mathcal{F}_T$ and $\gamma_0(Q^\eta) < +\infty$. U and u are the utility functions, and $\mathfrak{A}$ is the set of admissible strategies, which will be specified later. Our problem then constitutes two optimization subproblems. The first, the infimum, has been studied by Faidi et al. [3]. They have proven under the exponential integrability condition of random variables $U(X_T^{x,\pi,c})$ and $\int_0^T u(c_s)\mathrm{d}s$ that the infimum is reached in a unique probability measure $\mathbb{Q}^*$ that is equivalent to $\mathbb{P}$, and they characterized the value process of the dynamical optimization problem using a quadratic BSDE. More precisely, under the assumptions

(H1) δ is a uniformly bounded process, and
(H2) $U(X_T^{x,\pi,c}) \in L^{\exp}$ and $(u(c_t))_{0\leqslant t\leqslant T} \in D_1^{\exp}$, the process

$$Y_t^{x,\pi,c} = \inf_{Q^\eta\in\mathcal{Q}} \mathbb{E}_{Q^\eta}\left[\bar{\alpha}U(X_T^{x,\pi,c}) + \alpha\int_t^T e^{-\int_t^s \delta_u \mathrm{d}u}u(c_s)\mathrm{d}s + \beta\int_t^T e^{-\int_t^s \delta_u \mathrm{d}u}h\left(\eta_s\right)\mathrm{d}s|\mathcal{F}_t\right]$$

satisfied the following quadratic BSDE:

$$\begin{cases} \mathrm{d}Y_t^{x,\pi,c} = \left(\delta_t Y_t^{x,\pi,c} - \alpha u(c_t) + \beta h^*\left(\frac{1}{\beta} Z_t\right)\right) \mathrm{d}t - Z_t \mathrm{d}W_t, \\ Y_T^{x,\pi,c} = \bar{\alpha}\bar{U}(X_T^{x,\pi,c}), \end{cases} \tag{2.2}$$

where h^* is the Legendre-Fenchel transform of h. Thus, our problem is equivalent to finding

$$V_0(x) = \sup_{(\pi,c)\in\mathfrak{A}} Y_0^{x,\pi,c}. \tag{2.3}$$

To solve this problem, we use the notion of g-expectation introduced by Peng [6]. We then start with the main results of this class of nonlinear expectation.

3. g-Expectation

The concept of g-expectation was introduced in 1997 by Peng [6] for the Lipschitz generator g. This class of nonlinear expectation, closely related to BSDEs, has undergone considerable developments in both theory and applications. In this section, we present the basic notions related to this concept that will be used in the rest of the paper.

Let $\xi \in L^{\exp}$ a random variable and $g : [0,T] \times \Omega \times \mathbb{R}^d \to \mathbb{R}$ satisfy the following assumptions:

(A1) $\forall t \in [0,T], z \mapsto g(t,w,z)$ is a continuous convex (or concave) function.

(A2) There exists a positive constant β together with a progressively measurable nonnegative stochastic process $(\alpha_t)_{0\leqslant t\leqslant T} \in D_1^{\exp}$ such that

$$\forall (t,z) \in \mathbb{R}\times\mathbb{R}^d; \quad |g(t,.,z)| \leqslant \beta|z|^2 + \alpha_t, \quad \mathbb{P}\text{-a.s.}$$

According to Briand and Hu [15], the BSDE

$$Y_t = \xi + \int_t^T g(s,\ Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s \tag{3.4}$$

admits a unique solution $(Y,Z) \in D_0^{\exp} \times \mathscr{H}_T^p(\mathbb{R}^m),\ \forall p > 1$.

In the sequel, the BSDE (3.4) will be noted BSDE (g,ξ).

Remark 3.1 *Note that if (Y,Z) is the solution of the BSDE (g,ξ), then $(-Y,-Z)$ is solution of the BSDE $(\tilde{g},-\xi)$, where $\tilde{g}(t,w,z) = -g(t,w,-z)$. Therefore, the comparison theorem established by Briand and Hu [15] in the case of a convex generator remains valid in the case of a concave generator.*

If, in addition, g satisfies

(A3) $\forall t \in [0,T]; \quad g(t,w,0) = 0,\ \mathbb{P}$-a.s, then Y_0 (resp. Y_t) is called the g-expectation (resp. conditional g-expectation under $\mathcal{F}_t$) of ξ and is denoted by $\mathcal{E}_g[\xi]$ (resp.$\mathcal{E}_g[\xi|\mathcal{F}_t]$).

Definition 3.1 *Let g satisfy* (A1 − A3).

An $\mathbb{F}$-adapted stochastic process $(X_t)_{0\leqslant t\leqslant T}$ such that $X_t \in L^{\exp}$ for all $t \in [0,T]$ is called a g-martingale (resp.submartingale,supermartingale), if

$$\forall\ 0 \leqslant s \leqslant t \leqslant T, \mathcal{E}_g[X_t|\mathcal{F}_s] = (\mathit{resp.} \geqslant, \leqslant) X_s, \quad \mathbb{P}\text{-a.s.}$$

The following result is an immediate consequence of the comparison theorem.

Proposition 3.1 *Let g_1 satisfy* (A1 − A3) *and g_2 satisfy* (A1 − A2) *such that $g_1 \leqslant g_2$ (resp. $g_1 \geqslant g_2$) and $\xi \in L^{\exp}$.*

(1) *The first component of the solution* (Y, Z) *of the BSDE* (g_2, ξ) *is* g_1 *-supermartingale (resp. submartingale).*

(2) *If, in addition,* g_2 *satisfies* A3, *then any* g_2*-supermartingale (resp. submartingale) is* g_2 *-supermartingale (resp. submartingale).*

Proof We only deal with the case where $g_1 \leqslant g_2$; however, the other case is dealt with in a similar way.

(1) Let (Y, Z) be the solution of the BSDE (g_2, ξ). We have

$$\forall 0 \leqslant s \leqslant t \leqslant T; \quad Y_s = Y_t + \int_s^t g_2(u, Z_u)\mathrm{d}s - \int_s^t Z_u \mathrm{d}W_u.$$

For $0 \leqslant s \leqslant t$, let $\tilde{Y}_s = \mathcal{E}_{g_1}[Y_t|\mathcal{F}_s]$; then,

$$\tilde{Y}_s = Y_t + \int_s^t g_1(u, \tilde{Z}_u)\mathrm{d}s - \int_s^t \tilde{Z}_u \mathrm{d}W_u.$$

Using the comparison theorem, we obtain $\mathcal{E}_{g_1}[Y_t|\mathcal{F}_s] = \tilde{Y}_s \leqslant Y_s$, $\mathbb{P}$-a.s.

