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A maximum principle for robust optimal control problems of quadratic BSDEs

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Abstract The study investigates the necessary maximum principle for robust optimal control problems associated with quadratic backward stochastic differential equations (BSDEs). The system coefficients depend on parameter θ, while the generator of BSDEs exhibits quadratic growth with respect to z. To address the uncertainty present in the model, the variational inequality is derived using weak convergence techniques. Additionally, due to the generator being quadratic with respect to z, the forward adjoint equations are stochastic differential equations with unbounded coefficients, involving mean oscillation martingales. By using the reverse Hölder inequality and John−Nirenberg inequality, we demonstrate that the solutions are continuous with respect to parameter θ. Moreover, the necessary and sufficient conditions for robust optimal control are established using the linearization method.

Keywords Quadratic BSDE, Model uncertainty, Maximum principle, Robust optimal control

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1. Introduction

The stochastic maximum principle, specifically the necessary condition for optimality, is an important approach for studying stochastic optimal control problems. Peng [28] first proved a global maximum principle for classical stochastic control systems in 1990. Since then, this topic has been extensively explored across various stochastic systems. Notable contributions include the work of Fuhrman, Hu, and Tessitore [10] on infinite-dimensional dynamics, Buckdahn, Li, and Ma [4] on mean-field control systems, and Hu, Ji, and Xue [17] on fully coupled forward-backward stochastic control systems. The maximum principle for stochastic recursive optimal control problems has also garnered substantial attention. Peng [29] introduced a local maximum

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principle in 1993 for cases with convex control domains and diffusion coefficients dependent on control. Ji and Zhou [23] investigated a local maximum principle for stochastic optimal control with terminal state constraints. Xu [34] established a stochastic maximum principle for nonconvex control domains where diffusion coefficients are independent of control. Further developments include Hu and Ji's [15] studies of the stochastic maximum principle for stochastic recursive optimal control problems under volatility ambiguity. Notably, Hu [14] resolved Peng's open problem in 2017 by deriving a global stochastic maximum principle.

The motivation for this paper arises from scenarios where model uncertainty exists, and the generator of BSDEs exhibits quadratic growth in z. The study focuses on the control problem within such frameworks. Section 2 introduces two examples to illustrate the relevance of the concerned model. These examples are derived from risk-sensitive control problems and optimal strategy problems for large investors.

Inspired by the two examples in Section 2, this paper examines a robust optimal control problem with quadratic growth. We assume that $\theta \in \Theta$ represents different market conditions, where Θ is defined as a locally compact Polish space with distance Ξ. We also assume that V is a given nonempty convex subset of $\mathbb{R}^k$. The following forward-backward control system is then investigated:

$$
\begin{cases}
X_\theta^v(t) = x + \displaystyle\int_0^t b_\theta(s, X_\theta^v(s), \mathbb{E}[X_\theta^v(s)], v(s))\mathrm{d}s + \int_0^t \sigma_\theta(s, X_\theta^v(s), v(s))\mathrm{d}W(s), \quad t \in [0,T], \\
Y_\theta^v(t) = \Phi_\theta(X_\theta^v(T), \mathbb{E}[X_\theta^v(T)]) + \displaystyle\int_t^T f_\theta(s, X_\theta^v(s), \mathbb{E}[X_\theta^v(s)], Y_\theta^v(s), Z_\theta^v(s), v(s))\mathrm{d}s \\
\qquad\qquad - \displaystyle\int_t^T (Z_\theta^v(s))^\top \mathrm{d}W(s),
\end{cases}
\tag{1.1}
$$

where $b_\theta : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times V \to \mathbb{R}^n, \sigma_\theta : [0,T] \times \mathbb{R}^n \times V \to \mathbb{R}^{n\times d}$, $f_\theta : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times V \to \mathbb{R}$, $\Phi_\theta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. The function f_θ is of quadratic growth with respect to z. The control process $v(\cdot)$ takes values in V. The precise assumptions on $b_\theta, \sigma_\theta, f_\theta, \Phi_\theta$ are detailed in Assumption 1. To obtain a general result, we assume a forward stochastic differential equation (SDE) of mean-field type and consider the following cost functional:

$$
J(v(\cdot)) = \sup_{Q \in \mathcal{Q}} \int_\Theta \mathbb{E}\Big[\phi_\theta(X_\theta^v(T)) + \gamma_\theta(Y_\theta^v(0))\Big] Q(\mathrm{d}\theta).
$$

The assumptions regarding ϕ_θ and γ_θ are presented in Assumption 2. The robust optimal control problem with quadratic growth aims to minimize the cost function $J(v(\cdot))$ over $\mathcal{V}_{ad}$ (see Definition 3.1 and (3.3)). The objective of this study is to provide the necessary and sufficient conditions for achieving optimal control.

The solvability of BSDEs with quadratic growth has been intensively investigated over the past 20 years. For example, in 2000, Kobylanski [25] examined the existence and uniqueness of one-dimensional BSDEs with generator g exhibiting quadratic growth in z and bounded terminal values ξ. Briand and Hu [2, 3] established the existence and uniqueness of one-dimensional BSDEs with quadratic growth even when the terminal values are unbounded. For the multidimensional case, Hu and Tang [21] and Xing and Zitkovic [33] investigated scenarios with bounded terminal values, while Fan, Hu, and Tang [8] tackled the more complex multidimensional case with unbounded terminal values. Further advancements in concerning quadratic BSDEs have been documented in recent works, including those by Fan, Hu, and Tang [9], Hu, Li, and Wen [20], Hu, Wen, and Xiong [22], and Hao, Hu, Tang, and Wen [12], Hao, Wen, and Xiong [13]. Significant progress has also been achieved in stochastic optimal control problems involving quadratic BSDEs. For instance, Lim and Zhou [26] demonstrated their relevance in risk-sensitive control problems by incorporating a risk-sensitive parameter. Other notable applications include exponential utility maximization problems as explored by [6, 19, 27, 30–32].

Since the generator exhibits quadratic growth and the model is uncertain, the present work encounters three primary challenges:

(i) The quadratic growth of f in z leads to variational equations that take the form of linear BSDEs with stochastic unbounded Lipschitz coefficients involving BMO-martingales. The existing results for BSDEs with bounded Lipschitz coefficients (such as those provided by Hu and Wang [18, Lemma 3.2]) are not applicable in this context. To obtain the maximum principle, we prove a new estimate (Proposition 4.5), i.e., for $p \in (1 \vee 2p^{-1}_{\partial_z f_\theta}, 2)$,

$$\lim_{\lambda\to 0}\sup_{\theta\in\Theta}\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}\Gamma_\theta(t)|\delta Y^\lambda_\theta(t)|^p+\left(\int_0^T(\Gamma_\theta(t))^{\frac{2}{p}}|\delta Z^\lambda_\theta(t)|^2\mathrm{d}t\right)^{\frac{p}{2}}\right]=0,$$

where $\Gamma_\theta(t) := \mathcal{E}\left(\int_0^t \partial_z f_\theta(s)^\top \mathrm{d}W(s)\right),\ t\in[0,T]$.

(ii) The forward adjoint equations are linear SDEs with stochastic unbounded coefficients. We establish the wellposedness of such equations, building upon the foundational work of Gal'chuk [11]. The continuity of its solutions to θ, i.e., for $1 < p < (p_{\partial_z f_\theta} \wedge p_{(\partial_z f_{\tilde\theta} - \partial_z f_\theta)})$,

$$\lim_{\epsilon\to 0}\sup_{\Xi(\theta,\tilde\theta)\leqslant\epsilon}\mathbb{E}\left[\sup_{t\in[0,T]}|p_\theta(t)-p_{\tilde\theta}(t)|^p\right]=0$$

is proved by the theory of BMO-martingales (Lemma 5.2), which is necessary to prove that $\Lambda_\theta(\cdot)$ defined by (5.11) is $\mathcal{F}$-progressively measurable.

(iii) Given that we address robust optimal control problems, where the cost functional is defined as a supremum over a set of probability measures, the classical convex variational method proves inadequate. To tackle this, we adopt the weak convergence technique to establish the variational inequality.

Compared to the existing literature, our paper makes the following kehy contributions. First, our model incorporates uncertainty, and generator f exhibits quadratic growth with respect to z. Unlike classical optimal control problems, we address robust optimal control problems under these conditions. A new and nontrivial estimate is established (Proposition 4.5), extending the results of Hu and Wang [18] from the Lipschitz case to the quadratic case. Second, we establish the existence and uniqueness of L^p-solutions for linear SDEs with stochastic unbounded coefficients and prove the continuity of its solutions to θ using the reverse Hölder inequality and John–Nirenberg inequality. In addition, we extend this analysis to linear BSDEs with stochastic unbounded coefficients, proving the continuity of their solutions with respect to θ. Third, considering the uncertainty inherent in the model, we derive the necessary maximum principle using a combination of the linearization technique and weak convergence method. Under certain convexity assumptions, we also prove the sufficient maximum principle by employing Sion's minimax theorem.

The paper is structured as follows. Section 2 presents two examples to illustrate the motivations behind this work. The problem formulation is detailed in Section 3. Section 4 focuses on proving the variational inequality. Section 5 discusses the necessary and sufficient maximum principles.

2. Two examples

This section presents two examples to illustrate the applications of robust optimal control problems for quadratic BSDEs.

Example 2.1 *(Risk-sensitive control) We assume that a market can be broadly classified into two states, A and B (for example, a share market being either a bull market or a bear market). These states are associated with different coefficients in two states. We denote the states by $\theta=1$ and 2, representing state A and state B , respectively, which can be characterized as $\theta\in\Theta=\{1,2\}$. Let $\mathcal{Q}=\{Q^\lambda:\lambda\in[0,1]\}$, where Q^λ is the probability such that $Q^\lambda(\{1\})=\lambda$, $Q^\lambda(\{2\})=1-\lambda$. For simplicity, we only consider the 1-dimensional case. Suppose that there exist N-individual agents in a system. The state process of the i-th agent is described by*

$$\begin{cases}\mathrm{d}X_\theta^{i,v}(t)=b_\theta^i(t,X_\theta^{i,v}(t),\dfrac{1}{N}\displaystyle\sum_{j=1}^N X_\theta^{j,v}(t),v(t))\mathrm{d}t+\sigma_\theta^i(t,X_\theta^{i,v}(t),v(t))\mathrm{d}W^i(t),\\ X_\theta^{i,v}(t)=x_0,\end{cases}\tag{2.1}$$

where $W^i,i=1,2,\cdots,N$ are independent copies of a 1-dimensional standard Brownian motion W; and $v(\cdot)$ denotes the control process. $\mathcal{V}_{ad}$ represents the set of admissible controls (see Definition 3.1). The robust objective functional of the i-th agent is

$$J^i(v(\cdot))=\sup_{\lambda\in[0,1]}\left\{\lambda Y_1^{i,v}+(1-\lambda)Y_2^{i,v}\right\}=\sup_{Q^\lambda\in\mathcal{Q}}\int_\Theta Y_\theta^{i,v}Q^\lambda(\mathrm{d}\theta),\tag{2.2}$$

where $(Y_\theta^{i,v},Z_\theta^{i,v})$ is the solution of the following BSDE with quadratic generator (see El Karoui and Hamadène [7])

$$\begin{cases}\mathrm{d}Y_\theta^{i,v}(t)=-\dfrac{\kappa}{2}|Z_\theta^{i,v}(t)|^2\mathrm{d}t+Z_\theta^{i,v}(t)\mathrm{d}W^i(t),\\ Y_\theta^{i,v}(T)=\Phi_\theta^i(X_\theta^{i,v}(T),\dfrac{1}{N}\displaystyle\sum_{j=1}^N X_\theta^{j,v}(T)).\end{cases}\tag{2.3}$$

Parameter κ is called a risk-sensitive parameter. The i-th agent wants to minimize their objective functional. The above problem is considered a risk-sensitive robust optimal control problem for particle systems.

Now, we assume that the game is symmetric. In other words, let $b_\theta^i=b_\theta,\sigma_\theta^i=\sigma_\theta,\Phi_\theta^i=\Phi_\theta$. As $N\to\infty$ in (2.1) and (2.3), from the strong law of large numbers, we obtain the following FBSDE:

$$\begin{cases}\mathrm{d}X_\theta^v(t)=b_\theta(t,X_\theta^v(t),\mathbb{E}[X_\theta^v(t)],v(t))\mathrm{d}t+\sigma_\theta(t,X_\theta^v(t),v(t))\mathrm{d}W(t),\\ \mathrm{d}Y_\theta^v(t)=-\dfrac{\kappa}{2}|Z_\theta^v(t)|^2\mathrm{d}t+Z_\theta^v(t)\mathrm{d}W(t),\\ X_\theta^v(t)=x_0,\quad Y_\theta^v(T)=\Phi_\theta(X_\theta^v(T),\mathbb{E}[X_\theta^v(T)]),\end{cases}$$

and, meanwhile, the cost functional becomes $J(v(\cdot))=\sup\limits_{Q^\lambda\in\mathcal{Q}}\int_\Theta Y_\theta^v(0)Q^\lambda(\mathrm{d}\theta)$. Consequently, the control problem can be written as follows:

Problem R *Find an optimal control $\bar v(\cdot)$ such that*

$$J(\bar v(\cdot))=\inf_{v(\cdot)\in\mathcal{V}_{ad}}J(v(\cdot))=\inf_{v(\cdot)\in\mathcal{V}_{ad}}\sup_{Q^\lambda\in\mathcal{Q}}\int_\Theta Y_\theta^v(0)Q^\lambda(\mathrm{d}\theta).$$

Example 2.2 *(Optimal strategy for large investor) Let $K\subset\mathbb{R}$ be a closed convex cone. Assume that there are N investors in a market and the market consists of a bond and a stock whose prices denoted by $S_j(t), j=0,1$ are governed by the following ODE and SDE:*

$$\begin{cases}\mathrm{d}S_0(t)=S_0(t)\Big[r(t)+f_0^i\Big(t,x^{i,\pi}(t),\frac{1}{N}\sum_{j=1}^N x^{j,\pi}(t),\pi(t)\Big)\Big]\mathrm{d}t,\\ \mathrm{d}S_1(t)=S_1(t)\Big[\Big(\mu(t)-f_1^i(t,x^{i,\pi}(t),\frac{1}{N}\sum_{j=1}^N x^{j,\pi}(t),\pi(t)\Big)\mathrm{d}t+\sigma\mathrm{d}W(t)\Big],\end{cases}$$

where $(r,\mu):\Omega\times[0,T]\to\mathbb{R}\times\mathbb{R}$ are the interest rate and return rate, respectively; σ is the volatility of stock; (r,μ) are $\{\mathcal{F}_t\}_{t\geqslant 0}$-adapted and uniformly bounded processes and $\sigma\neq 0$; W is a 1-dimensional Brownian motion; for each $i=1,2,\cdots,N$, $f_\ell^i,\ell=0,1:[0,T]\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$ are two given functions representing the effect of the strategies chosen by the investors on the prices.

The i-th investor aims to allocate $\pi^i(t)$ toward stock investment at time t, with the remainder of their assets being invested in bonds. Operating under a self-financed portfolio regine, the wealth process of the i-th investor, starting with an initial wealth x_0^i, satisfies the following wealth equation:

$$\begin{cases}\mathrm{d}x^{i,\pi}(t)=\Big(r(t)x^{i,\pi}(t)+(x^{i,\pi}(t)-\pi^i(t))f_0^i\Big(t,x^{i,\pi}(t),\frac{1}{N}\sum_{j=1}^N x^{j,\pi}(t),\pi(t)\Big)\\ \qquad\qquad+\pi^i(t)\Big[\mu(t)-r(t)+f_1^i\Big(t,x^{i,\pi}(t),\frac{1}{N}\sum_{j=1}^N x^{j,\pi}(t),\pi(t)\Big)\Big]\Big)\mathrm{d}t+\pi^i(t)\sigma\mathrm{d}W(t),\quad t\in[0,T],\\ x^{i,\pi}(0)=x_0^i.\end{cases}$$

For $i=1,2,\cdots,N$, we assume that $\gamma^i>0$ represents a given parameter and C^i denotes the consumption at time T for the i-investor. We consider the following cost functional

$$J^i(\pi)=\frac{1}{\gamma^i}\ln\mathbb{E}\Big[\exp[-\gamma^i(x^{i,\pi}(T)-C^i)]\Big].$$

A strategy π taking values in K is called admissible if for any $t\in[0,T]$, $\mathbb{E}\left[\int_0^T|\pi(t)|^2\mathrm{d}t\right]<\infty$. By $\mathcal{K}$ we denote the set of all admissible strategies. The i-th investor intends to minimize their cost functional over $\mathcal{K}$. Next, we consider $N\to\infty$ and the asymptotic behavior of N investors. For simplicity, we set $\pi=\pi^i, x_0=x_0^i,\gamma=\gamma^i,f_0=f_0^i,f_1=f_1^i$ and $C=C^i$. This optimal strategy problem can be summarized as follows:

Problem O *Find an optimal strategy $\bar{\pi}(\cdot)$ such that*

$$J(\bar{\pi}(\cdot))=\inf_{\pi(\cdot)\in\mathcal{K}}\frac{1}{\gamma}\ln\mathbb{E}\Big[\exp[-\gamma(x^\pi(T)-C)]\Big],$$

subject to a mean-field SDE

$$\begin{cases}\mathrm{d}x^\pi(t)=\Big(r(t)x^\pi(t)+(x^\pi(t)-\pi(t))f_0(t,x^\pi(t),\mathbb{E}[x^\pi(t)],\pi(t))\\ \qquad\qquad+\pi(t)[\mu(t)-r(t)+f_1(t,x^\pi(t),\mathbb{E}[x^\pi(t)],\pi(t))]\Big)\mathrm{d}t+\pi(t)\sigma\mathrm{d}W(t),\quad t\in[0,T],\\ x^\pi(0)=x_0.\end{cases}$$

To illustrate our problem concisely, we set

$$f_0(t,x^\pi(t),\mathbb{E}[x^\pi(t)],\pi(t)):=\alpha(t),\quad f_1(t,x^\pi(t),\mathbb{E}[x^\pi(t)],\pi(t)):=\beta(t).$$

Here $\alpha(\cdot)$ and $\beta(\cdot)$ are two bounded functions. To determine the forward value of a portfolio, we consider a zero-coupon bond Λ with a maturity of T. The financial asset generates a cash flow of 1 at time T. Specifically, there is an $\mathbb{R}$-valued progressively measurable process Γ, such that the zero-coupon price Λ satisfies BSDE

$$\begin{cases} \mathrm{d}\Lambda(t) = \Lambda(t)\Big[\Big(r(t) + \Gamma(t)(\mu(t) - r(t) + \beta(t) - \alpha(t))\Big)\mathrm{d}t + \Gamma(t)\sigma \mathrm{d}W(t)\Big], \quad t \in [0,T], \\ \Lambda(T) = 1. \end{cases}$$

Here, for simplicity, we assume that Γ is $\{\mathcal{F}_t\}_{t\geqslant 0}$-adapted and uniformly bounded for any $(t,\omega) \in [0,T]\times\Omega$. According to Rouge and El Karoui [30, Theorem 2.1], one can see

$$J(\bar{\pi}) = \inf_{\pi(\cdot)\in\mathcal{K}} \frac{1}{\gamma} \ln \mathbb{E}\Big[\exp[-\gamma(x^\pi(T) - C)]\Big] = -\frac{x_0}{\Lambda(0)} + \sup_{v(\cdot)\in\mathcal{V}} Y^v(0), \tag{2.4}$$

where (Y^v, Z^v) satisfies BSDE

$$\begin{cases} \mathrm{d}Y^v(t) = -\Big(-\frac{1}{2\gamma}|\delta(t) + \Pi_{\sigma^{-1}\tilde{K}}(-\delta(t) - \gamma Z^v(t))|^2 - (\delta(t) + \Pi_{\sigma^{-1}\tilde{K}}(-\delta(t) - \gamma Z^v(t)))Z^v(t)\Big)\mathrm{d}t \\ \qquad\qquad + Z^v(t)\mathrm{d}W(t), \quad t \in [0,T], \\ Y^v(T) = C. \end{cases} \tag{2.5}$$

Here $\delta(t) = \sigma^{-1}(\mu(t) - r(t) + \beta(t) - \alpha(t) - \sigma^2\Gamma(t))$; $\widetilde{K} = \{x \in \mathbb{R} | \sup_{\pi\in K}(-\pi x) < \infty\}$; $\Pi_{\sigma^{-1}\tilde{K}}(u)$ is the projection of $u \in \mathbb{R}$ on $\sigma^{-1}\tilde{K}$; and $\mathcal{V} \triangleq \{v : \Omega\times[0,T]\to\sigma^{-1}\widetilde{K} |$ v is $\{\mathcal{F}_t\}_{t\geqslant 0}$-adapted and bounded$\}$. *Since $|\Pi_{\sigma^{-1}\tilde{K}}(u)| \leqslant |u|$, for $\forall u \in \mathbb{R}$, there exists a constant $L > 0$ such that the generator of (2.5) is dominated by $L(1 + |Z^v(t)|^2)$.*

