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Backward stochastic differential equations with central value reflection

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Abstract In this study, we investigate the well-posedness of a backward stochastic differential equation with jumps and a central value reflection constraint. The reflection condition is imposed on the real-valued function obtained by solving the equation $\mathbb{E}(\arctan(Y_t - x)) = 0$ at each time $t \in [0,T]$. The driver depends on the distribution of the solution process Y and follows a general quadratic-exponential structure. The terminal value is assumed to be bounded. Using a fixed-point argument and Bounded Mean Oscillation (BMO in short) martingale theory, we establish the existence and uniqueness of local solutions, which are then extended to construct a global solution over the entire time interval $[0,T]$.

Keywords Central value reflection, Bounded mean oscillation martingales, Jumps

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1. Introduction

Despite the pointwise constraint for solutions of continuous BSDEs first established by El Karoui et al. [8], Bouchard et al. [3] introduced a novel class of BSDEs with a weak terminal condition. In their framework, instead of imposing a fixed terminal condition, they considered a constraint on the distribution of the random variable Y_T. Specifically, they required that $\mathbb{E}\left(\Psi(Y_T)\right) \geqslant m$ for some nondecreasing (possibly random) map Ψ and threshold m. Building on this work, Briand et al. [5] extended the idea to a dynamic setting by studying BSDEs with an evolving expectation constraint of the form $\mathbb{E}\left(l(t,Y_t)\right) \geqslant 0$ for each $t \in [0,T]$, where $(l(t,.))_{t\in[0,T]}$ is a family of nondecreasing (possibly random) real-valued functions satisfying certain intermediate assumptions, particularly the bi-Lipschitz condition with respect to the second variable. To establish the existence and uniqueness of the solution (Y,Z,K) for the BSDE with mean reflection, Briand et al. assumed a Lipschitz driver f and a square-integrable terminal value. They also considered K as a deterministic, nondecreasing process that satisfies a Skorokhod-type condition:

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$\int_0^T \mathbb{E}\left[\ell\left(t, Y_t\right)\right] \mathrm{d}K_t = 0$. As K is deterministic, Briand et al. [5] provided the following explicit representation, for each $t \in [0,T]$,

$$K_t = \sup_{0 \leqslant s \leqslant T} L_s\left(\mathbb{E}\left[\xi + \int_t^T f\left(r, Y_r, Z_r\right) \mathrm{d}r / \mathcal{F}_s\right]\right) - \sup_{t \leqslant s \leqslant T} L_s\left(\mathbb{E}\left[\xi + \int_t^T f\left(r, Y_r, Z_r\right) \mathrm{d}r / \mathcal{F}_s\right]\right), \tag{1.1}$$

where L is a map defined as $L_t : L^1 \to \mathbb{R}$ for each $t \in [0,T]$,

$$L_t(\eta) = \inf\{x \geqslant 0 : \mathbb{E}[\ell(t, x+\eta)] \geqslant 0\}, \quad \forall \eta \in L^1. \tag{1.2}$$

The well-posedness of this problem was further generalized by Hibon et al. [11] with a quadratic generator and a bounded terminal condition. Due to the lack of comparison results when dealing with mean reflection (see counterexample 3.4 in their paper), a fixed-point argument was the appropriate approach. The key idea in their method is that the solution to the BSDE with mean reflection, expressed as $(Y_t - (K_T - K_t), Z_t)$, coincides with the solution of a classical BSDE for each $t \in [0,T]$ when the driver is independent of y. In their case, the process K follows the form given in (1.1). Leveraging this observation, they established the existence of a unique solution to the BSDE with mean reflection when the driver depends only on Z, i.e., $f(Z)$. By applying a contraction mapping, they proved the well-posedness of the problem in its general form. However, the solution was initially constructed only on small time intervals, with the maximal length determined by the bound of the component Y. To extend the solution over the entire time interval $[0,T]$, they first derived a uniform estimate for these local solutions under the intermediate assumption that $f(t,y,0)$ remains uniformly bounded. By systematically combining these local solutions, they ultimately established the existence and uniqueness of the solution on the full interval.

Hu et al. [12] considered a more general case, studying the well-posedness of the BSDE with mean reflection where the driver depends on the distribution of the solution term Y. Their analysis covered various scenarios based on different assumptions. First, they examined the case of a Lipschitz driver with an L^p-terminal condition. Second, they relaxed the Lipschitz condition on z, replacing it with a quadratic growth condition while assuming a bounded terminal condition. Finally, within this latter framework, they extended their results to the case of an unbounded terminal condition by imposing an additional convexity assumption on the driver with respect to z. Their study relied on BMO martingale theory and fixed-point arguments, which played a crucial role in analyzing the BSDE with mean reflection. A key distinction from the work by Hibon et al. [11] was their removal of the assumption that $f(t,y,0)$ is uniformly bounded, as this condition is unrelated to the solvability of the classical BSDE in the quadratic case. To achieve this, they proposed an alternative formulation of the process K, defined for each $t \in [0,T]$ as

$$K_t = \sup_{0 \leqslant s \leqslant T} L_s\left(y_s\right) - \sup_{t \leqslant s \leqslant T} L_s\left(y_s\right), \tag{1.3}$$

where y is the first component of the solution to the following BSDE

$$y_t = \xi + \int_t^T f\left(s, Y_s, \mathbf{P}_{Y_s}, z_s\right) \mathrm{d}s - \int_t^T z_s \mathrm{d}W_s,$$

and the operator L is as defined in (1.2).

Several generalizations of mean reflection in the continuous case have been proposed in the literature. Among them, we highlight the study by Briand et al. [7], where the authors approximated the solution using an interacting particle system and analysed its well-posedness.

They proved that this approximation converges to the solution of the BSDE with mean reflection and provided an estimate of the convergence rate of the particle system toward the square-integrable, deterministic flat solution of the mean-reflected BSDE. Additionally, Luo [16] investigated the existence and uniqueness of solutions for mean-field BSDEs with mean reflection and nonlinear resistance. Using a fixed-point approach, they addressed both cases: one with a Lipschitz driver and another with a driver exhibiting quadratic growth. For studies within the context of mean-reflected SDEs, we refer to the works by [4, 6, 19].

For the framework involving jumps, particularly in the case of quadratic BSDEs, we refer to previous studies [1, 2, 9, 17, 18]. Additionally, for mean reflection in BSDEs with jumps, we cite the study by Gu et al. [10], where the authors examined the well-posedness of a class of mean-reflected BSDEs driven by a marked point process and a Brownian motion. Using a fixed-point approach, they established the existence and uniqueness of solutions within the Lipschitz framework under a terminal condition satisfying certain integrability constraints. They also provided an application by solving a super-hedging problem with insurance repayment under a risk management constraint. In a recent study [15], Lin and Xu investigated mean-reflected BSDEs driven by a Poisson martingale measure with a quadratic generator. Furthermore, they extended the results of Briand and Hibon on the propagation of chaos for mean-reflected BSDEs [7] to include the jump setting.

Building on the studies by Briand et al., Hibon et al., and Hu et. al [5, 11, 12] and extending to the jump setting, we consider a BSDE with a generator that depends on the distribution of the solution term Y. Our focus is on the solvability of this BSDE under central value reflection, where the constraint is imposed on the mapping $t \mapsto \gamma_{Y_t}$, defined as the solution to the equation $\mathbb{E}(\arctan(Y_t - x)) = 0$ for each $t \in [0,T]$. The well-posedness of the operator $\gamma : L^2(\mathcal{F}_T) \to \mathbb{R}$ is analyzed in Section 3. The problem is formulated as follows:

$$\begin{cases} Y_t = \xi + \displaystyle\int_t^T f\left(s, Y_s, \mathbf{P}_{Y_s}, Z_s, V_s\right) \mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s - \int_t^T\int_U V_s(e)\tilde{N}(\mathrm{d}s, \mathrm{d}e) + K_T - K_t, \quad t \in [0,T], \\ \gamma_{Y_t} \geqslant 0, \quad \forall t \in [0,T], \end{cases} \tag{1.4}$$

where W is a d-dimensional Brownian motion and $\tilde{N}(\mathrm{d}s, \mathrm{d}e)$ is a Poisson martingale measure independent of W. The objective of this study is to establish the existence and uniqueness of a solution to this problem under a bounded terminal condition and a Lipschitz quadratic-exponential driver that satisfies a monotonicity condition with respect to its last variable. Inspired by arguments from papers mentioned above, we aim to construct a deterministic flat solution, where deterministic means that the process K is nonrandom, and flat implies that the Skorokhod condition $\int_0^T \gamma_{Y_{t-}} \mathrm{d}K_t = 0$ is satisfied. Our proof approach is based on the fixed-point method.

Starting with a driver independent of the components Y and $\mathbf{P}_Y$, we observe that the solution $(Y_t - (K_T - K_t), Z_t, V_t)$ coincides with the solution of the classical BSDE studied by Possamai et al. [18]. This allows us to derive the components Z and V accordingly. Given that K is deterministic, it takes the explicit form

$$K_t = \sup_{0 \leqslant s \leqslant T} \left(\gamma_{y_s}\right)^- - \sup_{t \leqslant s \leqslant T} \left(\gamma_{y_s}\right)^-, \tag{1.5}$$

where y is the first component of the solution to the following BSDE:

$$y_t = \xi + \int_t^T f\left(s, Y_s, \mathbf{P}_{Y_s}, z_s, v_s\right) \mathrm{d}s - \int_t^T z_s \mathrm{d}W_s - \int_t^T \int_U v_s(e)\tilde{N}(\mathrm{d}s, \mathrm{d}e).$$

This, in turn, determines the component Y and thereby the entire solution. Next, using a contractive map defined solely in terms of Y, we establish the solvability of the problem over small-sized intervals, from which we recover local solutions to the BSDE with central value reflection. These local solutions are then stitched together to construct a global solution over the entire time interval. Given that we work with bounded solutions, the space of BMO martingales plays a crucial role in our analysis. In previous studies on mean reflection, a key assumption on the collection $(l(t,.))_{t\in[0,T]}$ defining the constraint was bi-Lipschitz continuity with respect to the second variable. This condition ensures that the operator L remains Lipschitz continuous uniformly in time, facilitating the estimation of the term $K_T - K_t$. In our case, to obtain a similar estimate, we leverage the boundedness of solutions and the fact that the function arctan is locally bi-Lipschitz. These properties, together with the definition of our operator γ, demonstrate that γ forms a local isometry in the L^1-norm, enabling the required estimation.

The remainder of the paper is organized as follows. Section 2 introduces basic notations and provides preliminaries on BMO martingales. Section 3 presents the problem setup and outlines the assumptions. The final section consists of four subsections: the first establishes the well-posedness of the problem for a driver that is a progressively measurable process satisfying certain integrability conditions; the second addresses the case where the driver is independent of y; the third focuses on local solutions; and the fourth contains the main result, demonstrating the solvability of the problem over the entire interval, along with a boundedness result under an intermediate assumption.

2. Preliminaries

Let $T > 0$ be a fixed time horizon and d, l a strictly positive integers. Throughout this paper, we consider a fixed probability space $(\Omega, \mathscr{F}, \mathbf{P})$ equipped with a complete filtration $\mathbb{F} = \{\mathcal{F}_t : t \in [0,T]\}$ generated by the following two mutually independent processes:

- a d-dimensional $\mathbb{F}$-Brownian motion $(W_t)_{0\leqslant t\leqslant T}$,
- an $\mathbb{F}$-Poisson random measure $N(\mathrm{d}t, \mathrm{d}e)$ on $\mathbb{R}^+ \times U$, with compensator $\mathrm{d}t \otimes \nu(\mathrm{d}e)$, where $(U, \mathcal{U}) := (\mathbb{R}^l \setminus \{0\}, \mathcal{B}(\mathbb{R}^l \setminus \{0\}))$ is a measurable space equipped with a σ-finite positive measure ν satisfying

$$\int_U \left(1 \wedge |e|^2\right) \nu(\mathrm{d}e) < \infty. \tag{2.1}$$

We denote $\tilde{N}(\mathrm{d}t, \mathrm{d}e) := N(\mathrm{d}t, \mathrm{d}e) - \nu(\mathrm{d}e)\mathrm{d}t$ the compensated measure of $N(\mathrm{d}t, \mathrm{d}e)$. Finally, let $\mathcal{P}$ denote the σ-algebra of all predictable subsets of $\Omega \times [0,T]$.

2.1 Notation

We introduce the following notations and functional spaces of processes:

- For $t \in [0,T]$, let $\mathcal{T}_{t,T}$ denote the set of stopping times τ such that $\mathbf{P}(t \leqslant \tau \leqslant T) = 1$.
- $\mathcal{S}^2$ and $\mathcal{S}^\infty$: the spaces of real-valued, progressively measurable càdlàg processes $Y = (Y_t)_{0\leqslant t\leqslant T}$ such that

$$\|Y\|^2_{\mathcal{S}^2} := \mathbb{E}\big(\sup_{t\in[0,T]} \mid Y_t \mid^2\big) < \infty, \quad \text{and} \quad \|Y\|_{\mathcal{S}^\infty} := \left\| \sup_{t\in[0,T]} |Y_t| \right\|_\infty < \infty, \quad \text{respectively.}$$

- $\mathcal{H}^2$: the space of $\mathbb{R}^d$-valued, progressively measurable processes $Z = (Z_t)_{0\leqslant t\leqslant T}$ such that

$$\|Z\|_{\mathcal{H}^2}^2 := \mathbb{E}\left(\int_0^T \mid Z_s \mid^2 \mathrm{d}s\right) < \infty.$$

• $\mathcal{H}_\nu^2$: the space of mappings $V : \Omega \times [0,T] \times U \to \mathbb{R}$ that are $(\mathcal{P} \otimes \mathcal{U}, \mathcal{B}(\mathbb{R}))$-measurable and satisfy

$$\|V\|_{\mathcal{H}_\nu^2}^2 := \mathbb{E}\left(\int_0^T \int_U \mid V_t(e) \mid^2 \nu(\mathrm{d}e)\mathrm{d}t\right) < \infty.$$

• $\mathcal{L}^2(U,\nu;\mathbb{R})$: the space of measurable functions $\phi : U \to \mathbb{R}$ satisfying

$$\|\phi\|_{\mathcal{L}^2(U,\nu;\mathbb{R})}^2 := \left(\int_U |\phi(e)|^2 \nu(\mathrm{d}e)\right) < \infty.$$

• $\mathcal{L}^\infty$: the space of real valued $\mathcal{F}_T$-measurable random variables ξ such that

$$\|\xi\|_{\mathcal{L}^\infty} := \operatorname*{esssup}_{\omega\in\Omega} \mid \xi(\omega) \mid < \infty.$$

• $\mathcal{J}^\infty$: the space of functions which are $\mathbf{P} \otimes \nu(\mathrm{d}e)$ essentially bounded, i.e.,

$$\| V \|_{\mathcal{J}^\infty} := \| \sup_{t\in[0,T]} \| V \|_{L^\infty(\nu)} \|_\infty < \infty,$$

where $L^\infty(\nu)$ is the space of $\mathbb{R}$-valued measurable functions $\nu(\mathrm{d}e)$ a.e.–bounded endowed with the usual essential sup-norm.