(2) Let $(X_t)_{\{0\leqslant t\leqslant T\}}$ be a g_2-supermartingale. For $0 \leqslant s \leqslant t$, we denote $Y_{s,t}^{(1)} := \mathcal{E}_{g_1}[X_t|\mathcal{F}_s]$ and $Y_{s,t}^{(2)} := \mathcal{E}_{g_2}[X_t|\mathcal{F}_s]$. By definition, the processes $(Y_{s,t}^{(1)})_{0\leqslant s\leqslant t}$ and $(Y_{s,t}^{(2)})_{0\leqslant s\leqslant t}$ respectively satisfy the following BSDEs:

$$Y_{s,t}^{(1)} = X_t + \int_s^t g_1(u,\ Z_u^{(1)})\mathrm{d}u - \int_s^t Z_u^{(1)} \mathrm{d}W_u,$$

and

$$Y_{s,t}^{(2)} = X_t + \int_s^t g_2(u,\ Z_u^{(2)})\mathrm{d}u - \int_s^t Z_u^{(2)} \mathrm{d}W_u.$$

Using the comparison theorem, we obtain

$$\forall\ 0 \leqslant s \leqslant t, \quad Y_{s,t}^{(1)} \leqslant Y_{s,t}^{(2)}, \quad \mathbb{P}\text{-a.s.}$$

Therefore, $\forall\ 0 \leqslant s \leqslant t$, $\mathcal{E}_{g_1}[X_t|\mathcal{F}_s] \leqslant X_s$, $\mathbb{P}$-a.s.

This completes the proof. □

Lemma 3.1 *Let* Y *satisfy (2.2). Then,*

$$Y_0^{x,\pi,c} = \mathcal{E}_g\left[\bar{\alpha}e^{-\int_0^T \delta_u \mathrm{d}u}\bar{U}(X_T^{x,\pi,c}) + \int_0^T \alpha e^{-\int_0^s \delta_u \mathrm{d}u}u(c_s)\mathrm{d}s\right],$$

where

$$g(w, t, z) = -\beta e^{-\int_0^t \delta_u \mathrm{d}u} h^*\left(-\frac{1}{\beta}e^{\int_0^t \delta_u \mathrm{d}u} z\right). \tag{3.5}$$

Proof The stochastic process $L^{x,\pi,c}$ defined by

$$L_t^{x,\pi,c} := e^{-\int_0^t \delta_u \mathrm{d}u} Y_t^{x,\pi,c} + \int_0^t \alpha e^{-\int_0^s \delta_u \mathrm{d}u}u(c_s)\mathrm{d}s$$

satisfies the following BSDE:

$$\begin{cases} \mathrm{d}L_t^{x,\pi,c} = \beta e^{-\int_0^t \delta_u \mathrm{d}u} h^*\left(-\dfrac{1}{\beta}e^{\int_0^t \delta_u \mathrm{d}u} Z_t\right)\mathrm{d}t + Z_t \mathrm{d}W_t, \\ L_T^{x,\pi,c} = \bar{\alpha}e^{-\int_0^T \delta_u \mathrm{d}u}\bar{U}(X_T^{x,\pi,c}) + \displaystyle\int_0^T \alpha e^{-\int_0^s \delta_u \mathrm{d}u}u(c_s)\mathrm{d}s. \end{cases} \tag{3.6}$$

Thus,

$$L_t^{x,\pi,c} = \mathcal{E}_g\Big[\bar{\alpha}e^{-\int_0^T \delta_u \mathrm{d}u}\bar{U}(X_T^{x,\pi,c}) + \int_0^T \alpha e^{-\int_0^s \delta_u \mathrm{d}u}u(c_s)\mathrm{d}s|\mathcal{F}_t\Big].$$

□

Remark 3.2 *The conditions mentioned above are verified by the generator g given by (3.5). Indeed,*

(1) *As $h(0) = 0$ and $h(x) \geqslant 0$ for all $x \in \mathbb{R}$, we have $h^*(0) = \sup_{y \leqslant 0} -h(y) = 0$. Therefore,* $\forall t \in [0,t]; g(t,w,0) = 0$, $\mathbb{P}$-a.s.

(2) *h is supposed to be a continuous and convex function, as is h^*, which implies that g is concave in z.*

(3) *The growth condition on h and the uniform boundedness of δ guarantee the existence of positive constants $\beta_1, \beta_2, \beta_3$, and β_4 such that*

$$\forall (t,w,z) \in [0,T] \times \Omega \times \mathbb{R}^d, \quad -\beta_1\|z\|^2 - \beta_2 \leqslant g(t,w,z) \leqslant -\beta_3\|z\|^2 + \beta_4. \tag{3.7}$$

Thus, our problem reduces to a utility maximization problem under g-expectation. In the sequel, we assume that

$$U(x) = u(x) = \ln(x), \quad \forall x > 0.$$

We will now specify the structure of the financial market and the set of admissible strategies.

4. Market Model

The financial market consists of one bond without interest rate and $m \leqslant d$ stocks. In the case of $m < d$, we face an incomplete market. The pricing of stock i evolves according to the following equation:

$$\frac{\mathrm{d}S_t^i}{S_t^i} = b_t^i \mathrm{d}t + \sigma_t^i \mathrm{d}W_t, \quad i = 1, \ldots, m,$$

where b^i (resp. $\sigma^i = (\sigma_1^i, \ldots, \sigma_d^i)$) is an $\mathbb{R}$-valued (resp. $\mathbb{R}^{1\times d}$-valued) predictable uniformly bounded stochastic process. The volatility matrix σ is the $m \times d$-matrix whose i^{th} line is given by the vector σ^i for i ranging from 1 to m. We assume that:

– $\sigma = \left(\sigma_j^i\right)_{i=1,\ldots,d,j=1,\ldots,m}$ has full rank.

– The matrix $\sigma\sigma^{tr}$ is uniformly elliptic, i.e., there are two constants $0 < \varepsilon < K$, such that $\varepsilon I_m \leqslant \sigma\sigma^{tr} \leqslant K I_m$ $\mathbb{P}$-a.s.

– The predictable $\mathbb{R}^m$-valued process

$$\theta_t = \sigma_t^{tr}\left(\sigma_t\sigma_t^{tr}\right)^{-1} b_t, \quad t \in [0,T]$$

is then also uniformly bounded.