The cost functional (2.4) indicates that solving an optimal strategy problem is equivalent to solving an optimal control problem of BSDE, whose generator is dominated by a quadratic generator. Building on this, a natural question arises: if γ and C depend on a parameter θ and are continuous in θ, how can we characterize the equivalent optimal control problem, where $\theta \in \Theta$ represents the different market states and Θ is a closed set of $\mathbb{R}$. Actually, in this case, we need to consider the following cost functional:

$$\widehat{J}(v(\cdot)) := \sup_{Q\in\mathcal{Q}} \int_\Theta Y_\theta^v(0)Q(\mathrm{d}\theta),$$

where $\mathcal{Q}$ is a weakly compact and convex set of probability measures on $(\Theta, \mathcal{B}(\Theta))$; (Y_θ^v, Z_θ^v) satisfies the following BSDE:

$$\begin{cases} \mathrm{d}Y_\theta^v(t) = -\Big(-\frac{1}{2\gamma_\theta}|\delta(t) + \Pi_{\sigma^{-1}\tilde{K}}(-\delta(t) - \gamma_\theta Z^v(t))|^2 - (\delta(t) + \Pi_{\sigma^{-1}\tilde{K}}(-\delta(t) - \gamma_\theta Z^v(t)))Z^v(t)\Big)\mathrm{d}t \\ \qquad\qquad + Z_\theta^v(t)\mathrm{d}W(t), \quad t \in [0,T], \\ Y_\theta^v(T) = C_\theta. \end{cases} \tag{2.6}$$

Similarly to (2.5), the generator of (2.6) is dominated by $L(1 + |Z_\theta^v(t)|^2)$, where L is a constant independent of θ. Consequently, the corresponding optimal control problem is formulated as follows:

Problem S *Find an optimal control $\bar{v}(\cdot)$ such that*

$$\widehat{J}(\bar{v}(\cdot)) = \sup_{v(\cdot)\in\mathcal{V}} \sup_{Q\in\mathcal{Q}} \int_\Theta Y_\theta^v(0)Q(\mathrm{d}\theta).$$

3. Problem formulation

3.1 Some properties for BMO-martingales

Denote by $\mathbb{N}$ the set of all natural numbers and by $\mathbb{R}^+$ the set of all positive real numbers, respectively. Let the superscript $\top$ denote the transpose of vectors or matrices. Let $M = (M_t, \mathscr{F}_t)$ be a uniformly integrable martingale with $M_0 = 0$. For $p \geqslant 1$, we define

$$\|M\|_{\mathrm{BMO}_p(\mathbb{P})} := \sup_{\tau} \left\| \mathbb{E}_\tau \left[(\langle M\rangle_\infty - \langle M\rangle_\tau)^{\frac{p}{2}} \right]^{\frac{1}{p}} \right\|_\infty,$$

where τ is a stopping time on $[0,T]$ and $\mathbb{E}_\tau[\cdot] = \mathbb{E}[\cdot|\mathcal{F}_\tau]$ is the conditional expectation. The class $\{M : \|M\|_{\mathrm{BMO}_p(\mathbb{P})} < \infty\}$ is denoted by $\mathrm{BMO}_p(\mathbb{P})$. Note that $\mathrm{BMO}_p(\mathbb{P})$ is a Banach space under the norm $\|\cdot\|_{\mathrm{BMO}_p(\mathbb{P})}$. For simplicity, $\mathrm{BMO}_2(\mathbb{P})$ is written as BMO.

Next, we list some properties for BMO-martingales. For more details, the reader can refer to Kazamaki [24].

- Denote by $\mathcal{E}(M)$ the Doléans–Dade exponential of a continuous local martingale M, i.e., $\mathcal{E}(M_t) = \exp\{M_t - \frac{1}{2}\langle M\rangle_t\}$, for any $t \in [0,T]$. If $M \in \mathrm{BMO}$, then $\mathcal{E}(M)$ is a uniformly integrable martingale.
- Let Ψ be the monotonically decreasing function defined on $(1,\infty)$ by

$$\Psi(p) = \left(1 + p^{-2} \ln \frac{2p-1}{2(p-1)}\right)^{\frac{1}{2}} - 1,$$

and then $\Psi(+\infty) := \lim_{x\to+\infty} \Psi(x) = 0$. For $M \in \mathrm{BMO}$, we can find a positive constant p_M which satisfies $\Psi(p_M) = \|M\|_{\mathrm{BMO}}$. In particular, set $p_M = +\infty$ if $\|M\|_{\mathrm{BMO}} = 0$. Then, p_M is uniquely determined.

- The reverse Hölder inequality: If $p \in (1, p_M)$, for any stopping time $\tau \in [0,T]$,

$$\mathbb{E}\Big[(\mathcal{E}(M_T))^p/(\mathcal{E}(M_\tau))^p|\mathcal{F}_\tau\Big] \leqslant K(p, ||M||_{\mathrm{BMO}}), \quad \text{a.s.},$$

where

$$K(p, ||M||_{\mathrm{BMO}}) = 2\left(1 - \frac{2p-2}{2p-1} \exp\left\{p^2\Big[||M||^2_{\mathrm{BMO}} + 2||M||_{\mathrm{BMO}}\Big]\right\}\right)^{-1}.$$

- John–Nirenberg inequality: For $M \in \mathrm{BMO}$, if $\theta \in (0, \|M\|^{-2}_{\mathrm{BMO}})$, for any stopping time $\tau \in [0,T]$,

$$\mathbb{E}\left[\exp\left\{\theta(\langle M\rangle_T - \langle M\rangle_\tau)\right\}\Big|\mathcal{F}_\tau\right] \leqslant \left(1 - \theta\|M\|^2_{\mathrm{BMO}}\right)^{-1}, \quad \text{a.s.}$$

- By p^*_M, we denote the conjugate exponent of p_M, i.e., $(p_M)^{-1} + (p^*_M)^{-1} = 1$.

For any $p \geqslant 1$, $t \in [0,T]$ and a filtration $\mathbb{F}$, we introduce some useful spaces:

$$\mathcal{S}^p_{\mathbb{F}}(t,T;\mathbb{R}^m) = \Bigg\{\varphi : \Omega \times [t,T] \to \mathbb{R}^m \Big| \varphi \text{ is } \mathbb{F}\text{-adapted, continuous,}$$
$$\|\varphi\|_{\mathcal{S}^p_{\mathbb{F}}(t,T)} \triangleq \left\{\mathbb{E}\left(\sup_{s\in[t,T]} \left|\varphi_s\right|^p\right)\right\}^{\frac{1}{p}} < \infty\Bigg\},$$

$$\mathcal{S}^\infty_{\mathbb{F}}(t,T;\mathbb{R}^m) = \Big\{\varphi : \Omega \times [t,T] \to \mathbb{R}^m \Big| \varphi \text{ is } \mathbb{F}\text{-adapted, continuous,}$$
$$\|\varphi\|_{\mathcal{S}^\infty_{\mathbb{F}}(t,T)} \triangleq \operatorname*{esssup}_{(s,\omega)\in[t,T]\times\Omega} |\varphi_s(\omega)| < \infty\Big\},$$

$$\mathcal{H}_{\mathbb{F}}^{2,p}(t,T;\mathbb{R}^m)=\bigg\{\varphi:\Omega\times[t,T]\to\mathbb{R}^m\bigg|\varphi \text{ is } \mathbb{F}\text{-progressively measurable}, \|\varphi\|_{\mathcal{H}_{\mathbb{F}}^{2,p}(t,T)}\triangleq\mathbb{E}\bigg[\bigg(\int_t^T|\varphi_s|^2\mathrm{d}s\bigg)^{\frac{p}{2}}\bigg]^{\frac{1}{p}}<\infty\bigg\}.$$

3.2 Formulate optimal control problem

In this subsection, we formulate the optimal control problem. To this end, we first introduce the set of admissible controls.

Definition 3.1 *An $\mathbb{F}$-adapted process v taking values in $V\subset\mathbb{R}^k$ is called an admissible control, if for any $\ell>0$, $\mathbb{E}[\int_0^T|v(t)|^\ell\mathrm{d}t]<\infty$. $\mathcal{V}_{ad}$ denotes the set of all admissible controls.*

The following assumptions on coefficients $b_\theta,\sigma_\theta,f_\theta,\Phi_\theta$ are forced.

Assumption 1 (i) *There exists a constant $C_0>0$ independent of θ such that, for $t\in[0,T]$, $x_1,x_2,x_1',x_2'\in\mathbb{R}^n,v,v'\in V$,*

$$\begin{aligned}&|b_\theta(t,0,0,v)|+|\sigma_\theta(t,0,v)|\leqslant C_0(1+|v|),\\&|b_\theta(t,x_1,x_1',v)-b_\theta(t,x_2,x_2',v')|\leqslant C_0(|x_1-x_2|+|x_1'-x_2'|+|v-v'|),\\&|\sigma_\theta(t,x_1,v)-\sigma_\theta(t,x_2,v')|\leqslant C_0(|x_1-x_2|+|v-v'|),\end{aligned}$$

b_θ,σ_θ is continuously differentiable in (x,x',v) and (x,v), respectively; $\partial_xb_\theta,\partial_{x'}b_\theta,\partial_vb_\theta,\partial_x\sigma_\theta$, $\partial_v\sigma_\theta$ are Lipschitz continuous in (x,x',v).

(ii) *$\Phi_\theta,\partial_x\Phi_\theta,\partial_{x'}\Phi_\theta$ are continuous in (x,x') and bounded.*

(iii) *f is continuously differentiable in (x,x',y,z,v); $\partial_xf_\theta,\partial_{x'}f_\theta,\partial_yf_\theta,\partial_zf_\theta,\partial_vf_\theta$ are Lipschitz continuous in (x,x',y,z,v); there exists a constant $C_1>0$ independent of θ such that, for $t\in[0,T]$, $x_1,x_2,x_1',x_2'\in\mathbb{R}^n,y_1,y_2\in\mathbb{R},z_1,z_2\in\mathbb{R}^d,v\in V$,*

$$\begin{aligned}&|f_\theta(t,x_1,x_1',0,0,v)|\leqslant C_1,\\&|f_\theta(t,x_1,x_1',y_1,z_1,v)-f_\theta(t,x_2,x_2',y_2,z_2,v)|\\\leqslant&\, C_1(|x_1-x_2|+|x_1'-x_2'|+|y_1-y_2|)+C_1(1+|z_1|+|z_2|)|z_1-z_2|,\\&|\partial_vf_\theta(t,x_1,x_1',y_1,z_1,v)|\leqslant C_1.\end{aligned}$$

(iv) *There exists a constant $C_2>0$ independent of θ such that, for any $t\in[0,T]$, $\theta,\bar\theta\in\Theta$, $x_1,x_1'\in\mathbb{R}^n,y_1\in\mathbb{R},z_1\in\mathbb{R}^d,v\in V$,*

$$|\varphi_\theta(t,x_1,x_1',y_1,z_1,v)-\varphi_{\bar\theta}(t,x_1,x_1',y_1,z_1,v)|\leqslant C_2\Xi(\theta,\bar\theta),$$

where φ_θ is $b_\theta,\sigma_\theta,f_\theta,\Phi_\theta$ and their derivatives w.r.t. their respective variables.

(v) *$\mathcal{Q}$ is a weakly compact and convex set of probability measures on $(\Theta,\mathcal{B}(\Theta))$.*

According to Briand and Hu [3, Proposition 3], and Hu and Tang [21, Lemma 2.1], we establish the existence and uniqueness of (Y_θ^v,Z_θ^v).

Theorem 3.2 *Under Assumption 1, for any $v(\cdot)\in\mathcal{V}_{ad}$ and $p>1$, equation (1.1) has a unique solution $(X_\theta^v,Y_\theta^v,Z_\theta^v)\in\mathcal{S}_{\mathbb{F}}^p(0,T;\mathbb{R}^n)\times\mathcal{S}_{\mathbb{F}}^\infty(0,T;\mathbb{R})\times\mathcal{H}_{\mathbb{F}}^{2,p}(0,T;\mathbb{R}^d)$. Moreover, the following estimates hold:*

$$\text{(i) } \|X^v_\theta\|^p_{\mathcal{S}^p_{\mathbb{F}}(0,T)} \leqslant C\bigg\{|x|^p + \mathbb{E}\bigg[\bigg(\int_0^T |b_\theta(s,0,0,v(s))|\mathrm{d}s\bigg)^p\bigg] + \bigg\{\mathbb{E}\bigg[\bigg(\int_0^T |\sigma_\theta(s,0,v(s))|^2\mathrm{d}s\bigg)^{\frac{p}{2}}\bigg]\bigg\},$$
$$\text{(ii) } \|Y^v_\theta\|_{\mathcal{S}^\infty_{\mathbb{F}}(0,T)} \leqslant M_1 \quad \textit{and} \quad \|Z^v_\theta \cdot W\|_{\mathrm{BMO}} \leqslant M_2, \tag{3.1}$$

where constant C depends on C_0, p, T, and constants M_1, M_2 depend on $C_0, C_1, T, \|\Phi\|_\infty$.

In the rest of this paper, we use C to represent a generic constant that only depends on the given parameters and could vary from line to line.

The continuity of $(X^v_\theta, Y^v_\theta, Z^v_\theta)$ concerning θ is proved in the following proposition.

Proposition 3.3 *Under Assumption 1, for any $v(\cdot) \in \mathcal{V}_{ad}$, $p > 1$ and $q > p^*_{\partial_z f_\theta}$,*

(i) $\lim\limits_{\epsilon\to 0} \sup\limits_{\Xi(\theta,\tilde\theta)\leqslant\epsilon} \mathbb{E}\bigg[\sup\limits_{0\leqslant t\leqslant T} |X^v_\theta(t) - X^v_{\tilde\theta}(t)|^p\bigg] = 0,$

(ii) $\lim\limits_{\epsilon\to 0} \sup\limits_{\Xi(\theta,\tilde\theta)\leqslant\epsilon} \mathbb{E}\bigg[\sup\limits_{0\leqslant t\leqslant T} |Y^v_\theta(t) - Y^v_{\tilde\theta}(t)|^q + \bigg(\int_0^T |Z^v_\theta(t) - Z^v_{\tilde\theta}(t)|^2\mathrm{d}t\bigg)^{\frac{q}{2}}\bigg] = 0.$

Proof As for (i), denote $\Delta X(s) = X^v_\theta(s) - X^v_{\tilde\theta}(s)$, and for $h = b, \sigma$,

$$\Delta h(s) := h_\theta(s, X^v_\theta(s), \mathbb{E}[X^v_\theta(s)], v(s)) - h_{\tilde\theta}(s, X^v_\theta(s), \mathbb{E}[X^v_\theta(s)], v(s)).$$

Then it follows that

$$\begin{aligned}\Delta X(t) = &\int_0^t (\Delta b(s) + b_{\tilde\theta}(s, X^v_{\tilde\theta}(s) + \Delta X(s), \mathbb{E}[X^v_{\tilde\theta}(s)] + \mathbb{E}[\Delta X(s)], v(s)) \\ &- b_{\tilde\theta}(s, X^v_{\tilde\theta}(s), \mathbb{E}[X^v_{\tilde\theta}(s)], v(s)))\mathrm{d}s \\ &+ \int_0^t \big(\Delta\sigma(s) + \sigma_{\tilde\theta}(s, X^v_{\tilde\theta}(s) + \Delta X(s), v(s)) - \sigma_{\tilde\theta}(s, X^v_{\tilde\theta}(s), v(s))\big)\,\mathrm{d}W(s).\end{aligned}$$

Thanks to Theorem 3.2, it yields, for $p > 1$,

$$\mathbb{E}\bigg[\sup_{t\in[0,T]} |\Delta X(t)|^p\bigg] \leqslant C\mathbb{E}\bigg[\bigg(\int_0^T |\Delta b(s)|\mathrm{d}s\bigg)^p + \bigg(\int_0^T |\Delta\sigma(s)|^2\mathrm{d}s\bigg)^{\frac{p}{2}}\bigg] \leqslant C\Xi(\theta,\tilde\theta)^p.$$

Then we turn to (ii). For simplicity, we denote:

$$\Delta Y(s) = Y^v_\theta(s) - Y^v_{\tilde\theta}(s), \quad \Delta Z(s) = Z^v_\theta(s) - Z^v_{\tilde\theta}(s),$$
$$\Delta\Phi(T) = \Phi_\theta(X^v_{\tilde\theta}(T), \mathbb{E}[X^v_{\tilde\theta}(T)]) - \Phi_{\tilde\theta}(X^v_{\tilde\theta}(T), \mathbb{E}[X^v_{\tilde\theta}(T)]),$$
$$\Delta f(s) = f_\theta(s, X^v_{\tilde\theta}(s), \mathbb{E}[X^v_{\tilde\theta}(s)], Y^v_{\tilde\theta}(s), Z^v_{\tilde\theta}(s), v(s)) - f_{\tilde\theta}(s, X^v_{\tilde\theta}(s), \mathbb{E}[X^v_{\tilde\theta}(s)], Y^v_{\tilde\theta}(s), Z^v_{\tilde\theta}(s), v(s)).$$

Then

$$\begin{aligned}\Delta Y(t) = &\Delta\Phi(T) + \Phi_\theta(X^v_\theta(T), \mathbb{E}[X^v_\theta(T)]) - \Phi_\theta(X^v_{\tilde\theta}(T), \mathbb{E}[X^v_{\tilde\theta}(T)]) \\ &+ \int_t^T (\Delta f(s) + f_\theta(s, X^v_\theta(s), \mathbb{E}[X^v_\theta(s)], Y^v_{\tilde\theta}(s) + \Delta Y(s), Z^v_{\tilde\theta}(s) + \Delta Z(s), v(s)) \\ &- f_\theta(s, X^v_{\tilde\theta}(s), \mathbb{E}[X^v_{\tilde\theta}(s)], Y^v_{\tilde\theta}(s), Z^v_{\tilde\theta}(s), v(s)))\mathrm{d}s \\ &- \int_t^T \Delta Z(s)\mathrm{d}W(s), \quad t \in [0,T].\end{aligned}$$

According to Briand and Confortola [1, Corollary 9], we know, for any $q > p^*_{\partial_z f_\theta}$,

$$\begin{aligned}&\mathbb{E}\bigg[\sup_{t\in[0,T]}|\Delta Y(t)|^q+\bigg(\int_0^T|\Delta Z(t)|^2\mathrm{d}t\bigg)^{\frac{q}{2}}\bigg]\\&\leqslant C\bigg(\mathbb{E}\Big[|\Delta\Phi(T)|^{q+1}+|\Delta X(T)|^{q+1}+|\mathbb{E}[\Delta X(T)]|^{q+1}\Big]\\&\quad+\mathbb{E}\bigg[\int_0^T(|\Delta f(t)|+|f_\theta(t,X_\theta^v(t),\mathbb{E}[X_\theta^v(t)],Y_{\tilde\theta}^v(t),Z_{\tilde\theta}^v(t),v(t))\\&\quad-f_\theta(t,X_{\tilde\theta}^v(t),\mathbb{E}[X_{\tilde\theta}^v(t)],Y_{\tilde\theta}^v(t),Z_{\tilde\theta}^v(t),v(t))|)^{q+1}\mathrm{d}t\bigg]\bigg)^{\frac{q}{q+1}}.\end{aligned}$$

Hence, from (i), we derive, for $q>p^*_{\partial_z f_\theta}$,

$$\mathbb{E}\bigg[\sup_{t\in[0,T]}|\Delta Y(t)|^q+\bigg(\int_0^T|\Delta Z(t)|^2\mathrm{d}t\bigg)^{\frac{q}{2}}\bigg]\leqslant C\Xi(\theta,\tilde\theta)^q.$$

This completes the proof. □

To introduce the cost functional, the following assumption for mappings $\phi_\theta:\mathbb{R}^n\to\mathbb{R}$ and $\gamma_\theta:\mathbb{R}\to\mathbb{R}$ are needed:

Assumption 2 (i) *ϕ_θ and γ_θ are continuously differentiable in their respective variables and bounded.*

(ii) *There exists a positive constant L_1 such that, for $x,x'\in\mathbb{R}$, $\theta\in\Theta$,*

$$|\phi_\theta(x)-\phi_\theta(x')|\leqslant L_1(1+|x|+|x'|)|x-x'|,\quad|\partial_x\phi_\theta(x)-\partial_x\phi_\theta(x')|\leqslant L_1|x-x'|.$$

(iii) *γ_θ and its derivative $\partial_y\gamma_\theta$ are Lipschitz continuous to y uniformly in θ and bounded.*

(iv) *There exists a positive constant L_2 such that, for $x\in\mathbb{R}^n$, $\theta,\theta'\in\Theta$,*

$$|h_\theta(x)-h_{\theta'}(x)|\leqslant L_2\Xi(\theta,\theta'),$$

where h denotes ϕ,γ and their derivatives w.r.t. their respective variables.