• $\mathcal{A}_D$ (resp. $\mathcal{A}_D^\infty$): the closed subset of $\mathcal{S}^2$ (resp. $\mathcal{S}^\infty$) consisting of deterministic, nondecreasing processes $(K_t)_{0\leqslant t\leqslant T}$ starting from the origin.

• $\mathcal{P}_p(\mathbb{R})$: The set of of all probability measures over $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with finite p^{th} moment, endowed with the p-Wasserstein distance W_p.

• For a given càdlàg process $(Y_t)_{0\leqslant t\leqslant T}$, we denote its jump process by $\Delta Y = Y_t - Y_{t^-}$, where Y_{t^-} is the left limit of Y at time t, with $Y_{0^-} = Y_0$.

• The symbol C denotes a generic positive constant, independent of time, whose value may change at each appearance.

• To avoid ambiguity, we use $|x|$ for both the absolute value of x when $x \in \mathbb{R}$ and the Euclidean norm when $x \in \mathbb{R}^d$. The sign function is defined as $sgn(x) := \frac{x}{|x|}\mathbf{1}_{|x|>0}$.

2.2 BMO martingales

This section provides a brief overview of essential properties of BMO martingales, which plays a crucial role throughout this paper. We begin by defining the relevant spaces:

• BMO: the space of square integrable martingale M such that

$$\|M\|_{\mathrm{BMO}}^2 := ess \sup_{\tau\in\mathcal{T}_{0,T}} \left\|\mathbb{E}\left((M_T - M_{\tau^-})^2 / \mathcal{F}_\tau\right)\right\|_\infty < \infty.$$

• $\mathcal{H}_{BMO}^2$: the space of $\mathbb{R}^d$-valued, progressively measurable processes $Z = (Z_t)_{0\leqslant t\leqslant T}$ such that

$$\|Z\|_{\mathcal{H}_{BMO}^2}^2 := \left\|\int_0^\cdot Z_s \mathrm{d}W_s\right\|_{BMO}^2 = \sup_{\tau\in\mathcal{T}_{0T}} \left\|\mathbb{E}\left(\int_\tau^T |Z_s|^2 \mathrm{d}s / \mathcal{F}_\tau\right)\right\|_\infty < \infty.$$

• $\mathcal{J}_{BMO}^2$: the space of mappings $V : \Omega \times [0,T] \times U \to \mathbb{R}$, which are $(\mathcal{P} \otimes \mathcal{U}, \mathcal{B}(\mathbb{R}))$-measurable and satisfy

$$\|V\|^2_{\mathcal{J}^2_{BMO}} := \left\| \int_0^{\cdot} \int_U V_s(e)\tilde{N}(\mathrm{d}s, \mathrm{d}e) \right\|^2_{BMO}$$
$$:= \sup_{\tau \in \mathcal{T}_{0,T}} \left\| \mathbb{E}\left(\int_\tau^T \int_U |V_s(e)|^2 N(\mathrm{d}e, \mathrm{d}s)/\mathcal{F}_\tau \right) + (\Delta M_\tau)^2 \right\|_\infty < \infty,$$

where ΔM_τ denotes the jump of $\int_0^\cdot \int_U V_s(e)\tilde{N}(\mathrm{d}s, \mathrm{d}e)$ at time τ.

- $\mathcal{J}^2_B$: the space of mappings $V : \Omega \times [0, T] \times U \to \mathbb{R}$, which are $(\mathcal{P} \otimes \mathcal{U}, \mathcal{B}(\mathbb{R}))$-measurable and satisfy

$$\|V\|^2_{\mathcal{J}^2_B} := \sup_{\tau \in \mathcal{T}_{0,T}} \left\| \mathbb{E}\left(\int_\tau^T \int_U |V_s(e)|^2 \nu(\mathrm{d}e)\mathrm{d}s/\mathcal{F}_\tau \right) \right\|_\infty < \infty.$$

Note that

$$\| V \|^2_{\mathcal{J}^2_B} \vee \| V \|^2_{\mathcal{J}^\infty} \leqslant \| V \|^2_{\mathcal{J}^2_{BMO}} \leqslant \| V \|^2_{\mathcal{J}^2_B} + \| V \|^2_{\mathcal{J}^\infty}, \tag{2.2}$$

which results from Lemma 2.1 in the study by Fujii and Takahashi [9]. We now introduce the following inequalities, commonly referred to as energy inequalities.

Lemma 2.1 *(Energy Inequalities [14]) Let $Z \in \mathcal{H}^2_{BMO}$ and $V \in \mathcal{J}^2_{BMO}$. Then, for any $p \in \mathbb{N}$, we have*

$$\mathbb{E}\left[\left(\int_0^T |Z_s|^2 \,\mathrm{d}s \right)^p \right] \leqslant p! \left(\|Z\|^2_{\mathcal{H}^2_{BMO}} \right)^p, \quad \mathbb{E}\left[\left(\int_0^T \int_U |V_s(x)|^2 N(\mathrm{d}s, \mathrm{d}x) \right)^p \right] \leqslant p! \left(\|V\|^2_{\mathcal{J}^2_{BMO}} \right)^p,$$

and

$$\mathbb{E}\left[\left(\int_0^T \int_U |V_s(x)|^2 \nu(\mathrm{d}x)\mathrm{d}s \right)^p \right] \leqslant p! \left(\|V\|^2_{\mathcal{J}^2_B} \right)^p \leqslant p! \left(\|V\|^2_{\mathcal{J}^2_{BMO}} \right)^p.$$

Now, we focus on some specific characteristics of BMO martingales. We define the following Doléans-Dade exponential of a square-integrable martingale M by

$$\mathcal{E}(M)_t = e^{M_t - \frac{1}{2}\langle M^c \rangle_t} \prod_{0 < s \leqslant t} (1 + \Delta M_s)\, e^{-\Delta M_s}, \quad \mathbf{P}\text{-a.s.},$$

which is the unique solution of the following SDE:

$$X_t = 1 + \int_0^t X_{s-} \mathrm{d}M_s, \quad \mathbf{P}\text{-a.s.}$$

In the next proposition, we establish a condition on the jumps of a BMO martingale M under which $\mathcal{E}(M)$ becomes a strictly positive, uniformly integrable martingale. This result is attributed to the study by Kazamaki [13].

Proposition 2.1 *Let M be a càdlàg BMO martingale such that there exists $\delta > 0$ with $\Delta M_t \geqslant -1 + \delta$, for all $t \in [0.T]$, $\boldsymbol{P}$-a.s. Then, $\mathcal{E}(M)$ is a strictly positive, uniformly integrable martingale.*

We conclude this section with a result on Girsanov's Theorem, which will be applied throughout this paper.

Proposition 2.2 *([18]) Consider the following càdlàg martingale*

$$M_t := \int_0^t \beta_s \mathrm{d}W_s + \int_0^t \int_U \gamma_s(e)\tilde{N}(\mathrm{d}s, \mathrm{d}e), \quad \mathbf{P}\text{-a.s.},$$

where γ is bounded, $(\beta,\gamma)\in\mathcal{H}^2_{BMO}\times\mathcal{J}^2_{BMO}$, and there exists $\delta>0$ such that $\gamma_t\geqslant -1+\delta$ $\boldsymbol{P}\times \mathrm{d}\nu$ a.e. for all $t\in[0,T]$. Then, the probability $\boldsymbol{Q}$, defined as $\frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}\boldsymbol{P}}=\mathcal{E}(M_.)$, is well defined. Furthermore, any $\boldsymbol{P}$-martingale is a $\boldsymbol{Q}$-martingale by adequately modifying its drift and jump intensity.

After these preliminaries established, we now present the framework of our problem in the next section.

3. Formulation of the problem and assumptions

First, we define the following operator

$$\gamma : L^0(\mathscr{F}) \longrightarrow \mathbb{R},\qquad X\longmapsto \gamma_X,$$

where γ_X is the unique solution of the equation $\mathbb{E}(\arctan(X-x))=0$. The well-posedness of γ_X follows from the Intermediate Value Theorem, applied to the function

$$\varphi:\ \mathbb{R}\longrightarrow]-\pi/2,\pi/2[\qquad x\longmapsto \mathbb{E}(\arctan(X-x)).$$

We refer to γ_X as the Doob constant associated with the random variable X or the central value of X.

The main objective of this paper is to construct a solution $(Y,Z,V,K)\in S^\infty\times\mathcal{H}^2_{BMO}\times\mathcal{J}^2_{BMO}\times\mathcal{A}^\infty_D$ to the following BSDE with central value reflection:

$$\begin{cases} Y_t=\xi+\int_t^T f(s,Y_s,\mathbf{P}_{Y_s},Z_s,V_s)\,\mathrm{d}s-\int_t^T Z_s\mathrm{d}W_s-\int_t^T\int_U V_s(e)\tilde{N}(\mathrm{d}s,\mathrm{d}e)+K_T-K_t, & t\in[0,T],\\ \gamma_{Y_t}\geqslant 0, \quad \forall t\in[0,T], \end{cases} \tag{3.1}$$

where for each $t\in[0,T]$, γ_{Y_t} is the unique solution to the equation $\mathbb{E}(\arctan(Y_t-x))=0$. When the process Y is càdlàg, the map $t\mapsto\gamma_{Y_t}$ is also càdlàg, a property that will be discussed in detail in subsequent sections.

In the context of the solution to BSDEs with mean reflection presented in the study by Briand et al. [5], we now provide the formal definition of the solution to problem (3.1).

Definition 3.1 *A quadriplet $(Y,Z,V,K)\in\mathcal{S}^\infty\times\mathcal{H}^2_{BMO}\times\mathcal{J}^2_{BMO}\times\mathcal{A}^\infty_D$ is said to be a deterministic solution to (3.1) with central value reflection if it satisfies (3.1). Moreover, it is said to be flat if K increases only when needed, i.e.,*

$$\int_0^T \gamma_{Y_{t-}}\mathrm{d}K_t=0. \tag{3.2}$$

The parameters of our BSDE include the terminal condition ξ, which is assumed to be bounded, and the generator f that satisfies a quadratic growth condition with other assumptions detailed below. Due to the imposed constraint on the central value of Y_t for all $t\in[0,T]$, this problem is referred to as a BSDE with central value reflection.

3.1 Assumptions

Throughout this work, we assume that the parameters of the BSDE (3.1) satisfy the following assumptions:

$(\mathbf{H}_\xi)$ The terminal value ξ is an $\mathcal{F}_T$-measurable random variable bounded by some constant $M>0$, i.e., $\|\xi\|_\infty \leqslant M$ and $\gamma_\xi \geqslant 0$.

$(\mathbf{H}_f)$ The generator $f:\Omega\times[0,T]\times\mathbb{R}\times\mathcal{P}_1(\mathbb{R})\times\mathbb{R}^d\times\mathcal{L}^2(U,\nu;\mathbb{R})\longrightarrow\mathbb{R}$, which is $\mathcal{P}\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathcal{P}_1(\mathbb{R}))\otimes\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{U}$-measurable, such that

- For each $t\in[0,T]$, $f(t,0,\delta_0,0,0)$ is bounded by some constant M **P**-a.s.
- There exists $(\beta,\kappa)\in\mathbb{R}^+\times\mathbb{R}^{*+}$ and a positive progressively, measurable essentially bounded process $(\alpha_t)_{t\in[0,T]}$ such that for all $(t,y,\mu,z,v)\in[0,T]\times\mathbb{R}\times\mathcal{P}_1(\mathbb{R})\times\mathbb{R}^d\times\mathcal{L}^2(U,\nu;\mathbb{R})$,

$$\begin{aligned}&-\alpha_t-\beta(|y|+W_1(\mu,\delta_0))-\frac{\kappa}{2}|z|^2-\int_U\mathbf{j}_\kappa(-v(e))\nu(\mathrm{d}e)\\&\leqslant f(t,y,\mu,z,v)\leqslant\alpha_t+\beta(|y|+W_1(\mu,\delta_0))+\frac{\kappa}{2}|z|^2+\int_U\mathbf{j}_\kappa(v(e))\nu(\mathrm{d}e),\end{aligned}\tag{3.3}$$

where $\mathbf{j}_\kappa(v):=\frac{1}{\kappa}\left(e^{\kappa v}-1-\kappa v\right)$.

- f is uniformly Lipschitz in y and μ, i.e.,

$$|f(t,y,\mu,z,v)-f(t,y',\mu',z,v)|\leqslant C_f\left(|y-y'|+W_1(\mu,\mu')\right)\text{ for all }(t,y,y',\mu,\mu',z,v).$$

- f is of class $\mathcal{C}^2$ in z and satisfies the following conditions: there exist constants $\theta>0$ and a process $(r_t)_{0\leqslant t\leqslant T}\in\mathcal{H}^2_{\mathrm{BMO}}$ such that for all (t,y,μ,z,v),

$$|D_zf(t,y,\mu,z,v)|\leqslant r_t+\theta|z|,\quad\text{and}\quad\left|D^2_{zz}f(t,y,\mu,z,v)\right|\leqslant\theta.$$

- For every (y,μ,z,v,v'), there exists a predictable and $\mathcal{U}$ -measurable process $\left(\psi_t^{v,v'}\right)_{0\leqslant t\leqslant T}$ such that

$$f(t,y,\mu,z,v)-f(t,y,\mu,z,v')\leqslant\int_U\psi_t^{v,v'}(e)\left(v-v'\right)(e)\nu(\mathrm{d}e),\tag{3.4}$$

where there exist constants $C_2>0$ and $C_1\geqslant-1+\delta$ for some $\delta>0$ such that

$$C_1(1\wedge|e|)\leqslant\psi_t^{v,v'}(e)\leqslant C_2(1\wedge|e|).\tag{3.5}$$

3.2 Regularity of the map γ_{Y_t}

Let $Y\in\mathcal{S}^2$, and consider the process $(\bar{Y}_t):=(Y_{t^-})$. Let $t\in[0,T]$ and denote the function $\varphi:x\longmapsto\mathbb{E}(\arctan(\bar{Y}_t-x))$. It is known that there exists a unique $\gamma_{\bar{Y}_t}\in\mathbb{R}$ such that

$$\mathbb{E}\left(\arctan\left(\bar{Y}_t-\gamma_{\bar{Y}_t}\right)\right)=0.\tag{3.6}$$

Following the work by Briand et al. [5], we choose $x,y\in\mathbb{R}$ such that

$$x<\gamma_{\bar{Y}_t}<y.\tag{3.7}$$

By the dominated convergence theorem, the continuity of the arctan function, and the strict nonincreasing property of φ, we conclude that

$$\begin{aligned}\lim_{s\uparrow t}\mathbb{E}\left(\arctan\left(Y_s-x\right)\right)&=\mathbb{E}\left(\arctan\left(Y_{t^-}-x\right)\right)\\&=\mathbb{E}\left(\arctan\left(\bar{Y}_t-x\right)\right)\\&>\mathbb{E}\left(\arctan\left(\bar{Y}_t-\gamma_{\bar{Y}_t}\right)=0\right.\\&>\mathbb{E}\left(\arctan\left(\bar{Y}_t-y\right)\right)\\&=\mathbb{E}\left(\arctan\left(Y_{t^-}-y\right)\right)\\&=\lim_{s\uparrow t}\mathbb{E}\left(\arctan\left(Y_s-y\right)\right).\end{aligned}$$

Therefore, as n tends to infinity, for $0 < t - s < \frac{1}{n}$, we have

$$\mathbb{E}\left(\arctan\left(Y_s - x\right)\right) > 0 > \mathbb{E}\left(\arctan\left(Y_s - y\right)\right).$$

Thus, for sufficiently large n and by the intermediate value theorem, there exists a unique $z_s \in]x, y[$ such that $\mathbb{E}\left(\arctan\left(Y_s - z_s\right)\right) = 0$. As γ_{Y_s} is unique such that $\mathbb{E}\left(\arctan\left(Y_s - \gamma_{Y_s}\right)\right) = 0$, we obtain $z_s = \gamma_{Y_s}$. Consequently, for sufficiently large n,

$$x < \gamma_{Y_s} < y.$$

Combining this with (3.7) and the uniqueness in (3.6), we deduce that $\lim_{s\uparrow t} \gamma_{Y_s} = \gamma_{\bar{Y}_t}$. This establishes the left-limit property of the map $t \mapsto \gamma_{Y_t}$. The right continuity follows similarly.