An economic agent investing in the financial market can consume part of his wealth at intermediate times. Let $\pi_t^i, 1 \leqslant i \leqslant d$, be the proportion of the investor's wealth invested in the i^{th} high-risk stock S^i and let c_t be the consumed proportion rate at time t. We assume that (π, c) belong to space

$$\mathscr{H}_T := \left\{(\pi,c) \in \mathcal{P}^{1\times d} \times \mathcal{P} \text{ such that } c_t > 0 \ \mathrm{d}t \times \mathrm{d}\mathbb{P} \text{ a.e. and } \int_0^T (|\pi_t|^2 + c_t)\mathrm{d}t < \infty, \mathbb{P}\text{-a.s.} \right\}.$$

By the self-financing condition, the investor's wealth $X_t^{x,\pi,c}$ at time t, starting from the positive initial capital x, satisfies

$$X_t^{x,\pi,c} = x + \sum_{i=1}^{m} \int_0^t X_t^{x,\pi,c} \pi_s^i \frac{\mathrm{d}S_s^i}{S_s^i} - \int_0^t c_s X_s^{x,\pi,c} \mathrm{d}s,$$

or equivalently

$$\frac{\mathrm{d}X_t^{x,\pi,c}}{X_t^{x,\pi,c}} = \pi_t \sigma_t (\mathrm{d}W_t + \theta_t \mathrm{d}t) - c_t \mathrm{d}t; \quad X_0^{\pi,c} = x.$$

The investor's wealth process can be written as

$$X_t^{x,\pi,c} = x \exp\Big(\int_0^t \pi_s \sigma_s \mathrm{d}W_s + \int_0^t \Big(\pi_s \sigma_s \theta_s - \frac{1}{2} |\pi_s \sigma_s|^2 - c_s \Big) \mathrm{d}s \Big) > 0, \tag{4.8}$$

and the problem (2.3) is equivalent to

$$V_0(x) = \sup_{(\pi,c)\in\mathfrak{A}} \mathcal{E}_g \Big[\bar{\alpha} e^{-\int_0^T \delta_u \mathrm{d}u} \ln(X_T^{x,\pi,c}) + \int_0^T \alpha e^{-\int_0^u \delta_s \mathrm{d}s} \ln(c_u X_u^{x,\pi,c}) \mathrm{d}u \Big]. \tag{4.9}$$

To ensure that

$$(\ln(c_t X_t^{x,\pi,c}))_{t\in[0,T]} \in D_1^{\exp} \ \text{ and } \ \ln(X_T^{x,\pi,c}) \in L^{\exp},$$

we need the following integrability conditions:

$$\|\pi\|^2 + c + |\ln(c)| \in D_1^{\exp} \ \text{ and } \ \Big(\int_0^t \pi_s \sigma_s \mathrm{d}W_s \Big)_{t\in[0,T]} \in D_0^{\exp}.$$

We also recall the following definitions related to conditional analysis. For more details, see Cheridito et al. [7, 8]. A subset A of $\mathcal{P}^{1\times k}$ is sequentially closed if it contains every process a that is the $\nu \otimes \mathbb{P}$-a.e. limit of a sequence $(a^n)_{n\geqslant 1}$ of processes in A. We call it $\mathcal{P}$-stable if it contains $1_B a + 1_{B^c} a'$ for all $a, a' \in A$, and every predictable set $B \subset [0,T] \times \Omega$. We say A is $\mathcal{P}$-convex if it contains $\lambda a + (1-\lambda) a'$ for all $a, a' \in A$, and every process $\lambda \in \mathcal{P}$ with values in $[0,1]$.

Definition 4.1 *The set $\mathfrak{A} = \mathscr{A} \times \mathscr{C}$ of admissible strategies consists of non-empty sequentially closed and $\mathcal{P}$-stable subsets on $\mathcal{P}^{1\times d} \times \mathcal{P}$ such that $\mathscr{A}$ is convex and for all $(\pi, c) \in \mathscr{A} \times \mathscr{C}$,*

$$\|\pi\|^2 + c + |\ln(c)| \in D_1^{\exp} \ \textit{ and } \ \Big(\int_0^t \pi_s \sigma_s \mathrm{d}W_s \Big)_{t\in[0,T]} \in D_0^{\exp}.$$

We conclude this section by providing an example of an admissible strategies set.

Example 4.1 *Consider*

$$\mathscr{A} = \{\pi \in \mathcal{P}^{1\times d}; \forall t \in [0,T]; \ \ \|\pi_t(w)\| \leqslant \vartheta(t) \ \ \mathbb{P}\text{-a.s.}\}$$

and

$$\mathscr{C} = \{c \in \mathcal{P}; \forall t \in [0,T]; \ \ \psi(t) \leqslant c_t(w) \leqslant \varphi(t) \ \ \mathbb{P}\text{-a.s.}\},$$

where $\vartheta, \varphi, \psi : [0,T] \longrightarrow \mathbb{R}_+$ are measurable deterministic functions such that

$$\int_0^T \vartheta^2(s) + \varphi(s) + |\ln(\psi(s))| \mathrm{d}s < \infty.$$

It is obvious that $\mathscr{A} \times \mathscr{C}$ *is sequentially closed;* $\mathcal{P}$*-stable,* $\mathscr{A}$ *is convex; and* $\forall(\pi, c) \in \mathscr{A} \times \mathscr{C}$,

$$\|\pi\|^2 + c + |\ln(c)| \in D_1^{\exp}.$$

Moreover, there is a positive constant λ *such that* $\|\pi_t\sigma_t\| \leqslant \lambda\vartheta(t)$ *for all* $0 \leqslant t \leqslant T$*. Consequently, according to Yong [16], we have for all* $\gamma > 0, \mathbb{E}_{\mathbb{P}}\Big[e^{\gamma \sup_{0\leqslant t\leqslant T} |\int_0^t \pi_s\sigma_s \mathrm{d}W_s|}\Big] < \infty$.