Since the control system (1.1) is model uncertainty, we consider the robust cost functional

$$J(v(\cdot))=\sup_{Q\in\mathcal{Q}}\int_\Theta\mathbb{E}\Big[\phi_\theta(X_\theta^v(T))+\gamma_\theta(Y_\theta^v(0))\Big]Q(\mathrm{d}\theta).\tag{3.2}$$

The optimal control problem is:

Problem (PQU) We identify an optimal control $\bar v(\cdot)$ such that

$$J(\bar v(\cdot))=\inf_{v(\cdot)\in\mathcal{V}_{ad}}J(v(\cdot)),\tag{3.3}$$

subject to (1.1) and (3.2).

The control problem described above will remain the focus for the rest of this paper. We aim to establish the necessary and sufficient conditions for achieving optimal control.

4. Variational inequality

In this section, we investigate the variational equations and variational inequality, which are essential materials to study the stochastic maximum principle.

Let $\bar v(\cdot)$ be an optimal control and $(\bar X_\theta,\bar Y_\theta,\bar Z_\theta)$ be the corresponding state processes of (1.1). Notably, $\mathcal{V}_{ad}$ is convex; therefore, for any $v(\cdot)\in\mathcal{V}_{ad}$ and $0<\lambda<1$, $v^\lambda(\cdot):=\bar v(\cdot)+\lambda(v(\cdot)-\bar v(\cdot))\in\mathcal{V}_{ad}$. By $(X_\theta^\lambda,Y_\theta^\lambda,Z_\theta^\lambda)$ we denote the corresponding trajectories of (1.1) with $v^\lambda(\cdot)$ for each $\theta \in \Theta$. To avoid heavy notation, we set

$$b_\theta(t) = b_\theta(t, \bar{X}_\theta(t), \mathbb{E}[\bar{X}_\theta(t)], \bar{v}(t)),$$
$$f_\theta(t) = f_\theta(t, \bar{X}_\theta(t), \mathbb{E}[\bar{X}_\theta(t)], \bar{Y}_\theta(t), \bar{Z}_\theta(t), \bar{v}(t)),$$

$\sigma_\theta(t), \partial_x b_\theta(t), \partial_{x'} b_\theta(t), \partial_v b_\theta(t), \partial_x \sigma_\theta(t), \partial_{x'} \sigma_\theta(t), \partial_v \sigma_\theta(t), \partial_x f_\theta(t), \partial_{x'} f_\theta(t), \partial_y f_\theta(t), \partial_z f_\theta(t), \partial_v f_\theta(t)$ can be set similarly.

4.1 Variational equation

Consider the following variational SDE on $[0,T]$: for each $\theta \in \Theta$,

$$\begin{cases} \mathrm{d}X^1_\theta(t) = \Big(\partial_x b_\theta(t) X^1_\theta(t) + \partial_{x'} b_\theta(t) \mathbb{E}[X^1_\theta(t)] + \partial_v b_\theta(t)(v(t) - \bar{v}(t))\Big)\mathrm{d}t \\ \qquad\qquad + \displaystyle\sum_{i=1}^d \Big(\partial_x \sigma^i_\theta(t) X^1_\theta(t) + \partial_v \sigma^i_\theta(t)(v(t) - \bar{v}(t))\Big)\mathrm{d}W^i(t), \\ X^1_\theta(0) = 0. \end{cases} \tag{4.1}$$

The equation (4.1) is a linear mean-field SDE. Since the coefficients $\partial_x b_\theta(t), \partial_{x'} b_\theta(t), \partial_u b_\theta(t)$, $\partial_x \sigma_\theta(t), \partial_{x'} \sigma_\theta(t), \partial_u \sigma_\theta(t)$ are bounded, according to Theorem 3.2, SDE (4.1) has a unique solution $X^1_\theta \in \mathcal{S}^p_{\mathbb{F}}(0,T;\mathbb{R}^n)$ for $p > 1$. Furthermore, we have, for $p > 1$,

$$\mathbb{E}\Big[\sup_{0 \leqslant t \leqslant T} |X^1_\theta(t)|^p\Big] \leqslant C\Big\{\mathbb{E}\Big[\int_0^T (|v(t)|^p + |\bar{v}(t)|^p)\mathrm{d}t\Big] + \Big(\mathbb{E}\int_0^T |v(t)|^2 \mathrm{d}t\Big)^{\frac{p}{2}} + \Big(\mathbb{E}\int_0^T |\bar{v}(t)|^2 \mathrm{d}t\Big)^{\frac{p}{2}}\Big\}. \tag{4.2}$$

The following lemma shows that X^1_θ is continuous in θ.

Proposition 4.1 *Under Assumption 1, we have for $p > 1$,*

$$\lim_{\epsilon \to 0} \sup_{\Xi(\theta, \tilde{\theta}) \leqslant \epsilon} \mathbb{E}\Big[\sup_{0 \leqslant t \leqslant T} |X^1_\theta(t) - X^1_{\tilde{\theta}}(t)|^p\Big] = 0.$$

Proof Notice

$$\begin{cases} \mathrm{d}\left(X^1_\theta(t) - X^1_{\tilde{\theta}}(t)\right) = \partial_x b_\theta(t)(X^1_\theta(t) - X^1_{\tilde{\theta}}(t)) + \partial_{x'} b_\theta(t)\mathbb{E}[X^1_\theta(t) - X^1_{\tilde{\theta}}(t)] + I_{1,\theta,\tilde{\theta}}(t)\mathrm{d}t \\ \qquad\qquad + \displaystyle\sum_{i=1}^d \left(\partial_x \sigma^i_\theta(t)(X^1_\theta(t) - X^1_{\tilde{\theta}}(t)) + I^i_{2,\theta,\tilde{\theta}}(t)\right)\mathrm{d}W^i(t), \\ X^1_\theta(t) - X^1_{\tilde{\theta}}(t) = 0, \end{cases}$$

where

$$\begin{aligned} I_{1,\theta,\tilde{\theta}}(t) &= \left(\partial_x b_\theta(t) - \partial_x b_{\tilde{\theta}}(t)\right) X^1_{\tilde{\theta}}(t) + \left(\partial_{x'} b_\theta(t) - \partial_{x'} b_{\tilde{\theta}}(t)\right)\mathbb{E}[X^1_{\tilde{\theta}}(t)] \\ &\quad + \left(\partial_v b_\theta(t) - \partial_v b_{\tilde{\theta}}(t)\right)(v(t) - \bar{v}(t)), \\ I^i_{2,\theta,\tilde{\theta}}(t) &= \left(\partial_x \sigma^i_\theta(t) - \partial_x \sigma^i_{\tilde{\theta}}(t)\right) X^1_{\tilde{\theta}}(t) + \left(\partial_v \sigma^i_\theta(t) - \partial_v \sigma^i_{\tilde{\theta}}(t)\right)(v(t) - \bar{v}(t)). \end{aligned}$$

Then, we can prove Proposition 4.1 by Theorem 3.2 and Buckholder-Davis-Gundy inequality, following a standard proof. □

Set $\delta X^\lambda_\theta(t) = \frac{1}{\lambda}(X^\lambda_\theta(t) - \bar{X}_\theta(t)) - X^1_\theta(t)$, and we have the following estimate on it.

Lemma 4.2 *Under Assumption 1, there exists a constant $C > 0$ depending on C_0 and T such that, for each $\theta \in \Theta$ and $p > 1$,*

(i) $\mathbb{E}\Big[\sup\limits_{0\leqslant t\leqslant T}|\delta X_\theta^\lambda(t)|^p\Big]\leqslant C\Big\{\mathbb{E}\Big[\int_0^T(|v(t)|^p+|\bar v(t)|^p)\mathrm{d}t\Big]+\Big(\mathbb{E}\int_0^T|v(t)|^2\mathrm{d}t\Big)^{\frac{p}{2}}+\Big(\mathbb{E}\int_0^T|\bar v(t)|^2\mathrm{d}t\Big)^{\frac{p}{2}}\Big\}.$

(ii) $\lim\limits_{\lambda\to 0}\sup\limits_{\theta\in\Theta}\mathbb{E}\Big[\sup\limits_{0\leqslant t\leqslant T}|\delta X_\theta^\lambda(t)|^p\Big]=0.$

Proof Set $b_\theta^\lambda(s)=b_\theta(s,X_\theta^\lambda(s),\mathbb{E}[X_\theta^\lambda(s)],v^\lambda(s)),\sigma_\theta^{i,\lambda}(s)=\sigma_\theta^i(s,X_\theta^\lambda(s),v^\lambda(s))$. According to the definition of δX_θ^λ, we have

$$\begin{aligned}\delta X_\theta^\lambda(t)=&\int_0^t\Big\{\frac{1}{\lambda}\big(b_\theta^\lambda(s)-b_\theta(s)\big)-\Big[\partial_x b_\theta(s)X_\theta^1(s)+\partial_{x'}b_\theta(s)\mathbb{E}[X_\theta^1(s)]+\partial_v b_\theta(s)(v(s)-\bar v(s))\Big]\Big\}\mathrm{d}s\\&+\int_0^t\sum_{i=1}^d\Big\{\frac{1}{\lambda}\Big(\sigma_\theta^{\lambda,i}(s)-\sigma_\theta^i(s)\Big)-\Big[\partial_x\sigma_\theta^i(s)X_\theta^1(s)+\partial_v\sigma_\theta^i(s)(v(s)-\bar v(s))\Big]\Big\}\mathrm{d}W^i(s).\end{aligned}\tag{4.3}$$

Denote

$$\begin{aligned}&\pi_\theta^{\rho\lambda}(t)=(\bar X_\theta(t)+\rho\lambda(\delta X_\theta^\lambda(t)+X_\theta^1(t)),\mathbb{E}[\bar X_\theta(t)]+\rho\lambda(\mathbb{E}[\delta X_\theta^\lambda(t)]+\mathbb{E}[X_\theta^1(t)]),\bar v(t)+\rho\lambda(v(t)-\bar v(t))),\\&\tilde\pi_\theta^{\rho\lambda}(t)=(\bar X_\theta(t)+\rho\lambda(\delta X_\theta^\lambda(t)+X_\theta^1(t)),\bar v(t)+\rho\lambda(v(t)-\bar v(t))),\\&A_\theta^\lambda(t)=\int_0^1\partial_x b_\theta(t,\pi_\theta^{\rho\lambda}(t))\mathrm{d}\rho,\quad B_\theta^\lambda(t)=\int_0^1\partial_{x'}b_\theta(t,\pi_\theta^{\rho\lambda}(t))\mathrm{d}\rho,\\&C_\theta^\lambda(t)=\int_0^1[\partial_v b_\theta(t,\pi_\theta^{\rho\lambda}(t))-\partial_v b_\theta(t)](v(t)-\bar v(t))\mathrm{d}\rho\\&\qquad\quad+[A_\theta^\lambda(t)-\partial_x b_\theta(t)]X_\theta^1(t)+[B_\theta^\lambda(t)-\partial_{x'}b_\theta(t)]\mathbb{E}[X_\theta^1(t)],\\&D_\theta^{\lambda,i}(t)=\int_0^1\partial_x\sigma_\theta^i(t,\tilde\pi_\theta^{\rho\lambda}(t))\mathrm{d}\rho,\\&F_\theta^{\lambda,i}(t)=\int_0^1[\partial_v\sigma_\theta^i(t,\tilde\pi_\theta^{\rho\lambda}(t))-\partial_v\sigma_\theta^i(t)](v(t)-\bar v(t))\mathrm{d}\rho+[D_\theta^{\lambda,i}(t)-\partial_x\sigma_\theta^i(t)]X_\theta^1(t).\end{aligned}$$

Then the equation (4.3) can be written as

$$\begin{aligned}\delta X_\theta^\lambda(t)=&\int_0^t\big(A_\theta^\lambda(s)\delta X_\theta^\lambda(s)+B_\theta^\lambda(s)\mathbb{E}[\delta X_\theta^\lambda(s)]+C_\theta^\lambda(s)\big)\,\mathrm{d}s\\&+\sum_{i=1}^d\int_0^t\Big(D_\theta^{\lambda,i}(s)\delta X_\theta^\lambda(s)+F_\theta^{\lambda,i}(s)\Big)\,\mathrm{d}W^i(s).\end{aligned}$$

Notice that $\partial_x b_\theta(t),\partial_{x'}b_\theta(t),\partial_v b_\theta(t),\partial_x\sigma_\theta^i(t),\partial_v\sigma_\theta^i(t)$ are bounded. Thanks to Theorem 3.2 and (4.2), it yields, for $p>1$,

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T}|\delta X_\theta^\lambda(t)|^p\Big]&\leqslant C\mathbb{E}\Big(\int_0^T|C_\theta^\lambda(s)|\mathrm{d}s\Big)^p+C\mathbb{E}\Big(\int_0^T\sum_{i=1}^d|F_\theta^{\lambda,i}(s)|^2\mathrm{d}s\Big)^{\frac{p}{2}}\\&\leqslant C\Big\{\mathbb{E}\Big[\int_0^T(|v(t)|^p+|\bar v(t)|^p)\mathrm{d}t\Big]+\Big(\mathbb{E}\int_0^T|v(t)|^2\mathrm{d}t\Big)^{\frac{p}{2}}+\Big(\mathbb{E}\int_0^T|\bar v(t)|^2\mathrm{d}t\Big)^{\frac{p}{2}}\Big\}.\end{aligned}$$

Next, we turn to prove (ii). It is enough to prove

$$\begin{cases}\lim\limits_{\lambda\to0}\sup\limits_{\theta\in\Theta}\mathbb{E}\Big(\int_0^T|C_\theta^\lambda(s)|\mathrm{d}s\Big)^p=0,\\\lim\limits_{\lambda\to0}\sup\limits_{\theta\in\Theta}\mathbb{E}\Big(\int_0^T\sum_{i=1}^d|F_\theta^{\lambda,i}(s)|^2\mathrm{d}s\Big)^{\frac{p}{2}}=0.\end{cases}$$

On the one hand, we have

$$
\begin{aligned}
|C_\theta^\lambda(s)|^p \leqslant\ & \big(2^{p-1}\vee 1\big)\bigg\{\int_0^1 |\partial_v b_\theta(s,\pi_\theta^{\rho\lambda}(s))-\partial_v b_\theta(s)|^p|v(s)-\bar v(s)|^p\mathrm{d}\rho \\
&+\int_0^1 |\partial_x b_\theta(s,\pi_\theta^{\rho\lambda}(s))-\partial_x b_\theta(s)|^p|X_\theta^1(s)|^p\mathrm{d}\rho \\
&+\int_0^1 |\partial_{x'} b_\theta(s,\pi_\theta^{\rho\lambda}(s))-\partial_{x'} b_\theta(s)|^p|\mathbb{E}[X_\theta^1(s)]|^p\mathrm{d}\rho\bigg\}.
\end{aligned}
$$

From $ab\leqslant \frac12 a^2+\frac12 b^2, a,b\in\mathbb{R}^+$, it follows

$$
\begin{aligned}
|C_\theta^\lambda(s)|^p \leqslant C_p\lambda^p\bigg\{ & |\delta X_\theta^\lambda(s)+X_\theta^1(s)|^{2p}+|\mathbb{E}[\delta X_\theta^\lambda(s)+X_\theta^1(s)]|^{2p} \\
&+|v(s)-\bar v(s)|^{2p}+|X_\theta^1(s)|^{2p}+|\mathbb{E}[X_\theta^1(s)]|^{2p}\bigg\},
\end{aligned}
$$

where the constant C_p depends on p. By (4.2) and (i) in Lemma 4.2 $\lim\limits_{\lambda\to 0}\sup\limits_{\theta\in\Theta}\mathbb{E}(\int_0^T|C_\theta^\lambda(s)|\mathrm{d}s)^p=0$ follows immediately. On the other hand, based on the fact that

$$
\begin{aligned}
|F_\theta^{\lambda,i}(s)|^2 \leqslant C\lambda^2\bigg\{ & |\delta X_\theta^\lambda(s)+X_\theta^1(s)|^{4}+|\mathbb{E}[\delta X_\theta^\lambda(s)+X_\theta^1(s)]|^{4} \\
&+|v(s)-\bar v(s)|^{4}+|X_\theta^1(s)|^{4}+|\mathbb{E}[X_\theta^1(s)]|^{4}\bigg\},
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\mathbb{E}\bigg(\int_0^T\sum_{i=1}^d|F_\theta^{\rho\lambda,i}(s)|^2\mathrm{d}s\bigg)^{\frac p2} \leqslant C_{p,d}\lambda^p\bigg\{ & \mathbb{E}\bigg(\int_0^T|v(s)-\bar v(s)|^4\mathrm{d}s\bigg)^{\frac p2} \\
&+\mathbb{E}\left[\sup_{0\leqslant s\leqslant T}|\delta X_\theta^\lambda(s)|^{2p}\right]+\mathbb{E}\left[\sup_{0\leqslant s\leqslant T}|X_\theta^1(s)|^{2p}\right]\bigg\},
\end{aligned}
$$

where the constant $C_{p,d}$ depends on p and d. Notice that by Hölder inequality we deduce

$$
\text{if } 1<p<2,\quad \mathbb{E}\bigg(\int_0^T|v(s)-\bar v(s)|^4\mathrm{d}s\bigg)^{\frac p2}\leqslant\bigg(\mathbb{E}\int_0^T|v(s)-\bar v(s)|^4\mathrm{d}s\bigg)^{\frac p2}
$$

and

$$
\text{if } p>2,\quad \mathbb{E}\bigg(\int_0^T|v(s)-\bar v(s)|^4\mathrm{d}s\bigg)^{\frac p2}\leqslant T^{\frac{p}{p-2}}\mathbb{E}\int_0^T|v(s)-\bar v(s)|^{2p}\mathrm{d}s.
$$

Finally, by (4.2) and (i) in Lemma 4.2 again, it yields

$$
\lim_{\lambda\to 0}\sup_{\theta\in\Theta}\mathbb{E}\bigg(\int_0^T\sum_{i=1}^d|F_\theta^{\rho\lambda,i}(s)|^2\mathrm{d}s\bigg)^{\frac p2}=0.
$$

This completes the proof. □

We next introduce the variational BSDE on $[0,T]$: for each $\theta\in\Theta$,

$$
\begin{cases}
-\mathrm{d}Y_\theta^1(t)=\Big(\partial_x f_\theta(t)X_\theta^1(t)+\partial_{x'}f_\theta(t)\mathbb{E}[X_\theta^1(t)]+\partial_y f_\theta(t)Y_\theta^1(t)+\partial_z f_\theta(t)Z_\theta^1(t) \\
\qquad\qquad +\partial_v f_\theta(t)(v(t)-\bar v(t))\Big)\mathrm{d}t-Z_\theta^1(t)\mathrm{d}W(t), \\
Y_\theta^1(T)=\partial_x\Phi_\theta(T)X_\theta^1(T)+\partial_{x'}\Phi_\theta(T)\mathbb{E}[X_\theta^1(T)].
\end{cases}
\tag{4.4}
$$

Since $|\partial_z f_\theta(t)| \leqslant C_2(1+|\bar{Z}_\theta(t)|)$, we know from (ii) in (3.1) that $\partial_z f_\theta \cdot W$ is a BMO martingale and equation (4.4) is a linear BSDE with unbounded coefficients. Owing to $|\partial_x \Phi_\theta(T)| \leqslant C$ and $|\partial_{x'} \Phi_\theta(T)| \leqslant C$, by Hu et al. [16, Proposition 3.1], it has a unique solution $(Y,Z) \in \bigcap_{p>1}(\mathcal{S}_{\mathbb{F}}^p(0,T;\mathbb{R}) \times \mathcal{H}_{\mathbb{F}}^{2,\frac{p}{2}}(0,T;\mathbb{R}^d))$, and we further have for all $\bar{p} > p^*_{\partial_z f_\theta}$ and $p^*_{\partial_z f_\theta} < p < \bar{p}$, there exists a constant $C>0$ depending on $p, \bar{p}, T, \|\partial_x f_\theta\|_\infty, \|\partial_{x'} f_\theta\|_\infty, \|\partial_y f_\theta\|_\infty$ and $\|\partial_z f_\theta \cdot W\|_{\mathrm{BMO}}$ such that,

$$
\begin{aligned}
&\mathbb{E}\bigg[\sup_{0\leqslant t\leqslant T}|Y^1_\theta(t)|^p + \bigg(\int_0^T |Z^1_\theta(t)|^2 \mathrm{d}t\bigg)^{\frac{p}{2}}\bigg] \\
&\leqslant C\bigg\{\mathbb{E}\bigg[\int_0^T (|v(t)|^{\bar{p}} + |\bar{v}(t)|^{\bar{p}})\mathrm{d}t\bigg] + \bigg(\mathbb{E}\int_0^T |v(t)|^2\mathrm{d}t\bigg)^{\frac{\bar{p}}{2}} + \bigg(\mathbb{E}\int_0^T |\bar{v}(t)|^2\mathrm{d}t\bigg)^{\frac{\bar{p}}{2}}\bigg\}^{\frac{p}{\bar{p}}}. \qquad (4.5)
\end{aligned}
$$

Next, we consider the continuity of the pair (Y^1_θ, Z^1_θ) concerning θ.