4. Well-posedness of the BSDE with central value reflection

In this section, we present the main result of this paper, establishing the well-posedness of the BSDE (3.1) with central value reflection, as defined in Definition 3.1. Our approach relies on a fixed-point argument to establish the existence and uniqueness of the solution in small time intervals. These solutions are then combined to demonstrate the solvability of the BSDE with central value reflection over the entire time interval $[0, T]$. To achieve this, we proceed through the intermediate subsections addressing the specific cases of the generator.

4.1 The constant driver case

In this section, we establish the well-posedness of the problem under a constant driver. The solution in this case belongs to the product of spaces $\mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{H}^2_\nu \times \mathcal{A}_D$.

Proposition 4.1 *Let $(f_t)_{t\in[0,T]}$ be in the space $L^2\left(\Omega; L^1([0,T])\right)$, and ξ a terminal condition satisfying $(\boldsymbol{H}_\xi)$. Then, the BSDE with central value reflection*

$$\begin{cases} Y_t = \xi + \int_t^T f_s \mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s - \int_t^T \int_U V_s(e)\tilde{N}(\mathrm{d}s, \mathrm{d}e) + K_T - K_t, \quad t \in [0,T], \\ \gamma_{Y_t} \geqslant 0, \quad \forall t \in [0,T], \end{cases} \tag{4.1}$$

admits a unique square-integrable deterministic flat solution, i.e., $(Y, Z, V, K) \in \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{H}^2_\nu \times \mathcal{A}_D$ and satisfies (4.1) as well as the flatness condition (3.2). With $L^2\left(\Omega; L^1([0,T])\right) := \{$progressively measurable processes $(f_t)_t$ such that $\mathbb{E}(|\int_0^T |f_t|\mathrm{d}t|^2) < \infty\}$.

Proof Let $(\bar{Y}, \bar{Z}, \bar{V}) \in \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{H}^2_\nu$ be the solution of the following BSDE:

$$\bar{Y}_t = \xi + \int_t^T f_s \mathrm{d}s - \int_t^T \bar{Z}_s \mathrm{d}W_s - \int_t^T \int_U \bar{V}_s(e)\tilde{N}(\mathrm{d}s, \mathrm{d}e), \tag{4.2}$$

the existence and uniqueness of this solution follow from martingale representation theorem.

From the previous section, we know that for all $t \in [0, T]$, there exists a unique $\gamma_{\bar{Y}_t} \in \mathbb{R}$ such that

$$\mathbb{E}\left(\arctan\left(\bar{Y}_t - \gamma_{\bar{Y}_t}\right)\right) = 0, \tag{4.3}$$

and the map $t \mapsto \gamma_{\bar{Y}_t}$ is right continuous left limited.

Now, we introduce the following deterministic nondecreasing right-continuous left-limited process K defined y

$$K_t = \sup_{0\leqslant s\leqslant T} \left(\gamma_{\bar{Y}_s}\right)^- - \sup_{t\leqslant s\leqslant T} \left(\gamma_{\bar{Y}_s}\right)^-,$$

where we have $K_0 = 0$. Given K, we define

$$(Y_t, Z_t, V_t, K_t) = \left(\bar{Y}_t + (K_T - K_t), \bar{Z}_t, \bar{V}_t, K_t\right) \in \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{H}^2_\nu \times \mathcal{A}_D,$$

as the unique solution to the first equation of (4.1).

To address the central value reflection in (4.1), we consider $t \in [0,T]$. Using $(\boldsymbol{H}_\xi)$, we obtain

$$\begin{aligned}\mathbb{E}\left(\arctan\left(Y_t\right)\right) = \mathbb{E}\left(\arctan\left(\bar{Y}_t + K_T - K_t\right)\right) &= \mathbb{E}\left(\arctan\left(\bar{Y}_t + \sup_{t \leqslant s \leqslant T}\left(\gamma_{\bar{Y}_s}\right)^- - (\gamma_\xi)^-\right)\right)\\ &\geqslant \mathbb{E}\left(\arctan\left(\bar{Y}_t - \gamma_{\bar{Y}_t}\right)\right) = 0 = \mathbb{E}\left(\arctan\left(Y_t - \gamma_{Y_t}\right)\right).\end{aligned}$$

Applying the inverse function of $x \mapsto \mathbb{E}(\arctan(Y_t - x))$ to both sides of the previous inequality, we obtain $\gamma_{Y_t} \geqslant 0$ for all $t \in [0,T]$.

For the flatness of the solution, we observe that

$$K_T - K_{t^-} = \sup_{t \leqslant s \leqslant T} (\gamma_{\bar{Y}_{s^-}})^- = (\gamma_{\bar{Y}_{t^-}})^- \quad \mathrm{d}K \text{ a.e.},$$

and $(\gamma_{\bar{Y}_{t^-}})^- > 0$ $\mathrm{d}K$ a.e., then,

$$\begin{aligned}\int_0^T \mathbb{E}\left(\arctan\left(Y_{t^-}\right)\right)\mathrm{d}K_t &= \int_0^T \mathbb{E}\left(\arctan\left(\bar{Y}_{t^-} + K_T - K_{t^-}\right)\right)\mathrm{d}K_t\\ &= \int_0^T \mathbb{E}\left(\arctan\left(\bar{Y}_{t^-} + (\gamma_{\bar{Y}_{t^-}})^-\right)\right)\mathbf{1}_{(\gamma_{\bar{Y}_{t^-}})^- > 0}\mathrm{d}K_t\\ &= 0\\ &= \int_0^T \mathbb{E}\left(\arctan\left(Y_{t^-} - \gamma_{Y_{t^-}}\right)\right)\mathrm{d}K_t.\end{aligned} \tag{4.4}$$

The last equality follows because the integrand is zero. Therefore, since $x \mapsto \mathbb{E}\left(\arctan\left(Y_{t^-} - x\right)\right)$ is nonincreasing and $\gamma_{Y_{t^-}} \geqslant 0$, we have $\mathbb{E}\left(\arctan\left(Y_{t^-}\right)\right) = \mathbb{E}\left(\arctan\left(Y_{t^-} - \gamma_{Y_{t^-}}\right)\right)$ $\mathrm{d}K$ a.e. Similarly, using the inverse function, it holds that $\gamma_{Y_{t^-}} = 0$ $\mathrm{d}K$ a.e. Consequently, we have $\int_0^T \gamma_{Y_{t^-}}\mathrm{d}K_t = 0$. Hence, the quadruplet $(Y, Z, V, K) \in \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{H}^2_\nu \times \mathcal{A}_D$ is a deterministic flat solution to the BSDE (4.1) with central value reflection.

It remains to show the uniqueness of the solution. Let $\left(Y^1, Z^1, V^1, K^1\right)$ and $\left(Y^2, Z^2, V^2, K^2\right)$ two different deterministic flat solutions to the BSDE with central value reflection (4.1). As both $\left(Y^1_t - (K^1_T - K^1_t), Z^1_t, V^1_t\right)$ and $\left(Y^2_t - (K^2_T - K^2_t), Z^2_t, V^2_t\right)$ are solutions to the standard BSDE (4.2), uniqueness of this solution implies that for all $t \in [0,T]$

$$Y^1_t - \left(K^1_T - K^1_t\right) = Y^2_t - \left(K^2_T - K^2_t\right); \;\; Z^1_t = Z^2_t \text{ and } V^2_t = V^2_t.$$

By contradiction, we first prove that $K^1 = K^2$, then we deduce that $Y^1 = Y^2$. Assume that there exists $t_1 \in [0,T]$ such that $K^1_T - K^1_{t_1} > K^2_T - K^2_{t_1}$, and let

$$t_2 := \inf\left\{t \geqslant t_1; \quad K^1_T - K^1_t \leqslant K^2_T - K^2_t\right\}.$$

We have $K^1_T - K^1_{t^-} \geqslant K^2_T - K^2_{t^-}$ for each $t \in (t_1, t_2]$.

- For $t_2 \leqslant T$, we define

$$\bar{t} := \inf\left\{t \in (t_1, t_2]; \quad K^1_T - K^1_{t^-} = K^2_T - K^2_{t^-}\right\},$$

– if $\bar{t} \leqslant t_2$, we obtain $K^1_T - K^1_{t^-} > K^2_T - K^2_{t^-}$ for all $t \in (t_1, \bar{t})$. Then, $Y^1_{t^-} > Y^2_{t^-}$ on $(t_1, \bar{t})$. As a result, for all $t \in (t_1, \bar{t})$

$$\mathbb{E}\left(\arctan\left(Y^1_{t^-}\right)\right) > \mathbb{E}\left(\arctan\left(Y^2_{t^-}\right)\right) \geqslant \mathbb{E}(\arctan(Y^2_{t^-} - \gamma_{Y^2_{t^-}})) = \mathbb{E}(\arctan(Y^1_{t^-} - \gamma_{Y^1_{t^-}})),$$

where we used the fact that $\gamma_{Y^2_{t-}} \geqslant 0$ and that the last two terms are equal to zero. Then, we conclude that $\gamma_{Y^1_{t-}} > 0$ for each $t \in (t_1, \bar{t})$. By the flatness of the solution Y^1, we obtain $\mathrm{d}K^1 = 0$ on $(t_1, \bar{t})$. Finally, combining this result with the definition of $\bar{t}$, we deduce that

$$K_T^2 - K_{t_1}^2 < K_T^1 - K_{t_1}^1 = K_T^1 - K_{t_1^+}^1 = K_T^1 - K_{\bar{t}-}^1 = K_T^2 - K_{\bar{t}-}^2,$$

which contradicts the fact that $K_T^2 - K_t^2$ is nonincreasing.

– if $\bar{t} = \infty$, we obtain $K_T^1 - K_{t-}^1 > K_T^2 - K_{t-}^2$ for all $t \in (t_1, t_2]$, then $Y_{t-}^1 > Y_{t-}^2$ on $(t_1, t_2]$. Similar arguments as above yield $\gamma_{Y^1_{t-}} > 0$ for all $t \in (t_1, t_2]$. Thus, by the flatness of the solution Y^1, we deduce that $\mathrm{d}K^1 = 0$ on $(t_1, t_2]$. Combing this result and the definition of t_2, we conclude that

$$K_T^2 - K_{t_1}^2 < K_T^1 - K_{t_1}^1 = K_T^1 - K_{t_1^+}^1 = K_T^1 - K_{t_2}^1 \leqslant K_T^2 - K_{t_2}^2,$$

which contradicts the fact that $K_T^2 - K_t^2$ is nonincreasing.

- For $t_2 = \infty$, we have $K_T^1 - K_{t-}^1 \geqslant K_T^2 - K_{t-}^2$ for each $t \in (t_1, T]$. We define

$$\bar{t} := \inf\left\{t \in (t_1, T]; \quad K_T^1 - K_{t-}^1 = K_T^2 - K_{t-}^2\right\},$$

– if $\bar{t} \leqslant T$, we proceed exactly as in the previous case, leading to a contradiction.

– if $\bar{t} = \infty$, we have $K_T^1 - K_{t-}^1 > K_T^2 - K_{t-}^2$ for each $t \in (t_1, T]$, then $Y_{t-}^1 > Y_{t-}^2$ on $(t_1, T]$. Similarly to the previous cases, we obtain $\mathrm{d}K^1 = 0$ on $(t_1, T]$, so that

$$0 \leqslant K_T^2 - K_{t_1}^2 < K_T^1 - K_{t_1}^1 = K_T^1 - K_{t_1^+}^1 = K_T^1 - K_T^1 = 0,$$

which leads to a contradiction.

Finally, there exists a unique deterministic flat solution $(Y, Z, V, K) \in \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{H}_\nu^2 \times \mathcal{A}_D$ to the BSDE (4.1) with central value reflection. □

4.2 Driver independent of Y

In this section, we address the well-posedness of the BSDE with central value reflection when the generator is independent of both Y and $\mathbf{P}_Y$. This specific case is considered because our fixed-point argument relies solely on the first component of the solution. We now consider the following BSDE with central value reflection:

$$\begin{cases} Y_t = \xi + \int_t^T f\left(s, Z_s, V_s\right) \mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s - \int_t^T \int_U V_s(e) \tilde{N}(\mathrm{d}s, \mathrm{d}e) + K_T - K_t, \quad t \in [0, T], \\ \gamma_{Y_t} \geqslant 0, \quad \forall t \in [0, T]. \end{cases} \tag{4.5}$$

Theorem 4.1 *Assume that $(\boldsymbol{H}_\xi)$ and $(\boldsymbol{H}_f)$ hold, with f independent of y and μ. Then, the BSDE (4.5) with central value reflection has a unique deterministic flat solution $(Y, Z, V, K) \in \mathcal{S}^\infty \times \mathcal{H}_{BMO}^2 \times \mathcal{J}_{BMO}^2 \times \mathcal{A}_D^\infty$.*

Proof Let $(\bar{Y}, \bar{Z}, \bar{V}) \in \mathcal{S}^\infty \times \mathcal{H}_{BMO}^2 \times \mathcal{J}_{BMO}^2$ be the solution to the following quadratic BSDE:

$$\bar{Y}_t = \xi + \int_t^T f\left(s, \bar{Z}_s, \bar{V}_s\right) \mathrm{d}s - \int_t^T \bar{Z}_s \mathrm{d}W_s - \int_t^T \int_U \bar{V}_s(e) \tilde{N}(\mathrm{d}s, \mathrm{d}e), \tag{4.6}$$

Theorem 6.1 in the study by Possamai et al. [18] ensures that such a solution exists and is unique. Given that $\bar{Z} \in \mathcal{H}_{BMO}^2$, and $\bar{V} \in \mathcal{J}_{BMO}^2$, Lemma 2.1 ensures that

$$\mathbb{E}\left[\left(\int_0^T \left|\bar{Z}_s\right|^2 \mathrm{d}s\right)^2\right] < \infty \text{ and } \mathbb{E}\left[\left(\int_0^T \int_U \left|\bar{V}_s(x)\right|^2 \nu(\mathrm{d}x)\mathrm{d}s\right)^2\right] < \infty. \tag{4.7}$$