Remark 4.1 *This example shows that the constraints that we consider are less weak than the closeness (or compactness) constraint widely used in related studies, such as those by Imkeller and Hu [5] and Jiang et al. [4].*

5. Optimal Investment and Consumption Strategies

To identify the optimal investment and consumption strategies, we adopt the same approach as Imkeller and Hu [5] and Jiang et al. [4]. Let us first start with the following verification lemma:

Lemma 5.1 *If there exists a family of RCLL-adapted processes* $R^{x,\pi,c}$ *such that*

- $\forall(\pi, c) \in \mathscr{A} \times \mathscr{C}; R_T^{x,\pi,c} := \bar{\alpha}e^{-\int_0^T \delta_u \mathrm{d}u} \ln(X_T^{x,\pi,c}) + \int_0^T \alpha e^{-\int_0^u \delta_s \mathrm{d}s} \ln(c_u X_u^{x,\pi,c})\mathrm{d}u$,
- $R_0^{x,\pi,c} = R_0$ *is constant* $\forall(\pi, c) \in \mathscr{A} \times \mathscr{C}$,
- $\forall(\pi, c) \in \mathscr{A} \times \mathscr{C}; R^{x,\pi,c}$ *is a* g *-supermartingale,*
- *There exists* $(\pi^*, c^*) \in \mathscr{A} \times \mathscr{C}$ *such that* R^{x,π^*,c^*} *is a g-martingale.*

Then, (π^*, c^*) *is an optimal strategy for the problem (2.3).*
If, in addition, R is coherent (i.e., $R_t^{x,\pi,c} = R_t^{x,\tilde{\pi},\tilde{c}}$ *if* $(\pi, c) = (\tilde{\pi}, \tilde{c})$ *on* $[0, t]$*), then*

$$\begin{aligned}R_t^{x,\pi^*,c^*} = \sup_{(\hat{\pi},\hat{c})} \mathcal{E}_g\Big[&\bar{\alpha}e^{-\int_0^T \delta_u \mathrm{d}u} \ln(X_t^{x,\pi^*,c^*}\bar{X}_{t,T}^{(\hat{\pi},\hat{c})}) + \int_0^t \alpha e^{-\int_0^u \delta_s \mathrm{d}s} \ln(c_u X_u^{x,\pi^*,c^*})\mathrm{d}u\\ &+ \int_t^T \alpha e^{-\int_0^u \delta_s \mathrm{d}s} \ln(\hat{c}_u X_t^{x,\pi^*,c^*}\bar{X}_{t,T}^{(\hat{\pi},\hat{c})})\mathrm{d}u|\mathcal{F}_t\Big],\end{aligned}$$

where $\bar{X}_{t,T}^{(\hat{\pi},\hat{c})} = \dfrac{X_T^{(\hat{\pi},\hat{c})}}{X_t^{(\hat{\pi},\hat{c})}}$ *is the increase in wealth if the investor changes their strategy from* (π, c) *to* $(\hat{\pi}, \hat{c})$.

Proof We have

$$\mathcal{E}_g[R_T^{x,\pi,c}] \leqslant R_0^{x,\pi,c} = R_0 = R_0^{x,\pi^*,c^*} = \mathcal{E}_g[R_T^{x,\pi^*,c^*}],$$

which implies that (π^*, c^*) is the optimal strategy.

Given $t \in [0, T]$ and two admissible strategies (π, c) and $(\hat{\pi}, \hat{c})$, we introduce $\tilde{\pi}_u = \pi_u I_{u\leqslant t} + \hat{\pi}_u I_{u>t}$, and $\tilde{c}_u = c_u I_{u\leqslant t} + \hat{c}_u I_{u>t}$. As $\mathscr{A} \times \mathscr{C}$ is $\mathcal{P}$*-stable*, one can see that $(\tilde{\pi}, \tilde{c})$ is also an admissible strategy and $R_t^{x,\tilde{\pi},\tilde{c}} = R_t^{x,\pi,c}$. As $R^{x,\tilde{\pi},\tilde{c}}$ is a g-supermartingale, we have

$$\begin{aligned}R_t^{x,\pi,c} &\geqslant \mathcal{E}_g[\bar{\alpha}e^{-\int_0^T \delta_u \mathrm{d}u}\Big(\ln(X_T^{x,\tilde{\pi},\tilde{c}})\Big) + \int_0^T \alpha e^{-\int_0^u \delta_s \mathrm{d}s} \ln(\tilde{c}_u X_u^{x,\tilde{\pi},\tilde{c}})\mathrm{d}u|\mathcal{F}_t]\\ &= \mathcal{E}_g\Big[\bar{\alpha}e^{-\int_0^T \delta_u \mathrm{d}u}\Big(\ln\Big(X_t^{x,\pi,c}\frac{X_T^{x,\hat{\pi},\hat{c}}}{X_t^{x,\hat{\pi},\hat{c}}}\Big)\Big) + \int_0^t \alpha e^{-\int_0^u \delta_s \mathrm{d}s} \ln(c_u X_u^{x,\pi,c})\mathrm{d}u\\ &\quad + \int_t^T \alpha e^{-\int_0^u \delta_s \mathrm{d}s} \ln\Big(\hat{c}_u X_t^{x,\pi,c}\frac{X_u^{x,\hat{\pi},\hat{c}}}{X_t^{x,\hat{\pi},\hat{c}}}\Big)\mathrm{d}u|\mathcal{F}_t\Big].\end{aligned}$$

Thus,

$$\begin{aligned}R_t^{x,\pi,c} \geqslant \sup_{(\hat\pi,\hat c)} \mathcal{E}_g\Big[&\bar\alpha e^{-\int_0^T \delta_u \mathrm{d}u} \ln\Big(X_t^{x,\pi,c}\frac{X_T^{x,\hat\pi,\hat c}}{X_t^{x,\hat\pi,\hat c}}\Big) + \int_0^t \alpha e^{-\int_0^u \delta_s \mathrm{d}s} \ln(c_u X_u^{x,\pi,c})\mathrm{d}u \\ &+ \int_t^T \alpha e^{-\int_0^u \delta_s \mathrm{d}s} \ln\Big(\hat c_u X_t^{x,\pi,c}\frac{X_u^{x,\hat\pi,\hat c}}{X_t^{x,\hat\pi,\hat c}}\Big)\mathrm{d}u|\mathcal{F}_t\Big].\end{aligned}$$

Furthermore, as for (π^*, c^*), $R_t^{\pi^*}$ is a g-martingale.