Proposition 4.3 *Under Assumption 1, for* $q > p^*_{\partial_z f_\theta}$,

$$
\lim_{\epsilon\to 0}\sup_{\Xi(\theta,\tilde{\theta})\leqslant\epsilon}\mathbb{E}\bigg[\sup_{0\leqslant t\leqslant T}|Y^1_\theta(t) - Y^1_{\tilde{\theta}}(t)|^q + \bigg(\int_0^T |Z^1_\theta(t) - Z^1_{\tilde{\theta}}(t)|^2\bigg)^{\frac{q}{2}}\bigg] = 0.
$$

The proof is similar to that of Proposition 3.3. Thus, we omit it to save space.

Remark 4.4 *The main difficulty in the proof of Proposition 4.3 is that* $\partial_z f_\theta(t)$ *is unbounded. Hence, the classical argument is not applicable to this case. However, by observing that* $\partial_z f_\theta \cdot W$ *is a BMO martingale, we can leverage BMO-martingale theory to obtain the stated quality for some* p. *Proposition 4.3 plays an essential role in establishing the variational inequality (see Theorem 4.9 below).*

We set

$$
\begin{aligned}
&\delta Y^\lambda_\theta(t) = \frac{1}{\lambda}(Y^\lambda_\theta(t) - \bar{Y}_\theta(t)) - Y^1_\theta(t), \quad \delta Z^\lambda_\theta(t) = \frac{1}{\lambda}(Z^\lambda_\theta(t) - \bar{Z}_\theta(t)) - Z^1_\theta(t), \\
&\Gamma_\theta(t) := \mathcal{E}\bigg(\int_0^t \partial_z f_\theta(s)^\top \mathrm{d}W(s)\bigg), \quad t\in[0,T].
\end{aligned} \qquad (4.6)
$$

The following proposition demonstrates the continuity of $(\delta Y^\lambda_\theta, \delta Z^\lambda_\theta)$ concerning θ.

Proposition 4.5 *Under Assumption 1, for* $p \in (1 \vee 2p^{-1}_{\partial_z f_\theta}, 2)$,

$$
\lim_{\lambda\to 0}\sup_{\theta\in\Theta}\mathbb{E}\bigg[\sup_{0\leqslant t\leqslant T}\Gamma_\theta(t)|\delta Y^\lambda_\theta(t)|^p + \bigg(\int_0^T (\Gamma_\theta(t))^{\frac{2}{p}}|\delta Z^\lambda_\theta(t)|^2\mathrm{d}t\bigg)^{\frac{p}{2}}\bigg] = 0.
$$

Proof Set

$$
\begin{aligned}
\kappa^{\lambda\rho}(T) &:= \Big(\bar{X}_\theta(T) + \lambda\rho(X^1_\theta(T) + \delta X^\lambda_\theta(T)), \mathbb{E}[\bar{X}_\theta(T)] + \lambda\rho(\mathbb{E}[X^1_\theta(T)] + \mathbb{E}[\delta X^\lambda_\theta(T)])\Big), \\
\pi^{\lambda\rho}_\theta(t) &:= \Big(\bar{X}_\theta(t) + \lambda\rho(X^1_\theta(t) + \delta X^\lambda_\theta(t)), \mathbb{E}[\bar{X}_\theta(t)] + \lambda\rho(\mathbb{E}[X^1_\theta(t)] + \mathbb{E}[\delta X^\lambda_\theta(t)]) \\
&\qquad \bar{Y}_\theta(t) + \lambda\rho(Y^1_\theta(t) + \delta Y^\lambda_\theta(t)), \bar{Z}_\theta(t) + \lambda\rho(Z^1_\theta(t) + \delta Z^\lambda_\theta(t)), \bar{v}(t) + \lambda\rho(v(t) - \bar{v}(t))\Big), \\
A^\lambda_\theta(t) &:= \int_0^1 \partial_x f_\theta(t, \pi^{\lambda\rho}_\theta(t))\mathrm{d}\rho, \quad \bar{A}^\lambda_\theta(t) := \int_0^1 \partial_{x'} f_\theta(t, \pi^{\lambda\rho}_\theta(t))\mathrm{d}\rho, \\
B^\lambda_\theta(t) &:= \int_0^1 \partial_y f_\theta(t, \pi^{\lambda\rho}_\theta(t))\mathrm{d}\rho, \quad C^\lambda_\theta(t) := \int_0^1 \partial_z f_\theta(t, \pi^{\lambda\rho}_\theta(t))\mathrm{d}\rho.
\end{aligned}
$$

By the definitions of $\delta Y_\theta^\lambda, \delta Z_\theta^\lambda$, we arrive at

$$\begin{aligned}\delta Y_\theta^\lambda(t) &= I_\theta^{1,\lambda} + I_\theta^{2,\lambda}\\ &\quad+\int_t^T \Big(\partial_x f(s)\delta X_\theta^\lambda(s) + \partial_{x'} f(s)\mathbb{E}[\delta X_\theta^\lambda(s)] + \partial_y f(s)\delta Y_\theta^\lambda(s) + \partial_z f(s)\delta Z_\theta^\lambda(s) + D_\theta^\lambda(s)\Big)\mathrm{d}s\\ &\quad-\int_t^T \delta Z_\theta^\lambda(s)\mathrm{d}W(s),\quad t\in[0,T],\end{aligned}$$

where

$$\begin{aligned}I_\theta^{1,\lambda} &:= \int_0^1 \partial_x\Phi_\theta(\kappa^{\lambda\rho}(T))\mathrm{d}\rho\cdot\delta X_\theta^\lambda(T) + \int_0^1 \partial_{x'}\Phi_\theta(\kappa^{\lambda\rho}(T))\mathrm{d}\rho\cdot\mathbb{E}[\delta X_\theta^\lambda(T)],\\ I_\theta^{2,\lambda} &:= \Big(\int_0^1 \partial_x\Phi_\theta(\kappa^{\lambda\rho}(T))\mathrm{d}\rho - \partial_x\Phi_\theta(T)\Big)X_\theta^1(T) + \Big(\int_0^1 \partial_{x'}\Phi_\theta(\kappa^{\lambda\rho}(T))\mathrm{d}\rho - \partial_{x'}\Phi_\theta(T)\Big)\mathbb{E}[X_\theta^1(T)],\\ D_\theta^\lambda(t) &:= \int_0^1 \Big(\partial_v f_\theta(t,\pi_\theta^{\lambda\rho}(t)) - \partial_v f_\theta(t)\Big)(v(t)-\bar v(t))\mathrm{d}\rho + \Big(A_\theta^\lambda(t) - \partial_x f_\theta(t)\Big)(X_\theta^\lambda(t) - \bar X_\theta(t))\\ &\quad+\Big(\bar A_\theta^\lambda(t) - \partial_{x'} f_\theta(t)\Big)\mathbb{E}[X_\theta^\lambda(t) - \bar X_\theta(t)] + \Big(B_\theta^\lambda(t) - \partial_y f_\theta(t)\Big)(Y_\theta^\lambda(t) - \bar Y_\theta(t))\\ &\quad+\Big(C_\theta^\lambda(t) - \partial_z f_\theta(t)\Big)(Z_\theta^\lambda(t) - \bar Z_\theta(t)).\end{aligned}$$

By Hu et al. [16, Proposition 3.2], it yields, for any $p\in(1\vee 2p_{\partial_z f_\theta}^{-1}, 2)$ and $p<\bar p<2$,

$$\begin{aligned}&\mathbb{E}\Big[\sup_{t\in[0,T]}\Gamma_\theta(t)|\delta Y_\theta^\lambda(t)|^p + \Big(\int_0^T (\Gamma_\theta(t))^{\frac{2}{p}}|\delta Z_\theta^\lambda(t)|^2\mathrm{d}t\Big)^{\frac{p}{2}}\Big]\\ &\leqslant C_0\Big(\mathbb{E}\Big[(\Gamma_\theta(T))^{\frac{\bar p}{p}}|I_\theta^{1,\lambda}+I_\theta^{2,\lambda}|^{\bar p} + \Big(\int_0^T (\Gamma_\theta(t))^{\frac{1}{p}}|\partial_x f(t)\delta X_\theta^\lambda(t) + \partial_{x'} f(t)\mathbb{E}[\delta X_\theta^\lambda(t)] + D_\theta^\lambda(t)|\mathrm{d}t\Big)^{\bar p}\Big]\Big)^{\frac{p}{\bar p}}.\end{aligned}\tag{4.7}$$

First, reverse Hölder inequality leads to that, for $p\in(1\vee 2p_{\partial_z f_\theta}^{-1}, 2)$,

$$\mathbb{E}\Big[(\Gamma_\theta(T))^{\frac{2}{p}}\Big] \leqslant K\Big(\frac{2}{p}, \|\partial_z f_\theta\cdot W\|\Big).$$

Consequently, from Lemma 4.2, we know, for all $p\in(1\vee 2p_{\partial_z f_\theta}^{-1}, 2)$ and $p<\bar p<2$,

$$\begin{aligned}&\mathbb{E}\left[(\Gamma_\theta(T))^{\frac{\bar p}{p}}|I_\theta^{1,\lambda}|^{\bar p}\right]\\ &\leqslant C\mathbb{E}\left[(\Gamma_\theta(T))^{\frac{\bar p}{p}}\cdot(|\delta X_\theta^\lambda(T)|^{\bar p} + |\mathbb{E}[\delta X_\theta^\lambda(T)]|^{\bar p})\right]\\ &\leqslant C\Big\{\mathbb{E}\left[(\Gamma_\theta(T))^{\frac{2}{p}}\right]\Big\}^{\frac{\bar p}{2}}\cdot\Big\{\mathbb{E}\left[\sup_{t\in[0,T]}|\delta X_\theta^\lambda(t)|^{\frac{2\bar p}{2-\bar p}}\right]\Big\}^{\frac{(2-\bar p)}{2}}\\ &\leqslant C\left(K(\frac{2}{p}, \|\partial_z f_\theta\cdot W\|)\right)^{\frac{\bar p}{2}}\left\{\mathbb{E}\left[\sup_{t\in[0,T]}|\delta X_\theta^\lambda(t)|^{\frac{2\bar p}{2-\bar p}}\right]\right\}^{\frac{(2-\bar p)}{2}}\to 0,\quad\text{as }\lambda\to 0.\end{aligned}$$

We can similarly deduce that, for all $p\in(1\vee 2p_{\partial_z f_\theta}^{-1}, 2)$ and $p<\bar p<2$,

$$\lim_{\lambda\to 0}\mathbb{E}\Big[(\Gamma_\theta(T))^{\frac{\bar p}{p}}|I_\theta^{2,\lambda}|^{\bar p} + \Big(\int_0^T (\Gamma_\theta(t))^{\frac{1}{p}}|\partial_x f(t)\delta X_\theta^\lambda(t) + \partial_{x'} f(t)\mathbb{E}[\delta X_\theta^\lambda(t)]|\mathrm{d}t\Big)^{\bar p}\Big] = 0.$$

Next, we focus on $\mathbb{E}\left[\left(\int_0^T (\Gamma_\theta(t))^{\frac{1}{p}}|D_\theta^\lambda(t)|\mathrm{d}t\right)^{\bar p}\right]$. Notice

$$
\begin{aligned}
|C_\theta^\lambda(t)-\partial_z f_\theta(t)| \leqslant C\lambda\Big(&|\delta X_\theta^\lambda(t)+X_\theta^1(t)|+|\mathbb{E}[\delta X_\theta^\lambda(t)+X_\theta^1(t)]| \\
&+|\delta Y_\theta^\lambda(t)+Y_\theta^1(t)|+|\delta Z_\theta^\lambda(t)+Z_\theta^1(t)|+|v(t)-\bar v(t)|\Big).
\end{aligned}
$$

We only deal with the most difficult term $\mathbb{E}\bigg[\Big(\int_0^T(\Gamma_\theta(t))^{\frac{1}{p}}\cdot|\bar Z_\theta(t)|\cdot|Z_\theta^1(t)|\mathrm{d}t\Big)^{\bar p}\bigg]$. The other terms can be estimated similarly. Actually, by Hölder inequality, one has

$$
\begin{aligned}
&\mathbb{E}\bigg[\Big(\int_0^T\{\Gamma_\theta(t)\}^{\frac{1}{p}}\cdot|\bar Z_\theta(t)|\cdot|Z_\theta^1(t)|\mathrm{d}t\Big)^{\bar p}\bigg] \\
\leqslant&\bigg\{\mathbb{E}\bigg[\int_0^T\{\Gamma_\theta(t)\}^{\frac{2}{p}}\cdot|\bar Z_\theta(t)|^2dt\bigg]\bigg\}^{\frac{\bar p}{2}}\bigg\{\mathbb{E}\bigg[\Big(\int_0^T|Z_\theta^1(t)|^2\mathrm{d}t\Big)^{\frac{\bar p}{2-\bar p}}\bigg]\bigg\}^{\frac{2-\bar p}{2}}.
\end{aligned}
$$

By Hölder inequality again, together with Doob's inequality and reverse Höler inequality, we further have for fixed $p_1\in(1\vee 2p_{\partial_z f_\theta}^{-1},2)$ and any $p>p_1$,

$$
\begin{aligned}
\mathbb{E}\bigg[\int_0^T(\Gamma_\theta(t))^{\frac{2}{p}}\cdot|\bar Z_\theta(t)|^2\mathrm{d}t\bigg]&\leqslant\bigg\{\mathbb{E}\bigg[\sup_{t\in[0,T]}(\Gamma_\theta(t))^{\frac{2}{p_1}}\bigg]\bigg\}^{\frac{p_1}{p}}\cdot\bigg\{\mathbb{E}\bigg[\Big(\int_0^T|\bar Z_\theta(t)|^2\mathrm{d}t\Big)^{\frac{p}{p-p_1}}\bigg]\bigg\}^{\frac{p-p_1}{p}} \\
&\leqslant\Big\{\mathbb{E}\Big[(\Gamma_\theta(T))^{\frac{2}{p_1}}\Big]\Big\}^{\frac{p_1}{p}}\cdot\bigg\{\mathbb{E}\bigg[\Big(\int_0^T|\bar Z_\theta(t)|^2\mathrm{d}t\Big)^{\frac{p}{p-p_1}}\bigg]\bigg\}^{\frac{p-p_1}{p}}<\infty.
\end{aligned}
$$

The above and (4.5) implies

$$
\mathbb{E}\bigg[\Big(\int_0^T(\Gamma_\theta^\lambda(t))^{\frac{1}{p}}\cdot|\bar Z_\theta(t)|\cdot|Z_\theta^1(t)|\mathrm{d}t\Big)^{\bar p}\bigg]<\infty.
$$

The proof is completed. □

Remark 4.6 *On the one hand, from (4.7), we also have*

$$
\mathbb{E}\bigg[\sup_{t\in[0,T]}\Gamma_\theta^\lambda(t)|\delta Y_\theta^\lambda(t)|^p+\Big(\int_0^T(\Gamma_\theta^\lambda(t))^{\frac{2}{p}}|\delta Z_\theta^\lambda(t)|^2\mathrm{d}t\Big)^{\frac{p}{2}}\bigg]<\infty,
$$

which implies $|\delta Y_\theta^\lambda(0)|<\infty$. On the other hand, by taking $t=0$, it follows from Proposition 4.5 that, for $p\in(1\vee 2p_{\partial_z f_\theta}^{-1},2)$,

$$
\lim_{\lambda\to0}\sup_{\theta\in\Theta}|\delta Y_\theta^\lambda(0)|^p=0.
$$

Remark 4.7 *If coefficient f depends on the mean-field terms $\mathbb{E}[Y_s]$ and $\mathbb{E}[Z_s]$, the present method encounters a difficulty in proving the above Proposition 4.5. We leave it for further work.*

4.2 Variational inequality

This subsection is devoted to studying variational inequality. For given $v(\cdot)\in\mathcal{V}_{ad}$, define

$$
\mathcal{Q}^v=\bigg\{Q\in\mathcal{Q}\bigg|J(v(\cdot))=\int_\Theta\mathbb{E}\Big[\phi_\theta(X_\theta^v(T))+\gamma_\theta(Y_\theta^v(0))\Big]Q(\mathrm{d}\theta)\bigg\}.
$$

Lemma 4.8 *Under Assumption 2, $\mathcal{Q}^v$ is nonempty.*

Proof From the definition of $J(v(\cdot))$, there exists a sequence $Q^n\in\mathcal{Q}$ such that, for all $v(\cdot)\in\mathcal{V}_{ad}$,

$$\int_\Theta \mathbb{E}\Big[\phi_\theta(X_\theta^v(T)) + \gamma_\theta(Y_\theta^v(0))\Big]Q^n(\mathrm{d}\theta) \geqslant J(v(\cdot)) - \frac{1}{n}.$$

Since $\mathcal{Q}$ is weakly compact, there exists some $Q^v \in \mathcal{Q}$ such that, if necessary, a subsequence of Q^n converges weakly to Q^v. Thanks to Theorem 3.2, Proposition 3.3 and Assumption 2, the mapping $\theta \mapsto \mathbb{E}[\phi_\theta(X_\theta^v(T)) + \gamma_\theta(Y_\theta^v(0))]$ is continuous and bounded. Hence, we obtain

$$\begin{aligned} J(v(\cdot)) &\geqslant \int_\Theta \mathbb{E}\Big[\phi_\theta(X_\theta^v(T)) + \gamma_\theta(Y_\theta^v(0))\Big]Q^v(\mathrm{d}\theta) \\ &= \lim_{n\to\infty}\int_\Theta \mathbb{E}\Big[\phi_\theta(X_\theta^v(T)) + \gamma_\theta(Y_\theta^v(0))\Big]Q^n(\mathrm{d}\theta) \geqslant J(v(\cdot)). \end{aligned}$$

Thereby, $\mathcal{Q}^v$ is nonempty. □

We now give the variational inequality.

Theorem 4.9 *Under Assumption 1 and Assumption 2, we have*

$$\lim_{\lambda\to 0}\frac{1}{\lambda}\Big(J(v^\lambda(\cdot)) - J(\bar{v}(\cdot))\Big) = \sup_{Q\in\mathcal{Q}^{\bar{v}}}\int_\Theta \mathbb{E}\Big[\partial_x\phi_\theta(\bar{X}_\theta(T))X_\theta^1(T) + \partial_y\gamma_\theta(\bar{Y}_\theta(0))Y_\theta^1(0)\Big]Q(\mathrm{d}\theta).$$

Proof The proof is split into three steps.

