Probability, Uncertainty and Quantitative Risk
Vol. 10, No. 2, 2025, 265–292
https://www.aimsciences.org/PUQR
DOI:10.3934/puqr.2025012

Optimal stopping under G-expectation

Hanwu Li[1,2]

[1]*Research Center for Mathematics and Interdisciplinary Sciences, Shandong University, Qingdao* 266237, *China*
[2]*Frontiers Science Center for Nonlinear Expectations* (*Ministry of Education*), *Shandong University, Qingdao* 266237, *China*

Email: *lihanwu@sdu.edu.cn*

Abstract In this study, we develop a theory of optimal stopping problems within the G-expectation framework. To address this problem, we first introduce a type of random times, called G-stopping times, which are specifically suited for this setting. In the discrete-time case with a finite horizon, we define the value function backward and show that it is the smallest G-supermartingale that dominates the payoff process, ensuring the existence of an optimal stopping time. We then extend these results to both the infinite-horizon case and the continuous-time setting. Moreover, we establish the relationship between the value function and the solution of the reflected backward stochastic differential equation driven by G-Brownian motion.

Keywords Optimal stopping, G-expectation, G-stopping time, Knightian uncertainty

2020 Mathematics Subject Classification 60G40

1. Introduction

Consider a filtered probability space $(\Omega, \mathcal{F}, P, \mathbb{F}=\{\mathcal{F}_t\}_{t\in[0,T]})$, the objective of the optimal stopping problem is to find a stopping time τ^* to maximize the expectation of X_τ over all stopping times. Here, X is a given progressively measurable and integrable process, called the payoff process. In financial markets, X can be regarded as the gain of an option. An agent has the right to stop this option at any time t and then get the reward X_t, or to wait in the hope that he would obtain a bigger reward if he stops in the future. This problem has wide applications in finance and economics, such as pricing for American contingent claims and the decision of a firm to enter a new market. Note that there is an implicit hypothesis in the above examples that the agent knows the probability distribution of the payoff process. This assumption excludes the case in which the agent faces Knightian uncertainty. In this study, we investigate the optimal stopping problem under Knightian uncertainty, especially volatility uncertainty.

Optimal stopping problems under Knightian uncertainty have attracted significant attention due to their theoretical and practical importance. Relevant studies in this field include [1–5, 7, 8,

Received 12 December 2023; Accepted 24 February 2025; Early access 14 April 2025

13, 19, 25]. For example, Riedel [25] examined optimal stopping problems within a multiple priors framework, where a traditional linear expectation is replaced by a nonlinear one. Cheng and Riedel [5] and Grigorova et al. [8] investigated the optimal stopping problem under g-expectation induced by a backward stochastic differential equation (BSDE) for the case of continuous and irregular rewards, respectively. Bayraktar and Yao [1, 2] further explored this problem under what they termed filtration-consistent nonlinear expectations. In these studies, the authors imposed assumptions on either the set of multiple priors $\mathcal{P}$ or the nonlinear expectation $\mathcal{E}$ to ensure that the associated conditional expectation remains time-consistent and that the optional sampling theorem holds. Similar to the classical case, the value function is an $\mathcal{E}$-supermartingale that dominates the payoff process X. Moreover, the first hitting time τ^* is optimal, and X is an $\mathcal{E}$-martingale up to time τ^*. However, in the above studies, all probability measures in $\mathcal{P}$ are assumed to be equivalent to a reference measure P, implying that these models primarily captures drift uncertainty. However, addressing volatility uncertainty requires $\mathcal{P}$ to be a family of non-dominated probability measures, making the problem more complex. Bayraktar and Yao [3, 4], Ekren, Touzi and Zhang [7], and Nutz and Zhang [19] investigated the optimal stopping problem under a non-dominated family of probability measures. Specifically, [4, 7] studied the problem $\sup_\tau \sup_P E^P[X_\tau]$, which can be interpreted as a control problem, whereas [3, 19] studied $\inf_\tau \sup_P E^P[X_\tau]$, which can be viewed as a game problem. Notably, in these studies, the value function is defined pathwise. They also obtained the optimality of the first hitting time τ^* and the nonlinear martingale property of the value function stopped at τ^*. Recently, Huang and Yu [13] considered a more general case under an α-maxmin nonlinear expectation, which is time-inconsistent.

According to the above studies, it is obvious that we need some nonlinear expectations to study the optimal stopping problem under Knightian uncertainty. Recently, Peng systematically established a time-consistent nonlinear expectation theory, called G-expectation theory (see [21, 22]). As the counterpart of the classical linear expectation case, the notions of G-Brownian motion, G-martingale and G-Itô's integral were also introduced. A basic mathematical tool for the analysis is BSDEs driven by G-Brownian motion (G-BSDE) studied by Hu et al. In [10, 11], they proved the existence and uniqueness of solutions to G-BSDE and established the corresponding comparison theorem, Girsanov transformation and Feynman–Kac formula. The G-expectation theory is convincing as a useful tool for developing a theory of financial markets under volatility uncertainty. Therefore, in this study, we aimed to investigate the optimal stopping problem under the G-expectation framework.

In the classical setting, the value function is typically defined as the essential supremum over a set of random variables. However, in the G-framework, the essential supremum must be defined in the quasi-surely sense, which may not always exist. Additionally, random variables in the G-framework must satisfy certain continuity and monotonicity properties. For a given random time τ and process X, X_τ may not belong to an appropriate space where the conditional G-expectation is well-defined. These challenges make the study of optimal stopping under G-expectation significantly more complex and far from being fully understood.

In this study, we first address the optimal stopping problems under G-expectation in the discrete-time case, considering both finite- and infinite-time horizon. To facilitate our analysis, we introduce a new type of random time, called the G-stopping time. The key advantage of this approach is that it allows the definition of the conditional G-expectation to be extended to the random variable X stopped at a G-stopping time τ. Moreover, within this broader framework, many essential properties, such as time consistency, remain valid. For finite-time case, we define the value function V backward. It is straightforward to verify that the value function is the minimal G-supermartingale that dominates the payoff process X. Although our approach is limited to a specific class of stopping times, this restriction does not result in a significant loss of information. Notably, the first hitting time after time t serves as an optimal G-stopping time for the value function defined at time t. Then, we extend the theory to the infinite horizon case, where the backward induction cannot be applied directly. The value function is defined by the limit of the one in the finite-time case. We show that it is still the minimal G-supermartingale, which is greater than the payoff process and satisfies the recursive equations similar to the finite-time case. This problem has not been addressed by the previous studies [3, 7, 19]. Recall that Li, Peng and Soumana Hima [16] studied the reflected BSDE driven by G-Brownian motion, which implies that the solution is required to be above an obstacle process. The solution of the reflected G-BSDE is the minimal nonlinear supermartingale that dominates the obstacle process. We show that it coincides with the value function of the optimal stopping problem in the continuous-time case when the payoff process X equals to the obstacle process.

From a mathematical perspective, the optimal stopping problem introduced in [7] is the most closely related to our work. Compared with their results, the advantage of considering this problem under G-expectation lies in the following aspects. First, we do not need to assume the boundedness of the payoff process, and we can analyze this problem in the setting of infinite-time horizon. In the continuous-time case, we show that the value function can be obtained as the limit of the discrete-time value functions, which is particularly useful for numerical approximation. Additionally, similar to the result in [5], we demonstrate that the value function, defined by the Snell envelope, coincides with the solution of a reflected BSDE driven by G-Brownian motion, which improves the results in [18] since they can only get the ε-optimality. At last, the case that the payoff process is Markovian can be involved and similar results as the classical case still hold.

The remainder of this paper is organized as follows. We first recall some basic results of G-expectation and reflected G-BSDEs in Section 2. In Section 3, we introduce the G-stopping times and the essential supremum in the quasi-surely sense. Section 4 presents the study of optimal stopping problems under G-expectation in finite- and infinite-time horizon. Then, we extend the results to the continuous-time case and show that the value function of optimal stopping problem corresponds to the solution of the reflected G-BSDE in Section 5. Finally, Section 6 presents some results of optimal stopping when the payoff process is Markovian.

2. Preliminaries

In this section, we review some notations and results in the G-expectation framework. For simplicity, we only consider the one-dimensional case. For more details, refer to the papers [12, 16, 17, 21, 22, 23].