$$\begin{aligned}R_t^{x,\pi^*,c^*} &= \mathcal{E}_g\Big[\bar\alpha e^{-\int_0^T \delta_u \mathrm{d}u} \ln\Big(X_t^{x,\pi^*,c^*}\frac{X_T^{x,\pi^*,c^*}}{X_t^{x,\pi^*,c^*}}\Big) + \int_0^t \alpha e^{-\int_0^u \delta_s \mathrm{d}s} \ln(c_u^* X_u^{x,\pi^*,c^*})\mathrm{d}u \\ &\quad + \int_t^T \alpha e^{-\int_0^u \delta_s \mathrm{d}s} \ln\Big(c_u^* X_t^{x,\pi^*,c^*}\frac{X_u^{x,\pi^*,c^*}}{X_t^{x,\pi^*,c^*}}\Big)\mathrm{d}u|\mathcal{F}_t\Big] \\ &\leqslant \sup_{(\hat\pi,\hat c)} \mathcal{E}_g\Big[\bar\alpha e^{-\int_0^T \delta_u \mathrm{d}u} \ln\Big(X_t^{x,\pi^*,c^*}\frac{X_T^{x,\hat\pi,\hat c}}{X_t^{x,\hat\pi,\hat c}}\Big) + \int_0^t \alpha e^{-\int_0^u \delta_s \mathrm{d}s} \ln(c_u^* X_u^{x,\pi^*,c^*})\mathrm{d}u \\ &\quad + \int_t^T \alpha e^{-\int_0^u \delta_s \mathrm{d}s} \ln\Big(\hat c_u X_t^{x,\pi^*,c^*}\frac{X_u^{x,\hat\pi,\hat c}}{X_t^{x,\hat\pi,\hat c}}\Big)\mathrm{d}u|\mathcal{F}_t\Big].\end{aligned}$$

Thus

$$\begin{aligned}R_t^{x,\pi^*,c^*} = \sup_{(\hat\pi,\hat c)} \mathcal{E}_g\Big[&\bar\alpha e^{-\int_0^T \delta_u \mathrm{d}u} \ln\Big(X_t^{x,\pi^*,c^*}\frac{X_T^{x,\hat\pi,\hat c}}{X_t^{x,\hat\pi,\hat c}}\Big) + \int_0^t \alpha e^{-\int_0^u \delta_s \mathrm{d}s} \ln(c_u X_u^{x,\pi^*,c^*})\mathrm{d}u \\ &+ \int_t^T \alpha e^{-\int_0^u \delta_s \mathrm{d}s} \ln\Big(\hat c_u X_t^{x,\pi^*,c^*}\frac{X_u^{x,\hat\pi,\hat c}}{X_t^{x,\hat\pi,\hat c}}\Big)\mathrm{d}u|\mathcal{F}_t\Big].\end{aligned}$$

□

In the sequel, we assume that the process δ is deterministic. We define the process ℓ as follows:

$$\ell_t = \bar\alpha e^{-\int_0^T \delta_u \mathrm{d}u} + \alpha \int_t^T e^{-\int_0^s \delta_u \mathrm{d}u}\mathrm{d}s. \tag{5.10}$$

For a process q in $\mathcal{P}^{n\times k}$ and P, a non-empty sequentially closed and $\mathcal{P}$-stable subset of $\mathcal{P}^{n\times k}$, we denote by $\mathrm{dist}(q,P)$ the predictable process

$$\mathrm{dist}(q,P) := \operatorname*{ess\,inf}_{p\in P} \|q-p\|,$$

where ess inf denotes the greatest lower bound with respect to the $\nu\otimes\mathbb{P}$-a.e. order. Cheridito et al. [8] show that there exists a process $p\in P$ satisfying $|q-p| = \mathrm{dist}(q,P)$ and that it is unique (up to $\nu\otimes\mathbb{P}$-a.e. equality) if P is $\mathcal{P}$-convex. We denote the set of these processes by $\Pi_P(q)$.

Lemma 5.2 (1) *$\forall Z\in\mathcal{P}; \arg\min_{\pi\in\mathscr{A}}(\ell\pi\sigma\theta - g(t, Z+\ell\pi\sigma))$ is a non-empty subset of $\mathscr{A}$. Moreover, $\forall(t,w)\in[0,T]\times\Omega$, the functional $z\longmapsto \operatorname*{ess\,inf}_{\pi\in\mathscr{A}}(\ell\pi\sigma\theta - g(t,z+\ell\pi\sigma))$ is convex on $\mathbb{R}^d$, and there exist a positive constant c_1 together with a progressively measurable nonnegative stochastic process $(\rho_t)_{0\leqslant t\leqslant T}\in D_1^{\exp}$ such that*

$$\forall Z\in\mathcal{P}; |\operatorname*{ess\,inf}_{\pi\in\mathscr{A}}(\ell\pi\sigma\theta - g(t,Z+\ell\pi\sigma))| \leqslant c_1\|Z\|^2 + \rho. \tag{5.11}$$

(2) *$\arg\min_{c\in\mathscr{C}}(\ell c - \alpha e^{-\int_0^t \delta_s \mathrm{d}s}\ln(c))$ is a non-empty subset from $\mathscr{C}$ and $\operatorname*{ess\,inf}_{c\in\mathscr{C}}(\ell c - \alpha e^{-\int_0^t \delta_s \mathrm{d}s}\ln(c)) \in D_1^{\exp}$.*

Proof (1) For all $Z \in \mathcal{P}$, the functional $f_Z : \pi \longmapsto \ell\pi\sigma\theta - g(t, Z + \ell\pi\sigma)$ is sequentially continuous stable on $\mathscr{A}$.

Let $(\pi, \tilde{\pi}) \in \mathscr{A}^2$ such that $f_Z(\pi) \leqslant f_Z(\tilde{\pi})$, then inequalities (3.7) yield

$$\beta_3\|Z + \ell\pi\sigma\|^2 - \beta_4 + \ell\pi\sigma\theta \leqslant f_Z(\tilde{\pi}).$$

Using Cauchy–Schwartz inequality, we obtain for all $\epsilon > 0$,

$$\frac{\beta_3 - \epsilon^2}{2}\|\ell\pi\sigma\|^2 \leqslant f_Z(\tilde{\pi}) - \beta_3\|Z\|^2 - \frac{1}{\epsilon^2}\|\theta\|^2 + \beta_4.$$

As ℓ is bounded below, by choosing a sufficiently small ϵ such that $\beta_3 - \epsilon^2 > 0$, there is a positive constant d_1 such that

$$\pi\sigma\sigma^{tr}\pi^{tr} = \|\pi\sigma\|^2 \leqslant d_1(f_Z(\tilde{\pi}) - \beta_3\|Z\|^2 - \frac{1}{\epsilon^2}\|\theta\|^2 + \beta_4).$$

The matrix $\sigma\sigma^{tr}$ is uniformly elliptic, which ensures the existence of a strictly positive constant d_2 such that

$$\|\pi\|^2 \leqslant d_2(f_Z(\tilde{\pi}) - \beta_3\|Z\|^2 - \frac{1}{\epsilon^2}\|\theta\|^2 + \beta_4).$$

Thus, the set $\{\pi \in \mathscr{A}; f_Z(\pi) \leqslant f_Z(\tilde{\pi})\}$ is L^0-bounded. According to Cheridito et al. [8], $\arg\min_{\pi\in\mathscr{A}}(\ell\pi\sigma\theta - g(t, Z + \ell\pi\sigma)) \neq \emptyset$.