Step 1 We show

$$\liminf_{\lambda\to 0}\frac{1}{\lambda}\Big(J(v^\lambda(\cdot)) - J(\bar{v}(\cdot))\Big) \geqslant \sup_{Q\in\mathcal{Q}^{\bar{v}}}\int_\Theta \mathbb{E}\Big[\partial_x\phi_\theta(\bar{X}_\theta(T))X_\theta^1(T) + \partial_y\gamma_\theta(\bar{Y}_\theta(0))Y_\theta^1(0)\Big]Q(\mathrm{d}\theta). \quad (4.8)$$

From the definition of $\mathcal{Q}^{\bar{v}}$, it yields that, for any $Q \in \mathcal{Q}^{\bar{v}}$,

$$\begin{aligned} J\Big(v^\lambda(\cdot)\Big) &\geqslant \int_\Theta \mathbb{E}\Big[\phi_\theta(X_\theta^\lambda(T)) + \gamma_\theta(Y_\theta^\lambda(0))\Big]Q(\mathrm{d}\theta), \\ J\Big(\bar{v}(\cdot)\Big) &= \int_\Theta \mathbb{E}\Big[\phi_\theta(\bar{X}_\theta(T)) + \gamma_\theta(\bar{Y}_\theta(0))\Big]Q(\mathrm{d}\theta). \end{aligned}$$

Consequently, for any $Q \in \mathcal{Q}^{\bar{v}}$,

$$\frac{1}{\lambda}\Big(J(v^\lambda(\cdot)) - J(\bar{v}(\cdot))\Big) \geqslant \int_\Theta \mathbb{E}\Big[\partial_x\phi_\theta(\bar{X}_\theta(T))X_\theta^1(T) + \partial_y\gamma_\theta(\bar{Y}_\theta(0))Y_\theta^1(0)\Big]Q(\mathrm{d}\theta) + J_1^\lambda + J_2^\lambda,$$

where

$$\begin{aligned} J_1^\lambda &= \int_\Theta \mathbb{E}\left[\int_0^1 \partial_x\phi_\theta^{\lambda\rho}(T)\mathrm{d}\rho\cdot\, \delta X_\theta^\lambda(T) + \int_0^1 \partial_y\gamma_\theta^{\lambda\rho}(0)\mathrm{d}\rho\cdot\, \delta Y_\theta^\lambda(0)\right]Q(\mathrm{d}\theta), \\ J_2^\lambda &= \int_\Theta \mathbb{E}\Big[\int_0^1 \Big(\partial_x\phi_\theta^{\lambda\rho}(T) - \partial_x\phi_\theta(\bar{X}_\theta(T))\Big)\,\mathrm{d}\rho\cdot\, X_\theta^1(T) \\ &\qquad + \int_0^1 \Big(\partial_y\gamma_\theta^{\lambda\rho}(0) - \partial_y\gamma_\theta(\bar{Y}_\theta(0))\Big)\,\mathrm{d}\rho\cdot\, Y_\theta^1(0)\Big]Q(\mathrm{d}\theta), \\ \partial_x\phi_\theta^{\lambda\rho}(T) &= \partial_x\phi_\theta(\bar{X}_\theta(T) + \lambda\rho(X_\theta^1(T) + \delta X_\theta^\lambda(T))), \\ \partial_y\gamma_\theta^{\lambda\rho}(0) &= \partial_y\gamma_\theta(\bar{Y}_\theta(0) + \lambda\rho(Y_\theta^1(0) + \delta Y_\theta^\lambda(0))). \end{aligned} \quad (4.9)$$

On the one hand, since $|\int_0^1 \partial_x\phi_\theta^{\lambda\rho}(T)\mathrm{d}\rho| \leqslant L_1(1 + |\bar{X}_\theta(T)| + |X_\theta^1(T)| + |\delta X_\theta^\lambda(T)|)$ and $|\partial_y\gamma_\theta^{\lambda\rho}(0)| \leqslant C$, it follows that, for any $\theta \in \Theta$,

$$\begin{aligned}&\mathbb{E}\left[\int_0^1 \partial_x\phi_\theta^{\lambda\rho}(T)\mathrm{d}\rho\cdot\ \delta X_\theta^\lambda(T)+\int_0^1 \partial_y\gamma_\theta^{\lambda\rho}(0)\mathrm{d}\rho\cdot\ \delta Y_\theta^\lambda(0)\right]\\ &\leqslant C\Big\{\mathbb{E}\Big[\sup_{t\in[0,T]}(1+|\bar{X}_\theta(t)|^2+|X_\theta^1(t)|^2+|\delta X_\theta^\lambda(t)|^2)\Big]\Big\}^{\frac{1}{2}}\cdot\Big\{\mathbb{E}\Big[\sup_{t\in[0,T]}|\delta X_\theta^\lambda(t)|^2\Big]\Big\}^{\frac{1}{2}}+|\delta Y_\theta^\lambda(0)|.\end{aligned}$$

Making use of (3.1), (4.2), Lemma 4.2 and Proposition 4.5 (or Remark 4.6) we have

$$0\leqslant \lim_{\lambda\to 0}|J_1^\lambda|\leqslant \lim_{\lambda\to 0}\sup_{\theta\in\Theta}\mathbb{E}\left[\int_0^1 \partial_x\phi_\theta^{\lambda\rho}(T)\mathrm{d}\rho\cdot\ \delta X_\theta^\lambda(T)+\int_0^1 \partial_y\gamma_\theta^{\lambda\rho}(0)\mathrm{d}\rho\cdot\ \delta Y_\theta^\lambda(0)\right]=0.$$

Moreover, based on (4.2), Lemma 4.2, (4.5), Remark 4.6, and the assumption that $\partial_y\gamma_\theta$ is Lipschitz continuous in y uniformly in θ, we have

$$\begin{aligned}|J_2^\lambda|\leqslant&\ \sup_{\theta\in\Theta}\mathbb{E}\left[\Big|\int_0^1(\partial_x\phi_\theta^{\lambda\rho}(T)-\partial_x\phi_\theta(\bar{X}_\theta(T)))\mathrm{d}\rho|\cdot\ |X_\theta^1(T)\Big|\right]\\ &+\int_0^1\sup_{\theta\in\Theta}|\partial_y\gamma_\theta^{\lambda\rho}(0)-\partial_y\gamma_\theta(\bar{Y}_\theta(0))|\mathrm{d}\rho\cdot\ \sup_{\theta\in\Theta}|Y_\theta^1(0)|\\ \leqslant&\ C\lambda\mathbb{E}\left[\sup_{t\in[0,T]}|X_\theta^1(t)|^2+\sup_{t\in[0,T]}|\delta X_\theta^\lambda(t)|^2\right]\\ &+\int_0^1\sup_{\theta\in\Theta}|\partial_y\gamma_\theta^{\lambda\rho}(0)-\partial_y\gamma_\theta(\bar{Y}_\theta(0))|\mathrm{d}\rho\cdot\ \sup_{\theta\in\Theta}|Y_\theta^1(0)|\to 0,\quad \text{as } \lambda\to 0.\end{aligned}$$

From the above estimate, (4.8) follows immediately.

Step 2 Find a subsequence $\lambda_n\to 0$ such that

$$\limsup_{\lambda\to 0}\frac{J(v^\lambda(\cdot))-J(\bar{v}(\cdot))}{\lambda}=\lim_{n\to\infty}\frac{J(v^{\lambda_n}(\cdot))-J(\bar{v}(\cdot))}{\lambda_n}.$$

For each $n\in\mathbb{N}$, due to $\mathcal{Q}^{v^{\lambda_n}}$ being nonempty, there exists a probability measure $Q^{\lambda_n}\in\mathcal{Q}^{v^{\lambda_n}}$ such that

$$\begin{aligned}J\big(v^{\lambda_n}(\cdot)\big)&=\int_\Theta\mathbb{E}\left[\phi_\theta(X_\theta^{\lambda_n}(T))+\gamma_\theta(Y_\theta^{\lambda_n}(0))\right]Q^{\lambda_n}(\mathrm{d}\theta),\\ J\big(\bar{v}(\cdot)\big)&\geqslant\int_\Theta\mathbb{E}\Big[\phi_\theta(\bar{X}_\theta(T))+\gamma_\theta(\bar{Y}_\theta(0))\Big]Q^{\lambda_n}(\mathrm{d}\theta).\end{aligned}$$

To see this, first notice

$$\begin{aligned}\frac{J\big(v^{\lambda_n}(\cdot)\big)-J\big(\bar{v}(\cdot)\big)}{\lambda_n}&\leqslant\int_\Theta\frac{\mathbb{E}[\phi_\theta(X_\theta^{\lambda_n}(T))-\phi_\theta(\bar{X}_\theta(T))]+\gamma_\theta(Y_\theta^{\lambda_n}(0))-\gamma_\theta(\bar{Y}_\theta(0))}{\lambda_n}Q^{\lambda_n}(\mathrm{d}\theta)\\ &=\int_\Theta\mathbb{E}\Big[\partial_x\phi_\theta(\bar{X}_\theta(T))X_\theta^1(T)+\partial_y\gamma_\theta(\bar{Y}_\theta(0))Y_\theta^1(0)\Big]Q^{\lambda_n}(\mathrm{d}\theta)+J_1^{\lambda_n}+J_2^{\lambda_n},\end{aligned}\tag{4.10}$$

where $J_1^{\lambda_n}, J_2^{\lambda_n}$ are given in (4.9). Similar to Step 1, we can show

$$\lim_{n\to\infty}\Big(|J_1^{\lambda_n}|+|J_2^{\lambda_n}|\Big)=0.$$

Since $\mathcal{Q}$ is weakly compact, there exists a probability measure $\widehat{Q}\in\mathcal{Q}$ such that the sequence $(Q^{\lambda_n})_{n\in\mathbb{N}}$, possibly by selecting a subsequence, converges weakly to $\widehat{Q}$. In addition, based on results from Theorem 3.2, Proposition 3.3, Proposition 4.1, Proposition 4.3, (4.2), (4.5), and Assumption 2, the mapping $\theta\mapsto\mathbb{E}[\partial_x\phi_\theta(\bar{X}_\theta(T))X_\theta^1(T)+\partial_y\gamma_\theta(\bar{Y}_\theta(0))Y_\theta^1(0)]$ is both continuous and bounded. As $n\to\infty$ in (4.10), it yields

$$\begin{aligned}\limsup_{\lambda\to 0}\frac{J(v^{\lambda}(\cdot))-J(\bar v(\cdot))}{\lambda}&=\lim_{n\to\infty}\frac{J(v^{\lambda_n}(\cdot))-J(\bar v(\cdot))}{\lambda_n}\\&\leqslant\int_\Theta\mathbb{E}\Big[\partial_x\phi_\theta(\bar X_\theta(T))X^1_\theta(T)+\partial_y\gamma_\theta(\bar Y_\theta(0))Y^1_\theta(0)\Big]\widehat{Q}(\mathrm{d}\theta).\end{aligned}$$

Step 3 We prove $\widehat{Q}\in\mathcal{Q}^{\bar v}$. For this, we need to show

$$J(\bar v(\cdot))=\int_\Theta\mathbb{E}\Big[\phi_\theta(\bar X_\theta(T))+\gamma_\theta(\bar Y_\theta(0))\Big]\widehat{Q}(\mathrm{d}\theta).$$

In fact, since $|\partial_x\phi_\theta^{\lambda_n\rho}(T)|\leqslant L_1(1+|\bar X_\theta(T)|+|X^1_\theta(T)|+|\delta X^\lambda_\theta(T)|)$ and $|\partial_y\gamma_\theta^{\lambda_n\rho}(0)|\leqslant C$, it follows from (3.1), (4.2), Lemma 4.2, (4.5) and Remark 4.6 that, for all $\theta\in\Theta$,

$$\begin{aligned}&\Big|\mathbb{E}[\phi_\theta(X^{\lambda_n}_\theta(T))-\phi_\theta(\bar X_\theta(T))]+\gamma_\theta(Y^{\lambda_n}_\theta(0))-\gamma_\theta(\bar Y_\theta(0))\Big|\\=&\lambda_n\Big|\mathbb{E}\Big[\int_0^1\partial_x\phi_\theta^{\lambda_n\rho}(T)\mathrm{d}\rho\cdot(X^1_\theta(T)+\delta X^{\lambda_n}_\theta(T))\Big]\Big|+\lambda_n\Big|\int_0^1\partial_y\gamma_\theta^{\lambda_n\rho}(0)\mathrm{d}\rho\cdot(Y^1_\theta(0)+\delta Y^{\lambda_n}_\theta(0))\Big|\\\leqslant&\lambda_n\Big\{\mathbb{E}\Big[1+|\bar X_\theta(T)|^2+|X^1_\theta(T)|^2+|\delta X^{\lambda_n}_\theta(T)|^2\Big]+|Y^1_\theta(0)+\delta Y^{\lambda_n}_\theta(0)|\Big\}\to 0,\quad\text{as}\quad\lambda_n\to 0.\end{aligned}$$

Above inequality leads to

$$\begin{aligned}&\lim_{n\to\infty}|J(v^{\lambda_n}(\cdot))-J(\bar v(\cdot))|\\&\leqslant\lim_{n\to\infty}\int_\Theta\Big|\mathbb{E}[\phi_\theta(X^{\lambda_n}_\theta(T))-\phi_\theta(\bar X_\theta(T))]\Big|+\Big|\gamma_\theta(Y^{\lambda_n}_\theta(0))-\gamma_\theta(\bar Y_\theta(0))\Big|Q^{\lambda_n}(\mathrm{d}\theta)\\&\leqslant\lim_{n\to\infty}\sup_{\theta\in\Theta}\mathbb{E}\Big[\Big|\phi_\theta(X^{\lambda_n}_\theta(T))-\phi_\theta(\bar X_\theta(T))\Big|\Big]+\Big|\gamma_\theta(Y^{\lambda_n}_\theta(0))-\gamma_\theta(\bar Y_\theta(0))\Big|=0\end{aligned}$$

and

$$\begin{aligned}&\lim_{n\to\infty}\Big|\int_\Theta\Big(\mathbb{E}[\phi_\theta(X^{\lambda_n}_\theta(T))+\gamma_\theta(Y^{\lambda_n}_\theta(0))]-\mathbb{E}[\phi_\theta(\bar X_\theta(T))+\gamma_\theta(\bar Y_\theta(0))]\Big)Q^{\lambda_n}(\mathrm{d}\theta)\Big|\\\leqslant&\lim_{n\to\infty}\int_\Theta\Big|\mathbb{E}[\phi_\theta(X^{\lambda_n}_\theta(T))-\phi_\theta(\bar X_\theta(T))]\Big|+\Big|\gamma_\theta(Y^{\lambda_n}_\theta(0))-\gamma_\theta(\bar Y_\theta(0))\Big|Q^{\lambda_n}(\mathrm{d}\theta)\\\leqslant&\lim_{n\to\infty}\sup_{\theta\in\Theta}\Big\{\mathbb{E}\Big[\Big|\phi_\theta(X^{\lambda_n}_\theta(T))-\phi_\theta(\bar X_\theta(T))\Big|\Big]+\Big|\gamma_\theta(Y^{\lambda_n}_\theta(0))-\gamma_\theta(\bar Y_\theta(0))\Big|\Big\}=0.\end{aligned}$$

Consequently, it yields

$$\begin{aligned}J(\bar v(\cdot))&=\lim_{n\to\infty}J(v^{\lambda_n}(\cdot))=\lim_{n\to\infty}\int_\Theta\mathbb{E}[\phi_\theta(X^{\lambda_n}_\theta(T))+\gamma_\theta(Y^{\lambda_n}_\theta(0))]Q^{\lambda_n}(\mathrm{d}\theta)\\&=\lim_{n\to\infty}\int_\Theta\mathbb{E}[\phi_\theta(\bar X_\theta(T))+\gamma_\theta(\bar Y_\theta(0))]Q^{\lambda_n}(\mathrm{d}\theta)=\int_\Theta\mathbb{E}[\phi_\theta(\bar X_\theta(T))+\gamma_\theta(\bar Y_\theta(0))]\widehat{Q}(\mathrm{d}\theta).\end{aligned}$$

□

From Theorem 4.9, we have the main theorem in this section.

Theorem 4.10 *Under Assumption 1 and Assumption 2, there exists a probability* $\bar Q\in\mathcal{Q}^{\bar v}$ *such that, for any* $v(\cdot)\in\mathcal{V}_{0,T}$,

$$\int_\Theta\mathbb{E}\Big[\partial_x\phi_\theta(\bar X_\theta(T))X^1_\theta(T)+\partial_y\gamma_\theta(\bar Y_\theta(0))Y^1_\theta(0)\Big]\bar Q(\mathrm{d}\theta)\geqslant 0.$$

Proof For a more precise expression, we write X^1_θ, Y^1_θ as $X^{1,v}_\theta, Y^{1,v}_\theta$. It follows Theorem 4.9 that, for any $v(\cdot)\in\mathcal{V}_{ad}$,

$$0 \leqslant \lim_{\lambda \to 0} \frac{1}{\lambda} \big(J(v^\lambda(\cdot)) - J(\bar{v}(\cdot)) \big) = \sup_{Q \in \mathcal{Q}^{\bar{v}}} \int_\Theta \mathbb{E}\left[\partial_x \phi_\theta(\bar{X}_\theta(T)) X^{1,v}_\theta(T) + \partial_y \gamma_\theta(\bar{Y}_\theta(0)) Y^{1,v}_\theta(0) \right] Q(\mathrm{d}\theta),$$

which implies

$$\inf_{v(\cdot)\in\mathcal{V}_{ad}} \sup_{Q \in \mathcal{Q}^{\bar{v}}} \int_\Theta \mathbb{E}\left[\partial_x \phi_\theta(\bar{X}_\theta(T)) X^{1,v}_\theta(T) + \partial_y \gamma_\theta(\bar{Y}_\theta(0)) Y^{1,v}_\theta(0) \right] Q(\mathrm{d}\theta) \geqslant 0.$$

Define $\Upsilon^v_\theta := \mathbb{E}\left[\partial_x \phi_\theta(\bar{X}_\theta(T)) X^{1,v}_\theta(T) + \partial_y \gamma_\theta(\bar{Y}_\theta(0)) Y^{1,v}_\theta(0) \right]$. From the linearity of $X^{1,v}, Y^{1,v}_\theta$ concerning v, we know that $v \mapsto \Upsilon^v_\theta$ is a linear mapping, i.e., for $v(\cdot), v'(\cdot) \in \mathcal{V}_{ad}, 0 < l < 1$ and $\theta \in \Theta$,

$$\Upsilon^{lv+(1-l)v'}_\theta = l\Upsilon^v_\theta + (1-l)\Upsilon^{v'}_\theta.$$

In addition, for $v(\cdot), v'(\cdot) \in \mathcal{V}_{ad}$ and $\theta \in \Theta$,

$$\begin{aligned} |\Upsilon^v_\theta - \Upsilon^{v'}_\theta| &\leqslant C\mathbb{E}\left[(1+|\bar{X}_\theta(T)|) \left| X^{1,v}_\theta(T) - X^{1,v'}_\theta(T) \right| \right] + \left| Y^{1,v}_\theta(0) - Y^{1,v'}_\theta(0) \right| \\ &\leqslant C\Big\{ \mathbb{E}[1+|\bar{X}_\theta(T)|^2] \Big\}^{\frac{1}{2}} \cdot \Big\{ \mathbb{E}\Big[\Big| X^{1,v}_\theta(T) - X^{1,v'}_\theta(T) \Big|^2 \Big] \Big\}^{\frac{1}{2}} + \Big| Y^{1,v}_\theta(0) - Y^{1,v'}_\theta(0) \Big|. \end{aligned}$$

This, together with (4.2) and (4.5), show that Υ^v_θ is continuous with respect to v uniformly in θ. Leveraging Sion's minimax theorem, we obtain

$$\begin{aligned} &\inf_{v(\cdot)\in\mathcal{V}_{ad}} \sup_{Q \in \mathcal{Q}^{\bar{v}}} \int_\Theta \mathbb{E}\left[\partial_x \phi_\theta(\bar{X}_\theta(T)) X^{1,v}_\theta(T) + \partial_y \gamma_\theta(\bar{Y}_\theta(0)) Y^{1,v}_\theta(0) \right] Q(\mathrm{d}\theta) \\ =& \sup_{Q \in \mathcal{Q}^{\bar{v}}} \inf_{v(\cdot)\in\mathcal{V}_{ad}} \int_\Theta \mathbb{E}\left[\partial_x \phi_\theta(\bar{X}_\theta(T)) X^{1,v}_\theta(T) + \partial_y \gamma_\theta(\bar{Y}_\theta(0)) Y^{1,v}_\theta(0) \right] Q(\mathrm{d}\theta) \geqslant 0. \end{aligned}$$

Consequently, for any $\epsilon > 0$, there exists a $Q^\epsilon \in \mathcal{Q}^{\bar{v}}$ such that

$$\inf_{v(\cdot)\in\mathcal{V}_{ad}} \int_\Theta \mathbb{E}\left[\partial_x \phi_\theta(\bar{X}_\theta(T)) X^{1,v}_\theta(T) + \partial_y \gamma_\theta(\bar{Y}_\theta(0)) Y^{1,v}_\theta(0) \right] Q^\epsilon(\mathrm{d}\theta) \geqslant -\epsilon.$$

Ultimately, the compactness of $\mathcal{Q}^{\bar{v}}$ enables us to select a subsequence $\epsilon_n \to 0$ such that Q^{ϵ_n} converges weakly to a probability measure $\bar{Q} \in \mathcal{Q}^{\bar{v}}$ and

$$\begin{aligned} &\int_\Theta \mathbb{E}\left[\partial_x \phi_\theta(\bar{X}_\theta(T)) X^{1,v}_\theta(T) + \partial_y \gamma_\theta(\bar{Y}_\theta(0)) Y^{1,v}_\theta(0) \right] \bar{Q}(\mathrm{d}\theta) \\ =& \lim_{\epsilon_n \to 0} \int_\Theta \mathbb{E}\left[\partial_x \phi_\theta(\bar{X}_\theta(T)) X^{1,v}_\theta(T) + \partial_y \gamma_\theta(\bar{Y}_\theta(0)) Y^{1,v}_\theta(0) \right] Q^{\epsilon_n}(\mathrm{d}\theta) \geqslant 0. \end{aligned}$$

□

5. Necessary and sufficient maximum principles

This section discusses the necessary and sufficient maximum principles. For this, we introduce the adjoint equation

$$\begin{cases} \mathrm{d}p_\theta(t) = \partial_y f_\theta(t) p_\theta(t) \mathrm{d}t + \partial_z f_\theta(t) p_\theta(t) \mathrm{d}W(t), \quad t \in [0,T], \\ -\mathrm{d}q_\theta(t) = \Big(-\partial_x f_\theta(t) p_\theta(t) + (\partial_x b_\theta(t))^\top q_\theta(t) + \displaystyle\sum_{i=1}^d (\partial_x^i \sigma_\theta(t))^\top r^i_\theta(t) - \mathbb{E}[\partial_{x'} f_\theta(t) p_\theta(t)] \\ \qquad\qquad + \mathbb{E}[(\partial_{x'} b_\theta(t))^\top q_\theta(t)] \Big) \mathrm{d}t - r_\theta(t) \mathrm{d}W(t), \quad t \in [0,T], \\ p_\theta(0) = -\partial_y \gamma_\theta(\bar{Y}_\theta(0)), \\ q_\theta(T) = -(\partial_x \Phi_\theta(\bar{X}_\theta(T), \mathbb{E}[\bar{X}_\theta(T)]))^\top p_\theta(T) - \mathbb{E}\Big[\partial_{x'} \Phi_\theta(\bar{X}_\theta(T), \mathbb{E}[\bar{X}_\theta(T)]))^\top p_\theta(T) \Big] \\ \qquad\qquad + \partial_x \phi_\theta(\bar{X}_\theta(T)). \end{cases} \tag{5.1}$$

The forward equation in (5.1) is a 1-dimensional SDE with unbounded coefficients. The solvability of (5.1) stems from the following lemma, which is an immediate adaptation of Gal'chuk [11, Lemma 3 and Lemma 4].