From the quadratic growth assumption on f, we obtain for each $t \in [0,T]$

$$|f(t,\bar{Z}_t,\bar{V}_t)| \leqslant \max\left(\left|\alpha_t + \frac{\kappa}{2}|\bar{Z}_t|^2 + \frac{\mathbf{j}_\kappa(\bar{V}_t)}{\kappa}\right|; \left|\alpha_t + \frac{\kappa}{2}|\bar{Z}_t|^2 + \frac{\mathbf{j}_\kappa(-\bar{V}_t)}{\kappa}\right|\right). \tag{4.8}$$

Additionally, by applying Taylor's expansion and mean value theorem, we obtain that

$$\mathbf{j}_\kappa(\bar{V}_t) \leqslant \frac{\kappa}{2} e^{\kappa\|\bar{V}\|_{\mathcal{J}^\infty}} \parallel \bar{V} \parallel^2_{L^2(\nu)} \quad \text{and } \mathbf{j}_\kappa(-\bar{V}_t) \leqslant \frac{\kappa}{2} e^{\kappa\|\bar{V}\|_{\mathcal{J}^\infty}} \parallel \bar{V} \parallel^2_{L^2(\nu)} .$$

Consequently,

$$|f(t,\bar{Z}_t,\bar{V}_t)| \leqslant \alpha_t + \frac{\kappa}{2}|\bar{Z}_t|^2 + \frac{\kappa}{2} e^{\kappa\|\bar{V}\|_{\mathcal{J}^\infty}} \parallel \bar{V} \parallel^2_{L^2(\nu)} .$$

Finally, because α is essentially bounded, we deduce that

$$\begin{aligned}
\mathbb{E}\left[\left|\int_0^T |f(t,\bar{Z}_t,\bar{V}_t)|\mathrm{d}t\right|^2\right] &\leqslant 3T^2 \parallel \alpha \parallel^2_{\mathcal{S}^\infty} + \frac{3\kappa^2}{4}\mathbb{E}\left[\left(\int_0^T |\bar{Z}_t|^2 \mathrm{d}t\right)^2\right] \\
&\quad + \frac{3\kappa^2}{4} e^{\kappa\|\bar{V}\|_{\mathcal{J}^\infty}} \mathbb{E}\left[\left(\int_0^T \parallel \bar{V}_t \parallel^2_{L^2(\nu)} \mathrm{d}t\right)^2\right] \\
&< \infty,
\end{aligned} \tag{4.9}$$

where we also use (2.2) to obtain $\parallel \bar{V} \parallel_{\mathcal{J}^\infty} < \infty$. Then, $f(t,\bar{Z}_t,\bar{V}_t) \in L^2\left(\Omega; L^1([0,T])\right)$, and Proposition 4.1 ensures that the following BSDE with central value reflection

$$\begin{cases} Y_t = \xi + \int_t^T f(s,\bar{Z}_s,\bar{V}_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s - \int_t^T \int_U V_s(e)\tilde{N}(\mathrm{d}s,\mathrm{d}e) + K_T - K_t, \quad t \in [0,T], \\ \gamma_{Y_t} \geqslant 0, \quad \forall t \in [0,T], \end{cases} \tag{4.10}$$

has a unique square integrable deterministic flat solution, i.e., $(Y,Z,V,K) \in \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{H}^2_\nu \times \mathcal{A}_D$, where

$$K_t = \sup_{0\leqslant s\leqslant T} \left(\gamma_{\bar{Y}_s}\right)^- - \sup_{t\leqslant s\leqslant T} \left(\gamma_{\bar{Y}_s}\right)^- .$$

Observing that $(Y_t - (K_T - K_t), Z_t, V_t)$ and $\left(\bar{Y}_t, \bar{Z}_t, \bar{V}_t\right)$ are both solutions to the following standard BSDE:

$$y_t = \xi + \int_t^T f\left(s,\bar{Z}_s,\bar{V}_s\right)\mathrm{d}s - \int_t^T z_s \mathrm{d}W_s - \int_t^T \int_U v_s(e)\tilde{N}(\mathrm{d}s,\mathrm{d}e).$$

By uniqueness of the solution of this equation, we deduce that

$$\bar{Y}_t = Y_t - (K_T - K_t), \quad \bar{Z}_t = Z_t \quad \text{and} \quad \bar{V}_t = V_t, \quad \text{for each} \quad t \in [0,T].$$

Finally, because K is deterministic, $(Y,Z,V,K) \in \mathcal{S}^\infty \times \mathcal{H}^2_{BMO} \times \mathcal{J}^2_{BMO} \times \mathcal{A}^\infty_D$ is a deterministic flat solution to the BSDE (4.5) with central value reflection. This completes the proof of the existence part.

For uniqueness, let (Y,Z,V,K) and $(\bar{Y},\bar{Z},\bar{V},\bar{K})$ be two different solutions to the BSDE (4.5) with central value reflection. Then, the processes $(Y_t - (K_T - K_t), Z_t, V_t)$ and $\left(\bar{Y}_t - \left(\bar{K}_T - \bar{K}_t\right),\right.$

$\bar{Z}_t, \bar{V}_t)$ are both solutions to the standard BSDE (4.6). By the uniqueness of the solution of this equation, we have $Z_t = \bar{Z}_t$ and $V_t = \bar{V}_t$ for all $[0,T]$.

Therefore, (Y, Z, V, K) and $(\bar{Y}, \bar{Z}, \bar{V}, \bar{K})$ are both solutions to the following BSDE with central value reflection:

$$\begin{cases} y_t = \xi + \int_t^T f(s, Z_s, V_s)\,\mathrm{d}s - \int_t^T z_s \mathrm{d}W_s - \int_t^T \int_U v_s(e)\tilde{N}(\mathrm{d}s, \mathrm{d}e) + k_T - k_t, \quad \forall t \in [0,T], \\ \gamma_{y_t \geqslant 0}, \quad \forall t \in [0,T], \end{cases}$$

By the uniqueness provided in Proposition 4.1, we have $(Y, Z, V, K) = (\bar{Y}, \bar{Z}, \bar{V}, \bar{K})$. This completes the proof of Theorem 4.1. □

4.3 Local solution

In this section, we construct the solution of the BSDE (3.1) with central value reflection over small time intervals. The maximal length of these intervals is determined by the universal constants defined in our assumptions, as well as the bound of the first component of the solution. This result is stated in Theorem 4.2 below and is derived using fixed-point arguments.

Let $h \in (0,1)$, and consider the time interval $[T-h, T]$. We define Φ as the mapping that associates to a process $\Gamma \in \mathcal{S}_h^\infty$ another process Y, which is the first component of the solution of the following BSDE with central value reflection:

$$\begin{cases} Y_t = \xi + \int_t^T f(s, \Gamma_s, \mathbf{P}_{\Gamma_s}, Z_s, V_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s - \int_t^T \int_U V_s(e)\tilde{N}(\mathrm{d}s, \mathrm{d}e) + K_T - K_t, \quad t \in [T-h, T], \\ \gamma_{Y_t} \geqslant 0, \quad \forall t \in [T-h, T]. \end{cases} \tag{4.11}$$

In view of Theorem 4.1, the map Φ is well defined from $\mathcal{S}_h^\infty$ to itself. For each $\Gamma \in \mathcal{S}_h^\infty$, the BSDE (4.11) with central value reflection has a unique deterministic flat solution $(Y, Z, V, K) \in \mathcal{S}_h^\infty \times \mathcal{H}^2_{BMO_h} \times \mathcal{J}^2_{BMO_h} \times \mathcal{A}^\infty_{D_h}$. Here, $\mathcal{S}_h^\infty$ refers to the space $\mathcal{S}^\infty$, where the supremum is taken over the interval $[T-h, T]$, and the same notation applies to other spaces.

To ensure the contractive property of the map Φ, we select a sufficiently large constant $\bar{C}$ and prove that Φ is a contraction from $\mathcal{B}_{\bar{C}}$ to itself, where

$$\mathcal{B}_{\bar{C}} := \left\{ \Gamma \in \mathcal{S}_h^\infty; \ \ \|\Gamma\|_{\mathcal{S}_h^\infty} \leqslant \bar{C} \right\}. \tag{4.12}$$

To achieve this, we first prove that $\Phi(\mathcal{B}_{\bar{C}}) \subset \mathcal{B}_{\bar{C}}$.

Lemma 4.1 *Assume that* $(\boldsymbol{H}_\xi)$ *and* $(\boldsymbol{H}_f)$ *hold. We define*

$$C_0 := 2(M + \| \alpha \|_{\mathcal{S}_h^\infty}). \tag{4.13}$$

Let $\bar{C} \geqslant C_0$*, then*

$$\exists h_{\bar{C}} > 0, \ \ \forall h \in (0, h_{\bar{C}}], \quad \|\Gamma\|_{\mathcal{S}_h^\infty} \leqslant \bar{C} \implies \|Y\|_{\mathcal{S}_h^\infty} \leqslant \bar{C}, \tag{4.14}$$

where Y *is the first component of the solution of the BSDE with central value reflection 4.11.*

Proof For $\Gamma \in \mathcal{B}_{\bar{C}}$, let $(Y, Z, V, K) \in \mathcal{S}_h^\infty \times \mathcal{H}^2_{BMO_h} \times \mathcal{J}^2_{BMO_h} \times \mathcal{A}^\infty_{D_h}$ be the unique solution of the BSDE with central value reflection (4.11), and let $(\bar{Y}, \bar{Z}, \bar{V}) \in \mathcal{S}_h^\infty \times \mathcal{H}^2_{BMO_h} \times \mathcal{J}^2_{BMO_h}$ be the solution of the following BSDE:

$$\bar{Y}_t = \xi + \int_t^T f\left(s, \Gamma_s, \mathbf{P}_{\Gamma_s} \bar{Z}_s, \bar{V}\right)\mathrm{d}s - \int_t^T \bar{Z}_s \mathrm{d}W_s - \int_t^T \int_U \bar{V}_s(e)\tilde{N}(\mathrm{d}s, \mathrm{d}e), \quad t \in [T-h, T], \quad \text{a.s.} \tag{4.15}$$

The existence and uniqueness of the solution follow from Theorem 6.1 in Possamai [18], as f satisfies the quadratic growth condition and $\Gamma \in \mathcal{S}_h^\infty$. For each $t \in [T-h, T]$, we define

$$K_t = \sup_{0 \leqslant s \leqslant T} \left(\gamma_{\bar{Y}_s}\right)^- - \sup_{t \leqslant s \leqslant T} \left(\gamma_{\bar{Y}_s}\right)^- . \tag{4.16}$$

It follows that for each $t \in [T-h, T]$,

$$Y_t - (K_T - K_t) = \bar{Y}_t, \quad Z_t = \bar{Z}_t \quad \text{and} \quad V_t = \bar{V}_t. \tag{4.17}$$

As a result,

$$\|Y\|_{\mathcal{S}_h^\infty} \leqslant \left\|\bar{Y}\right\|_{\mathcal{S}_h^\infty} + \sup_{T-h \leqslant t \leqslant T} \left(\gamma_{\bar{Y}_t}\right)^- . \tag{4.18}$$

The rest of the proof is performed in two steps. First, we estimate $\left\|\bar{Y}\right\|_{\mathcal{S}_h^\infty}$. Then, we estimate the term $\sup_{T-h \leqslant t \leqslant T} \left(\gamma_{\bar{Y}_t}\right)^-$.

Step 1 According to Tanaka's formula, we have, for each $t \in [T-h, T]$,

$$\begin{aligned} \left|\bar{Y}_t\right| = |\xi| &+ \int_t^T \operatorname{sgn}\left(\bar{Y}_{s^-}\right) f\left(s, \Gamma_s, \mathbf{P}_{\Gamma_s}, \bar{Z}_s, \bar{V}_s\right) \mathrm{d}s - \int_t^T \operatorname{sgn}\left(\bar{Y}_{s^-}\right) \bar{Z}_s \mathrm{d}W_s \\ &- \int_t^T \int_U \operatorname{sgn}\left(\bar{Y}_{s^-}\right) \bar{V}_s(e) \tilde{N}(\mathrm{d}e, \mathrm{d}s) - \int_t^T \mathrm{d}L_s^0 - \sum_{t<s \leqslant T} \left|\bar{Y}_s\right| - \left|\bar{Y}_{s-}\right| - \operatorname{sgn}\left(\bar{Y}_{s-}\right) \Delta\bar{Y}_s, \end{aligned} \tag{4.19}$$

where L^0 denotes the local time of $\bar{Y}$ at the origin. Consider the following process:

$$\Psi_t := \exp\left(\kappa|\bar{Y}_t| + \kappa\int_0^t \alpha_s \mathrm{d}s + \kappa\beta\int_0^t |\Gamma_s| + W_1(\mathbf{P}_{\Gamma_s}, \delta_0)\mathrm{d}s\right).$$

By Itô's formula applied to ψ_t, we obtain

$$\begin{aligned} \Psi_t = \Psi_T &+ \int_t^T \kappa\Psi_s \operatorname{sgn}\left(\bar{Y}_{s^-}\right) f\left(s, \Gamma_s, \mathbf{P}_{\Gamma_s}, \bar{Z}_s, \bar{V}_s\right) \mathrm{d}s \\ &- \int_t^T \kappa\Psi_s \left(\alpha_s + \beta(|\Gamma_s| + W_1\left(\mathbf{P}_{\Gamma_s}, \delta_0\right)) + \frac{\kappa}{2}\left|\bar{Z}_s\right|^2 + \int_U \mathbf{j}_\kappa\left(\operatorname{sgn}\left(\bar{Y}_{s^-}\right)\bar{V}_s(e)\right)\nu(\mathrm{d}e)\right)\mathrm{d}s \\ &- \int_t^T \kappa\Psi_s \operatorname{sgn}\left(\bar{Y}_{s^-}\right) \bar{Z}_s \mathrm{d}W_s - \int_t^T \int_U \Psi_s(\mathrm{e}^{\kappa \operatorname{sgn}\left(\bar{Y}_{s^-}\right)\bar{V}_s(e)} - 1)\tilde{N}(\mathrm{d}s, \mathrm{d}e) \\ &- \int_t^T \kappa\Psi_s \mathrm{d}L_s^0 - \sum_{t<s\leqslant T} \kappa\Psi_s\left(\left|\bar{Y}_s\right| - \left|\bar{Y}_{s-}\right| - \operatorname{sgn}\left(\bar{Y}_{s-}\right)\Delta\bar{Y}_s\right). \end{aligned} \tag{4.20}$$

By Assumption (3.3), we deduce that

$$\begin{aligned} &\operatorname{sgn}\left(\bar{Y}_{s^-}\right) f\left(s, \Gamma_s, \mathbf{P}_{\Gamma_s}, \bar{Z}_s, \bar{V}_s\right) \\ &\leqslant \alpha_s + \beta(|\Gamma_s| + W_1\left(\mathbf{P}_{\Gamma_s}, \delta_0\right)) + \frac{\kappa}{2}\left|\bar{Z}_s\right|^2 + \int_U \mathbf{j}_\kappa\left(\operatorname{sgn}\left(\bar{Y}_{s^-}\right)\bar{V}_s(e)\right)\nu(\mathrm{d}e). \end{aligned} \tag{4.21}$$