2.1 G-expectation and extended conditional G-expectation

Let $\Omega = C_0([0,\infty);\mathbb{R})$, the space of real-valued continuous functions starting from the origin, be endowed with the following norm:

$$\rho(\omega^1,\omega^2) := \sum_{i=1}^{\infty} 2^{-i}[(\max_{t\in[0,i]} |\omega_t^1 - \omega_t^2|) \wedge 1], \quad \text{for } \omega^1,\omega^2 \in \Omega.$$

Let B be the canonical process on Ω. Set

$$L_{ip}(\Omega) := \{\varphi(B_{t_1},...,B_{t_n}) : \ n \in \mathbb{N}, \ t_1,\cdots,t_n \in [0,\infty), \ \varphi \in C_{b,Lip}(\mathbb{R}^n)\},$$

where $C_{b,Lip}(\mathbb{R}^n)$ denotes the set of bounded Lipschitz functions on $\mathbb{R}^n$. Let $(\Omega, L_{ip}(\Omega), \hat{\mathbb{E}})$ be the G-expectation space, where the function $G:\mathbb{R}\to\mathbb{R}$ is defined by

$$G(a) := \frac{1}{2}\hat{\mathbb{E}}[aB_1^2] = \frac{1}{2}(\bar{\sigma}^2 a^+ - \underline{\sigma}^2 a^-).$$

Herein, we always assume that G is non-degenerate, i.e., $\underline{\sigma}^2 > 0$. The (conditional) G-expectation for $\xi \in L_{ip}(\Omega)$ can be calculated as follows. We assume that ξ can be represented as

$$\xi = \varphi(B_{t_1}, B_{t_2}, \cdots, B_{t_n}).$$

Then, for $t \in [t_{k-1}, t_k)$, $k = 1, \cdots, n$,

$$\hat{\mathbb{E}}_t[\varphi(B_{t_1}, B_{t_2}, \cdots, B_{t_n})] = u_k(t, B_t; B_{t_1}, \cdots, B_{t_{k-1}}),$$

where, for any $k = 1, \cdots, n$, $u_k(t, x; x_1, \cdots, x_{k-1})$ is a function of (t, x) parameterized by $(x_1, \cdots, x_{k-1})$ such that it solves the following fully nonlinear PDE defined on $[t_{k-1}, t_k) \times \mathbb{R}$:

$$\partial_t u_k + G(\partial_x^2 u_k) = 0$$

with terminal conditions

$$u_k(t_k, x; x_1, \cdots, x_{k-1}) = u_{k+1}(t_k, x; x_1, \cdots, x_{k-1}, x),\ k < n$$

and $u_n(t_n, x; x_1, \cdots, x_{n-1}) = \varphi(x_1, \cdots, x_{n-1}, x)$. Hence, the G-expectation of ξ is $\hat{\mathbb{E}}_0[\xi]$.

For each $p \geqslant 1$, the completion of $L_{ip}(\Omega)$ under the norm $\|\xi\|_{L_G^p} := (\hat{\mathbb{E}}[|\xi|^p])^{1/p}$ is denoted by $L_G^p(\Omega)$. The conditional G-expectation $\hat{\mathbb{E}}_t[\cdot]$ can be extended continuously to the completion $L_G^p(\Omega)$. Besides, Denis, Hu and Peng [6] proved that the G-expectation has the following representation.

Theorem 2.1 *([6]) There exists a weakly compact set $\mathcal{P}$ of probability measures on $(\Omega, \mathcal{B}(\Omega))$, such that*

$$\hat{\mathbb{E}}[\xi] = \sup_{P\in\mathcal{P}} E_P[\xi] \quad for\ all \quad \xi \in L_G^1(\Omega).$$

$\mathcal{P}$ is called the set that represents $\hat{\mathbb{E}}$.

Let $\mathcal{P}$ be a weakly compact set that represents $\hat{\mathbb{E}}$. For this $\mathcal{P}$, we define the capacity

$$c(A) := \sup_{P\in\mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega).$$

A set $A \in \mathcal{B}(\Omega_T)$ is called polar if $c(A) = 0$. A property holds "*quasi - surely*" (q.s.) if it holds outside a polar set. In the following, we do not distinguish the two random variables X and Y if $X = Y$, q.s.

The following notations (see [12]) are frequently used in this paper.

$$\begin{aligned}
&L^0(\Omega) := \{X : \Omega \to [-\infty, \infty] \text{ and } X \text{ is } \mathcal{B}(\Omega)\text{-measurable}\},\\
&\mathcal{L}(\Omega) := \{X \in L^0(\Omega) : E^P[X] \text{ exists for each } P \in \mathcal{P}\},\\
&\mathbb{L}^p(\Omega) := \{X \in L^0(\Omega) : \hat{\mathbb{E}}[|X|^p] < \infty\} \text{ for } p \geqslant 1,\\
&L_G^{1^*}(\Omega) := \{X \in \mathbb{L}^1(\Omega) : \exists\{X_n\} \subset L_G^1(\Omega) \text{ such that } X_n \downarrow X, \text{q.s.}\},\\
&L_G^{*1}(\Omega) := \{X - Y : X, Y \in L_G^{1^*}(\Omega)\},\\
&L_G^{1^*_*}(\Omega) := \{X \in \mathbb{L}^1(\Omega) : \exists\{X_n\} \subset L_G^{1^*}(\Omega) \text{ such that } X_n \uparrow X, \text{q.s.}\},\\
&\bar{L}_G^{1^*_*}(\Omega) := \{X \in \mathbb{L}^1(\Omega) : \exists\{X_n\} \subset L_G^{1^*_*}(\Omega) \text{ such that } \hat{\mathbb{E}}[|X_n - X|] \to 0\}.
\end{aligned}$$

Remark 2.2 (i) *It is easy to check that* $L_G^{1^*}(\Omega)\subset L_G^{*1}(\Omega)$ *and* $L_G^{1^*_*}(\Omega)\subset \bar{L}_G^{1^*_*}(\Omega)$. *Furthermore, we have* $L_G^{*1}(\Omega)\subset L_G^{1^*_*}(\Omega)$. *Let* $\xi\in L_G^{*1}(\Omega)$ *with representation* $\xi=X-Y$, *where* $X,Y\in L_G^{1^*}(\Omega)$. *By the definition of* $L_G^{1^*}(\Omega)$, *there exists a sequence* $\{Y_n\}\subset L_G^1(\Omega)$ *such that* $Y_n\downarrow Y$, *q.s. It follows that* $X-Y_n\uparrow X-Y(=\xi)$, *q.s. Noting that* $X-Y_n\in L_G^{1^*}(\Omega)$, *by the definition of* $L_G^{1^*_*}(\Omega)$, *we have* $\xi\in L_G^{1^*_*}(\Omega)$.

(ii) *Notably,* $L_G^{1^*}(\Omega)$, $L_G^{1^*_*}(\Omega)$, $\bar{L}_G^{1^*_*}(\Omega)$ *are not linear spaces and* $L_G^{*1}(\Omega)$ *is a linear space.*

Set $\Omega_t=\{\omega_{\cdot\wedge t}:\omega\in\Omega\}$ for $t>0$. Similarly, we can define $L^0(\Omega_t)$, $\mathcal{L}(\Omega_t)$, $\mathbb{L}^p(\Omega_t)$, $L_G^1(\Omega_t)$, $L_G^{1^*}(\Omega_t)$, $L_G^{1^*_*}(\Omega_t)$ and $\bar{L}_G^{1^*_*}(\Omega_t)$ respectively. As demonstrated in [12], we can extend the conditional G-expectation to space $\bar{L}_G^{1^*_*}(\Omega)$, and it satisfies the following property.

Proposition 2.3 *([12]) For each* $X\in\bar{L}_G^{1^*_*}(\Omega)$, *we have for each* $P\in\mathcal{P}$,

$$\hat{\mathbb{E}}_t[X]=\operatorname*{ess\,sup}_{Q\in\mathcal{P}(t,P)}{}^P E^Q[X|\mathcal{F}_t],\quad P\text{-a.s.},$$

where $\mathcal{P}(t,P)=\{Q\in\mathcal{P}:E_Q[X]=E_P[X],\forall X\in L_{ip}(\Omega_t)\}$.

Because the conditional expectation can be well-defined on the space $\bar{L}_G^{1^*_*}(\Omega)$, we can modify the definition of G-martingale (-sub, -supmartingale) slightly.

Definition 2.4 *A process* $M=\{M_t\}_{t\in[0,T]}$ *is called a G-martingale (-sub, -supermartingale, resp.), if for each* $t\in[0,T]$, $M_t\in\bar{L}_G^{1^*_*}(\Omega_t)$ *and* $\hat{\mathbb{E}}_s[M_t]=M_s$ $(\geqslant,\leqslant,\text{resp.})$, *for any* $0\leqslant s\leqslant t\leqslant T$.

The extended conditional G-expectation shares many properties with the classical conditional expectation, except the linearity. More precisely, we have the following proposition:

Proposition 2.5 *([12]) We have*

(1) $X,Y\in\bar{L}_G^{1^*_*}(\Omega),\ X\leqslant Y\ \Rightarrow\ \hat{\mathbb{E}}_t[X]\leqslant\hat{\mathbb{E}}_t[Y]$;

(2) $X\in\bar{L}_G^{1^*_*}(\Omega_t),\ Y\in\bar{L}_G^{1^*_*}(\Omega)\ \Rightarrow\ \hat{\mathbb{E}}_t[X+Y]=X+\hat{\mathbb{E}}_t[Y]$;

(3) $X,Y\in\bar{L}_G^{1^*_*}(\Omega)\ \Rightarrow\ \hat{\mathbb{E}}_t[X+Y]\leqslant\hat{\mathbb{E}}_t[X]+\hat{\mathbb{E}}_t[Y]$;

(4) $X\in\bar{L}_G^{1^*_*}(\Omega_t)$ *is bounded,* $X\geqslant 0$, $Y\in\bar{L}_G^{1^*_*}(\Omega)$, $Y\geqslant 0$, $\lim_{n\to\infty}\hat{\mathbb{E}}[YI_{\{Y\geqslant n\}}]=0\ \Rightarrow\ \hat{\mathbb{E}}_t[XY]=X\hat{\mathbb{E}}_t[Y]$;

(5) $X\in\bar{L}_G^{1^*_*}(\Omega)\ \Rightarrow\ \hat{\mathbb{E}}_s[\hat{\mathbb{E}}_t[X]]=\hat{\mathbb{E}}_{s\wedge t}[X]$;

(6) $\{X_n\}_{n=1}^\infty\subset L_G^{1^*}(\Omega)(L_G^{1^*_*}(\Omega))$, $X_n\downarrow X(X_n\uparrow X)$ *q.s.* $\Rightarrow X\in L_G^{1^*}(\Omega)(L_G^{1^*_*}(\Omega))$ *and* $\hat{\mathbb{E}}_t[X]=\lim_{n\to\infty}\hat{\mathbb{E}}_t[X_n]$.