Lemma 5.1 *We assume* $\mathbb{E}\left[\int_0^T |\varphi_1(t)|^p \mathrm{d}t\right] < \infty$, $\mathbb{E}\left(\int_0^T |\varphi_2(t)|^2 \mathrm{d}t\right)^{\frac{p}{2}} < \infty$, *and* $a_1 \cdot W$, $a_2 \cdot W \in \mathrm{BMO}$. *Then, for* $p > 1$, *the 1-dimensional SDE with unbounded coefficients below*

$$\begin{cases} \mathrm{d}X(t) = [a_1(t)X(t) + \varphi_1(t)]\mathrm{d}t + [a_2(t)X(t) + \varphi_2(t)]\mathrm{d}W(t), \quad t \in [0,T], \\ X(0) = x_0 \in \mathbb{R} \end{cases} \tag{5.2}$$

has a unique solution $X(\cdot) \in \mathcal{S}_{\mathbb{F}}^p([0,T];\mathbb{R})$.

Proof Define $\tau_0 = 0$,

$$\tau_{n+1} = \inf\left\{ t > \tau_n : \left(\int_{\tau_n}^t |a_2(s)|^2 \mathrm{d}s \right)^{\frac{p}{2}} \geqslant 2^{-p-1} C_p^{-1} \quad \text{or} \quad \int_{\tau_n}^t |a_1(s)|^p \mathrm{d}s \geqslant 2^{-p-1} \right\} \wedge T,$$

where C_p only depending on p is the constant in Buckholder−Davis−Gundy inequality. Clearly, $\tau_n \uparrow T$, as $n \uparrow \infty$.

The proof is split into two steps.

Step 1 We show that the equation (5.2) possesses a unique solution $X_1(\cdot) \in \mathcal{S}_{\mathbb{F}}^p([\tau_0,\tau_1];\mathbb{R})$ on the interval $[\tau_0,\tau_1]$ with the fixed point theorem. Let $X \in \mathcal{S}_{\mathbb{F}}^p([\tau_0,\tau_1];\mathbb{R})$. Define

$$\begin{cases} \Gamma(X)(t) := x_0 + \displaystyle\int_{\tau_0}^t \Big[a_1(s)X(s) + \varphi_1(s)\Big]\mathrm{d}s + \int_{\tau_0}^t \Big[a_2(s)X(s) + \varphi_2(s)\Big]\mathrm{d}W(s), \quad t \in [\tau_0,\tau_1], \\ \Gamma(X)(\tau_0) := x_0. \end{cases}$$

It is easy to check $\Gamma(X) \in \mathcal{S}_{\mathbb{F}}^p([\tau_0,\tau_1];\mathbb{R})$. For $X', X'' \in \mathcal{S}_{\mathbb{F}}^p([\tau_0,\tau_1];\mathbb{R})$, define

$$\Delta X = X' - X'', \quad \Gamma(\Delta X) = \Gamma(X') - \Gamma(X'').$$

Then it follows that

$$\begin{cases} \Gamma(\Delta X)(t) = \displaystyle\int_{\tau_0}^t a_1(s)\Delta X(s)\mathrm{d}s + \int_{\tau_0}^t a_2(s)\Delta X(s)\mathrm{d}W(s), \quad t \in [\tau_0,\tau_1], \\ \Gamma(\Delta X)(\tau_0) = 0. \end{cases}$$

By Buckholder−Davis−Gundy inequality and Hölder inequality, we have, for $p > 1$

$$\begin{aligned} \mathbb{E}\Big[\sup_{t\in[\tau_0,\tau_1]} |\Gamma(\Delta X)(t)|^p\Big] \leqslant{}& 2^{p-1}\mathbb{E}\Big[\int_{\tau_0}^{\tau_1} |a_1(s)|^p \mathrm{d}s \cdot \sup_{t\in[\tau_0,\tau_1]} |\Delta X(t)|^p\Big] \\ &+ 2^{p-1}C_p\mathbb{E}\bigg[\Big(\int_{\tau_0}^{\tau_1} |a_2(s)|^2\mathrm{d}s\Big)^{\frac{p}{2}} \cdot \sup_{t\in[\tau_0,\tau_1]} |\Delta X(t)|^p\bigg]. \end{aligned}$$

Recalling the definition of τ_1, we deduce

$$\mathbb{E}\Big[\sup_{t\in[\tau_0,\tau_1]} |\Gamma(\Delta X)(t)|^p\Big] \leqslant \frac{1}{2}\mathbb{E}\Big[\sup_{t\in[\tau_0,\tau_1]} |\Delta X(t)|^p\Big].$$

The fixed point theorem yields that SDE (5.2) has a unique solution $X_1(\cdot) \in \mathcal{S}_{\mathbb{F}}^p([\tau_0,\tau_1];\mathbb{R})$ on the interval $[\tau_0,\tau_1]$.

Step 2 Assuming that a solution X of (5.2) has been constructed on $[\tau_0, \tau_n]$, we construct it on $(\tau_n, \tau_{n+1}]$. For this, define

$$\Gamma(X)(t) := X(\tau_n) + \int_{\tau_n}^t \Big[a_1(s)X(s) + \varphi_1(s)\Big]\mathrm{d}s + \int_{\tau_n}^t \Big[a_2(s)X(s) + \varphi_2(s)\Big]\mathrm{d}W(s), \quad t \in [\tau_n, \tau_{n+1}]. \tag{5.3}$$

According to the definition of τ_{n+1}, and following an approach similar to **Step 1**, it can be determined that (5.3) has a unique solution $X_{n+1} \in \mathcal{S}_{\mathbb{F}}^p([\tau_n, \tau_{n+1}]; \mathbb{R})$. Repeating this process repeatedly, a solution for equation (5.2) on the interval $[0, T]$ can be constructed. Moreover, the uniqueness of this solution is guaranteed by the constructive nature of the procedure. □

From Lemma 5.1, the forward equation in (5.1) has a unique strong solution $p(\cdot) \in \mathcal{S}_{\mathbb{F}}^p([0, T]; \mathbb{R})$ for $p > 1$. The backward equation in (5.1) is a n-dimensional mean-field BSDE with a bounded Lipschitz coefficient. According to Chen et al. [5, Theorem 3.4], for $p > 1$, it has a unique adapted solution $(q(\cdot), r(\cdot)) \in \mathcal{S}_{\mathbb{F}}^p([0, T]; \mathbb{R}^n) \times \mathcal{H}_{\mathbb{F}}^{2,p}([0, T]; \mathbb{R}^{n\times d})$.

The following lemma shows that p_θ is continuous with respect to θ.

Lemma 5.2 *Under Assumption 1, for* $1 < p < (p_{\partial_z f_\theta} \wedge p_{(\partial_z f_{\tilde{\theta}} - \partial_z f_\theta)})$,

$$\lim_{\epsilon\to 0} \sup_{\Xi(\theta,\tilde{\theta})\leqslant\epsilon} \mathbb{E}\Big[\sup_{t\in[0,T]} |p_\theta(t) - p_{\tilde{\theta}}(t)|^p\Big] = 0.$$

Proof Since $|p_\theta(t)|^2 > 0$ for $t \in [0, T]$, applying Itô's formula to $\ln |p_\theta(t)|^2$ on $[0, T]$, it leads to

$$p_\theta(t) = \pm p_\theta(0)\cdot e^{\int_0^t \partial_y f_\theta(s)\mathrm{d}s}\cdot\mathcal{E}\left(\int_0^t \partial_z f_\theta(s)^\top \mathrm{d}W(s)\right).$$

We only show $p_\theta(t) = -p_\theta(0)\cdot e^{\int_0^t \partial_y f_\theta(s)\mathrm{d}s}\cdot\mathcal{E}(\int_0^t \partial_z f_\theta(s)^\top \mathrm{d}W(s))$, i.e.,

$$p_\theta(t) = \partial_y\gamma_\theta(\bar{Y}_\theta(0))\cdot e^{\int_0^t \partial_y f_\theta(s)\mathrm{d}s}\cdot\mathcal{E}\left(\int_0^t \partial_z f_\theta(s)^\top \mathrm{d}W(s)\right).$$

Notice

$$p_\theta(t) - p_{\tilde{\theta}}(t) = I_1(t) + I_2(t) + I_3(t),$$

where

$$\begin{aligned}
I_1(t) &:= \Big[\partial_y\gamma_\theta(\bar{Y}_\theta(0)) - \partial_y\gamma_{\tilde{\theta}}(\bar{Y}_{\tilde{\theta}}(0))\Big]\cdot e^{\int_0^t \partial_y f_\theta(s)\mathrm{d}s}\cdot\mathcal{E}\left(\int_0^t \partial_z f_\theta(s)^\top \mathrm{d}W(s)\right),\\
I_2(t) &:= \partial_y\gamma_{\tilde{\theta}}(\bar{Y}_{\tilde{\theta}}(0))\cdot \left[e^{\int_0^t \partial_y f_\theta(s)\mathrm{d}s} - e^{\int_0^t \partial_y f_{\tilde{\theta}}(s)\mathrm{d}s}\right]\cdot\mathcal{E}\left(\int_0^t \partial_z f_\theta(s)^\top \mathrm{d}W(s)\right),\\
I_3(t) &:= \partial_y\gamma_{\tilde{\theta}}(\bar{Y}_{\tilde{\theta}}(0))\cdot e^{\int_0^t \partial_y f_{\tilde{\theta}}(s)\mathrm{d}s}\cdot \left[\mathcal{E}\left(\int_0^t \partial_z f_\theta(s)^\top \mathrm{d}W(s)\right) - \mathcal{E}\left(\int_0^t \partial_z f_{\tilde{\theta}}(s)^\top \mathrm{d}W(s)\right)\right].
\end{aligned}$$

For $I_1(t)$, since

$$\begin{aligned}
|\partial_y\gamma_\theta(\bar{Y}_\theta(0)) - \partial_y\gamma_{\tilde{\theta}}(\bar{Y}_{\tilde{\theta}}(0))| &\leqslant |\partial_y\gamma_\theta(\bar{Y}_\theta(0)) - \partial_y\gamma_{\tilde{\theta}}(\bar{Y}_\theta(0))| + |\partial_y\gamma_{\tilde{\theta}}(\bar{Y}_\theta(0)) - \partial_y\gamma_{\tilde{\theta}}(\bar{Y}_{\tilde{\theta}}(0))|\\
&\leqslant \Xi(\theta,\tilde{\theta}) + C|\bar{Y}_\theta(0) - \bar{Y}_{\tilde{\theta}}(0)|,
\end{aligned}$$

it yields from Doob's inequality and reverse Hölder inequality that

$$
\begin{aligned}
\mathbb{E}\Big[\sup_{t\in[0,T]}|I_1(t)|^p\Big] &\leqslant C_p\mathbb{E}\Big[\sup_{t\in[0,T]}\Big\{(\Xi^p(\theta,\tilde\theta)+C|\bar Y_\theta(0)-\bar Y_{\tilde\theta}(0)|^p)\cdot e^{pTC_1}\cdot\mathcal{E}\Big(\int_0^t\partial_z f_\theta(s)^\top \mathrm{d}W(s)\Big)^p\Big\}\Big]\\
&\leqslant C_p\Xi^p(\theta,\tilde\theta)\mathbb{E}\Big[\sup_{t\in[0,T]}\mathcal{E}\Big(\int_0^t\partial_z f_\theta(s)^\top \mathrm{d}W(s)\Big)^p\Big]\\
&\quad +C_p\Big\{\mathbb{E}\Big[\sup_{t\in[0,T]}|\bar Y_\theta(t)-\bar Y_{\tilde\theta}(t)|^{pq'}\Big]\Big\}^{\frac{1}{q'}}\cdot\Big\{\mathbb{E}\Big[\sup_{t\in[0,T]}\mathcal{E}\Big(\int_0^t\partial_z f_\theta(s)^\top \mathrm{d}W(s)\Big)^{pp'}\Big]\Big\}^{\frac{1}{p'}}\\
&\leqslant C_p\Xi^p(\theta,\tilde\theta)\mathbb{E}\Big[\mathcal{E}\Big(\int_0^T\partial_z f_\theta(s)^\top \mathrm{d}W(s)\Big)^p\Big]\\
&\quad +C_p\Big\{\mathbb{E}\Big[\sup_{t\in[0,T]}|\bar Y_\theta(t)-\bar Y_{\tilde\theta}(t)|^{pq'}\Big]\Big\}^{\frac{1}{q'}}\cdot\Big\{\mathbb{E}\Big[\mathcal{E}\Big(\int_0^T\partial_z f_\theta(s)^\top \mathrm{d}W(s)\Big)^{pp'}\Big]\Big\}^{\frac{1}{p'}}\\
&\leqslant C_p\Xi^p(\theta,\tilde\theta)\cdot K(p,\|\partial_z f_\theta\cdot W\|_{\mathrm{BMO}})\\
&\quad +\Big\{\mathbb{E}\Big[\sup_{t\in[0,T]}|\bar Y_\theta(t)-\bar Y_{\tilde\theta}(t)|^{pq'}\Big]\Big\}^{\frac{1}{q'}}\cdot K(pp',\|\partial_z f_\theta\cdot W\|_{\mathrm{BMO}}),
\end{aligned}
$$

where $p'=\frac{p+p_{\partial_z f_\theta}}{2p}$ and $q'=\frac{p'}{p'-1}$. Clearly $pq'>p^*_{\partial_z f_\theta}$, and by Proposition 3.3 we have, for $1<p<p_{\partial_z f_\theta}$,

$$\lim_{\epsilon\to0}\sup_{\Xi(\theta,\tilde\theta)\leqslant\epsilon}\mathbb{E}\left[\sup_{t\in[0,T]}|I_1(t)|^p\right]=0.$$

For $I_2(t)$, since

$$\left[e^{\int_0^t\partial_y f_\theta(s)\mathrm{d}s}-e^{\int_0^t\partial_y f_{\tilde\theta}(s)\mathrm{d}s}\right]\leqslant e^{C_1T}\cdot\left[e^{T\Xi(\theta,\tilde\theta)}-1\right],$$

we know, for $1<p<p_{\partial_z f_\theta}$,

$$
\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]}|I_2(t)|^p\right]&\leqslant Le^{C_1T}\cdot\left[e^{T\Xi(\theta,\tilde\theta)}-1\right]\cdot\mathbb{E}\left[\sup_{t\in[0,T]}\mathcal{E}\left(\int_0^t\partial_z f_\theta(s)^\top\mathrm{d}W(s)\right)^p\right]\\
&\leqslant Le^{C_1T}\cdot\left[e^{T\Xi(\theta,\tilde\theta)}-1\right]\cdot K(p,\|\partial_z f_\theta\cdot W\|_{\mathrm{BMO}}).
\end{aligned}
$$

Finally, we prove, for $1<p<(p_{\partial_z f_\theta}\wedge p_{(\partial_z f_{\tilde\theta}-\partial_z f_\theta)})$,

$$\lim_{\epsilon\to0}\sup_{\Xi(\theta,\tilde\theta)\leqslant\epsilon}\mathbb{E}\left[\sup_{t\in[0,T]}|I_3(t)|^p\right]=0.$$

In fact, due to $|\partial_y\gamma_{\tilde\theta}(\bar Y_{\tilde\theta}(0))\cdot e^{\int_0^t\partial_y f_{\tilde\theta}(s)\mathrm{d}s}|\leqslant C$, we only need to prove, for $1<p<(p_{\partial_z f_\theta}\wedge p_{(\partial_z f_{\tilde\theta}-\partial_z f_\theta)})$,

$$\lim_{\epsilon\to0}\sup_{\Xi(\theta,\tilde\theta)\leqslant\epsilon}\mathbb{E}\left[\sup_{t\in[0,T]}\left|\mathcal{E}\left(\int_0^t\partial_z f_\theta(s)^\top\mathrm{d}W(s)\right)-\mathcal{E}\left(\int_0^t\partial_z f_{\tilde\theta}(s)^\top\mathrm{d}W(s)\right)\right|^p\right]=0.$$

For convenience, set $X_\theta(t)=\mathcal{E}(\int_0^t\partial_z f_\theta(s)^\top\mathrm{d}W(s))$. It is easy to see that X_θ satisfies the following SDE

$$X_\theta(t)=1+\int_0^t\partial_z f_\theta(s)X_\theta(s)\mathrm{d}W(s),\quad t\in[0,T].$$

Set $\bar X(s)=X_\theta(s)-X_{\tilde\theta}(s)$ and $\varphi(s)=(\partial_z f_\theta(s)-\partial_z f_{\tilde\theta}(s))X_{\tilde\theta}(s)$. Then

$$\bar{X}(t)=\int_0^t\big(\partial_z f_\theta(s)\bar{X}(s)+\varphi(s)\big)\,\mathrm{d}W(s),\quad t\in[0,T].$$

Consider

$$\begin{cases}\mathrm{d}\Pi_\theta(t)=(\partial_z f_\theta(t))^2\Pi_\theta(t)\mathrm{d}t-\partial_z f_\theta(t)\Pi_\theta(t)\mathrm{d}W(t),\quad t\in[0,T],\\ \Pi_\theta(0)=1.\end{cases}$$

By Lemma 5.1, the above equation has a unique strong solution $\Pi_\theta\in\mathcal{S}^p([0,T];\mathbb{R})$ for $p>1$. Then it follows from Itô's formula to $\bar{X}(s)\Pi_\theta(s)$ on $[0,t]$ that

$$\bar{X}(t)\Pi_\theta(t)=\int_0^t(-\partial_z f_\theta(s)\Pi_\theta(s)\varphi(s))\mathrm{d}s+\int_0^t\Pi_\theta(s)\varphi(s)\mathrm{d}W(s),\quad t\in[0,T].\tag{5.4}$$

In addition, consider

$$\begin{cases}\mathrm{d}\pi_\theta(t)=\partial_z f_\theta(t)\pi_\theta(t)\mathrm{d}W(t),\quad t\in[0,T],\\ \pi_\theta(0)=1.\end{cases}$$

By Lemma 5.1 again, the above equation also has a unique strong solution $\pi_\theta\in\mathcal{S}^p([0,T];\mathbb{R})$ for $p>1$. Moreover, we know from Itô's formula that $\Pi_\theta(t)\pi_\theta(t)=1,\ t\in[0,T]$. Hence,

$$\Pi_\theta(t)^{-1}=\pi_\theta(t)=\mathcal{E}\left(\int_0^t\partial_z f_\theta(r)^\top\mathrm{d}W(r)\right).$$

From Hölder inequality one can see, for $1<p<(p_{\partial_z f_\theta}\wedge p_{(\partial_z f_{\tilde\theta}-\partial_z f_\theta)})$,

$$\begin{aligned}\mathbb{E}\Big[\sup_{t\in[0,T]}|\bar{X}(t)|^p\Big]&=\mathbb{E}\Big[\sup_{t\in[0,T]}|\bar{X}(t)\Pi_\theta(t)\pi_\theta(t)|^p\Big]\leqslant\mathbb{E}\Big[\sup_{t\in[0,T]}|\bar{X}(t)\Pi_\theta(t)|^p\cdot\sup_{t\in[0,T]}|\pi_\theta(t)|^p\Big]\\&\leqslant\Big\{\mathbb{E}\Big[\sup_{t\in[0,T]}|\bar{X}(t)\Pi_\theta(t)|^{pq'}\Big]\Big\}^{\frac{1}{q'}}\Big\{\mathbb{E}\Big[\sup_{t\in[0,T]}|\pi_\theta(t)|^{pp'}\Big]\Big\}^{\frac{1}{p'}},\end{aligned}$$

where $p'=\frac{p+(p_{\partial_z f_\theta}\wedge p_{(\partial_z f_{\tilde\theta}-\partial_z f_\theta)})}{2p}$ and $q'=\frac{p'}{p'-1}$. Since $pp'<p_{\partial_z f_\theta}$, it follows from Doob's inequality and reverse Hölder inequality that

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\pi_\theta(t)^{pp'}\Big]\leqslant C\mathbb{E}\Big[\pi_\theta(T)^{pp'}\Big]\leqslant CK(pp',\|\partial_z f_\theta\cdot W\|_{\mathrm{BMO}}).$$

Consequently, by Hölder inequality we obtain

$$\begin{aligned}\mathbb{E}\Big[\sup_{t\in[0,T]}|\bar{X}(t)|^p\Big]&\leqslant C\Big\{\mathbb{E}\Big[\sup_{t\in[0,T]}|\bar{X}(t)\Pi_\theta(t)|^{pq'}\Big]\Big\}^{\frac{1}{q'}}\\&\leqslant C\Big\{\mathbb{E}\Big[\sup_{t\in[0,T]}|\bar{X}(t)\Pi_\theta(t)|^{p\frac{p'}{p'-1}}\Big]\Big\}^{\frac{1}{q'}}\leqslant C\Big\{\mathbb{E}\Big[\sup_{t\in[0,T]}|\bar{X}(t)\Pi_\theta(t)|^{pp'}\Big]\Big\}^{\frac{1}{p'}},\end{aligned}\tag{5.5}$$

where C depends on pp' and $\|\partial_z f_\theta\cdot W\|_{\mathrm{BMO}}$. Besides, from (5.4) we arrive at

$$\begin{aligned}\mathbb{E}\Big[\sup_{t\in[0,T]}|\bar{X}(t)\Pi_\theta(t)|^{pp'}\Big]&\leqslant C\mathbb{E}\Big[\int_0^T|-\partial_z f_\theta(s)\Pi_\theta(s)\varphi(s))|^{pp'}\mathrm{d}s\Big]\\&\quad+C\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t\Pi_\theta(s)\varphi(s)\mathrm{d}W(s)\Big|^{pp'}\Big],\end{aligned}\tag{5.6}$$

where the constant $C>0$ depends on pp'.