To prove the convexity of $z \longmapsto \operatorname{ess\,inf}_{\pi\in\mathscr{A}}(\ell\pi\sigma\theta - g(t, z + \ell\pi\sigma))$, we consider $\lambda \in [0,1]$ and $(z, z') \in \mathbb{R}^d \times \mathbb{R}^d$. For all $(\pi, \pi') \in \mathscr{A} \times \mathscr{A}$, we have

$$\begin{aligned}
&\operatorname*{ess\,inf}_{\pi\in\mathscr{A}}(\ell\pi\sigma\theta - g(t, \lambda z + (1-\lambda)z' + \ell\pi\sigma)) \\
&\leqslant \ell(\lambda\pi + (1-\lambda)\pi')\sigma\theta - g(t, \lambda z + (1-\lambda)z' + \ell(\lambda\pi + (1-\lambda)\pi')\sigma) \\
&\leqslant \lambda(\ell\pi\sigma\theta - g(z + \ell\pi\sigma)) + (1-\lambda)(\ell\pi'\sigma\theta - g(z' + \ell\pi'\sigma)).
\end{aligned}$$

By taking the ess inf over π and π' in the right side term we obtain the desired result. Finally, using inequalities (3.7), we have

$$\operatorname*{ess\,inf}_{\pi\in\mathscr{A}}(\ell\pi\sigma\theta + \beta_3\|Z + \ell\pi\sigma\|^2 - \beta_4) \leqslant \operatorname*{ess\,inf}_{\pi\in\mathscr{A}}(\ell\pi\sigma\theta - g(t, Z + \ell\pi\sigma)) \leqslant \operatorname*{ess\,inf}_{\pi\in\mathscr{A}}(\ell\pi\sigma\theta + \beta_1\|Z + \ell\pi\sigma\|^2 + \beta_2),$$

$$\beta_3\|Z + \ell\pi\sigma\|^2 + \ell\pi\sigma\theta - \beta_4) = \beta_3\|Z + \ell\pi\sigma + \frac{\theta}{2\beta_3}\|^2 - \frac{\|\theta\|^2}{4\beta_3} - Z\theta - \beta_4.$$

Therefore, $\operatorname*{ess\,inf}_{\pi\in\mathscr{A}}(\ell\pi\sigma\theta + \beta_3\|Z + \ell\pi\sigma\|^2 - \beta_4) = \beta_3 \mathrm{dist}^2(Z + \frac{\theta}{2\beta_3}, \ell\mathscr{A}\sigma) - \frac{\|\theta\|^2}{4\beta_3} - Z\theta - \beta_4$, where $\ell\mathscr{A}\sigma = \{\ell\pi\sigma; \pi \in \mathscr{A}\}$. Note that the set $\ell\mathscr{A}\sigma$ is non-empty sequentially closed and $\mathcal{P}$-stable; therefore, $\mathrm{dist}(Z + \frac{\theta}{2\beta_3}, \ell\mathscr{A}\sigma)$ is well defined. A positive constant β_5 is obtained such that

$$\begin{aligned}
|\operatorname*{ess\,inf}_{\pi\in\mathscr{A}}(\ell\pi\sigma\theta + \beta_3\|Z + \ell\pi\sigma\|^2 - \beta_4)| &\leqslant \beta_3\|Z + \ell\bar{\pi}\sigma + \frac{\theta}{2\beta_3}\|^2 + \frac{\|\theta\|^2}{4\beta_3} + |Z\theta| + \beta_4 \\
&\leqslant \beta_5(\|Z\|^2 + \|\bar{\pi}\|^2 + 1),
\end{aligned}$$

where $\bar{\pi}$ is a given element of $\mathscr{A}$. In the same way, there are two positive constants β_6 such that

$$|\operatorname*{ess\,inf}_{\pi\in\mathscr{A}}(\ell\pi\sigma\theta + \beta_1\|Z + \ell\pi\sigma\|^2 + \beta_2)| \leqslant \beta_6(\|Z\|^2 + \|\bar{\pi}\|^2 + 1).$$

The result then follows from the inequality

$$|\underset{\pi\in\mathscr{A}}{\text{ess inf}}(\ell\pi\sigma\theta - g(t, Z+\ell\pi\sigma))| \leqslant |\underset{\pi\in\mathscr{A}}{\text{ess inf}}(\ell\pi\sigma\theta + \beta_3\|Z+\ell\pi\sigma\|^2 - \beta_4)| + |\underset{\pi\in\mathscr{A}}{\text{ess inf}}(\ell\pi\sigma\theta + \beta_1\|Z+\ell\pi\sigma\|^2 + \beta_2)|.$$

(2) The functional $c \longmapsto \ell c - \alpha e^{-\int_0^t \delta_s \mathrm{d}s}\ln(c)$ is sequentially continuous stable on $\mathscr{C}$. In contrast, for all $(t,\omega)\in[0,T]\times\Omega$, the real function $f: x\longmapsto \ell_t x - \alpha e^{-\int_0^t \delta_s \mathrm{d}s}\ln(x)$ is convex on $]0,+\infty[$. Thus, for $a>0$ such that $f'(a)>0$, we have

$$x \leqslant \frac{f(x)-h(a)}{f'(a)} + a \quad \forall x \in]0,+\infty[.$$

This yields that, for any $\tilde{c}\in\mathscr{C}$, the set

$$\{c\in\mathscr{C}; \ell c - \alpha e^{-\int_0^t \delta_s \mathrm{d}s}\ln(c) \leqslant \ell\tilde{c} - \alpha e^{-\int_0^t \delta_s \mathrm{d}s}\ln(\tilde{c})\}$$