Next, we present some examples of random variables that belong to the above spaces.

Proposition 2.6 *([12]) We have*

(1) *Let* X *be a bounded upper (resp. lower) semicontinuous function on* Ω. *Then,* $X\in L_G^{1^*}(\Omega)$ *(resp.* $X\in L_G^{*1}(\Omega)$ *);*

(2) *Let* $X\in L_G^1(\Omega,\mathbb{R}^n)$ *and let* f *be a bounded upper (resp. lower) semicontinuous function on* $\mathbb{R}^n$. *Then,* $f(X)\in L_G^{1^*}(\Omega)$ *(resp.* $f(X)\in L_G^{*1}(\Omega)$ *).*

Remark 2.7 *Let* $X\in L_G^1(\Omega)$, $a\in\mathbb{R}$. *Then, by the above proposition, we have* $I_{\{X\leqslant a\}}$, $I_{\{X\geqslant a\}}$, $I_{\{X=a\}}\in L_G^{1^*}(\Omega)$.

2.2 G-Itô calculus and reflected G-BSDEs

In this subsection, we recall some basic results about reflected BSDE driven by G-Brownian motion.

Definition 2.8 (i) *Let $M_G^0(0,T)$ be the collection of processes in the following form: for a given partition $\{t_0,\cdots,t_N\}=\pi_T$ of $[0,T]$,*

$$\eta_t(\omega)=\sum_{j=0}^{N-1}\xi_j(\omega)1_{[t_j,t_{j+1})}(t),$$

where $\xi_i\in L_{ip}(\Omega_{t_i})$, $i=0,1,2,\cdots,N-1$. For each $p\geqslant 1$ and $\eta\in M_G^0(0,T)$, let $\|\eta\|_{H_G^p}:=\{\hat{\mathbb{E}}[(\int_0^T|\eta_s|^2ds)^{p/2}]\}^{1/p}$, $\|\eta\|_{M_G^p}:=(\hat{\mathbb{E}}[\int_0^T|\eta_s|^p\mathrm{d}s])^{1/p}$ and denote by $H_G^p(0,T)$, $M_G^p(0,T)$ the completion of $M_G^0(0,T)$ under the norm $\|\cdot\|_{H_G^p}$, $\|\cdot\|_{M_G^p}$, respectively.

(ii) *Let $S_G^0(0,T)=\{h(t,B_{t_1\wedge t},\ldots,B_{t_n\wedge t}):t_1,\ldots,t_n\in[0,T],h\in C_{b,Lip}(\mathbb{R}^{n+1})\}$. For $p\geqslant 1$ and $\eta\in S_G^0(0,T)$, set $\|\eta\|_{S_G^p}=\{\hat{\mathbb{E}}[\sup_{t\in[0,T]}|\eta_t|^p]\}^{1/p}$. Denote by $S_G^p(0,T)$ the completion of $S_G^0(0,T)$ under the norm $\|\cdot\|_{S_G^p}$.*

We have the following continuity property for any $Y\in S_G^p(0,T)$ with $p>1$.

Lemma 2.9 *([15]) For $Y\in S_G^p(0,T)$ with $p>1$, we have, by setting $Y_s:=Y_T$ for $s>T$,*

$$F(Y):=\limsup_{\varepsilon\to 0}\left(\hat{\mathbb{E}}\left[\sup_{t\in[0,T]}\sup_{s\in[t,t+\varepsilon]}|Y_t-Y_s|^p\right]\right)^{\frac{1}{p}}=0.$$

The parameters of reflected G-BSDE consist of the following three parts: the generators f and g, the obstacle process $\{X_t\}_{t\in[0,T]}$ and the terminal value ξ, where f and g are maps

$$f(t,\omega,y,z):[0,T]\times\Omega_T\times\mathbb{R}^2\to\mathbb{R}.$$

We will make the following assumptions: There exists some $\beta>2$ such that

(H1) for any $y,z,\ f(\cdot,\cdot,y,z),g(\cdot,\cdot,y,z)\in M_G^\beta(0,T)$;

(H2) $|f(t,\omega,y,z)-f(t,\omega,y',z')|+|g(t,\omega,y,z)-g(t,\omega,y',z')|\leqslant L(|y-y'|+|z-z'|)$ for some $L>0$;

(H3) $\{X_t\}_{t\in[0,T]}\in S_G^\beta(0,T)$ and $X_t\leqslant I_t$, $t\in[0,T]$, $q.s.$, where I is a generalized G-Itô process of the following form

$$I_t=I_0+\int_0^t b^I(s)\mathrm{d}s+\int_0^t\sigma^I(s)\mathrm{d}B_s+K_t^I,$$

and $\{b^I(t)\}_{t\in[0,T]}\in M_G^\beta(0,T)$, $\{\sigma^I(t)\}_{t\in[0,T]}\in H_G^\beta(0,T)$, $K^I\in S_G^\beta(0,T)$ is a nonincreasing G-martingale;

(H4) $\xi\in L_G^\beta(\Omega_T)$ and $\xi\geqslant X_T$, $q.s.$

Let us now introduce the reflected G-BSDE with a lower obstacle. A triple of processes (Y,Z,L) is called a solution of the reflected G-BSDE with a lower obstacle if for some $2\leqslant\alpha\leqslant\beta$, the following properties hold:

(a) $(Y,Z,L)\in\mathcal{S}_G^\alpha(0,T)$ and $Y_t\geqslant X_t$, $0\leqslant t\leqslant T$;

(b) $Y_t=\xi+\int_t^T f(s,Y_s,Z_s)\mathrm{d}s+\int_t^T g(s,Y_s,Z_s)d\langle B\rangle_s-\int_t^T Z_s\mathrm{d}B_s+(L_T-L_t)$;

(c) $\{-\int_0^t(Y_s-X_s)\mathrm{d}L_s\}_{t\in[0,T]}$ is a nonincreasing G-martingale.

Here, we denote by $\mathcal{S}_G^\alpha(0,T)$ the collection of processes (Y,Z,L) such that $Y\in S_G^\alpha(0,T)$, $Z\in H_G^\alpha(0,T)$, L is a continuous nondecreasing process with $L_0=0$ and $L\in S_G^\alpha(0,T)$.

Theorem 2.10 *([16, 17]) If we assume that (ξ, f, g, X) satisfy (H1)–(H4), then the reflected G-BSDE with data (ξ, f, g, X) has a unique solution (Y, Z, L). Moreover, for any $2 \leqslant \alpha < \beta$, we have $Y \in S_G^\alpha(0,T)$, $Z \in H_G^\alpha(0,T)$ and $L \in S_G^\alpha(0,T)$.*

Proposition 2.11 *([14]) Let $f = g = 0$, and assume that (ξ, X) satisfy (H3) and (H4). Then, the first argument Y of the solution to the reflected G-BSDE with data $(\xi, 0, 0, X)$ is a G-supermartingale dominating the process X.*

3. Essential supremum in the quasi-surely sense

In this section, we introduce the essential supremum in the quasi-surely sense and a new type of random time, called G-stopping time, appropriate for the study of optimal stopping under G-expectation. We then investigate some properties of the extended (conditional) G-expectation.

Definition 3.1 *A random time $\tau : \Omega \to [0, \infty)$ is called a G-stopping time if $I_{\{\tau \leqslant t\}} \in L_G^{1^*}(\Omega_t)$ for each $t \geqslant 0$.*

Let $\mathcal{H} \subset \mathbb{L}^1(\Omega)$ be a set of random variables. We give the definition of essential supremum of $\mathcal{H}$ in the quasi-surely sense. Roughly speaking, we only need to replace the "almost-surely" in the classical definition by "quasi-surely."

Definition 3.2 *The essential supremum of $\mathcal{H}$, denoted by $\operatorname{ess\,sup}_{\xi \in \mathcal{H}} \xi$, is a random variable in $L^0(\Omega)$ such that:*

(i) *For any $\xi \in \mathcal{H}$, $\operatorname{ess\,sup}_{\xi \in \mathcal{H}} \xi \geqslant \xi$, q.s.*

(ii) *If there exists another random variable $\eta' \in L^0(\Omega)$ such that $\eta' \geqslant \xi$, q.s. for any $\xi \in \mathcal{H}$, then $\operatorname{ess\,sup}_{\xi \in \mathcal{H}} \xi \leqslant \eta'$, q.s.*

Remark 3.3 (i) *Similarly, we may define the essential infimum. The essential supremum and infimum in the quasi-surely sense (if exist) must be unique.*

(ii) *Suppose that $\eta \in \mathcal{H}$ and for any $\xi \in \mathcal{H}$, $\eta \geqslant \xi$, q.s. Then, we have $\operatorname{ess\,sup}_{\xi \in \mathcal{H}} \xi = \eta$.*

Remark 3.4 *In the classical case, the essential supremum can be constructed by countable many random variables; however, this does not hold true for the one in the quasi-surely sense. We may consider the following example. Let $1 = \underline{\sigma}^2 < \bar{\sigma}^2 = 2$. Consider $\mathcal{H} = \{I_{\{\langle B \rangle_1 = x\}}, x \in [1,2]\}$. If there exists $\widetilde{\mathcal{H}} = \{I_{\{\langle B \rangle_1 = x_n\}}, x_n \in [1,2], n \in \mathbb{N}\}$, such that*

$$\operatorname{ess\,sup}_{\xi \in \mathcal{H}} \xi = \sup_{n \in \mathbb{N}} I_{\{\langle B \rangle_1 = x_n\}}.$$

Then, there exists a constant $x_0 \in [1,2]$ such that $x_0 \neq x_n$, for any $n \in \mathbb{N}$. We have

$$c(\sup_{n \in \mathbb{N}} I_{\{\langle B \rangle_1 = x_n\}} < I_{\{\langle B \rangle_1 = x_0\}}) = c(\langle B \rangle_1 = x_0) = 1,$$

which is a contradiction.