Using this equation and taking the conditional expectation on both sides of (4.20), and considering that the last two terms are negative, we deduce that

$$\Psi_t \leqslant \mathbb{E}\left(\Psi_T / \mathcal{F}_t\right). \tag{4.22}$$

Then, for each $t \in [T-h, T]$, we have

$$\begin{aligned}\exp(\kappa|\bar{Y}_t|) &\leqslant \mathbb{E}\left(\exp\left(\kappa|\xi| + \kappa\int_t^T \alpha_s \mathrm{d}s + \kappa\beta\int_t^T |\Gamma_s| + W_1(\mathbf{P}_{\Gamma_s}, \delta_0)\mathrm{d}s\right)/\mathcal{F}_t\right) \\ &\leqslant \exp\left(\kappa \parallel \xi \parallel_\infty + \kappa h \parallel \alpha \parallel_{\mathcal{S}_h^\infty} + 2\kappa\beta h \parallel \Gamma \parallel_{\mathcal{S}_h^\infty}\right). \end{aligned} \tag{4.23}$$

Consequently, composing using ln on both sides and using the fact that Γ belongs to $\mathcal{B}_{\bar{C}}$, it follows that

$$\parallel \bar{Y} \parallel_{\mathcal{S}_h^\infty} \leqslant M + h\left(\parallel \alpha \parallel_{\mathcal{S}_h^\infty} + 2\beta\bar{C}\right). \tag{4.24}$$

Step 2 The estimation of the term $\sup_{T-h\leqslant t\leqslant T}\left(\gamma_{\bar{Y}_t}\right)^-$. From Step 1, we have

$$\begin{aligned}\parallel \bar{Y} \parallel_{\mathcal{S}_h^\infty} &\leqslant M + h\left(\parallel \alpha \parallel_{\mathcal{S}_h^\infty} + 2\beta\bar{C}\right) \\ &:= R_h. \end{aligned} \tag{4.25}$$

Then, for each $t \in [T-h, T]$,

$$\mathbb{E}\left(\arctan\left(\bar{Y}_t - R_h\right)\right)\mathbb{E}\left(\arctan\left(\bar{Y}_t + R_h\right)\right) \leqslant 0.$$

Then, from the Intermediate Value Theorem, there exists $z_t \in [-R_h, R_h]$ such that

$$\mathbb{E}\left(\arctan\left(\bar{Y}_t - z_t\right)\right) = 0,$$

and by uniqueness of $\gamma_{\bar{Y}_t}$ such that $\mathbb{E}\left(\arctan\left(\bar{Y}_t - \gamma_{\bar{Y}_t}\right)\right) = 0$, we get $z_t = \gamma_{\bar{Y}_t}$ for all $t \in [T-h, T]$. Then, we deduce that

$$\gamma_{\bar{Y}_t} \in [-R_h, R_h]. \tag{4.26}$$

Hence,

$$\sup_{T-h\leqslant t\leqslant T}\left(\gamma_{\bar{Y}_t}\right)^- \leqslant \sup_{T-h\leqslant t\leqslant T}|\gamma_{\bar{Y}_t}| \leqslant R_h.$$

This completes Step 2 of the proof.

Therefore, in view of Steps 1 and 2 and using (4.18), we deduce that

$$\|Y\|_{\mathcal{S}_h^\infty} \leqslant 2M + 2h(\parallel \alpha \parallel_{\mathcal{S}_h^\infty} + 2\beta\bar{C}).$$

If we define $h_{\bar{C}} := \min\left(\frac{\|\alpha\|_{\mathcal{S}_h^\infty}}{\|\alpha\|_{\mathcal{S}_h^\infty} + 2\beta\bar{C}}\ ,\ T\right)$, it is easy to see that, for each $h \in (0, h_{\bar{C}}]$,

$$\|Y\|_{\mathcal{S}_h^\infty} \leqslant 2(M + \parallel \alpha \parallel_{\mathcal{S}_h^\infty}) := C_0.$$

Finally, for $\bar{C} \geqslant C_0$,

$$\exists h_{\bar{C}} := \min\left(\frac{\parallel \alpha \parallel_{\mathcal{S}_h^\infty}}{\parallel \alpha \parallel_{\mathcal{S}_h^\infty} + 2\beta\bar{C}}\ ,\ T\right) > 0, \quad \forall h \in (0, h_{\bar{C}}], \quad \|\Gamma\|_{\mathcal{S}_h^\infty} \leqslant \bar{C} \implies \|Y\|_{\mathcal{S}_h^\infty} \leqslant \bar{C}. \tag{4.27}$$

This completes the proof of Lemma 4.1. □

We now state the theorem concerning the existence and uniqueness of a local solution to our problem (3.1), specifically a solution defined over small time intervals. The following remark, which is used in the proof below, will be useful in subsequent discussions.

Remark 4.1 *If we set $\phi_t := D_z f(t, y, \mu, 0, v)$ for each $(t, y, \mu, v) \in [0, T] \times \mathbb{R} \times \mathcal{P}^1(\mathbb{R}) \times \mathcal{L}^2(U, \nu; \mathbb{R})$, then by the assumptions on f, we have $|D_z f(t, y, \mu, 0, v)| \leqslant r_t$, and because $r \in \mathcal{H}^2_{BMO_h}$, it follows $\phi \in \mathcal{H}^2_{BMO_h}$. Additionally, we know that $\left|D^2_{zz} f(t, y, \mu, z, v)\right| \leqslant \theta$. Therefore, by* *the Mean Value Theorem, for* $\lambda \in [0,1]$, *and all* $z, z' \in \mathbb{R}^d$, *we obtain*

$$\begin{aligned}
&|f(t,y,\mu,z,v) - f(t,y,\mu,z',v) \ -\phi_t \cdot (z-z')| \\
&= |D_z f(t,y,\mu,\lambda z+(1-\lambda)z',v)\cdot(z-z') - \phi_t\cdot(z-z')| \\
&\leqslant |D_z f(t,y,\mu,\lambda z+(1-\lambda)z',v) - D_z f(t,y,\mu,0,v)|\,|z-z'| \\
&\leqslant \max_{\bar{z}} \left|D^2_{zz} f(t,y,\mu,\bar{z},v)\right| |\lambda z+(1-\lambda)z'|\,|z-z'| \\
&\leqslant \theta\,|\lambda z+(1-\lambda)z'|\,|z-z'| \\
&\leqslant \max(\theta\lambda, \theta(1-\lambda))\,(|z|+|z|')\,|z-z'|. \qquad (4.28)
\end{aligned}$$

Then,

$$\frac{|f(t,y,\mu,z,v) - f(t,y,\mu,z',v)|}{|z-z'|^2} \quad |z-z'|\,\mathbf{1}_{z\neq z'} \leqslant \bar{C}_\theta\,(|\phi_t|+|z|+|z'|), \qquad (4.29)$$

with $\bar{C}_\theta := \max(\theta\lambda, \theta(1-\lambda)) \vee 1$.

Theorem 4.2 *Assume that* $(\boldsymbol{H}_\xi)$ *and* $(\boldsymbol{H}_f)$ *hold. Then, there exists a sufficiently large* $\bar{C} > 0$ *and a constant* $0 < \bar{h}_{\bar{C}} \leqslant T$ *such that for any* $h \in (0, \bar{h}_{\bar{C}}]$, *the BSDE (3.1) with central value reflection admits a unique deterministic flat solution* $(Y,Z,V,K) \in \mathcal{S}_h^\infty \times \mathcal{H}^2_{BMO_h} \times \mathcal{J}^2_{BMO_h} \times \mathcal{A}^\infty_{D_h}$ *satisfying*

$$\parallel Y \parallel_{\mathcal{S}_h^\infty} \leqslant \bar{C}. \qquad (4.30)$$

Proof The aim of this proof is to demonstrate that Φ defined at the beginning of this section is a contraction from $\mathcal{B}_{\bar{C}}$ to itself, where $\bar{C}$ is chosen to be at least C_0.

To achieve this, let $\Gamma^1, \Gamma^2 \in \mathcal{B}_{\bar{C}}$. We define $Y^1 = \Phi(\Gamma^1)$ and $Y^2 = \Phi(\Gamma^2)$, where $(Y^i, Z^i, V^i, K^i) \in \mathcal{S}_h^\infty \times \mathcal{H}^2_{BMO_h} \times \mathcal{J}^2_{BMO_h} \times \mathcal{A}^\infty_{D_h}$ is the deterministic flat solution of BSDE (4.11) with central value reflection associated with the data Γ^i for $i=1,2$.

Let $(\bar{Y}^i, \bar{Z}^i, \bar{V}^i) \in \mathcal{S}_h^\infty \times \mathcal{H}^2_{BMO_h} \times \mathcal{J}^2_{BMO_h}$ the solution of the BSDE (4.15) associated to Γ^i. We know that for each $t \in [T-h, T]$, $i=1,2$,

$$Y_t^i = \bar{Y}_t^i + \left(K_T^i - K_t^i\right), \quad Z_t^i = \bar{Z}_t^i \quad \textit{and} \quad V_t^i = \bar{V}_t^i. \qquad (4.31)$$

Then, we deduce that

$$\begin{aligned}
\left\|Y^1 - Y^2\right\|_{\mathcal{S}_h^\infty} &\leqslant \left\|\bar{Y}^1 - \bar{Y}^2\right\|_{\mathcal{S}_h^\infty} + \sup_{T-h\leqslant t\leqslant T} \left|\left(K_T^1 - K_t^1\right) - \left(K_T^2 - K_t^2\right)\right| \\
&= \left\|\bar{Y}^1 - \bar{Y}^2\right\|_{\mathcal{S}_h^\infty} + \sup_{T-h\leqslant t\leqslant T} \left|\sup_{t\leqslant s\leqslant T}\left(\gamma_{\bar{Y}_s^1}\right)^- - \sup_{t\leqslant s\leqslant T}\left(\gamma_{\bar{Y}_s^2}\right)^-\right| \\
&\leqslant \left\|\bar{Y}^1 - \bar{Y}^2\right\|_{\mathcal{S}_h^\infty} + \sup_{T-h\leqslant t\leqslant T} \left|\left(\gamma_{\bar{Y}_t^1}\right)^- - \left(\gamma_{\bar{Y}_t^2}\right)^-\right| \\
&\leqslant \left\|\bar{Y}^1 - \bar{Y}^2\right\|_{\mathcal{S}_h^\infty} + \sup_{T-h\leqslant t\leqslant T} \left|\gamma_{\bar{Y}_t^1} - \gamma_{\bar{Y}_t^2}\right|. \qquad (4.32)
\end{aligned}$$

The rest of the proof proceeds via three steps. We start by estimating the first term in the right-hand side of (4.32), then proceed to estimate the second term, and finally we give our conclusion in the last step.

Step 1 We denote $\delta x = x^1 - x^2$, with $x^i = \bar{Y}^i, \bar{Z}^i, \bar{V}^i$ for $i=1,2$. By Tanaka's formula applied to $|\delta\bar{Y}|$, for each $t \in [T-h, T]$, we have

$$\begin{aligned}\left|\delta\bar{Y}_t\right| &= \int_t^T \operatorname{sgn}\left(\delta\bar{Y}_{s^-}\right)\left(f\left(s,\Gamma_s^1,\mathbf{P}_{\Gamma_s^1},\bar{Z}_s^1,\bar{V}_s^1\right)-f\left(s,\Gamma_s^2,\mathbf{P}_{\Gamma_s^2},\bar{Z}_s^2,\bar{V}_s^2\right)\right)\mathrm{d}s-\int_t^T \operatorname{sgn}\left(\delta\bar{Y}_{s^-}\right)\delta\bar{Z}_s\mathrm{d}W_s\\ &\quad -\int_t^T\int_U \operatorname{sgn}\left(\delta\bar{Y}_{s^-}\right)\delta\bar{V}_s(e)\tilde{N}(\mathrm{d}e,\mathrm{d}s)-\int_t^T \mathrm{d}L_s^0-\sum_{t<s\leqslant T}\left|\delta\bar{Y}_s\right|-\left|\delta\bar{Y}_{s-}\right|-\operatorname{sgn}\left(\delta\bar{Y}_{s-}\right)\Delta(\delta\bar{Y}_s),\end{aligned} \tag{4.33}$$

where L^0 denotes the local time of $\delta\bar{Y}$ at 0. We consider the following process:

$$\Psi_t := \exp\left(\kappa|\delta\bar{Y}_t|+\kappa C_f\int_0^t|\delta\Gamma_s|+W^1(\mathbf{P}_{\Gamma_s^1},\mathbf{P}_{\Gamma_s^2})\mathrm{d}s\right).$$

By applying Itô's formula to Ψ_t, for each $t\in[T-h,T]$, we obtain

$$\begin{aligned}\Psi_t &= \Psi_T-\int_t^T\kappa C_f\Psi_s\left(|\delta\Gamma_s|+W^1(\mathbf{P}_{\Gamma_s^1},\mathbf{P}_{\Gamma_s^2})\right)\mathrm{d}s-\int_t^T\kappa\Psi_s\operatorname{sgn}\left(\delta\bar{Y}_{s^-}\right)\delta\bar{Z}_s\mathrm{d}W_s\\ &\quad+\int_t^T\kappa\Psi_s\operatorname{sgn}(\delta\bar{Y}_{s^-})\left(f\left(s,\Gamma_s^1,\mathbf{P}_{\Gamma_s^1},\bar{Z}_s^1,\bar{V}_s^1\right)-f\left(s,\Gamma_s^1,\mathbf{P}_{\Gamma_s^1},\bar{Z}_s^2,\bar{V}_s^1\right)\right)\mathrm{d}s\\ &\quad+\int_t^T\kappa\Psi_s\operatorname{sgn}(\delta\bar{Y}_{s^-})\left(f\left(s,\Gamma_s^1,\mathbf{P}_{\Gamma_s^1},\bar{Z}_s^2,\bar{V}_s^1\right)-f\left(s,\Gamma_s^1,\mathbf{P}_{\Gamma_s^1},\bar{Z}_s^2,\bar{V}_s^2\right)\right)\mathrm{d}s\\ &\quad+\int_t^T\kappa\Psi_s\operatorname{sgn}(\delta\bar{Y}_{s^-})\left(f\left(s,\Gamma_s^1,\mathbf{P}_{\Gamma_s^1},\bar{Z}_s^2,\bar{V}_s^2\right)-f\left(s,\Gamma_s^2,\mathbf{P}_{\Gamma_s^2},\bar{Z}_s^2,\bar{V}_s^2\right)\right)\mathrm{d}s\\ &\quad-\int_t^T\int_U\kappa\Psi_s\operatorname{sgn}\left(\delta\bar{Y}_{s^-}\right)\delta\bar{V}_s(e)\tilde{N}(\mathrm{d}e,\mathrm{d}s)-\frac{1}{2}\int_t^T\kappa^2\Psi_s|\delta\bar{Z}|^2\mathrm{d}s-\int_t^T\kappa\Psi_s\mathrm{d}L_s^0\\ &\quad-\sum_{t<s\leqslant T}\kappa\Psi_s\left(\left|\delta\bar{Y}_s\right|-\left|\delta\bar{Y}_{s-}\right|-\operatorname{sgn}\left(\delta\bar{Y}_{s-}\right)\Delta(\delta\bar{Y}_s)\right)-\int_t^T\int_U\kappa\Psi_s\mathbf{j}_\kappa\left(\operatorname{sgn}\left(\delta\bar{Y}_{s^-}\right)\delta\bar{V}_s(e)\right)N(\mathrm{d}e,\mathrm{d}s).\end{aligned} \tag{4.34}$$