Next, we estimate the ds-term and the dW-term in the above inequality one by one. Regarding the ds-term, since

$$\begin{cases} |\partial_z f_\theta(s)| \leqslant C(1+|\bar{Z}_\theta(s)|), \quad |\varphi(s)| \leqslant \Xi(\theta,\tilde{\theta})\cdot\mathcal{E}\left(\int_0^s \partial_z f_{\tilde{\theta}}(r)^\top \mathrm{d}W(r)\right), \\ \Pi_\theta(s) = \exp\Big\{\int_0^s (-\partial_z f_\theta(r))^\top \mathrm{d}W(r) + \frac{1}{2}\int_0^s |\partial_z f_\theta(r)|^2 \mathrm{d}r\Big\}, \end{cases} \tag{5.7}$$

by Hölder inequality we have

$$\begin{aligned} &\mathbb{E}\Big[\int_0^T |-\partial_z f_\theta(s)\Pi_\theta(s)\varphi(s))|^{pp'}\mathrm{d}s\Big] \\ &\leqslant C\Xi(\theta,\tilde{\theta})^{pp'}\mathbb{E}\Big[\Big(\int_0^T (1+|\bar{Z}_\theta(s)|)\cdot\mathcal{E}\Big(\int_0^s (\partial_z f_{\tilde{\theta}}(r)-\partial_z f_\theta(r))^\top \mathrm{d}W(r)\Big) \\ &\quad \bullet e^{\int_0^s \partial_z f_\theta(r)^\top(\partial_z f_\theta(r)-\partial_z f_{\tilde{\theta}}(r))\mathrm{d}r}\mathrm{d}s\Big)^{pp'}\Big] \\ &\leqslant C\Xi(\theta,\tilde{\theta})^{pp'}\mathbb{E}\Big[\sup_{s\in[0,T]}\mathcal{E}\Big(\int_0^s (\partial_z f_{\tilde{\theta}}(r)-\partial_z f_\theta(r))^\top \mathrm{d}W(r)\Big)^{pp'} \\ &\quad \bullet e^{C\Xi(\theta,\tilde{\theta})pp'\int_0^T(1+|\bar{Z}_\theta(r)|)\mathrm{d}r}\cdot\Big(\int_0^T (1+|\bar{Z}_\theta(r)|)\mathrm{d}r\Big)^{pp'}\Big] \\ &\leqslant C\Xi(\theta,\tilde{\theta})^{pp'}\Big\{\mathbb{E}\Big[\sup_{s\in[0,T]}\mathcal{E}\Big(\int_0^s (\partial_z f_{\tilde{\theta}}(r)-\partial_z f_\theta(r))^\top \mathrm{d}W(r)\Big)^{pp'p''}\Big]\Big\}^{\frac{1}{p''}} \\ &\quad \bullet\Big\{\mathbb{E}\Big[e^{C\Xi(\theta,\tilde{\theta})pp'q''\int_0^T(1+|\bar{Z}_\theta(r)|)\mathrm{d}r}\cdot\Big(\int_0^T (1+|\bar{Z}_\theta(r)|)\mathrm{d}r\Big)^{pp'q''}\Big]\Big\}^{\frac{1}{q''}}, \end{aligned} \tag{5.8}$$

where $p'' = \frac{p'+(p_{\partial_z f_\theta}\wedge p_{(\partial_z f_{\tilde{\theta}}-\partial_z f_\theta)})}{2pp'}$ and $q'' = \frac{p''}{p''-1}$. Since $|\partial_z f_{\tilde{\theta}}(s)-\partial_z f_\theta(s)| \leqslant C(1+|\bar{Z}_\theta(s)|+|\bar{Z}_{\tilde{\theta}}(s)|)$ we know $(\partial_z f_{\tilde{\theta}}-\partial_z f_\theta)\cdot W \in \mathrm{BMO}$. Notably, $1 < p' < pp'p'' < p_{(\partial_z f_{\tilde{\theta}}-\partial_z f_\theta)}$. It follows from Doob's inequality and reverse Hölder inequality that

$$\mathbb{E}\left[\sup_{s\in[0,T]}\mathcal{E}\left(\int_0^s (\partial_z f_{\tilde{\theta}}(r)-\partial_z f_\theta(r))^\top \mathrm{d}W(r)\right)^{pp'p''}\right] \leqslant K(pp'p'', \|(\partial_z f_{\tilde{\theta}}-\partial_z f_\theta)\cdot W\|_{\mathrm{BMO}}).$$

On the other hand, by Hölder inequality and $\bar{Z}_\theta\cdot W \in \mathrm{BMO}$, we get

$$\begin{aligned} &\Big\{\mathbb{E}\Big[e^{C\Xi(\theta,\tilde{\theta})pp'q''\int_0^T(1+|\bar{Z}_\theta(r)|)\mathrm{d}r}\cdot\Big(\int_0^T (1+|\bar{Z}_\theta(r)|)\mathrm{d}r\Big)^{pp'q''}\Big]\Big\}^{\frac{1}{q''}} \\ &\leqslant C\left\{\mathbb{E}\left[e^{2C\Xi(\theta,\tilde{\theta})pp'q''\int_0^T(1+|\bar{Z}_\theta(r)|)\mathrm{d}r}\right]\right\}^{\frac{1}{2q''}}\cdot\Big\{\mathbb{E}\Big[\Big(\int_0^T (1+|\bar{Z}_\theta(r)|^2)\mathrm{d}r\Big)^{pp'q''}\Big]\Big\}^{\frac{1}{2q''}} \\ &\leqslant C\left\{\mathbb{E}\left[e^{2C\Xi(\theta,\tilde{\theta})pp'q''\int_0^T(1+|\bar{Z}_\theta(r)|)\mathrm{d}r}\right]\right\}^{\frac{1}{2q''}}. \end{aligned}$$

Without loss of generality, we assume $\Xi(\theta,\tilde{\theta}) < 1$. Consequently, by applying the John–Nirenberg inequality, we obtain

$$
\begin{aligned}
\mathbb{E}\left[e^{2C\Xi(\theta,\tilde\theta)pp'q''\int_0^T(1+|\bar Z_\theta(r)|)\mathrm{d}r}\right] &< \mathbb{E}\left[e^{2Cpp'q''\int_0^T(1+|\bar Z_\theta(r)|)\mathrm{d}r}\right]\\
&\leqslant e^{2Cpp'q''(T+\frac{T}{\delta})}\mathbb{E}\left[e^{2Cpp'q''\frac{\delta}{4}\int_0^T|\bar Z_\theta(r)|^2\mathrm{d}r}\right]\\
&\leqslant e^{2Cpp'q''(T+\frac{T}{\delta})}\left(1-2Cpp'q''\frac{\delta}{4}\|\bar Z_\theta\cdot W\|^2_{\mathrm{BMO}}\right)^{-1}\\
&=\frac{4}{3}e^{2Cpp'q''(T+\frac{T}{\delta})},
\end{aligned}
$$

where $\delta=\frac{\|\bar Z_\theta\cdot W\|^{-2}_{\mathrm{BMO}}}{2Cpp'q''}$. Combining the above inequalities, one has

$$
\mathbb{E}\bigg[\int_0^T|-\partial_z f_\theta(s)\Pi_\theta(s)\varphi(s))|^{pp'}\mathrm{d}s\bigg]\leqslant C\Xi(\theta,\tilde\theta)^{pp'}. \tag{5.9}
$$

Next, we analyze the $\mathrm{d}W$-term. Similarly to the $\mathrm{d}s$-term, based on the Buckholder–Davis–Gundy inequality, (5.7), Hölder inequality, and John–Nirenberg inequality, it follows

$$
\begin{aligned}
&\mathbb{E}\bigg[\sup_{t\in[0,T]}\Big|\int_0^t\Pi_\theta(s)\varphi(s)\mathrm{d}W(s)\Big|^{pp'}\bigg]\\
&\leqslant C\Xi(\theta,\tilde\theta)^{pp'}\mathbb{E}\bigg[\sup_{s\in[0,T]}\mathcal{E}\bigg(\int_0^s(\partial_z f_{\tilde\theta}(r)-\partial_z f_\theta(r))^\top\mathrm{d}W(r)\bigg)^{pp'}\cdot e^{Cpp'\int_0^T(1+|\bar Z_\theta(r)|)\Xi(\theta,\tilde\theta)\mathrm{d}r}\bigg]\\
&\leqslant C\Xi(\theta,\tilde\theta)^{pp'}\bigg\{\mathbb{E}\bigg[\sup_{s\in[0,T]}\mathcal{E}\bigg(\int_0^s(\partial_z f_{\tilde\theta}(r)-\partial_z f_\theta(r))^\top\mathrm{d}W(r)\bigg)^{pp'p''}\bigg]\bigg\}^{\frac{1}{p''}},\\
&\bigg\{\mathbb{E}\left[e^{Cpp'q''\int_0^T(1+|\bar Z_\theta(r)|)\Xi(\theta,\tilde\theta)\mathrm{d}r}\right]\bigg\}^{\frac{1}{q''}}\leqslant C\Xi(\theta,\tilde\theta)^{pp'},
\end{aligned} \tag{5.10}
$$

where C depends on $\|(\partial_z f_{\tilde\theta}-\partial_z f_\theta)\cdot W\|_{\mathrm{BMO}}$ and $\partial_z f_\theta\cdot W\|_{\mathrm{BMO}}$, and p'',q'' are given in (5.8). Finally, taking into account (5.5), (5.6), (5.9) and (5.10), we have $\mathbb{E}\left[\sup_{t\in[0,T]}|\bar X(t)|^p\right]\leqslant C\Xi(\theta,\tilde\theta)^p$. The proof is completed. □

Remark 5.3 *The forward equation in (5.1) is a* 1*-dimensional linear SDE with unbounded coefficients, which prevents us from obtaining the continuity of* $p_\theta(\cdot)$ *concerning* θ *using the classical method. However, since* $\partial_z f_\theta\cdot W$ *is a BMO-martingale, utilizing properties of BMO martingale, we can derive the above result. Lemma 5.2 is used in the proof of Lemma 5.5 below.*

Before we prove the stochastic maximum principle, we need to establish the continuity of q_θ, r_θ in θ.

Lemma 5.4 *Under Assumption 1, we obtain, for* $1<p<(p_{\partial_z f_\theta}\wedge p_{\partial_z f_{\tilde\theta}-\partial_z f_\theta})$,

$$
\lim_{\epsilon\to 0}\sup_{\Xi(\theta,\tilde\theta)\leqslant\epsilon}\mathbb{E}\bigg[\sup_{t\in[0,T]}|q_\theta(t)-q_{\tilde\theta}(t)|^p+\bigg(\int_0^T|r_\theta(t)-r_{\tilde\theta}(t)|^2\mathrm{d}t\bigg)^{\frac{p}{2}}\bigg]=0.
$$

Proof Notice that

$$
\begin{aligned}
q_\theta(t)-q_{\tilde\theta}(t)=[q_\theta(T)-q_{\tilde\theta}(T)]&+\int_t^T(\partial_x b_\theta(s)^\top(q_\theta(s)-q_{\tilde\theta}(s))+\sum_{i=1}^d\partial_x\sigma_\theta^i(s)^\top(r_\theta^i(s)-r_{\tilde\theta}^i(s))\\
&+\mathbb{E}\Big[\partial_{x'}b_\theta(s)^\top(q_\theta(s)-q_{\tilde\theta}(s))\Big]+I_{1,\theta,\tilde\theta}(s)+I_{2,\theta,\tilde\theta}(s))\mathrm{d}s-\int_t^T(r_\theta(s)-r_{\tilde\theta}(s))\mathrm{d}W(s),
\end{aligned}
$$

where

$$I_{1,\theta,\tilde{\theta}}(s) = \partial_x f_{\tilde{\theta}}(s)p_{\tilde{\theta}}(s) - \partial_x f_\theta(s)p_\theta(s) + \mathbb{E}\Big[\partial_{x'} f_{\tilde{\theta}}(s)p_{\tilde{\theta}}(s) - \partial_{x'} f_\theta(s)p_\theta(s)\Big],$$

$$I_{2,\theta,\tilde{\theta}}(s) = (\partial_x b_\theta(t) - \partial_x b_{\tilde{\theta}}(t))^\top q_{\tilde{\theta}}(t) + \sum_{i=1}^{d}(\partial_x \sigma^i_\theta(t) - \partial_x \sigma^i_{\tilde{\theta}}(t))^\top r^i_{\tilde{\theta}}(t) + \mathbb{E}\Big[(\partial_{x'} b_\theta(t) - \partial_{x'} b_{\tilde{\theta}}(t))^\top q_{\tilde{\theta}}(t)\Big].$$

Since $\partial_x b_\theta, \partial_x \sigma^i_\theta, \partial_{x'} b_\theta$ are bounded, the backward equation in (5.1) is a mean-field BSDE with Lipschitz coefficient. By Chen et al. [5, Theorem 3.3], it yields that for $p > 1$,

$$\begin{aligned}
&\mathbb{E}\Bigg[\sup_{t\in[0,T]} |q_\theta(t) - q_{\tilde{\theta}}(t)|^p + \left(\int_0^T |r_\theta(t) - r_{\tilde{\theta}}(t)|^2 \mathrm{d}t\right)^{\frac{p}{2}}\Bigg] \\
&\leqslant C\mathbb{E}\Bigg[|q_\theta(T) - q_{\tilde{\theta}}(T)|^p + \left(\int_0^T |I_{1,\theta,\tilde{\theta}}(t) + I_{2,\theta,\tilde{\theta}}(t)|\mathrm{d}t\right)^p\Bigg].
\end{aligned}$$

Since $\partial_x \Phi_\theta, \partial_{x'} \Phi_\theta, \partial_x b_\theta, \partial_x \sigma_\theta, \partial_{x'} b_\theta, \partial_x f_\theta, \partial_{x'} f_\theta, \partial_x \phi_\theta$ are bounded and Lipschitz continuous in θ, it follows from Lemma 5.2 and Proposition 3.3 that

$$\lim_{\epsilon\to 0} \sup_{\Xi(\theta,\tilde{\theta})\leqslant\epsilon} \mathbb{E}\Bigg[\sup_{t\in[0,T]} |q_\theta(t) - q_{\tilde{\theta}}(t)|^p + \left(\int_0^T |r_\theta(t) - r_{\tilde{\theta}}(t)|^2 \mathrm{d}t\right)^{\frac{p}{2}}\Bigg] = 0.$$

This completes the proof. □

Define the Hamiltonian: for $t \in [0,T], x, x' \in \mathbb{R}^n, y \in \mathbb{R}, z \in \mathbb{R}^d, p \in \mathbb{R}, q \in \mathbb{R}^n, r \in \mathbb{R}^{n\times d}, v \in V$,

$$H_\theta(t,x,x',y,z,v,p,q,r) := q^\top b_\theta(t,x,x',v) + \sum_{i=1}^{d}(r^i)^{\top i}_\theta(t,x,v) - pf_\theta(t,x,x',y,z,v)$$

and its partial derivative with respect to v:

$$\Lambda_\theta(t,\omega) := \partial_v H_\theta(t, \bar{X}_\theta(t), \mathbb{E}[\bar{X}_\theta(t)], \bar{Y}_\theta(t), \bar{Z}_\theta(t), \bar{v}(t), p_\theta(t), q_\theta(t), r_\theta(t)). \tag{5.11}$$

Lemma 5.5 *Under Assumption 1 and Assumption 2, the above* Λ *is an* $\mathcal{F}$*-progressively measurable process, i.e., for each* $t \in [0,T]$*, the function* $\Lambda_\theta(t,\omega) : \Theta\times[0,t]\times\Omega\to\mathbb{R}$ *is* $\mathscr{B}(\Theta)\times\mathscr{B}([0,t])\times\mathcal{F}_t$*-measurable.*

Proof Since Θ is a Polish space, for each $M > 1$, there exists a compact subset $C^M \subset \Theta$ such that $\bar{Q}(\theta \notin C^M) \leqslant \frac{1}{M}$. Then we can find a subsequence of open neighborhoods $\big(B(\theta_l, \frac{1}{2M})\big)_{l=1}^{L_M}$ such that $C^M \subset \bigcup_{l=1}^{L_M}\big(B(\theta_l, \frac{1}{2M})\big)$. By the locally compact property of Θ and using partitions of unity, there exists a sequence of continuous functions $h_l : \Theta\to\mathbb{R}$ with values in $[0,1]$ such that

$$h_l(\theta) = 0, \text{ if } \theta \notin B\left(\theta_l, \frac{1}{2M}\right), \ l = 1,\cdots,L_M, \text{ and } \sum_{l=1}^{L_M} h_l(\theta) = 1, \text{ if } \theta \in C^M.$$

We choose some θ^*_l satisfying $h_l(\theta^*_l) > 0$ and define

$$\Lambda^M_\theta(t) := \sum_{l=1}^{L_M} \Lambda_{\theta^*_l}(t) h_l(\theta) \mathbf{1}_{\{\theta\in C^M\}}.$$

Notice that $\mathbb{E}\left[\int_0^T |\Lambda_\theta(t)|\mathrm{d}t\right] \leqslant L$. It yields that

$$
\begin{aligned}
&\int_\Theta \mathbb{E}\Big[\int_0^T \Big|\Lambda_\theta^M(t)-\Lambda_\theta(t)\Big|\mathrm{d}t\Big]\bar{Q}(\mathrm{d}\theta)\\
\leqslant\ &\int_\Theta \sum_{l=1}^{L_M}\mathbb{E}\Big[\int_0^T |\Lambda_\theta^M(t)-\Lambda_\theta(t)|\mathrm{d}t\Big]h_l(\theta)\mathbf{1}_{\{\theta\in C^M\}}+\mathbb{E}\Big[\int_0^T |\Lambda_\theta(t)|\mathrm{d}t\Big]h_l(\theta)\mathbf{1}_{\{\theta\notin C^M\}}\bar{Q}(\mathrm{d}\theta)\\
\leqslant\ &\sup_{\Xi(\theta,\tilde\theta)\leqslant\frac{1}{M}}\mathbb{E}\Big[\int_0^T |\Lambda_{\tilde\theta}(t)-\Lambda_\theta(t)|\mathrm{d}t\Big]+L\bar{Q}(\theta\notin C^M)\\
\leqslant\ &\sup_{\Xi(\theta,\tilde\theta)\leqslant\frac{1}{M}}\mathbb{E}\Big[\int_0^T |\Lambda_{\tilde\theta}(t)-\Lambda_\theta(t)|\mathrm{d}t\Big]+\frac{L}{M}.
\end{aligned}
$$

Here we use the fact that $h_l(\theta)=0$ whenever $\Xi(\theta,\tilde\theta)\geqslant\frac{1}{2M}$.