is L^0-bounded. Therefore, $\underset{c\in\mathscr{C}}{\arg\min}(\ell c - \alpha e^{-\int_0^t \delta_s \mathrm{d}s}\ln(c)) \neq \emptyset$. Furthermore,

$$\alpha e^{-\int_0^t \delta_u \mathrm{d}u}\ln\left(\frac{e\ell_t}{\alpha e^{-\int_0^t \delta_u \mathrm{d}u}}\right) \leqslant \underset{c\in\mathscr{C}}{\text{ess inf}}(\ell c - \alpha e^{-\int_0^t \delta_s \mathrm{d}s}\ln(c)) \leqslant \ell\bar{c} - \alpha e^{-\int_0^t \delta_s \mathrm{d}s}\ln(\bar{c}),$$

where $\bar{c}$ is a given element of $\mathscr{C}$. The left and right terms of the previous inequality are both in $D_1^{\exp}$, as is $\underset{c\in\mathscr{C}}{\text{ess inf}}(\ell c - \alpha e^{-\int_0^t \delta_s \mathrm{d}s}\ln(c))$.

□

Consider now the BSDE

$$Y_t = \int_t^T f(s, Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s, \tag{5.12}$$

where

$$f(t,z) = -\underset{\pi\in\mathscr{A}}{\text{ess inf}}(\ell_t\pi_t\sigma_t\theta_t - g(t, z+\ell_t\pi_t\sigma_t)) - \underset{c\in\mathscr{C}}{\text{ess inf}}(\ell_t c_t - \alpha e^{-\int_0^t \delta_s \mathrm{d}s}\ln(c_t)) \ \ \mathbb{P}\text{-a.s.} \tag{5.13}$$

Using Lemma 5.2, f satisfies assumptions $(\mathrm{A1}-\mathrm{A2})$. It follows from Briand and Hu [15] that equation (5.12) has a unique solution $(Y,Z)\in D_0^{\exp}\times\mathscr{H}_T^p(\mathbb{R}^m)$, $\forall p>1$.

Theorem 5.1 *The optimal value of the problem (2.3) is given by*

$$V_0(x) = (\bar{\alpha}e^{-\int_0^T \delta_u \mathrm{d}u} + \alpha\int_0^T e^{-\int_0^s \delta_u \mathrm{d}u}\mathrm{d}s)\ln(x) + Y_0,$$

where (Y,Z) is the solution of the following BSDE (5.12). Moreover, (π^*,c^*) is an optimal admissible strategy if and only if

$$c^*\in\underset{c\in\mathscr{C}}{\arg\min}\left(\ell c - \alpha e^{-\int_0^{\cdot} \delta_s \mathrm{d}s}\ln(c)\right) \quad \text{and} \quad \pi^*\in\underset{\pi\in\mathscr{A}}{\arg\min}[\ell\pi\sigma\theta - g(.,Z+\ell\pi\sigma)].$$

Proof The idea consists of constructing a family of processes $R^{x,\pi,c}$ that satisfy the conditions of the Lemma 5.1. We seek $R^{x,\pi,c}$ in the form

$$R_t^{x,\pi,c} := \ell_t\ln(X_t^{x,\pi,c}) + Y_t + \int_0^t \alpha e^{-\int_0^u \delta_s \mathrm{d}s}\ln(c_u X_u^{x,\pi,c})\mathrm{d}u,$$

where the process Y satisfies the quadratic BSDE (5.12) and the function ℓ is given by (5.10). Note that the function ℓ is the unique solution of the ordinary Cauchy problem.

$$\forall 0 \leqslant t \leqslant T; \quad \frac{d\ell_t}{dt} = -\alpha e^{-\int_0^t \delta_u du} \text{ and } \ell_T = \bar{\alpha} e^{-\int_0^T \delta_u du}. \tag{5.14}$$

Evidently, $\forall (\pi, c) \in \mathscr{A} \times \mathscr{C};\ R_T^{x,\pi,c} := \bar{\alpha} e^{-\int_0^T \delta_u du} \ln(X_T^{x,\pi,c}) + \int_0^T \alpha e^{-\int_0^u \delta_s ds} \ln(c_u X_u^{x,\pi,c}) du$, $R_0^{x,\pi,c} = Y_0$ does not depend on (π, c) and $R^{x,\pi,c} \in D_0^{\exp}$. Using the Itô formula, we have

$$\begin{aligned}
dR_t^{x,\pi,c} = & -\alpha e^{-\int_0^t \delta_u du} \ln(X_t^{x,\pi,c}) dt + \ell_t(\pi_t \sigma_t dW_t - \pi_t \sigma_t \theta_t dt - c_t dt) - f(t, Z_t) dt + Z_t dW_t \\
& + \alpha e^{-\int_0^u \delta_s ds} \ln(c_t) dt + \alpha e^{-\int_0^u \delta_s ds} \ln(X_t^{x,\pi,c}) dt \\
= & (-f(t, Z_t) - \ell_t \pi_t \sigma_t \theta_t - \ell_t c_t + \alpha e^{-\int_0^u \delta_s ds} \ln(c_t)) dt + (\ell_t \pi_t \sigma_t + Z_t) dW_t \\
= & (-f(t, Z_t^{x,\pi,c} - \ell_t \pi_t \sigma_t) - \ell_t \pi_t \sigma_t \theta_t - \ell_t c_t + \alpha e^{-\int_0^t \delta_s ds} \ln(c_t)) dt + Z_t^{x,\pi,c} dW_t,
\end{aligned}$$

where $Z_t^{x,\pi,c} := \ell_t \pi_t \sigma_t + Z_t;\ 0 \leqslant t \leqslant T$. Therefore, $R^{x,\pi,c}$ is solution of the BSDE

$$\begin{cases}
dR_t^{x,\pi,c} = -F^{\pi,c}(t, Z_t^{x,\pi,c}) dt + Z_t^{x,\pi,c} dW_t, \\
R_T^{x,\pi,c} := \bar{\alpha} e^{-\int_0^T \delta_u du} \ln(X_T^{x,\pi,c}) + \displaystyle\int_0^T \alpha e^{-\int_0^u \delta_s ds} \ln(c_u X_u^{x,\pi,c}) du,
\end{cases}$$

where

$$F^{\pi,c}(t, z) = f(t, z - \ell_t \pi_t \sigma_t) + \ell_t \pi_t \sigma_t \theta_t + \ell_t c_t - \alpha e^{-\int_0^t \delta_s ds} \ln(c_t).$$