First, by the Lipschitz continuity assumption on f, we have

$$\begin{aligned}&\int_t^T\kappa\Psi_s\operatorname{sgn}(\delta\bar{Y}_{s^-})\left(f\left(s,\Gamma_s^1,\mathbf{P}_{\Gamma_s^1},\bar{Z}_s^2,\bar{V}_s^2\right)-f\left(s,\Gamma_s^2,\mathbf{P}_{\Gamma_s^2},\bar{Z}_s^2,\bar{V}_s^2\right)\right)\mathrm{d}s\\ &\leqslant\int_t^T\kappa C_f\Psi_s\left(|\delta\Gamma_s|+W^1(\mathbf{P}_{\Gamma_s^1},\mathbf{P}_{\Gamma_s^2})\right)\mathrm{d}s.\end{aligned} \tag{4.35}$$

Second, using Remark 4.1, with $\phi_s = D_z f(s,\Gamma_s^1,\mathbf{P}_{\Gamma_s^1},0,\bar{V}_s^1)$, we see that there exists $b\in\mathcal{H}^2_{BMO}$ such that

$$f\left(s,\Gamma_s^1,\mathbf{P}_{\Gamma_s^1},\bar{Z}_s^1,\bar{V}_s^1\right)-f\left(s,\Gamma_s^1,\mathbf{P}_{\Gamma_s^1},\bar{Z}_s^2,\bar{V}_s^1\right)=b_s\,.\,\delta\bar{Z}_s. \tag{4.36}$$

Finally, by assumption (3.4), there exist predictable $\mathcal{U}$-measurable processes $\psi^{\bar{V}_s,0}$ and $\psi^{0,\bar{V}_s}$ such that

$$\begin{aligned}&\operatorname{sgn}(\delta\bar{Y}_{s^-})\left(f\left(s,\Gamma_s^1,\mathbf{P}_{\Gamma_s^1},\bar{Z}_s^2,\bar{V}_s^1\right)-f\left(s,\Gamma_s^1,\mathbf{P}_{\Gamma_s^1},\bar{Z}_s^2,\bar{V}_s^2\right)\right)\\ &\leqslant\int_U\left(\psi_s^{\bar{V}_s^1,\bar{V}_s^2}(e)\mathbf{1}_{\{\operatorname{sgn}(\delta\bar{Y}_{s^-})=1\}}+\psi_s^{\bar{V}_s^2,\bar{V}_s^1}(e)\mathbf{1}_{\{\operatorname{sgn}(\delta\bar{Y}_{s^-})=-1\}}\right)\operatorname{sgn}(\delta\bar{Y}_{s^-})\delta\bar{V}_s(e)\nu(\mathrm{d}e).\end{aligned} \tag{4.37}$$

We define the following probability measure

$$\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}} := \mathcal{E}\left(\int_t^T b_s \mathrm{d}W_s + \int_t^T \int_U \psi_s^{\bar{V}_s^1,\bar{V}_s^2}(e)\mathbf{1}_{\{\mathrm{sgn}(\delta\bar{Y}_{s-})=1\}} + \psi_s^{\bar{V}_s^2,\bar{V}_s^1}(e)\mathbf{1}_{\{\mathrm{sgn}(\delta\bar{Y}_{s-})=-1\}}\tilde{N}(\mathrm{d}s,\mathrm{d}e)\right), \tag{4.38}$$

which is well-defined because b, $\psi^{\bar{V}^1,\bar{V}^2}$ and $\psi^{\bar{V}^2,\bar{V}^1}$ satisfy the condition of Proposition 2.2.

Using (4.35), (4.36), and (4.37), then taking the conditional expectation with respect to the probability measure $\mathbf{Q}$ on both sides of (4.34), and considering that the last four terms in (4.34) are negative, we conclude that, for each $t \in [T-h,T]$,

$$\Psi_t \leqslant \mathbb{E}^{\mathbf{Q}}\left(\Psi_T/\mathcal{F}_t\right). \tag{4.39}$$

Consequently,

$$\exp\left(\kappa|\delta\bar{Y}_t|\right) \leqslant \exp\left(\kappa C_f \int_t^T |\delta\Gamma_s| + W^1(\mathbf{P}_{\Gamma_s^1},\mathbf{P}_{\Gamma_s^2})\mathrm{d}s\right). \tag{4.40}$$

Using the fact that $W^1(\mathbf{P}_{\Gamma_s^1},\mathbf{P}_{\Gamma_s^2}) \leqslant \mathbb{E}\left(|\delta\Gamma_s|\right)$, we obtain

$$\exp\left(\kappa|\delta\bar{Y}_t|\right) \leqslant \exp\left(2\kappa h C_f \parallel \delta\Gamma \parallel_{\mathcal{S}_h^\infty}\right).$$

Finally, by applying the function ln on both sides and taking the supremum, we deduce that

$$\parallel \bar{Y}^1 - \bar{Y}^2 \parallel_{\mathcal{S}_h^\infty} \leqslant 2C_f h \parallel \Gamma^1 - \Gamma^2 \parallel_{\mathcal{S}_h^\infty}. \tag{4.41}$$

Step 2 Following similar arguments to those applied in (4.25) and (4.26), we derive that, for $i=1,2$,

$$\begin{aligned}\|\bar{Y}^i\|_{\mathcal{S}_h^\infty} &\leqslant M + h\left(\parallel \alpha \parallel_{\mathcal{S}_h^\infty} + 2\beta\bar{C}\right)\\ &:= R_h\end{aligned} \tag{4.42}$$

and

$$\gamma_{\bar{Y}_t^i} \in [-R_h, R_h] \text{ for each } t \in [T-h,T]. \tag{4.43}$$

Suppose $\bar{R}_h := 2R_h$, we observe that by applying the mean value theorem to the function $g : x \mapsto \arctan(x)$ within the interval $[-\bar{R}_h, \bar{R}_h]$, g is locally bi-Lipschitz. Therefore, we conclude that for all $x,y \in [-\bar{R}_h,\bar{R}_h]$,

$$\frac{1}{1+\bar{R}_h^2}|x-y| \leqslant |\arctan(x)-\arctan(y)| \leqslant |x-y|.$$

By taking this into account, using equations (4.42) and (4.43), and noting that

$$\Theta_t := \arctan(\bar{Y}_t^1 - \gamma_{\bar{Y}_t^1}) - \arctan(\bar{Y}_t^2 - \gamma_{\bar{Y}_t^2}),$$

we conclude that for each $t \in [T-h,T]$,

$$\frac{1}{1+\bar{R}_h^2}\left|\bar{Y}_t^1 - \bar{Y}_t^2 - (\gamma_{\bar{Y}_t^1} - \gamma_{\bar{Y}_t^2})\right| \leqslant |\Theta_t| \leqslant \left|\bar{Y}_t^1 - \bar{Y}_t^2 - (\gamma_{\bar{Y}_t^1} - \gamma_{\bar{Y}_t^2})\right|. \tag{4.44}$$

Then,

$$\begin{cases} \dfrac{1}{1+\bar{R}_h^2}\left|\bar{Y}_t^1 - \bar{Y}_t^2 - (\gamma_{\bar{Y}_t^1} - \gamma_{\bar{Y}_t^2})\right| \leqslant \Theta_t \leqslant \left|\bar{Y}_t^1 - \bar{Y}_t^2 - (\gamma_{\bar{Y}_t^1} - \gamma_{\bar{Y}_t^2})\right| \\ \text{or} \\ -\left|\bar{Y}_t^1 - \bar{Y}_t^2 - (\gamma_{\bar{Y}_t^1} - \gamma_{\bar{Y}_t^2})\right| \leqslant \Theta_t \leqslant -\dfrac{1}{1+\bar{R}_h^2}\left|\bar{Y}_t^1 - \bar{Y}_t^2 - (\gamma_{\bar{Y}_t^1} - \gamma_{\bar{Y}_t^2})\right|. \end{cases} \tag{4.45}$$

By taking the expectation, we obtain

$$\begin{cases} \dfrac{1}{1+\bar{R}_h^2}\mathbb{E}\left(\left|\bar{Y}_t^1-\bar{Y}_t^2-(\gamma_{\bar{Y}_t^1}-\gamma_{\bar{Y}_t^2})\right|\right)\leqslant 0\leqslant \mathbb{E}\left(\left|\bar{Y}_t^1-\bar{Y}_t^2-(\gamma_{\bar{Y}_t^1}-\gamma_{\bar{Y}_t^2})\right|\right) \\ \text{or} \\ -\mathbb{E}\left(\left|\bar{Y}_t^1-\bar{Y}_t^2-(\gamma_{X_t^1}-\gamma_{\bar{Y}_t^2})\right|\right)\leqslant 0\leqslant \dfrac{-1}{1+R^2}\mathbb{E}\left(\left|\bar{Y}_t^1-\bar{Y}_t^2-(\gamma_{\bar{Y}_t^1}-\gamma_{\bar{Y}_t^2})\right|\right), \end{cases} \tag{4.46}$$

so that

$$\mathbb{E}\left(\left|\bar{Y}_t^1-\bar{Y}_t^2-(\gamma_{\bar{Y}_t^1}-\gamma_{\bar{Y}_t^2})\right|\right)=0.$$

Thus,

$$\mid \gamma_{\bar{Y}_t^1}-\gamma_{\bar{Y}_t^2}\mid=\mathbb{E}\left(\mid \bar{Y}_t^1-\bar{Y}_t^2\mid\right). \tag{4.47}$$

In conclusion, the operator γ is a local isometry in the L^1-norm uniformly with respect to t. Specifically, for $\|\bar{Y}^1\|_{\mathcal{S}_h^\infty}\leqslant R_h$ and $\|\bar{Y}^2\|_{\mathcal{S}_h^\infty}\leqslant R_h$,

$$\begin{aligned} \sup_{t\in[T-h,T]}\left|\gamma_{\bar{Y}_t^1}-\gamma_{\bar{Y}_t^2}\right| &= \sup_{t\in[T-h,T]}\mathbb{E}\left(\mid \bar{Y}_t^1-\bar{Y}_t^2\mid\right) \\ &\leqslant \|\bar{Y}^1-\bar{Y}^2\|_{\mathcal{S}_h^\infty} \\ &\leqslant 2C_f h\,\|\Gamma^1-\Gamma^2\|_{\mathcal{S}_h^\infty}. \end{aligned} \tag{4.48}$$

Step 3 Combining (4.41) and (4.48) and recalling (4.32), we deduce

$$\left\|Y^1-Y^2\right\|_{\mathcal{S}_h^\infty}\leqslant 4C_f h\,\|\Gamma^1-\Gamma^2\|_{\mathcal{S}_h^\infty}. \tag{4.49}$$

Then, by considering

$$\bar{h}_{\bar{C}}:=\min\left(\frac{1}{8C_f}\ ;\ h_{\bar{C}}\right), \tag{4.50}$$

we obtain, for all $h\in(0,\bar{h}_{\bar{C}}]$,

$$\left\|\Phi\left(\Gamma^1\right)-\Phi\left(\Gamma^2\right)\right\|_{\mathcal{S}_h^\infty}\leqslant\frac{1}{2}\left\|\Gamma^1-\Gamma^2\right\|_{\mathcal{S}_h^\infty},\quad \forall\,\Gamma^1,\Gamma^2\in B_{\bar{C}}. \tag{4.51}$$

We conclude that Φ is a contraction map from $\mathcal{B}_{\bar{C}}$ to itself over the time interval $[T-h,T]$ for any $h\in(0,\bar{h}_{\bar{C}}]$ with respect to the norm $\|\,.\,\|_{\mathcal{S}_h^\infty}$. Therefore, for any $h\in(0,\bar{h}_{\bar{C}}]$, the function Φ has a unique fixed point $\bar{Y}\in\mathcal{B}_{\bar{C}}$. Let $(Y,Z,V,K)\in\mathcal{S}_h^\infty\times\mathcal{H}^2_{BMO_h}\times\mathcal{J}^2_{BMO_h}\times\mathcal{A}^\infty_{D_h}$ be the unique solution to the following BSDE with central value reflection:

$$\begin{cases} Y_t=\xi+\displaystyle\int_t^T f(s,\bar{Y}_s,\mathbf{P}_{\bar{Y}_s},Z_s,V_s)\mathrm{d}s-\int_t^T Z_s\mathrm{d}W_s-\int_t^T\int_U V_s(e)\tilde{N}(\mathrm{d}s,\mathrm{d}e)+K_T-K_t,\quad t\in[T-h,T], \\ \gamma_{Y_t}\geqslant 0,\quad \forall t\in[T-h,T], \end{cases} \tag{4.52}$$

this solution exists and is unique by Theorem 4.1. Consequently, we have $Y=\Phi(\bar{Y})=\bar{Y}$, and by Lemma 4.1, it follows that $Y\in\mathcal{B}_{\bar{C}}$. Therefore, $(Y,Z,V,K)\in\mathcal{S}_h^\infty\times\mathcal{H}^2_{BMO_h}\times\mathcal{J}^2_{BMO_h}\times\mathcal{A}^\infty_{D_h}$ is the unique solution to BSDE (3.1) with central value reflection, and we have $\|Y\|_{\mathcal{S}_h^\infty}\leqslant\bar{C}$. This completes the proof of Theorem 4.2. □

Remark 4.2 *The condition on the process* α *can be relaxed to* $\|\int_0^T|\alpha_s|\mathrm{d}s\|_\infty<\infty$. *With some minor adjustments to the constants in the proofs, the result of Theorem 4.2 remains valid.*

4.4 Global solution

The main result of this paper is presented in this section. It establishes the well-possedness of BSDE (3.1) with central value reflection over the entire time interval $[0,T]$.

Theorem 4.3 *Assume that* $(\boldsymbol{H}_\xi)$ *and* $(\boldsymbol{H}_f)$ *hold. Then, the quadratic BSDE (3.1) with central value reflection admits a unique deterministic flat solution* $(Y,Z,V,K)\in\mathcal{S}^\infty\times\mathcal{H}^2_{BMO}\times\mathcal{J}^2_{BMO}\times\mathcal{A}^\infty_D$.

Proof Let $\bar{C}\geqslant C_0$. We then choose $h\in(0,\bar{h}_{\bar{C}}]$. Because the fixed-point argument can be applied for any terminal condition on a time interval of size h, we divide the time interval $[0,T]$ into n small intervals of length h, where n is a suitable integer.