We now list some typical situations under which the essential supremum exists.

Proposition 3.5 *If there are only countable many random variables in $\mathcal{H}$, then the essential supremum exists.*

Proof Without loss of generality, we may assume $\mathcal{H} = \{\xi_n, n \in \mathbb{N}\}$. We then define

$$\eta(\omega) := \sup_{n\in\mathbb{N}} \xi_n(\omega).$$

It is easy to check that η is the essential supremum of $\mathcal{H}$. □

Definition 3.6 *A set $\widetilde{\mathcal{H}}$ is said to be dense in $\mathcal{H}$, if for any $\xi \in \mathcal{H}$, there exists a sequence $\{\xi_n, n\in\mathbb{N}\} \subset \widetilde{\mathcal{H}}$ such that $\hat{\mathbb{E}}[|\xi_n - \xi|] \to 0$ as $n\to\infty$.*

Proposition 3.7 *If $\mathcal{H}$ has a countable dense subset, then the essential supremum of $\mathcal{H}$ exists.*

Proof Without loss of generality, set $\widetilde{\mathcal{H}} := \{\xi_m, m\in\mathbb{N}\}$ is the countable dense subset of $\mathcal{H}$. Denote

$$\eta(\omega) := \sup_{m\in\mathbb{N}} \xi_m(\omega).$$

We claim that η is the essential supremum of $\mathcal{H}$. It is sufficient to prove for any $\xi\in\mathcal{H}$, $\eta \geqslant \xi$, q.s. For any $\xi\in\mathcal{H}$, there exists a sequence $\{\hat{\xi}_n, n\in\mathbb{N}\}\subset\widetilde{\mathcal{H}}$ such that $\hat{\mathbb{E}}[|\hat{\xi}_n - \xi|]\to 0$ as $n\to\infty$. By Proposition 6.1.21 in [23], there exists a subsequence $\{\hat{\xi}_{n_k}\}_{k=1}^{\infty}$ such that,

$$\xi = \lim_{k\to\infty} \hat{\xi}_{n_k}, \quad q.s.$$

Since for any k, $\hat{\xi}_{n_k} \leqslant \eta$, we have $\xi\leqslant\eta$, q.s. □

Proposition 3.8 *Assume that $\mathcal{H}\subset L_G^{1^*}(\Omega)$ is upwards directed and*

$$\sup_{\xi\in\mathcal{H}} \hat{\mathbb{E}}[\xi] = \sup_{\xi\in\mathcal{H}} -\hat{\mathbb{E}}[-\xi].$$

Then, the essential supremum of $\mathcal{H}$ exists.

Proof Since the family $\mathcal{H}$ is upwards directed, there exist two increasing sequences $\{\xi_n^i, n\in\mathbb{N}\}$ $\subset\mathcal{H}$, $i=1,2$, such that

$$\lim_{n\to\infty}\hat{\mathbb{E}}[\xi_n^1] = \sup_{\xi\in\mathcal{H}}\hat{\mathbb{E}}[\xi], \quad \lim_{n\to\infty} -\hat{\mathbb{E}}[-\xi_n^2] = \sup_{\xi\in\mathcal{H}} -\hat{\mathbb{E}}[-\xi]. \tag{3.1}$$

We claim that $\eta := \sup_n \eta_n$ is the essential supremum of $\mathcal{H}$, where $\eta_n = \xi_n^1\vee\xi_n^2$. Obviously, the second statement in Definition 3.2 holds. We now prove the first statement. It is easy to check that

$$\sup_{\xi\in\mathcal{H}}\hat{\mathbb{E}}[\xi] \geqslant \hat{\mathbb{E}}[\eta_n] \geqslant \hat{\mathbb{E}}[\xi_n^1],$$
$$\sup_{\xi\in\mathcal{H}} -\hat{\mathbb{E}}[-\xi] \geqslant -\hat{\mathbb{E}}[-\eta_n] \geqslant -\hat{\mathbb{E}}[-\xi_n^2].$$

Letting $n\to\infty$, it follows that

$$\sup_{\xi\in\mathcal{H}}\hat{\mathbb{E}}[\xi] = \lim_{n\to\infty}\hat{\mathbb{E}}[\eta_n] = \lim_{n\to\infty} -\hat{\mathbb{E}}[-\eta_n] = \sup_{\xi\in\mathcal{H}} -\hat{\mathbb{E}}[-\xi].$$

Applying Proposition 28 (7) in [12], we have

$$\hat{\mathbb{E}}[\eta] = \lim_{n\to\infty}\hat{\mathbb{E}}[\eta_n], \quad -\hat{\mathbb{E}}[-\eta] = \lim_{n\to\infty} -\hat{\mathbb{E}}[-\eta_n],$$

which implies that η has no mean uncertainty. For any $\xi\in\mathcal{H}$, using the monotone convergence theorem, we obtain that

$$\hat{\mathbb{E}}[\eta\vee\xi] = \lim_{n\to\infty}\hat{\mathbb{E}}[\eta_n\vee\xi] \leqslant \sup_{\xi\in\mathcal{H}}\hat{\mathbb{E}}[\xi].$$

Then, we conclude that

$$0 \leqslant \hat{\mathbb{E}}[\xi \vee \eta - \eta] = \hat{\mathbb{E}}[\xi \vee \eta] - \hat{\mathbb{E}}[\eta] \leqslant 0,$$

which indicates that $\xi \vee \eta - \eta = 0$, q.s. The proof is complete. □

In the following, we list some properties of the extended (conditional) G-expectation. It is natural to extend the definition of G-expectation $\hat{\mathbb{E}}$ to the space $\mathcal{L}(\Omega)$, still denoted by $\hat{\mathbb{E}}$. For each $X \in \mathcal{L}(\Omega)$, the extended G-expectation has the following representation

$$\hat{\mathbb{E}}[X] = \sup_{P \in \mathcal{P}} E^P[X].$$

Lemma 3.9 *Let $\{X_n, n \in \mathbb{N}\} \subset \mathcal{L}(\Omega)$. Suppose that there exists a random variable $Y \in \mathcal{L}(\Omega)$ with $-\hat{\mathbb{E}}[-Y] > -\infty$ such that, for any $n \geqslant 1$, $X_n \geqslant Y$ q.s. Then, $\liminf_{n\to\infty} X_n \in \mathcal{L}(\Omega)$ and*

$$\hat{\mathbb{E}}[\liminf_{n\to\infty} X_n] \leqslant \liminf_{n\to\infty} \hat{\mathbb{E}}[X_n].$$

Proof By the classical monotone convergence theorem and Fatou's Lemma, we have for each $P \in \mathcal{P}$, $E^P[\liminf_{n\to\infty} X_n]$ exists and

$$E^P[\liminf_{n\to\infty} X_n] \leqslant \liminf_{n\to\infty} E^P[X_n] \leqslant \liminf_{n\to\infty} \hat{\mathbb{E}}[X_n].$$

Taking supremum over all $P \in \mathcal{P}$, we obtain the desired result. □

Remark 3.10 *Notably, the Fatou Lemma of the "$\limsup$" type does not hold under G-expectation. For example, set $0 < \underline{\sigma}^2 < \bar{\sigma}^2 = 1$. Consider the sequence $\{X_n, n \in \mathbb{N}\}$, where $X_n = I_{\{\langle B\rangle_1 \in (1-\frac{1}{n}, 1)\}}$. It is easy to check that $X_n \to 0$ and $\hat{\mathbb{E}}[X_n] = 1$ for any $n \in \mathbb{N}$. Therefore, we have*

$$0 = \hat{\mathbb{E}}[\limsup_{n\to\infty} X_n] < \limsup_{n\to\infty} \hat{\mathbb{E}}[X_n] = 1.$$

Next, we extend the definition of conditional G-expectation. For this purpose, we need the following lemma, which generalizes Lemma 2.4 in [10].