Then we prove $\lim\limits_{M\to\infty}\sup\limits_{\Xi(\theta,\tilde\theta)\leqslant\frac{1}{M}}\mathbb{E}\Big[\int_0^T|\Lambda_{\tilde\theta}(t)-\Lambda_\theta(t)|\mathrm{d}t\Big]=0$. Recall that

$$
\begin{aligned}
\Lambda_\theta(t,\omega)=\ &\partial_v b_\theta(t,\bar X_\theta(t),\mathbb{E}[\bar X_\theta(t)],\bar v(t))q_\theta(t)+\sum_{i=1}^d\partial_v\sigma_\theta^i(t,\bar X_\theta(t),\bar v(t))r_\theta^i(t)\\
&-\partial_v f_\theta(t,\bar X_\theta(t),\mathbb{E}[\bar X_\theta(t)],\bar Y_\theta(t),\bar Z_\theta(t),\bar v(t))p_\theta(t).
\end{aligned}
$$

Since $\partial_v b_\theta,\partial_v\sigma_\theta^i,\partial_v f_\theta$ are bounded and Lipschitz continuous in θ, the desired result comes from Lemma 5.2 and Lemma 5.4. □

Now, we are ready to derive the necessary conditions for achieving optimal control.

Theorem 5.6 *(Stochastic Maximum Principle) Let $\bar v(\cdot)$ be an optimal control to the problem (3.3), $(\bar X_\theta(\cdot),\bar Y_\theta(\cdot),\bar Z_\theta(\cdot))$ be the solution to the state equation (1.1) with $\bar v(\cdot)$, and $(p_\theta(\cdot)$, $q_\theta(\cdot),r_\theta(\cdot))$ be the solution to the adjoint equation (5.1) with $\bar v(\cdot)$. Under Assumption 1 and Assumption 2, there exists a* probability measure $\bar Q\in\mathcal{Q}^{\bar v}$ *such that, for $v\in V$, $\mathrm{d}t\times\mathrm{d}\mathbb{P}$-a.e.,*

$$
\int_\Theta\langle\partial_v H_\theta(t,\bar X_\theta(t),\mathbb{E}[\bar X_\theta(t)],\bar Y_\theta(t),\bar Z_\theta(t),\bar v(t),p_\theta(t),q_\theta(t),r_\theta(t)),v-\bar v(t)\rangle\bar Q(\mathrm{d}\theta)\geqslant 0.
$$

Proof Applying Itô's formula to $(X_\theta^1(\cdot))^\top q_\theta(\cdot)+Y_\theta^1(\cdot)p_\theta(\cdot)$, one has

$$
\begin{aligned}
&\mathbb{E}\Big[(\partial_x\phi_\theta(\bar X_\theta(T)))^\top X_\theta^1(T)+\partial_y\gamma_\theta(\bar Y_\theta(0))Y_\theta^1(0)\Big]\\
=\ &\mathbb{E}\Big[\int_0^T\langle(\partial_v b_\theta(t))^\top q_\theta(t)+\sum_{i=1}^d(\partial_v\sigma_\theta^i(t))^\top r_\theta^i(t)-\partial_v f_\theta(t)p_\theta(t),v(t)-\bar v(t)\rangle\mathrm{d}t\Big]\\
=\ &\mathbb{E}\Big[\int_0^T\langle\partial_v H_\theta(t,\bar X_\theta(t),\mathbb{E}[\bar X_\theta(t)],\bar Y_\theta(t),\bar Z_\theta(t),\bar v(t),p_\theta(t),q_\theta(t),r_\theta(t)),v(t)-\bar v(t)\rangle\mathrm{d}t\Big].
\end{aligned}
$$

Thanks to the above equality and Theorem 4.10, there exists some probability $\bar Q\in\mathcal{Q}^{\bar v}$ such that, for all $v(\cdot)\in\mathcal{V}_{ad}$,

$$
\int_\Theta\mathbb{E}\Big[\int_0^T\langle\partial_v H_\theta(t,\bar X_\theta(t),\mathbb{E}[\bar X_\theta(t)],\bar Y_\theta(t),\bar Z_\theta(t),\bar v(t),p_\theta(t),q_\theta(t),r_\theta(t)),v(t)-\bar v(t)\rangle\mathrm{d}t\Big]\bar Q(\mathrm{d}\theta)\geqslant 0.
$$

Thanks to Lemma 5.5, the process $\Lambda_\theta(\cdot)$ is $\mathcal{F}$-progressively measurable. Then, it follows from Fubini's Theorem that

$$\mathbb{E}\bigg[\int_0^T\int_\Theta \langle \partial_v H_\theta(t,\bar{X}_\theta(t),\mathbb{E}[\bar{X}_\theta(t)],\bar{Y}_\theta(t),\bar{Z}_\theta(t),\bar{v}(t),p_\theta(t),q_\theta(t),r_\theta(t)),v(t)-\bar{v}(t)\rangle \bar{Q}(\mathrm{d}\theta)\mathrm{d}t\bigg] \geqslant 0,$$

which implies for $v\in V$, $\mathrm{d}t\times\mathrm{d}\mathbb{P}$-a.e.,

$$\int_\Theta \langle \partial_v H_\theta(t,\bar{X}_\theta(t),\mathbb{E}[\bar{X}_\theta(t)],\bar{Y}_\theta(t),\bar{Z}_\theta(t),\bar{v}(t),p_\theta(t),q_\theta(t),r_\theta(t)),v(t)-\bar{v}(t)\rangle \bar{Q}(\mathrm{d}\theta) \geqslant 0.$$

This completes the proof. □

Remark 5.7 *As stated in the introduction, since the model is uncertain, we use a weak convergence technique to prove the variational inequality, see Theorem 4.10. Notably, the proof of Theorem 5.5 does not use the weak convergence technique.*

Then, we turn to the sufficient condition of optimal control.

Theorem 5.8 *Based on the assumptions and the settings in Theorem 5.6, if H_θ is convex in (x,x',y,z,v) and continuous in t, ϕ_θ is convex in x, γ_θ is convex in y and Φ_θ is concave in (x,x'), and there exists an admissible control $\bar{v}(\cdot)$ and a probability measure $\bar{Q}\in\mathcal{Q}^{\bar{v}}$ such that for $v\in V$, $\mathrm{d}t\times\mathrm{d}\mathbb{P}$-a.e.,*

$$\int_\Theta \langle \partial_v H_\theta(t,\bar{X}_\theta(t),\mathbb{E}[\bar{X}_\theta(t)],\bar{Y}_\theta(t),\bar{Z}_\theta(t),\bar{v}(t),p_\theta(t),q_\theta(t),r_\theta(t)),v-\bar{v}(t)\rangle \bar{Q}(\mathrm{d}\theta) \geqslant 0,$$

then $\bar{v}(\cdot)$ is an optimal control.

Proof For $\theta\in\Theta$ and $v(\cdot)\in\mathcal{V}_{ad}$, by $(X_\theta^v,Y_\theta^v,Z_\theta^v)$ we denote the solution to the equation (1.1). Set

$$(\alpha_\theta^\lambda,\beta_\theta^\lambda,\zeta_\theta^\lambda)=(X_\theta^\lambda-\bar{X}_\theta,Y_\theta^\lambda-\bar{Y}_\theta,Z_\theta^\lambda-\bar{Z}_\theta).$$

Then

$$\begin{cases}\alpha_\theta^\lambda(t)=\displaystyle\int_0^t(\partial_x b_\theta(s)\alpha_\theta^\lambda(s)+\partial_{x'}b_\theta(s)\mathbb{E}[\alpha_\theta^\lambda(s)]+S_\theta(s))\mathrm{d}s\\ \qquad\qquad+\displaystyle\sum_{i=1}^d\int_0^t(\partial_x\sigma_\theta^i(s)\alpha_\theta^\lambda(s)+J_\theta^i(s))\mathrm{d}W^i(s),\\ \beta_\theta^\lambda(t)=K_\theta(T)+\displaystyle\int_t^T(\partial_x f_\theta(s)^\top\alpha_\theta^\lambda(s)+\partial_{x'}f_\theta(s)^\top\mathbb{E}[\alpha_\theta^\lambda(s)]+\partial_y f_\theta(s)\beta_\theta^\lambda(s)\\ \qquad\qquad+\partial_z f_\theta(s)\zeta_\theta^\lambda(s)+L_\theta(s))\mathrm{d}s-\displaystyle\int_t^T\zeta_\theta(s)\mathrm{d}W(s),\end{cases}$$

where

$$\begin{aligned}&S_\theta(s)=b_\theta(s,X_\theta^v(s),\mathbb{E}[X_\theta^v(s)],v(s))-b_\theta(s)-\partial_x b_\theta(s)\alpha_\theta^\lambda(s)-\partial_{x'}b_\theta(s)\mathbb{E}[\alpha_\theta^\lambda(s)],\\&J_\theta^i(s)=\sigma_\theta^i(s,X_\theta^v(s),v(s))-\sigma_\theta^i(s)-\partial_x\sigma_\theta^i(s)\alpha_\theta^\lambda(s),\\&K_\theta(T)=\Phi_\theta(X_\theta^v(T),\mathbb{E}[X_\theta^v(T)])-\Phi_\theta(\bar{X}_\theta(T),\mathbb{E}[\bar{X}_\theta(T)]),\\&L_\theta(s)=f_\theta(s,X_\theta^v(s),\mathbb{E}[X_\theta^v(s)],Y_\theta^v(s),Z_\theta^v(s),v(s))-f_\theta(s)-\partial_x f_\theta(s)\alpha_\theta^\lambda(s)\\&\qquad\quad-\partial_{x'}f_\theta(s)\mathbb{E}[\alpha_\theta^\lambda(s)]-\partial_y f_\theta(s)\beta_\theta^\lambda(s)-\partial_z f_\theta(s)\zeta_\theta^\lambda(s).\end{aligned}$$

Applying Ito's formula to $\alpha_\theta^\lambda(\cdot)^\top q_\theta(\cdot)+\beta_\theta^\lambda(\cdot)p_\theta(\cdot)$ and then integrating from 0 to T, we have

$$\begin{aligned}&\mathbb{E}[\partial_x\phi_\theta(\bar{X}_\theta(T))^\top\alpha_\theta^\lambda(T)+\partial_y\gamma_\theta(\bar{Y}_\theta(0))\beta_\theta^\lambda(0)]\\=&-\mathbb{E}[p_\theta(T)(\Phi_\theta(X_\theta^v(T),\mathbb{E}[X_\theta^v(T)])-\Phi_\theta(\bar{X}_\theta(T),\mathbb{E}[\bar{X}_\theta(T)]))\\&-p_\theta(T)\partial_x\Phi(T)\alpha_\theta^\lambda(T)-\mathbb{E}[p_\theta(T)\partial_{x'}\Phi(T)]\alpha_\theta^\lambda(T)]\\&+\mathbb{E}\bigg[\int_0^T\bigg(q_\theta(t)S_\theta(t)+\sum_{i=1}^d r_\theta^i(t)J_\theta^i(t)-p_\theta(t)L_\theta(t)\bigg)\mathrm{d}t\bigg].\end{aligned}$$

Since Φ_θ is concave in (x,x') and the Hamiltonian H_θ is convex in (x,x',y,z,v), it yields that

$$\begin{aligned}&\mathbb{E}\Big[\partial_x\phi_\theta(\bar{X}_\theta(T))^\top\alpha_\theta^\lambda(T)+\partial_y\gamma_\theta(\bar{Y}_\theta(0))\beta_\theta^\lambda(0)\Big]\\\geqslant&\,\mathbb{E}\bigg[\int_0^T\langle\partial_vH_\theta(t,\bar{X}_\theta(t),\mathbb{E}[\bar{X}_\theta(t)],\bar{Y}_\theta(t),\bar{Z}_\theta(t),\bar{v}(t),p_\theta(t),q_\theta(t),r_\theta(t)),v^\lambda(t)-\bar{v}(t)\rangle\mathrm{d}t\bigg].\end{aligned}$$

Finally, recalling that $\bar{Q}\in\mathcal{Q}^{\bar{v}}$ and $\phi_\theta,\gamma_\theta$ are convex in x,y, respectively, it follows from Sion's minimax theorem that

$$\begin{aligned}&J(v^\lambda(\cdot))-J(v(\cdot))\\=&\int_\Theta\mathbb{E}\Big[\phi_\theta(X_\theta^\lambda(T))-\phi_\theta(\bar{X}_\theta(T))+\gamma_\theta(Y_\theta^\lambda(0))-\gamma_\theta(\bar{Y}_\theta(0))\Big]\bar{Q}(\mathrm{d}\theta)\\\geqslant&\int_\Theta\mathbb{E}\Big[\partial_x\phi_\theta(\bar{X}_\theta(T))^\top\alpha_\theta^\lambda(T)+\partial_y\gamma_\theta(\bar{Y}_\theta(0))\beta_\theta^\lambda(0)\Big]\bar{Q}(\mathrm{d}\theta)\\\geqslant&\int_\Theta\mathbb{E}\bigg[\int_0^T\langle\partial_vH_\theta(t,\bar{X}_\theta(t),\mathbb{E}[\bar{X}_\theta(t)],\bar{Y}_\theta(t),\bar{Z}_\theta(t),\bar{v}(t),p_\theta(t),q_\theta(t),r_\theta(t)),v^\lambda(t)-\bar{v}(t)\rangle\mathrm{d}t\bigg]\bar{Q}(\mathrm{d}\theta)\\\geqslant&\,\mathbb{E}\bigg[\int_0^T\int_\Theta\langle\partial_vH_\theta(t,\bar{X}_\theta(t),\mathbb{E}[\bar{X}_\theta(t)],\bar{Y}_\theta(t),\bar{Z}_\theta(t),\bar{v}(t),p_\theta(t),q_\theta(t),r_\theta(t)),v^\lambda(t)-\bar{v}(t)\rangle\bar{Q}(\mathrm{d}\theta)\mathrm{d}t\bigg]\geqslant 0.\end{aligned}$$

The proof is completed. □

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References

[1] Briand, P. and Confortola, F., BSDEs with stochastic Lipschitz condition and quadratic PDEs in Hilbert spaces, Stochastic Processes and their Applications, 2008, 118(5): 818–838.

[2] Briand, P. and Hu, Y., BSDE with quadratic growth and unbounded terminal value, Probability Theory and Related Fields, 2006, 136: 604–618.

[3] Briand, P. and Hu, Y., Quadratic BSDEs with convex generators and unbounded terminal conditions, Probability Theory and Related Fields, 2008, 141: 543–567.

[4] Buckdahn, R., Li, J. and Ma, J., A stochastic maximum principle for general mean-field systems, Applied Mathematics and Optimization, 2016, 74: 507–534.

[5] Chen, Y., Xing, C. and Zhang, X., L^p solution of general mean-field BSDEs with continuous coefficients, Acta

Mathematica Scientia, 2020, 40: 1116–1140.

[6] Delbaen, F., Grandits, P., Rheinländer, T., Samperi, D., Schweizer, M. and Stricker, C., Exponential hedging and entropic penalties, Mathematical Finance, 2002, 12(2): 99–123.

[7] El Karoui, N. and Hamadène, S., BSDEs and risk-sensitive control, zero-sum and nonzero-sum game problems of stochastic functional differential equations, Stochastic Processes and their Applications, 2003, 107(1): 145–169.

[8] Fan, S., Hu, Y. and Tang, S., On the uniqueness of solutions to quadratic BSDEs with non-convex generators and unbounded terminal conditions, Comptes Rendus Mathématique de l'Académie des Sciences Paris, 2020, 358: 227–235.

[9] Fan, S., Hu, Y. and Tang, S., Multi-dimensional backward stochastic differential equations of diagonally quadratic generators: The general result, Journal of Differential Equations, 2023, 368: 105–140.

[10] Fuhrman, M., Hu, Y. and Tessitore, G., Stochastic maximum principle for optimal control of SPDEs, Applied Mathematics and Optimization, 2013, 68: 181–217.

[11] Gal'chuk, I. L., Existence and uniqueness of a solution for stochastic equations with respect to semimartingales, Theory of Probability and its Applications, 1978, 23(4): 751–763.

[12] Hao, T., Hu, Y., Tang, S. and Wen, J., Mean-field backward stochastic differential equations and nonlocal PDEs with quadratic growth, Annals of Applied Probability, In press, 2025.

[13] Hao, T., Wen, J. and Xiong, J., Solvability of a class of mean-field BSDEs with quadratic growth, Statistics & Probability Letters, 2022, 191: 109652.

[14] Hu, M., Stochastic global maximum principle for optimization with recursive utilities, Probability, Uncertainty and Quantitative Risk, 2017, 2(1): 1–20.

[15] Hu, M. and Ji, S., Stochastic maximum principle for stochastic recursive optimal control problem under volatility ambiguity, SIAM Journal on Control and Optimization, 2016, 54(2): 918–945.

[16] Hu, M., Ji, S. and Xu, R., A global stochastic maximum principle for forward-backward stochastic control systems with quadratic generators, SIAM Journal on Control and Optimization, 2022, 60(3): 1791–1818.

[17] Hu, M., Ji, S. and Xue, X., A global stochastic maximum principle for fully coupled forward-backward stochastic systems, SIAM Journal on Control and Optimization, 2018, 56(6): 4309–4335.

[18] Hu, M. and Wang, F., Maximum principle for stochastic recursive optimal control problem under model uncertainty, SIAM Journal on Control and Optimization, 2020, 58(3): 1341–1370.

[19] Hu, Y., Imkeller, P. and Müller, M., Utility maximization in incomplete markets, Annals of Applied Probability, 2005, 15(3): 1691–1712.

[20] Hu, Y., Li, X. and Wen, J., Anticipated backward stochastic differential equations with quadratic growth, Journal of Differential Equations, 2021, 270: 1298–1331.

[21] Hu, Y. and Tang, S., Multi-dimensional backward stochastic differential equations of diagonally quadratic generators, Stochastic Processes and their Applications, 2016, 126(4): 1066–1086.

[22] Hu, Y., Wen, J. and Xiong, J., Backward doubly stochastic differential equations and SPDEs with quadratic growth, Stochastic Processes and their Applications, 2024, 175: 104405.

[23] Ji, S. and Zhou, X., A maximum principle for stochastic optimal control with terminal state constraints and its applications, Communications in Information and Systems, 2006, 6(4): 321–338.

[24] Kazamaki, N., Continuous Exponential Martingales and BMO, Springer, New York, 1994.

[25] Kobylanski, M., Backward stochastic differential equations and partial differential equations with quadratic growth, The Annals of Probability, 2000, 28(2): 558–602.

[26] Lim, A. E. B. and Zhou, X., A new risk-sensitive maximum principle, IEEE Transactions on Automatic Control, 2005, 50(7): 958–966.

[27] Morlais, M. A., Quadratic BSDEs driven by a continuous martingale and applications to the utility maximization

problem, Finance and Stochastics, 2009, 13: 121–150.

[28] Peng, S., A general stochastic maximum principle for optimal control problems, SIAM Journal on Control and Optimization, 1990, 28(4): 966–979.

[29] Peng, S., Backward stochastic differential equations and applications to optimal control, Applied Mathematics and Optimization, 1993, 27: 125–144.

[30] Rouge, R. and El Karoui, N., Pricing via utility maximization and entropy, Mathematical Finance, 2000, 10(2): 259–276.

[31] Shi, Y., Wen, J. and Yang, Z., Large deviation principle for backward stochastic differential equations with a stochastic Lipschitz condition on z, Stochastics and Dynamics, 2024, 24(6): 2450043.

[32] Wen, J., Li, X., Xiong, J. and Zhang, X., Stochastic linear-quadratic optimal control problems with random coefficients and Markovian regime switching system, SIAM Journal on Control and Optimization, 2023, 61(2): 949–979.

[33] Xing, H. and Žitković, G., A class of globally solvable Markovian quadratic BSDE systems and applications, The Annals of Probability, 2018, 46(1): 491–550.

[34] Xu, W., Stochastic maximum principle for optimal control problem of forward and backward system, ANZIAM Journal, 1995, 37(2): 172–185.