From Theorem 4.2, there exists a unique deterministic flat solution to the BSDE (3.1) with central value reflection on the time interval $[T-h,T]$. We denote this solution by $(Y^n,Z^n,V^n,K^n)\in\mathcal{S}^\infty_h\times\mathcal{H}^2_{BMO_h}\times\mathcal{J}^2_{BMO_h}\times\mathcal{A}^\infty_{D_h}$. Next, we consider $T-h$ as the terminal time and Y^n_{T-h} as the terminal condition. From the same theorem, there exists a unique deterministic flat solution to the BSDE with central reflection (3.1), denoted as $(Y^{n-1},Z^{n-1},V^{n-1},K^{n-1})$ on the time interval $[T-2h,T-h]$.

By applying a similar approach on each time interval $[T-ih,T-(i-1)h]$, with a similar dynamic but with terminal condition $Y^{n-(i-2)}_{T-(i-1)h}$ at time $T-ih$ for $i=1,...,n$, we recursively construct n solutions $(Y^{n-(i-1)},Z^{n-(i-1)},V^{n-(i-1)},K^{n-(i-1)})$ on $[T-ih,T-(i-1)h]$ for $i=1,...,n$.

By appropriately concatenating these processes together, we derive a deterministic flat solution $(Y,Z,V,K)\in\mathcal{S}^\infty\times\mathcal{H}^2_{BMO}\times\mathcal{J}^2_{BMO}\times\mathcal{A}^\infty_D$ that satisfies our BSDE (3.1) with central value reflection over the entire time interval $[0,T]$.

For the uniqueness of the solution on the entire time interval $[0,T]$, we have that (Y,Z,V,K) is recursively uniquely defined on each subinterval by the fixed point contraction. Therefore, it is defined uniquely over the entire interval $[0,T]$.

Remark 4.3 *Having established the existence of a unique deterministic flat solution to the BSDE (3.1) with central value reflection, we now discuss its minimality. In cases where the generator is either independent of* Y *or linear with respect to the component* Y*, our problem admits a minimal solution. Specifically, if another solution* $(\bar{Y},\bar{Z},\bar{V},\bar{K})$ *exists, it holds that* $Y_t\leqslant\bar{Y}_t$ *for each* $t\in[0,T]$*. The proof of this is similar to that of the uniqueness in Proposition 4.1. As the generator is independent of* Y*, we can infer that* $(Y_t-(K_T-K_t),Z_t,V_t)$ *and* $(\bar{Y}_t-(\bar{K}_T-\bar{K}_t),\bar{Z}_t,\bar{V}_t)$ *are both solutions to the same BSDE. Consequently, for each* $t\in[0,T]$*, it holds that* $Y_t-(K_T-K_t)=\bar{Y}_t-(\bar{K}_T-\bar{K}_t)$*. Therefore, proving that* $Y_t\leqslant\bar{Y}_t$ *reduces to showing that* $K_T-K_t\leqslant\bar{K}_T-\bar{K}_t$*, which can be established by contradiction, following the same argument as in Proposition 4.1.*

With an additional assumption, the following proposition proves that the solution of the BSDE (3.1) with central value reflection is bounded in $\mathcal{S}^\infty\times\mathcal{H}^2_{BMO}\times\mathcal{J}^2_{BMO}\times\mathcal{A}^\infty_D$.

Proposition 4.2 *Assume that* $(\boldsymbol{H}_\xi)$ *and* $(\boldsymbol{H}_f)$ *hold. Additionally, if*

$(\bar{H}_f)$ *for each* $(t,y,\mu)\in[0,T]\times\mathbb{R}\times\mathcal{P}^1(\mathbb{R})$, $f(t,y,\mu,0,0)$ *is bounded by M, P a.s., is also valid. Then, the solution of the BSDE (3.1) is bounded in* $\mathcal{S}^\infty\times\mathcal{H}^2_{BMO}\times\mathcal{J}^2_{BMO}\times\mathcal{A}^\infty_D$.

Proof Let $(Y,Z,V,K)\in\mathcal{S}^\infty\times\mathcal{H}^2_{BMO}\times\mathcal{J}^2_{BMO}\times\mathcal{A}^\infty_D$ be the solution of the BSDE with central value reflection (3.1). Consider $(\bar{Y},\bar{Z},\bar{V})\in\mathcal{S}^\infty\times\mathcal{H}^2_{BMO}\times\mathcal{J}^2_{BMO}$ the solution of the following BSDE:

$$\bar{Y}_t = \xi + \int_t^T f\left(s, Y_s, \mathbf{P}_{Y_s}, \bar{Z}_s, \bar{V}_s\right) \mathrm{d}s - \int_t^T \bar{Z}_s \mathrm{d}W_s - \int_t^T \int_U \bar{V}_s(e) \tilde{N}(\mathrm{d}s, \mathrm{d}e).$$

Notice that

$$Y_t - (K_T - K_t) = \bar{Y}_t, \quad Z_t = \bar{Z}_t \quad \text{and} \quad V_t = \bar{V}_t \quad \text{for all } t \in [0,T]. \tag{4.53}$$

The proof is divided into two steps. The first step involves providing the bound of Y in $\mathcal{S}^\infty$, and the second step concerns the bounds of Z and V in $\mathcal{H}^2_{BMO}$ and $\mathcal{J}^2_{BMO}$, respectively.

Step1 According to (4.53), we have that

$$\|Y\|_{\mathcal{S}^\infty} \leqslant \left\|\bar{Y}\right\|_{\mathcal{S}^\infty} + \sup_{0 \leqslant t \leqslant T} \left(\gamma_{\bar{Y}_t}\right)^- . \tag{4.54}$$

Applying Tanaka's formula to $\left|\bar{Y}_t\right|$, we have that for each $t \in [0,T]$,

$$\begin{aligned} \left|\bar{Y}_t\right| = |\xi| &+ \int_t^T \operatorname{sgn}\left(\bar{Y}_{s-}\right) f\left(s, Y_s, \mathbf{P}_{Y_s}, \bar{Z}_s, \bar{V}_s\right) \mathrm{d}s - \int_t^T \operatorname{sgn}\left(\bar{Y}_{s-}\right) \bar{Z}_s \mathrm{d}W_s \\ &- \int_t^T \int_U \operatorname{sgn}\left(\bar{Y}_{s-}\right) \bar{V}_s(e) \tilde{N}\left(\mathrm{d}e, \mathrm{d}s\right) - \int_t^T \mathrm{d}L_s^0 - \sum_{t<s\leqslant T} \left|\bar{Y}_s\right| - \left|\bar{Y}_{s-}\right| - \operatorname{sgn}\left(\bar{Y}_{s-}\right) \Delta \bar{Y}_s, \end{aligned} \tag{4.55}$$

where L^0 denotes the local time of $\bar{Y}$ at 0. Consider the following process:

$$\Psi_t := \exp\left(\kappa|\bar{Y}_t| + \kappa \int_0^t \operatorname{sgn}(\bar{Y}_s) f(s, Y_s, \mathbf{P}_{Y_s}, 0, 0) \mathrm{d}s\right),$$

by applying Itô's formula to Ψ_t, we obtain, for each $t \in [0,T]$,

$$\begin{aligned} \Psi_t = \Psi_T &- \int_t^T \kappa \Psi_s \operatorname{sgn}(\bar{Y}_{s-}) f\left(s, Y_s, \mathbf{P}_{Y_s}, 0, 0\right) \mathrm{d}s \\ &+ \int_t^T \kappa \Psi_s \operatorname{sgn}(\bar{Y}_{s-}) \left(f\left(s, Y_s, \mathbf{P}_{Y_s}, \bar{Z}_s, \bar{V}_s\right) - f\left(s, Y_s, \mathbf{P}_{Y_s}, 0, \bar{V}_s\right)\right) \mathrm{d}s \\ &+ \int_t^T \kappa \Psi_s \operatorname{sgn}(\bar{Y}_{s-}) \left(f\left(s, Y_s, \mathbf{P}_{Y_s}, 0, \bar{V}_s\right) - f\left(s, Y_s, \mathbf{P}_{Y_s}, 0, 0\right)\right) \mathrm{d}s \\ &+ \int_t^T \kappa \Psi_s \operatorname{sgn}(\bar{Y}_{s-}) f\left(s, Y_s, \mathbf{P}_{Y_s}, 0, 0\right) \mathrm{d}s - \int_t^T \kappa \Psi_s \operatorname{sgn}\left(\bar{Y}_{s-}\right) \bar{Z}_s \mathrm{d}W_s \\ &- \int_t^T \int_U \kappa \Psi_s \operatorname{sgn}\left(\bar{Y}_{s-}\right) \bar{V}_s(e) \tilde{N}\left(\mathrm{d}e, \mathrm{d}s\right) - \frac{1}{2} \int_t^T \kappa^2 \Psi_s |\bar{Z}|^2 \mathrm{d}s - \int_t^T \kappa \Psi_s \mathrm{d}L_s^0 \\ &- \sum_{t<s\leqslant T} \kappa \Psi_s \left(\left|\bar{Y}_s\right| - \left|\bar{Y}_{s-}\right| - \operatorname{sgn}\left(\bar{Y}_{s-}\right) \Delta \bar{Y}_s\right) - \int_t^T \int_U \kappa \Psi_s \mathbf{j}_\kappa \left(\operatorname{sgn}\left(\bar{Y}_{s-}\right) \bar{V}_s(e)\right) N\left(\mathrm{d}e, \mathrm{d}s\right). \end{aligned} \tag{4.56}$$

Using Remark 4.1, with $\phi_s = D_z f(s, Y_s, \mathbf{P}_{Y_s}, 0, \bar{V}_s)$, we conclude that there exists $b \in \mathcal{H}^2_{BMO}$ such that

$$f\left(s, Y_s, \mathbf{P}_{Y_s}, \bar{Z}_s, \bar{V}_s\right) - f\left(s, Y_s, \mathbf{P}_{Y_s}, 0, \bar{V}_s\right) = b_s \,.\, \bar{Z}_s. \tag{4.57}$$

Otherwise, by assumption (3.4), there exist predictable $\mathcal{U}$-measurable processes $\psi^{\bar{V}_s,0}$ and $\psi^{0,\bar{V}_s}$

such that

$$\begin{aligned}&\operatorname{sgn}(\bar{Y}_{s-})\left(f\left(s,Y_s,\mathbf{P}_{Y_s},0,\bar{V}_s\right)-f\left(s,Y_s,\mathbf{P}_{Y_s},0,0\right)\right)\\&\leqslant\int_U\left(\psi_s^{\bar{V}_s,0}(e)\mathbf{1}_{\{\operatorname{sgn}(\bar{Y}_{s-})=1\}}+\psi_s^{0,\bar{V}_s}(e)\mathbf{1}_{\{\operatorname{sgn}(\bar{Y}_{s-})=-1\}}\right)\operatorname{sgn}(\bar{Y}_{s-})\bar{V}_s(e)\nu(\mathrm{d}e).\end{aligned}\tag{4.58}$$

We define the following probability measure:

$$\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}:=\mathcal{E}\left(\int_t^T b_s\mathrm{d}W_s+\int_t^T\int_U\psi_s^{\bar{V}_s,0}(e)\mathbf{1}_{\{\operatorname{sgn}(\bar{Y}_{s-})=1\}}+\psi_s^{0,\bar{V}_s}(e)\mathbf{1}_{\{\operatorname{sgn}(\bar{Y}_{s-})=-1\}}\tilde{N}(\mathrm{d}s,\mathrm{d}e)\right),\tag{4.59}$$

which is well defined because b, $\psi^{\bar{V},0}$, and $\psi^{0,\bar{V}}$ satisfy the condition of Proposition 2.2.

Using (4.57) and (4.58), taking the conditional expectation with respect to the probability measure $\mathbf{Q}$ on both sides of (4.56), and considering that the last four terms in (4.56) are negative, we conclude that for each $t\in[0,T]$,

$$\Psi_t\leqslant\mathbb{E}^{\mathbf{Q}}\left(\Psi_T/\mathcal{F}_t\right).\tag{4.60}$$

Consequently,

$$\exp\left(\kappa|\bar{Y}_t|\right)\leqslant\exp\left(\kappa|\xi|+\kappa\int_t^T\operatorname{sgn}(\bar{Y}_s)f(s,Y_s,\mathbf{P}_{Y_s},0,0)\mathrm{d}s\right).\tag{4.61}$$

Finally, using $(\bar{\boldsymbol{H}}_f)$, we deduce that

$$\|\bar{Y}\|_{\mathcal{S}^\infty}\leqslant\|\xi\|_\infty+TM.\tag{4.62}$$

Using similar techniques as in (4.26), with $\|\xi\|_\infty+TM$ instead of R_h, we obtain

$$\sup_{0\leqslant t\leqslant T}\left(\gamma_{\bar{Y}_t}\right)^-\leqslant\sup_{0\leqslant t\leqslant T}|\gamma_{\bar{Y}_t}|\leqslant\|\xi\|_\infty+TM,$$

which gives

$$\|Y\|_{\mathcal{S}^\infty}\leqslant2\left(\|\xi\|_\infty+TM\right).\tag{4.63}$$

Step 2 Due to equation (4.53), determining the bounds for $\|Z\|^2_{\mathcal{H}^2_{BMO}}$ and $\|V\|^2_{\mathcal{J}^2_{BMO}}$ is equivalent to estimating $\bar{Z}$ and $\bar{V}$ in the spaces $\mathcal{H}^2_{BMO}$ and $\mathcal{J}^2_{BMO}$, respectively.

Using Remark (4.28) with $\phi_t:=D_zf(t,Y_t,\mathbf{P}_{Y_t},0,\bar{V}_t)$, and the monotonicity condition (3.4), we have

$$\begin{aligned}f(s,Y_s,\mathbf{P}_{Y_s},\bar{Z}_s,\bar{V}_s)&=f(s,Y_s,\mathbf{P}_{Y_s},\bar{Z}_s,\bar{V}_s)-f(s,Y_s,\mathbf{P}_{Y_s},0,\bar{V}_s)+f(s,Y_s,\mathbf{P}_{Y_s},0,\bar{V}_s)\\&\quad-f(s,Y_s,\mathbf{P}_{Y_s},0,0)+f(s,Y_s,\mathbf{P}_{Y_s},0,0)\\&\leqslant\frac{\bar{C}_\theta}{2}r_s^2+\frac{3\bar{C}_\theta}{2}|\bar{Z}_s|^2+\int_U\psi_s^{\bar{V}_s,0}(e)\bar{V}_s(e)\nu(\mathrm{d}e)+f(s,Y_s,\mathbf{P}_{Y_s},0,0),\end{aligned}\tag{4.64}$$

where we also use the elementary inequality $ab\leqslant\frac{1}{2}a^2+\frac{1}{2}b^2$ and the fact that $|\phi_t|\leqslant r_t$ for each $t\in[0,T]$.