Lemma 3.11 *For each $\xi, \eta \in L_G^{*1}(\Omega)$ and $A \in \mathcal{B}(\Omega_t)$, if $\xi I_A \geqslant \eta I_A$ q.s., then $\hat{\mathbb{E}}_t[\xi] I_A \geqslant \hat{\mathbb{E}}_t[\eta] I_A$ q.s.*

Proof Otherwise, we may choose a compact set $K \subset A$ with $c(K) > 0$ such that $(\hat{\mathbb{E}}_t[\xi] - \hat{\mathbb{E}}_t[\eta])^- > 0$ on K. Noting that K is compact, there exists a sequence of nonnegative functions $\{\zeta_n\}_{n=1}^{\infty} \subset C_b(\Omega_t)$ such that $\zeta_n \downarrow I_K$, which implies that $I_K \in L_G^{1^*}(\Omega_t)$. Since $\xi, \eta \in L_G^{*1}(\Omega)$, there exist $\xi_i, \eta_i \in L_G^{1^*}(\Omega)$ and $\{\xi_i^n\}_{n=1}^{\infty}, \{\eta_i^n\}_{n=1}^{\infty} \subset L_G^1(\Omega)$, $i = 1, 2$ such that $\xi_i^n \downarrow \xi$, $\eta_i^n \downarrow \eta$ and

$$\xi = \xi_1 - \xi_2, \quad \eta = \eta_1 - \eta_2.$$

Set $X_n = \xi_1^n + \eta_2^n$, $Y_n = \xi_2^n + \eta_1^n$. Then, $\{X_n\}_{n=1}^{\infty}, \{Y_n\}_{n=1}^{\infty} \subset L_G^1(\Omega)$ and they are decreasing in n. We denote by X, Y the limit of $\{X_n\}_{n=1}^{\infty}, \{Y_n\}_{n=1}^{\infty}$, respectively. It is easy to check that $X, Y \in L_G^{1^*}(\Omega)$ and $\xi - \eta = X - Y$. For each fixed $l, m, n \in \mathbb{N}$, we have

$$\hat{\mathbb{E}}[\zeta_l (X_n - Y_m)^-] \downarrow \hat{\mathbb{E}}[I_K (X_n - Y_m)^-], \quad \text{as } l \to \infty,$$

and

$$\hat{\mathbb{E}}[\zeta_l \hat{\mathbb{E}}_t[(X_n - Y_m)^-]] \downarrow \hat{\mathbb{E}}[I_K \hat{\mathbb{E}}_t[(X_n - Y_m)^-]], \quad \text{as } l \to \infty,$$

Noting that

$$\hat{\mathbb{E}}[\zeta_l (X_n - Y_m)^-] = \hat{\mathbb{E}}[\zeta_l \hat{\mathbb{E}}_t[(X_n - Y_m)^-]],$$

it follows that

$$\hat{\mathbb{E}}[I_K(X_n-Y_m)^-]=\hat{\mathbb{E}}[I_K\hat{\mathbb{E}}_t[(X_n-Y_m)^-]].$$

For each fixed $m,n\in\mathbb{N}$, we have $I_K(X_n-Y_m)^-\in L_G^{1^*}(\Omega)$. First letting $m\to\infty$, we obtain $I_K(X_n-Y_m)^-\downarrow I_K(X_n-Y)^-$ and $I_K(X_n-Y)^-\in L_G^{1^*}(\Omega)$. Then, letting $n\to\infty$, we obtain $I_K(X_n-Y)^-\uparrow I_K(X-Y)^-$ and $I_K(X-Y)^-\in L_G^{1^*_*}(\Omega)$. Therefore, we can calculate that

$$\lim_{n\to\infty}\lim_{m\to\infty}\hat{\mathbb{E}}[I_K(X_n-Y_m)^-]=\hat{\mathbb{E}}[I_K(X-Y)^-]=\hat{\mathbb{E}}[I_K(\xi-\eta)^-]=0.$$

By a similar analysis, we have

$$\lim_{n\to\infty}\lim_{m\to\infty}\hat{\mathbb{E}}[I_K\hat{\mathbb{E}}_t[(X_n-Y_m)^-]]=\hat{\mathbb{E}}[I_K\hat{\mathbb{E}}_t[(X-Y)^-]]=\hat{\mathbb{E}}[I_K\hat{\mathbb{E}}_t[(\xi-\eta)^-]].$$

By Proposition 34 (4) and (8) in [12], we can check that

$$(\hat{\mathbb{E}}_t[\xi]-\hat{\mathbb{E}}_t[\eta])^-\leqslant\hat{\mathbb{E}}_t[(\xi-\eta)^-],$$

which yields $\hat{\mathbb{E}}_t[(\xi-\eta)^-]>0$ on K. Recall that $c(K)>0$, which implies that $\hat{\mathbb{E}}[I_K\hat{\mathbb{E}}_t[(\xi-\eta)^-]]>0$. This is a contradiction, and the proof is complete. □

Lemma 3.11 allows us to extend the definition of conditional G-expectation. For each $t\geqslant 0$, set

$$L_G^{*1,0,t}(\Omega)=\left\{\xi=\sum_{i=1}^n\eta_iI_{A_i}:\{A_i\}_{i=1}^n\text{ is a partition of }\mathcal{B}(\Omega_t),\eta_i\in L_G^{*1}(\Omega),n\in\mathbb{N}\right\}.$$

Definition 3.12 *For each $\xi\in L_G^{*1,0,t}(\Omega)$ with representation $\xi=\sum_{i=1}^n\eta_iI_{A_i}$, we define the conditional expectation, still denoted by $\hat{\mathbb{E}}_s$, by setting*

$$\hat{\mathbb{E}}_s[\xi]:=\sum_{i=1}^n\hat{\mathbb{E}}_s[\eta_i]I_{A_i},\quad for\ s\geqslant t. \tag{3.2}$$

Remark 3.13 *Although the conditional expectation is well-defined for any $\xi\in\bar{L}_G^{1^*_*}(\Omega)$ as in [12], we do not know if the space $L_G^{*1,0,t}(\Omega)$ is contained in $\bar{L}_G^{1^*_*}(\Omega)$. This is why we give the definition as (3.2). If $\xi\in\bar{L}_G^{1^*_*}(\Omega)\cap L_G^{*1,0,t}(\Omega)$, then the extended conditional G-expectation defined by (3.2) coincides with the one in Proposition 2.3. For any $\xi\in L_G^{*1,0,t}(\Omega)$ with representation $\xi=\sum_{i=1}^n\eta_iI_{A_i}$ and $s\geqslant t$, we obtain that*

$$\begin{aligned}\sum_{i=1}^n\hat{\mathbb{E}}_s[\eta_i]I_{A_i}&=\sum_{i=1}^nI_{A_i}\operatorname*{ess\,sup}_{Q\in\mathcal{P}(s,P)}{}^PE^Q[\eta_i|\mathcal{F}_s]=\operatorname*{ess\,sup}_{Q\in\mathcal{P}(s,P)}{}^P\left(\sum_{i=1}^nE^Q[\eta_i|\mathcal{F}_s]I_{A_i}\right)\\&=\operatorname*{ess\,sup}_{Q\in\mathcal{P}(s,P)}{}^PE^Q\left[\sum_{i=1}^n\eta_iI_{A_i}|\mathcal{F}_s\right]=\operatorname*{ess\,sup}_{Q\in\mathcal{P}(s,P)}{}^PE^Q[\xi|\mathcal{F}_s],\quad P\text{-}a.s.\end{aligned}$$

for any $P\in\mathcal{P}$.

4. Optimal stopping in discrete-time case

This section provides insights into the optimal stopping problem under G-expectation for the discrete-time case, i.e. the G-stopping time τ takes values in some discrete set. We first investigate the finite-time case by applying the method of backward induction and then extend the results to the infinite-time case.

4.1 Finite-time horizon case

We assume that the payoff process X satisfies the following condition.

Assumption 4.1 $\{X_n, n = 0, 1, \cdots, N\}$ *is a sequence of random variables such that for any* n, $X_n \in L_G^1(\Omega_n)$.

Theorem 4.2 *Suppose Assumption 4.1 holds. We define the following sequence* $\{V_n, n = 0, 1, \cdots, N\}$ *backward: let* $V_N = X_N$ *and*

$$V_n = \max\{X_n, \hat{\mathbb{E}}_n[V_{n+1}]\},\ n \leqslant N-1.$$

Then, we have

(1) $\{V_n, n = 0, 1, \cdots, N\}$ *is the smallest G-supermartingale that dominates* $\{X_n, n = 0, 1, \cdots, N\}$;

(2) *Denote by* $\mathcal{T}_{j,N}$ *the set of all G-stopping time taking values in* $\{j, \cdots, N\}$. *Set* $\tau_j = \inf\{l \geqslant j : V_l = X_l\}$. *Then,* τ_j *is a G-stopping time and* $V_{n\wedge\tau_j} \in L_G^{*1}(\Omega_n)$ *, for any* $j \leqslant N$ *and* $n \leqslant N$. *Furthermore,* $\{V_{n\wedge\tau_j}, n = j, \cdots, N\}$ *is a G-martingale and for any* $j \leqslant N$,

$$V_j = \hat{\mathbb{E}}_j[X_{\tau_j}] = \underset{\tau\in\mathcal{T}_{j,N}}{ess\sup}\, \hat{\mathbb{E}}_j[X_\tau].$$

Proof (1) It is easy to check that for any $n = 0, 1, \cdots, N$,

$$V_n \geqslant X_n,\ \text{and}\ V_n \geqslant \hat{\mathbb{E}}_n[V_{n+1}],$$

which implies $\{V_n, n = 0, 1, \cdots, N\}$ is a G-supermartingale dominating $\{X_n, n = 0, 1, \cdots, N\}$. If $\{U_n, n = 0, 1, \cdots, N\}$ is another G-supmartingale dominating $\{X_n, n = 0, 1, \cdots, N\}$, we have $U_N \geqslant X_N = V_N$ and

$$U_{N-1} \geqslant \hat{\mathbb{E}}_{N-1}[U_N] \geqslant \hat{\mathbb{E}}_{N-1}[V_N],\ U_{N-1} \geqslant X_{N-1}.$$

It follows that $U_{N-1} \geqslant V_{N-1}$. By induction, we can prove that for all $n = 0, 1, \cdots, N$, $V_n \leqslant U_n$.