By the definition of f we have for all $t \in [0, +\infty[, z \in \mathbb{R}^d$ and for all $(\pi, c) \in \mathscr{A} \times \mathscr{C}$,

$$f(t, z) \geqslant g(t, z + \ell_t \pi_t \sigma_t) - \ell_t \pi_t \sigma_t \theta_t + \alpha e^{-\int_0^t \delta_s ds} \ln(c_t) - \ell_t c_t; \ \mathbb{P}\text{.a.s.}$$

This implies that for all $t \in [0, +\infty[, z \in \mathbb{R}^d$ and for all $(\pi, c) \in \mathscr{A} \times \mathscr{C}$,

$$F^{\pi,c}(t, z) \geqslant g(t, z) \ \mathbb{P}\text{-a.s.}$$

$F^{\pi,c}$ and g are both concave functions in z, and $F^{\pi,c}$ satisfies $(A1 - A2)$, therefore, based on Proposition 3.1, $R^{x,\pi,c}$ is g-supermartingale for all $(\pi, c) \in \mathscr{A} \times \mathscr{C}$. Moreover, let (Y, Z) be the unique solution of BSDE (5.12), $c^* \in \arg\min_{c \in \mathscr{C}} (\ell c - \alpha e^{-\int_0^t \delta_s ds} \ln(c))$ and $\pi^* \in \arg\min_{\pi \in \mathscr{A}} (\ell \pi \sigma \theta - g(t, Z + \ell \pi \sigma))$, then we have

$$dR_t^{x,\pi^*,c^*} = -g(t, Z_t^{x,\pi^*,c^*}) dt + Z_t^{x,\pi^*,c^*} dW_t.$$

So, the process R^{x,π^*,c^*} is a g-martingale. This completes the proof of the theorem. □

6. Example

In this section, we suppose that $h(x) = \frac{1}{2}\|x\|^2; \forall x \in \mathbb{R}^d$, which matches the entropic penalty case. Then, we have $h^*(x) = \frac{1}{2}\|x\|^2$ and therefore $g(w, t, z) = -\frac{1}{2\beta} e^{\int_0^t \delta_u du} \|z\|^2$. We also assume that the process $(\frac{\alpha e^{-\int_0^t \delta_u du}}{\ell_t})_{t \in [0,T]} \in \mathscr{C}$.

Proposition 6.1 *The generator (5.13) is given by*

$$\begin{aligned}
f(t, z) = & -\frac{1}{2\beta} e^{\int_0^t \delta_u du} \operatorname{dist}^2(z + \beta e^{-\int_0^t \delta_u du} \theta, \ell \mathscr{A} \sigma)_t + z\theta_t \\
& + \frac{\beta}{2} e^{-\int_0^t \delta_u du} \|\theta_t\|^2 + \alpha e^{-\int_0^t \delta_u du} \ln(\frac{\alpha e^{-\int_0^t \delta_u du}}{e \ell_t}),
\end{aligned}$$

and the optimal strategy (π^*, c^*) satisfies $c_t^* = \frac{\alpha e^{-\int_0^t \delta_u du}}{\ell_t}$ and $\ell \pi^* \sigma \in \Pi_{\ell \mathscr{A} \sigma}(Z + \beta e^{-\int_0^{\cdot} \delta_u du} \theta)$.

Proof The function $x \mapsto \ell_t x - \alpha e^{-\int_0^t \delta_s \mathrm{d}s} \ln(x)$ reaches its minimum in $\dfrac{\alpha e^{-\int_0^t \delta_s ds}}{\ell_t} \in \mathscr{C}$, and we have

$$\operatorname*{ess\,inf}_{c\in\mathscr{C}}(\ell_t c_t - \alpha e^{-\int_0^t \delta_s \mathrm{d}s} \ln(c_t)) = \alpha e^{-\int_0^t \delta_s \mathrm{d}s}\Big(1 - \ln\Big(\frac{\alpha e^{-\int_0^t \delta_u \mathrm{d}u}}{\ell_t}\Big)\Big) = \alpha e^{-\int_0^t \delta_s \mathrm{d}s} \ln\Big(\frac{e\ell_t}{\alpha e^{-\int_0^t \delta_u \mathrm{d}u}}\Big).$$

Furthermore, $\forall \pi \in \mathscr{A}$, we have

$$\begin{aligned}
&\ell_t\pi_t\sigma_t\theta_t - g\left(t, z + \ell_t\pi_t\sigma_t\right)\\
&= \frac{1}{2\beta}e^{\int_0^t \delta_u \mathrm{d}u}\|z + \ell_t\pi_t\sigma_t\|^2 + \ell_t\pi_t\sigma_t\theta_t\\
&= \frac{1}{2\beta}e^{\int_0^t \delta_u \mathrm{d}u}\|z + \beta e^{-\int_0^t \delta_u \mathrm{d}u}\theta_t + \ell_t\pi_t\sigma_t\|^2 - z\theta_t - \frac{\beta}{2}e^{-\int_0^t \delta_u \mathrm{d}u}\|\theta_t\|^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\operatorname*{ess\,inf}_{\pi\in\mathscr{A}} \ell_t\pi_t\sigma_t\theta_t - g\left(t, z + \ell_t\pi_t\sigma_t\right))\\
&= \frac{1}{2\beta}e^{\int_0^t \delta_u \mathrm{d}u} \operatorname*{ess\,inf}_{\pi\in\mathscr{A}} \|z + \beta e^{-\int_0^t \delta_u \mathrm{d}u}\theta_t + \ell_t\pi_t\sigma_t\|^2 - z\theta_t - \frac{\beta}{2}e^{-\int_0^t \delta_u \mathrm{d}u}\|\theta_t\|^2\\
&= \frac{1}{2\beta}e^{\int_0^t \delta_u \mathrm{d}u} \operatorname{dist}^2(z + \beta e^{-\int_0^{\cdot} \delta_u \mathrm{d}u}\theta, \ell\mathscr{A}\sigma)_t - z\theta_t - \frac{\beta}{2}e^{-\int_0^t \delta_u \mathrm{d}u}\|\theta_t\|^2.
\end{aligned}$$

□

7. Conclusion

In this paper, we studied the robust utility maximization problem in the logarithmic utility framework and in an incomplete market with general constraint. Using a nonlinear martingale, we characterized the optimal strategies using a quadratic BSDE. Although our study is limited to the case of logarithmic utility, it generalizes the work of Cheridito et al. [7] in the case of robust utility maximization and Jiang et al. [4] in the case of a model with consumption and general constraints.

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