Now, applying Itô's formula to $e^{4\bar{C}_\theta\bar{Y}_s}$, we obtain, for each $\tau\in\mathcal{T}_{0,T}$,

$$
\begin{aligned}
e^{4\bar{C}_\theta \bar{Y}_\tau} \leqslant\, & e^{4\bar{C}_\theta \xi} + 2\bar{C}_\theta^2 \int_\tau^T e^{4\bar{C}_\theta \bar{Y}_s} |r_s|^2 \mathrm{d}s + 6\bar{C}_\theta^2 \int_\tau^T e^{4\bar{C}_\theta \bar{Y}_s} |\bar{Z}_s|^2 \mathrm{d}s - 8\bar{C}_\theta^2 \int_\tau^T e^{4\bar{C}_\theta \bar{Y}_s} |\bar{Z}_s|^2 \mathrm{d}s \\
& + 4\bar{C}_\theta \int_\tau^T \int_U e^{4\bar{C}_\theta \bar{Y}_s} \psi_s^{\bar{V}_s,0}(e) \bar{V}(e) \nu(\mathrm{d}e) ds + 4\bar{C}_\theta \int_\tau^T e^{4\bar{C}_\theta \bar{Y}_s} f(s, Y_s, \mathbf{P}_{Y_s}, 0, 0) \mathrm{d}s \\
& - M_\tau - \int_\tau^T \int_U e^{4\bar{C}_\theta \bar{Y}_s} \left((e^{2\bar{C}_\theta \bar{V}_s(e)} - 1)^2 + 4\bar{C}_\theta \mathbf{j}_{2\bar{C}_\theta}(\bar{V}_s(e)) \right) \nu(\mathrm{d}e) \mathrm{d}s, \qquad (4.65)
\end{aligned}
$$

where we also use

$$
4\bar{C}_\theta \mathbf{j}_{4\bar{C}_\theta}(\bar{V}_s(e)) = (e^{2\bar{C}_\theta \bar{V}_s(e)} - 1)^2 + 4\bar{C}_\theta \mathbf{j}_{2\bar{C}_\theta}(\bar{V}_s(e)),
$$

and

$$
M_\tau := 4\bar{C}_\theta \int_\tau^T e^{4\bar{C}_\theta \bar{Y}_s} \bar{Z}_s \mathrm{d}W_s + \int_\tau^T \int_U e^{4\bar{C}_\theta (\bar{Y}_{s-} + \bar{V}_s(e))} - e^{4\bar{C}_\theta \bar{Y}_{s-}} \tilde{N}(\mathrm{d}s, \mathrm{d}e). \qquad (4.66)
$$

Therefore, using Hölder's inequality, we have

$$
\begin{aligned}
& 2\bar{C}_\theta^2 \int_\tau^T e^{4\bar{C}_\theta \bar{Y}_s} |\bar{Z}_s|^2 ds + \int_\tau^T \int_U e^{4\bar{C}_\theta \bar{Y}_s} (e^{2\bar{C}_\theta \bar{V}_s(e)} - 1)^2 \nu(\mathrm{d}e) \mathrm{d}s \\
& \leqslant e^{4\bar{C}_\theta \xi} + 2\bar{C}_\theta^2 \int_\tau^T e^{4\bar{C}_\theta \bar{Y}_s} |r_s|^2 \mathrm{d}s + 4 \int_\tau^T \int_U e^{8\bar{C}_\theta \bar{Y}_s} (\psi_s^{\bar{V}_s,0}(e))^2 \nu(\mathrm{d}e) \mathrm{d}s \\
& \quad + \bar{C}_\theta^2 \int_\tau^T \int_U (\bar{V}_s(e))^2 \nu(\mathrm{d}e) \mathrm{d}s + 4M\bar{C}_\theta \int_\tau^T e^{4\bar{C}_\theta \bar{Y}_s} \mathrm{d}s - M_\tau. \qquad (4.67)
\end{aligned}
$$

Now, by taking the conditional expectation on both sides, we obtain

$$
\begin{aligned}
& 2\bar{C}_\theta^2 \mathbb{E}\left(\int_\tau^T e^{4\bar{C}_\theta \bar{Y}_s} |\bar{Z}_s|^2 ds / \mathcal{F}_\tau \right) + \mathbb{E}\left(\int_\tau^T \int_U e^{4\bar{C}_\theta \bar{Y}_s} (e^{2\bar{C}_\theta \bar{V}_s(e)} - 1)^2 \nu(\mathrm{d}e) \mathrm{d}s / \mathcal{F}_\tau \right) \\
& - \bar{C}_\theta^2 \mathbb{E}\left(\int_\tau^T \int_U (\bar{V}_s(e))^2 \nu(\mathrm{d}e) \mathrm{d}s / \mathcal{F}_\tau \right) \\
& \leqslant e^{4\bar{C}_\theta \|\xi\|_\infty} + 4MT\bar{C}_\theta e^{4\bar{C}_\theta \|\bar{Y}\|_{\mathcal{S}^\infty}} + 2\bar{C}_\theta^2 e^{4\bar{C}_\theta \|\bar{Y}\|_{\mathcal{S}^\infty}} \mathbb{E}\left(\int_\tau^T |r_s|^2 \mathrm{d}s / \mathcal{F}_\tau \right) \\
& \quad + 4 e^{8\bar{C}_\theta \|\bar{Y}\|_{\mathcal{S}^\infty}} \mathbb{E}\left(\int_\tau^T \int_U (\psi_s^{\bar{V}_s,0}(e))^2 \nu(\mathrm{d}e) \mathrm{d}s / \mathcal{F}_\tau \right). \qquad (4.68)
\end{aligned}
$$

By applying the above estimate to the solution $(-\bar{Y}, -\bar{Z}, -\bar{V})$ of the BSDE with terminal condition $-\xi$ and generator $-f(s, Y_s, \mathbf{P}_{Y_s}, -z, -v)$, which satisfies assumptions (H_f), we deduce that

$$
\begin{aligned}
& 2\bar{C}_\theta^2 \mathbb{E}\left(\int_\tau^T e^{-4\bar{C}_\theta \bar{Y}_s} |\bar{Z}_s|^2 \mathrm{d}s / \mathcal{F}_\tau \right) + \mathbb{E}\left(\int_\tau^T \int_U e^{-4\bar{C}_\theta \bar{Y}_s} (e^{-2\bar{C}_\theta \bar{V}_s(e)} - 1)^2 \nu(\mathrm{d}e) \mathrm{d}s / \mathcal{F}_\tau \right) \\
& - \bar{C}_\theta^2 \mathbb{E}\left(\int_\tau^T \int_U (\bar{V}_s(e))^2 \nu(\mathrm{d}e) \mathrm{d}s / \mathcal{F}_\tau \right) \\
& \leqslant e^{4\bar{C}_\theta \|\xi\|_\infty} + 4MT\bar{C}_\theta e^{4\bar{C}_\theta \|\bar{Y}\|_{\mathcal{S}^\infty}} + 2\bar{C}_\theta^2 e^{4\bar{C}_\theta \|\bar{Y}\|_{\mathcal{S}^\infty}} \mathbb{E}\left(\int_\tau^T |r_s|^2 \mathrm{d}s / \mathcal{F}_\tau \right) \\
& \quad + 4 e^{8\bar{C}_\theta \|\bar{Y}\|_{\mathcal{S}^\infty}} \mathbb{E}\left(\int_\tau^T \int_U (\psi_s^{0,-\bar{V}_s}(e))^2 \nu(\mathrm{d}e) \mathrm{d}s / \mathcal{F}_\tau \right). \qquad (4.69)
\end{aligned}
$$

By adding (4.68) to (4.69) and using the fact that

$$2 \leqslant e^{x}+e^{-x} \quad \text{and} \quad x^{2} \leqslant a\left(e^{x}-1\right)^{2}+\frac{\left(e^{-x}-1\right)^{2}}{a}, \quad \forall(a, x) \in \mathbb{R}_{+}^{*} \times \mathbb{R},$$

we obtain

$$\begin{aligned}
&4 \bar{C}_{\theta}^{2} \mathbb{E}\left(\int_{\tau}^{T}|\bar{Z}_{s}|^{2} \mathrm{d} s / \mathcal{F}_{\tau}\right)+2 \bar{C}_{\theta}^{2} \mathbb{E}\left(\int_{\tau}^{T} \int_{U}|\bar{V}_{s}(e)|^{2} \nu(\mathrm{d} e) \mathrm{d} s / \mathcal{F}_{\tau}\right) \\
&\leqslant 2 e^{4 \bar{C}_{\theta}\|\xi\|_{\infty}}+8 M T \bar{C}_{\theta} e^{4 \bar{C}_{\theta}\|\bar{Y}\|_{\mathcal{S}^{\infty}}}+4 \bar{C}_{\theta}^{2} e^{4 \bar{C}_{\theta}\|\bar{Y}\|_{\mathcal{S}^{\infty}}} \mathbb{E}\left(\int_{\tau}^{T}|r_{s}|^{2} \mathrm{d} s / \mathcal{F}_{\tau}\right) \\
&\quad+4 e^{8 \bar{C}_{\theta}\|\bar{Y}\|_{\mathcal{S}^{\infty}}} \mathbb{E}\left(\int_{\tau}^{T} \int_{U}(\psi_{s}^{\bar{V}_{s}, 0}(e))^{2} \nu(\mathrm{d} e) \mathrm{d} s / \mathcal{F}_{\tau}\right) \\
&\quad+4 e^{8 \bar{C}_{\theta}\|\bar{Y}\|_{\mathcal{S}^{\infty}}} \mathbb{E}\left(\int_{\tau}^{T} \int_{U}(\psi_{s}^{0,-\bar{V}_{s}}(e))^{2} \nu(\mathrm{d} e) \mathrm{d} s / \mathcal{F}_{\tau}\right). \qquad (4.70)
\end{aligned}$$

By taking a supremum over stopping times, we find

$$\begin{aligned}
4 \bar{C}_{\theta}^{2} \parallel \bar{Z} \parallel_{\mathcal{H}_{BMO}^{2}}^{2}+2 \bar{C}_{\theta}^{2} \parallel \bar{V} \parallel_{\mathcal{J}_{B}^{2}}^{2} \leqslant 2 e^{4 \bar{C}_{\theta}\|\xi\|_{\infty}}+8 M T \bar{C}_{\theta} e^{4 \bar{C}_{\theta}\|\bar{Y}\|_{\mathcal{S}^{\infty}}}+4 \bar{C}_{\theta}^{2} e^{4 \bar{C}_{\theta}\|\bar{Y}\|_{\mathcal{S}^{\infty}}} \parallel r \parallel_{\mathcal{H}_{BMO}^{2}}^{2} \\
+4 e^{8 \bar{C}_{\theta}\|\bar{Y}\|_{\mathcal{S}^{\infty}}}\left(\parallel \psi^{\bar{V}, 0} \parallel_{\mathcal{J}_{B}^{2}}^{2}+\parallel \psi^{0,-\bar{V}} \parallel_{\mathcal{J}_{B}^{2}}^{2}\right). \qquad (4.71)
\end{aligned}$$

From (2.1) and (3.5), we know that $\psi^{\bar{V},0}$ and $\psi^{0,-\bar{V}}$ belong to $\mathcal{J}_B^2$. From (4.71) and (4.62), we obtain that

$$\begin{aligned}
\parallel \bar{Z} \parallel_{\mathcal{H}_{BMO}^{2}}^{2} \leqslant & \frac{e^{4 \bar{C}_{\theta} M}}{2 \bar{C}_{\theta}^{2}}+e^{4 \bar{C}_{\theta}(\|\xi\|_{\infty}+T M)}\left(\frac{2 M T}{\bar{C}_{\theta}}+\parallel r \parallel_{\mathcal{H}_{BMO}^{2}}^{2}\right) \\
&+\frac{e^{8 \bar{C}_{\theta}(\|\xi\|_{\infty}+T M)}}{\bar{C}_{\theta}^{2}}\left(\parallel \psi^{\bar{V}, 0} \parallel_{\mathcal{J}_{B}^{2}}^{2}+\parallel \psi^{0,-\bar{V}} \parallel_{\mathcal{J}_{B}^{2}}^{2}\right), \qquad (4.72)
\end{aligned}$$

and

$$\begin{aligned}
\parallel \bar{V} \parallel_{\mathcal{J}_{B}^{2}}^{2} \leqslant & \frac{e^{4 \bar{C}_{\theta} M}}{\bar{C}_{\theta}^{2}}+e^{4 \bar{C}_{\theta}(\|\xi\|_{\infty}+T M)}\left(\frac{4 M T}{\bar{C}_{\theta}}+2 \parallel r \parallel_{\mathcal{H}_{BMO}^{2}}^{2}\right) \\
&+2 \frac{e^{8 \bar{C}_{\theta}(\|\xi\|_{\infty}+T M)}}{\bar{C}_{\theta}^{2}}\left(\parallel \psi^{\bar{V}, 0} \parallel_{\mathcal{J}_{B}^{2}}^{2}+\parallel \psi^{0,-\bar{V}} \parallel_{\mathcal{J}_{B}^{2}}^{2}\right). \qquad (4.73)
\end{aligned}$$

Since $\bar{Y}$ is bounded, its jumps are also bounded. Therefore, there exists a version of $\bar{V}$ such that

$$\parallel \bar{V} \parallel_{L^{\infty}(\nu)} \leqslant 2 \parallel \bar{Y} \parallel_{\mathcal{S}^{\infty}} . \qquad (4.74)$$

This result together with (2.2), yields

$$\parallel \bar{V} \parallel_{\mathcal{J}_{BMO}^{2}}^{2} \leqslant \parallel \bar{V} \parallel_{\mathcal{J}_{B}^{2}}^{2}+\parallel \bar{V} \parallel_{\mathcal{J}^{\infty}}^{2} \leqslant \parallel \bar{V} \parallel_{\mathcal{J}_{B}^{2}}^{2}+4 \parallel \bar{Y} \parallel_{\mathcal{S}^{\infty}}^{2} . \qquad (4.75)$$

By adding $4 \parallel \bar{Y} \parallel_{\mathcal{S}^{\infty}}^2$ to both sides of (4.73), we deduce that

$$\| \bar{V} \|^2_{\mathcal{J}^2_{BMO}} \leqslant 4(\| \xi \|_\infty + TM) + \frac{e^{4\bar{C}_\theta M}}{\bar{C}^2_\theta} + e^{4\bar{C}_\theta(\|\xi\|_\infty + TM)} \left(\frac{4MT}{\bar{C}_\theta} + 2 \| r \|^2_{\mathcal{H}^2_{BMO}} \right)$$
$$+ 2\frac{e^{8\bar{C}_\theta(\|\xi\|_\infty + TM)}}{\bar{C}^2_\theta} \left(\| \psi^{\bar{V},0} \|^2_{\mathcal{J}^2_B} + \| \psi^{0,-\bar{V}} \|^2_{\mathcal{J}^2_B} \right). \tag{4.76}$$

Using equations (4.63), (4.72), and (4.76) and considering (4.53), we conclude the result of Theorem 4.3. □

Remark 4.4 *It is easy to see that the constraint of the BSDE (3.1) can be extended to a more general form. Specifically, for each* $t \in [0, T]$*, the map in the constraint can be defined as the unique solution to the equation* $\mathbb{E}\left(g(Y_t - x)\right) = 0$*, where* g *is a function exhibiting properties similar to those of the* arctan *function.*

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