(2) For any $n = j, \cdots, N$, we can check that

$$\{\tau_j \leqslant n\} = \cup_{k=j}^n\{\tau_k = k\} = \cup_{k=j}^n\{V_k = X_k\}$$

and

$$I_{\{\tau_j\leqslant n\}} = \max_{j\leqslant k\leqslant n} I_{\{V_k - X_k = 0\}}.$$

By Remark 2.7, $I_{\{V_k - X_k=0\}} \in L_G^{1^*}(\Omega_k)$, for any $j \leqslant k \leqslant n$. It follows that $I_{\{\tau_j\leqslant n\}} \in L_G^{1^*}(\Omega_n)$. That is, τ_j is a G-stopping time.

It is easy to check that, for any $j < n \leqslant N$,

$$\begin{aligned} V_{n\wedge\tau_j} &= \sum_{k=j}^{n-1}(V_k - V_{k+1})I_{\{\tau_j\leqslant k\}} + V_n \\ &= \sum_{k=j}^{n-1}(V_k - V_{k+1})^+ I_{\{\tau_j\leqslant k\}} - \sum_{k=j}^{n-1}(V_k - V_{k+1})^- I_{\{\tau_j\leqslant k\}} + V_n. \end{aligned} \tag{4.1}$$

We conclude that $V_{n\wedge\tau_j} \in L_G^{*1}(\Omega_n)$. Note that

$$V_{(n+1)\wedge\tau_j} - V_{n\wedge\tau_j} = I_{\{\tau_j\geqslant n+1\}}(V_{n+1} - V_n) = I_{\{\tau_j\leqslant n\}^c}(V_{n+1} - \hat{\mathbb{E}}_n[V_{n+1}]). \tag{4.2}$$

Since $\{\tau_j \leqslant n\} \in \mathcal{B}(\Omega_n)$, applying Lemma 3.11 and Equation (3.2), we have

$$\hat{\mathbb{E}}_n[I_{\{\tau_j\leqslant n\}^c}(V_{n+1}-\hat{\mathbb{E}}_n[V_{n+1}])]=I_{\{\tau_j\leqslant n\}^c}\hat{\mathbb{E}}_n[(V_{n+1}-\hat{\mathbb{E}}_n[V_{n+1}])]=0. \tag{4.3}$$

By combining (4.2) and (4.3), we obtain

$$0=\hat{\mathbb{E}}_n[V_{(n+1)\wedge\tau_j}-V_{n\wedge\tau_j}]=\hat{\mathbb{E}}_n[V_{(n+1)\wedge\tau_j}]-V_{n\wedge\tau_j},$$

which shows that $\{V_{n\wedge\tau_j}, n=j,j+1,\cdots,N\}$ is a G-martingale. Consequently, we have

$$V_j=\hat{\mathbb{E}}_j[V_{\tau_j}]=\hat{\mathbb{E}}_j[X_{\tau_j}].$$

By Remark 3.3, it remains to prove that, for any $\tau\in\mathcal{T}_{j,N}$,

$$V_j\geqslant\hat{\mathbb{E}}_j[X_\tau]. \tag{4.4}$$

First, similar to (4.1), we have $X_\tau\in L_G^{*1}(\Omega_N)$. Then,

$$\begin{aligned}\hat{\mathbb{E}}_{N-1}[X_\tau]\leqslant\hat{\mathbb{E}}_{N-1}[V_\tau]&=\hat{\mathbb{E}}_{N-1}\left[\sum_{k=j}^{N-1}(V_k-V_{k+1})I_{\{\tau\leqslant k\}}+V_N\right]\\&=\sum_{k=j}^{N-2}(V_k-V_{k+1})I_{\{\tau\leqslant k\}}+V_{N-1}I_{\{\tau\leqslant N-1\}}+\hat{\mathbb{E}}_{N-1}[V_NI_{\{\tau=N\}}]\\&=\sum_{k=j}^{N-2}(V_k-V_{k+1})I_{\{\tau\leqslant k\}}+V_{N-1}I_{\{\tau\leqslant N-1\}}+\hat{\mathbb{E}}_{N-1}[V_N]I_{\{\tau=N\}}\\&\leqslant\sum_{k=j}^{N-2}(V_k-V_{k+1})I_{\{\tau\leqslant k\}}+V_{N-1}=V_{(N-1)\wedge\tau},\end{aligned}$$

where we use Equation (3.2) again in the last equality. By repeating this procedure, we obtain that (4.4) holds. The proof is complete. □

Remark 4.3 *If $\{V_n\}_{n=0}^N$ is defined by*

$$V_N=X_N,\quad V_n=\min\{X_n,\hat{\mathbb{E}}_n[V_{n+1}]\},\quad n\leqslant N-1.$$

By a similar analysis, we have:

(1) *$\{V_n, n=0,1,\cdots,N\}$ is the largest G-submartingale dominated by $\{X_n, n=0,1,\cdots,N\}$;*

(2) *Set $\tau_j=\inf\{l\geqslant j: V_l=X_l\}$. Then, τ_j is a G-stopping time and $V_{n\wedge\tau_j}\in L_G^{*1}(\Omega_n)$, for any $j\leqslant N$ and $n\leqslant N$. Furthermore, $\{V_{n\wedge\tau_j}, n=j,\cdots,N\}$ is a G-martingale and for any $j\leqslant N$,*

$$V_j=\hat{\mathbb{E}}_j[X_{\tau_j}]=\underset{\tau\in\mathcal{T}_{j,N}}{ess\inf}\,\hat{\mathbb{E}}_j[X_\tau].$$

4.2 Infinite-time horizon case

In this subsection, we study the infinite-time case. The conditions on the payoff function are more restrictive than those for the finite-time case, because the order of the right-hand side of Doob's inequality under G-expectation is strictly larger than the one of the left-hand side (see Theorem 3.4 in [26]). In addition, the value function for the infinite-time case is defined as the limit of the one for the finite-time case in the sequel. When verifying that it belongs to an appropriate space, we need to ensure the positivity of the value function (see the proof of Proposition 4.8 below), which can be guaranteed by the positivity of the payoff function. Due to the translation invariance property of G-expectation, it suffices to assume that the payoff function is bounded from below.

Assumption 4.4 *$\{X_n, n\in\mathbb{N}\}$ is a sequence of random variables bounded from below and for any $n\in\mathbb{N}$, $X_n\in L_G^{\beta}(\Omega_n)$, where $\beta>1$. Furthermore,*

$$\hat{\mathbb{E}}[\sup_{n\in\mathbb{N}}|X_n|^{\beta}]<\infty.$$

For each fixed $N\in\mathbb{N}$, we define the following sequence $\{\tilde{V}_n^N, n=0,1,\cdots,N\}$ backward: let $\tilde{V}_N^N=X_N$ and

$$\tilde{V}_n^N=\max\{X_n,\hat{\mathbb{E}}_n[\tilde{V}_{n+1}^N]\},\quad n\leqslant N-1. \tag{4.5}$$

It is easy to check that for any $n\leqslant N\leqslant M$, $\tilde{V}_n^N\leqslant\tilde{V}_n^M$. We may define

$$\tilde{V}_n^{\infty}=\lim_{N\geqslant n, N\to\infty}\tilde{V}_n^N. \tag{4.6}$$

Proposition 4.5 *The sequence $\{\tilde{V}_n^{\infty}, n\in\mathbb{N}\}$ defined by* (4.6) *is the smallest G-supermartingale that dominates the process $\{X_n, n\in\mathbb{N}\}$.*

Proof By monotone convergence theorem, letting $N\to\infty$ in (4.5), we have

$$\tilde{V}_n^{\infty}=\max\{X_n,\hat{\mathbb{E}}_n[\tilde{V}_{n+1}^{\infty}]\},$$

which implies that $\{\tilde{V}_n^{\infty}, n\in\mathbb{N}\}$ is a G-supermartingale which dominates the process $\{X_n, n\in\mathbb{N}\}$. Let $\{U_n, n\in\mathbb{N}\}$ be a G-supermartingale dominating the process $\{X_n, n\in\mathbb{N}\}$. By Theorem 4.2, $\{\tilde{V}_n^N, n=0,1,\cdots,N\}$ is the smallest G-supermartingale that dominates $\{X_n, n=0,1,\cdots,N\}$. Then, for each $n\in\mathbb{N}$ and $N\geqslant n$, we have $\tilde{V}_n^N\leqslant U_n$. It follows that

$$U_n\geqslant\lim_{N\to\infty}\tilde{V}_n^N=\tilde{V}_n^{\infty},$$

which yields that $\{\tilde{V}_n^{\infty}, n=0,1,\cdots,N\}$ is the smallest G-supermartingale dominating $\{X_n, n=0,1,\cdots,N\}$. □

For each $j\in\mathbb{N}$, denote by $\mathcal{T}_j$ the collection of all G-stopping times taking values in $\{j,j+1,\cdots\}$ such that

$$\lim_{N\to\infty}c(\tau>N)=0. \tag{4.7}$$

Set

$$\tilde{V}_0=\sup_{\tau\in\mathcal{T}_0}\hat{\mathbb{E}}[X_\tau].$$

Remark 4.6 *If a G-stopping time τ satisfies condition* (4.7)*, noting that $\{\tau=\infty\}\subset\{\tau>N\}$ for any $N\in\mathbb{N}$, we obtain that*

$$0\leqslant c(\tau=\infty)\leqslant\lim_{N\to\infty}c(\tau>N)=0,$$

which implies that τ is finite quasi-surely. However, the inverse does not hold. Consider the following example. Let $0<\underline{\sigma}^2<\bar{\sigma}^2=1$. Set

$$\tau=\begin{cases}1, & if\ \langle B\rangle_1=0,\\ N, & if\ \langle B\rangle_1\in(1-\frac{1}{N-1},1-\frac{1}{N}],\quad N\geqslant 2.\end{cases}$$

It is easy to check that τ is a G-stopping time and $c(\tau=\infty)=0$. However, for any fixed $N\in\mathbb{N}$, we have

$$c(\tau > N) = c\left(\langle B\rangle_1 > 1 - \frac{1}{N}\right) = 1.$$

Proposition 4.7 *Under Assumption 4.4, we have*

$$\tilde{V}_0 = \tilde{V}_0^{\infty}.$$

Proof By Theorem 4.2, it is obvious that $\tilde{V}_0 \geqslant \tilde{V}_0^N$, for any $N \in \mathbb{N}$, which implies that $\tilde{V}_0 \geqslant \tilde{V}_0^{\infty}$. We then prove the inverse inequality. For any $\tau \in \mathcal{T}_0$ and $\varepsilon > 0$, there exists some N such that $c(\tau > N) \leqslant \varepsilon$. By Assumption 4.4 and the Hölder inequality, we can calculate that

$$\hat{\mathbb{E}}[|X_\tau - X_{\tau\wedge N}|] \leqslant \hat{\mathbb{E}}[2\sup_{n\in\mathbb{N}}|X_n|I_{\{\tau>N\}}] \leqslant C(\hat{\mathbb{E}}[\sup_{n\in\mathbb{N}}|X_n|^\beta])^{\frac{1}{\beta}}(\hat{\mathbb{E}}[I_{\{\tau>N\}}])^{\frac{\beta-1}{\beta}} \leqslant C\varepsilon^{\frac{\beta-1}{\beta}}. \tag{4.8}$$

It follows that

$$\hat{\mathbb{E}}[X_\tau] \leqslant \hat{\mathbb{E}}[X_{\tau\wedge N}] + C\varepsilon^{\frac{\beta-1}{\beta}} \leqslant \tilde{V}_0^N + C\varepsilon^{\frac{\beta-1}{\beta}} \leqslant \tilde{V}_0^{\infty} + C\varepsilon^{\frac{\beta-1}{\beta}}.$$

Let $\varepsilon \to \infty$; since τ is arbitrarily chosen, we finally obtain the desired result. □

Proposition 4.8 *Assume that*

$$\tau_j = \inf\{l \geqslant j : \tilde{V}_l^{\infty} = X_l\}$$

satisfies condition (4.7). *Then, we have*

(i) $\tau_j \in \mathcal{T}_j$ *and* $\tilde{V}_{n\wedge\tau_j}^{\infty} \in L_G^{1^*}(\Omega_n)$ *, for each* $n \in \mathbb{N}$;

(ii) $\{\tilde{V}_{n\wedge\tau_j}^{\infty}, n = j, j+1, \cdots\}$ *is a G-martingale;*

(iii) *For any* $j \in \mathbb{N}$,

$$\tilde{V}_j^{\infty} = \hat{\mathbb{E}}_j[X_{\tau_j}] = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_j} \hat{\mathbb{E}}_j[X_\tau].$$

Proof (i) Noting that for any $k \in \mathbb{N}$ and $N \geqslant k$, $\tilde{V}_k^N \geqslant X_k$ and $\tilde{V}_k^N \uparrow \tilde{V}_k^{\infty}$ as $N \to \infty$, then we have $\{\tilde{V}_k^{\infty} = X_k\} = \cap_{N\geqslant k}^{\infty}\{\tilde{V}_k^N = X_k\}$, which implies

$$I_{\{\tilde{V}_k^{\infty}=X_k\}} = \inf_{N\geqslant k} I_{\{\tilde{V}_k^N=X_k\}}.$$

By the proof of Theorem 4.2, we have $I_{\{\tilde{V}_k^N=X_k\}} \in L_G^{1^*}(\Omega_k)$. By applying Proposition 2.5, we confirm that $I_{\{\tilde{V}_k^{\infty}=X_k\}} \in L_G^{1^*}(\Omega_k)$. Since

$$I_{\{\tau_j\leqslant n\}} = \max_{j\leqslant k\leqslant n} I_{\{\tilde{V}_k^{\infty}=X_k\}},$$

it follows that $I_{\{\tau_j\leqslant n\}} \in L_G^{1^*}(\Omega_n)$. Without loss of generality, we assume $X_n \geqslant 0$ for any $n \in \mathbb{N}$. It is easy to check that

$$\tilde{V}_{n\wedge\tau_j}^{\infty} = \sum_{k=j}^{n-1}(\tilde{V}_k^{\infty} - \tilde{V}_{k+1}^{\infty})I_{\{\tau_j\leqslant k\}} + \tilde{V}_n^{\infty}.$$

Since $I_{\{\tau_j\leqslant k\}} \in L_G^{1^*}(\Omega_k)$, there exists a bounded sequence $\{\xi_n^{j,k}\}_{n=1}^{\infty} \subset L_G^1(\Omega_k)$ such that $\xi_n^{j,k} \downarrow I_{\{\tau_j\leqslant k\}}$. Note that

$$\begin{aligned}
&-\tilde{V}_{k+1}^N\xi_n^{j,k} \downarrow -\tilde{V}_{k+1}^{\infty}\xi_n^{j,k}, \quad \text{as } N \to \infty,\\
&-\tilde{V}_{k+1}^{\infty}\xi_n^{j,k} \uparrow -\tilde{V}_{k+1}^{\infty}I_{\{\tau_j\leqslant k\}}, \quad \text{as } n \to \infty,\\
&\tilde{V}_k^N\xi_n^{j,k} \downarrow \tilde{V}_k^N I_{\{\tau_j\leqslant k\}}, \quad \text{as } n \to \infty,\\
&\tilde{V}_k^N I_{\{\tau_j\leqslant k\}} \uparrow \tilde{V}_k^{\infty} I_{\{\tau_j\leqslant k\}}, \quad \text{as } N \to \infty.
\end{aligned}$$

It follows that $-\tilde{V}^{\infty}_{k+1}I_{\{\tau_j\leqslant k\}}\in L^{1^*_*}_G(\Omega_{k+1})$ and $\tilde{V}^{\infty}_{k}I_{\{\tau_j\leqslant k\}}\in L^{1^*_*}_G(\Omega_{k})$. Hence, $\tilde{V}^{\infty}_{n\wedge\tau_j}\in L^{1^*_*}_G(\Omega_{n})$.

(ii) Note that

$$\tilde{V}^{\infty}_{(n+1)\wedge\tau_j}-\tilde{V}^{\infty}_{n\wedge\tau_j}=I_{\{\tau_j\geqslant n+1\}}(\tilde{V}^{\infty}_{n+1}-\tilde{V}^{\infty}_{n})=I_{\{\tau_j\leqslant n\}^c}(\tilde{V}^{\infty}_{n+1}-\hat{\mathbb{E}}_n[\tilde{V}^{\infty}_{n+1}]). \tag{4.9}$$

Since $\{\tau_j\leqslant n\}\in\mathcal{B}(\Omega_n)$ and $\tilde{V}^{\infty}_{n+1}$, $-\hat{\mathbb{E}}_n[\tilde{V}^{\infty}_{n+1}]\in L^{*1}_G(\Omega_{n+1})$, applying Lemma 3.11 and (3.2), we have

$$\hat{\mathbb{E}}_n[I_{\{\tau_j\leqslant n\}^c}(\tilde{V}^{\infty}_{n+1}-\hat{\mathbb{E}}_n[\tilde{V}^{\infty}_{n+1}])]=I_{\{\tau_j\leqslant n\}^c}\hat{\mathbb{E}}_n[\tilde{V}^{\infty}_{n+1}-\hat{\mathbb{E}}_n[\tilde{V}^{\infty}_{n+1}]]=0. \tag{4.10}$$

By a similar analysis as Step (i), we obtain that $-\tilde{V}^{\infty}_{n\wedge\tau_j}\in L^{1^*_*}_G(\Omega_n)$. The above two equalities imply that

$$0=\hat{\mathbb{E}}_n[\tilde{V}^{\infty}_{(n+1)\wedge\tau_j}-\tilde{V}^{\infty}_{n\wedge\tau_j}]=\hat{\mathbb{E}}_n[\tilde{V}^{\infty}_{(n+1)\wedge\tau_j}]-\tilde{V}^{\infty}_{n\wedge\tau_j},$$

which shows that $\{\tilde{V}^{\infty}_{n\wedge\tau_j},n=j,j+1,\cdots\}$ is a G-martingale.

(iii) First, we claim that there exists some $1<p<\beta$ such that

$$\hat{\mathbb{E}}\left[\sup_{n\in\mathbb{N}}|\tilde{V}^{\infty}_n|^p\right]<\infty.$$

By Theorem 4.2, we have $\tilde{V}^N_j=\hat{\mathbb{E}}_j[X_{\tau^N_j}]$, where $\tau^N_j=\inf\{l\geqslant j:\tilde{V}^N_l=X_l\}$. It is easy to check that $|\tilde{V}^N_j|\leqslant\hat{\mathbb{E}}_j[\sup_{n\in\mathbb{N}}|X_n|]$ and

$$\hat{\mathbb{E}}\left[\sup_{1\leqslant j\leqslant N}|\tilde{V}^N_j|^p\right]\leqslant\hat{\mathbb{E}}\left[\sup_{1\leqslant j\leqslant N}\hat{\mathbb{E}}_j\left[\sup_{n\in\mathbb{N}}|X_n|^p\right]\right].$$

Since $\hat{\mathbb{E}}[\sup_{n\in\mathbb{N}}|X_n|^\beta]<\infty$, by Theorem 3.4 in [26], there exists a constant C independent of N such that $\hat{\mathbb{E}}[\sup_{1\leqslant j\leqslant N}|\tilde{V}^N_j|^p]\leqslant C$. By monotone convergence theorem, we have

$$\hat{\mathbb{E}}\left[\sup_{n\in\mathbb{N}}|\tilde{V}^{\infty}_n|^p\right]=\lim_{N\to\infty}\hat{\mathbb{E}}\left[\sup_{1\leqslant j\leqslant N}|\tilde{V}^N_j|^p\right]\leqslant C.$$

We then show that

$$\tilde{V}^{\infty}_j=\hat{\mathbb{E}}_j[\tilde{V}^{\infty}_{\tau_j}]=\hat{\mathbb{E}}_j[X_{\tau_j}].$$

Indeed, by Step (ii), we have for any $n\geqslant j$

$$\tilde{V}^{\infty}_j=\hat{\mathbb{E}}_j[\tilde{V}^{\infty}_{n\wedge\tau_j}]. \tag{4.11}$$

For any $\varepsilon>0$, there exists some $N>0$ such that, for any $n\geqslant N$, $c(\tau_j>n)\leqslant\varepsilon$. It is easy to check that

$$\begin{aligned}\hat{\mathbb{E}}[|\hat{\mathbb{E}}_j[\tilde{V}^{\infty}_{n\wedge\tau_j}]-\hat{\mathbb{E}}_j[\tilde{V}^{\infty}_{\tau_j}]|]&\leqslant\hat{\mathbb{E}}[|\tilde{V}^{\infty}_{n\wedge\tau_j}-\tilde{V}^{\infty}_{\tau_j}|]\\&\leqslant\hat{\mathbb{E}}\left[2\sup_{n\in\mathbb{N}}|\tilde{V}^{\infty}_n|I_{\{\tau_j>n\}}\right]\leqslant C\left(\hat{\mathbb{E}}[\sup_{n\in\mathbb{N}}|\tilde{V}^{\infty}_n|^p]\right)^{1/p}\varepsilon^{1/q},\end{aligned} \tag{4.12}$$

where $\frac{1}{p}+\frac{1}{q}=1$. First, letting $n\to\infty$, since ε is arbitrarily small, (4.11) and (4.12) yield that $\tilde{V}^{\infty}_j=\hat{\mathbb{E}}_j[\tilde{V}^{\infty}_{\tau_j}]$. In the following, we show that for any $\tau\in\mathcal{T}_j$, $\tilde{V}^{\infty}_j\geqslant\hat{\mathbb{E}}_j[X_\tau]$. For any $\tau\in\mathcal{T}_j$ and $\varepsilon>0$, there exists some N such that $c(\tau>N)\leqslant\varepsilon$. We obtain that

$$\hat{\mathbb{E}}[|X_\tau-X_{\tau\wedge N}|]\leqslant\hat{\mathbb{E}}\left[2\sup_{n\in\mathbb{N}}|X_n|I_{\{\tau>N\}}\right]\leqslant C\varepsilon^{\frac{\beta-1}{\beta}}.$$

It follows that

$$\hat{\mathbb{E}}_j[X_\tau] = \lim_{N\to\infty} \hat{\mathbb{E}}_j[X_{\tau\wedge N}].$$

Recalling Theorem 4.2, for each $N \geqslant j$, we have $\tilde{V}_j^\infty \geqslant \tilde{V}_j^N \geqslant \hat{\mathbb{E}}_j[X_{\tau\wedge N}]$. Letting $N \to \infty$, we deduce that $\tilde{V}_j^\infty \geqslant \hat{\mathbb{E}}_j[X_\tau]$. This completes the proof. □

Remark 4.9 *For each fixed $N \in \mathbb{N}$, we may define the sequence $\{\underline{V}_n^N, n = 0, 1, \cdots, N\}$ recursively: Let $\underline{V}_N^N = X_N$ and*

$$\underline{V}_n^N = \min\{X_n, \hat{\mathbb{E}}_n[\underline{V}_{n+1}^N]\}, \quad n \leqslant N-1.$$

Evidently, for any $n \leqslant N \leqslant M$, $\underline{V}_n^N \leqslant \underline{V}_n^M$. Thus, we define

$$\underline{V}_n^\infty = \lim_{N\geqslant n, N\to\infty} \underline{V}_n^N.$$

Then, similar results still hold for the sequence $\{\underline{V}^\infty, n \in \mathbb{N}\}$. More precisely, set

$$\tau_j = \inf\{l \geqslant j : \underline{V}_l^\infty = X_l\}.$$

Assume that τ_j is finite quasi-surely (i.e. $c(\tau_j > N) \to 0$, as $N \to \infty$). Then, we have

(i) *$\tau_j \in \mathcal{T}_j$ and $\underline{V}_{n\wedge\tau_j}^\infty \in L_G^{*1}(\Omega_n)$, for each $n \in \mathbb{N}$;*

(ii) *$\{\underline{V}_{n\wedge\tau_j}^\infty, n = j, j+1, \cdots\}$ is a G-martingale;*

(iii) *For any $j \in \mathbb{N}$,*

$$\underline{V}_j^\infty = \hat{\mathbb{E}}_j[X_{\tau_j}] = ess\inf_{\tau\in\mathcal{T}_j} \hat{\mathbb{E}}_j[X_\tau];$$

(iv) *The sequence $\{\underline{V}_n^\infty, n \in \mathbb{N}\}$ is the largest G-submartingale dominated by the process $\{X_n, n \in \mathbb{N}\}$.*

5. Optimal stopping in continuous-time

5.1 Finite-time horizon case

In this subsection, we present the relationship between the value function of the optimal stopping problem and the solution of the reflected G-BSDE. For simplicity, assume the time horizon is $[0,1]$. We need to consider the following payoff process $\{X_t\}_{t\in[0,1]}$.

Assumption 5.1 *The payoff process $\{X_t\}_{t\in[0,1]} \in S_G^\beta(0,1)$, where $\beta > 1$.*

Denote by $\mathcal{T}_{s,t}^\infty$ the collection of all G-stopping times τ such that $s \leqslant \tau \leqslant t$ and by $\mathcal{T}_{s,t}^n$ the collection of all G-stopping times taking values in $\mathcal{I}_n$ such that $s \leqslant \tau \leqslant t$, where $0 \leqslant s < t \leqslant 1$ and $\mathcal{I}_n = \{k/2^n, k = 0, 1, \cdots, 2^n\}$. Set

$$V_0 = \sup_{\tau\in\mathcal{T}_{0,1}^\infty} \hat{\mathbb{E}}[X_\tau]. \tag{5.1}$$

For each $n \in \mathbb{N}$, we define the following sequence $\{V_{t_k^n}^n, k = 0, 1, \cdots, 2^n\}$ backward: let $V_1^n = X_1$ and

$$V_{t_k^n}^n = \max(X_{t_k^n}, \hat{\mathbb{E}}_{t_k^n}[V_{t_{k+1}^n}^n]), \ k = 0, 1, \cdots, 2^n - 1,$$

where $t_k^n = k/2^n$. By Theorem 4.2, for any $n \in \mathbb{N}$ and $k = 0, 1, \cdots, 2^n$, we have

$$V_{t_k^n}^n = ess\sup_{\tau\in\mathcal{T}_{0,1}^n, \tau\geqslant t_k^n} \hat{\mathbb{E}}_{t_k^n}[X_\tau].$$

It is easy to check that for any $n \in \mathbb{N}$ and $k = 0, 1, \cdots, 2^n$, $V^n_{t^n_k} \leqslant V^{n+1}_{t^n_k}$. Then, we define

$$V^{\infty}_{t^n_k} = \lim_{m \geqslant n, m \to \infty} V^m_{t^n_k}. \tag{5.2}$$

Proposition 5.2 *Let $\mathcal{I} = \cup_n \mathcal{I}_n$. For each $t \in \mathcal{I}$, we have $V^{\infty}_t \in L^{*1}_G(\Omega_t)$. Morevoer, the sequence $\{V^{\infty}_t, t \in \mathcal{I}\}$ is the smallest G-supermartingale that dominates the process $\{X_t, t \in \mathcal{I}\}$.*

Proof Let $t^n_k, t^m_l \in \mathcal{I}$ and $t^n_k < t^m_l$, where $m, n \in \mathbb{N}$. It is easy to check that

$$\begin{aligned} \hat{\mathbb{E}}_{t^n_k}\left[V^{\infty}_{t^m_l}\right] &= \hat{\mathbb{E}}_{t^n_k}\left[\lim_{M \geqslant m, M \to \infty} V^M_{t^m_l}\right] = \hat{\mathbb{E}}_{t^n_k}\left[\lim_{M \geqslant (m \vee n), M \to \infty} V^M_{t^m_l}\right] \\ &= \lim_{M \geqslant (m \vee n), M \to \infty} \hat{\mathbb{E}}_{t^n_k}\left[V^M_{t^m_l}\right] \leqslant \lim_{M \geqslant (m \vee n), M \to \infty} V^M_{t^n_k} = V^{\infty}_{t^n_k}. \end{aligned}$$

Now, let $\{U_t, t \in \mathcal{I}\}$ be a G-supermartingale that dominates $\{X_t, t \in \mathcal{I}\}$. By Theorem 4.2, we know that $\{V^n, t \in \mathcal{I}_n\}$ is the smallest G-supermartingale that dominates $\{X_t, t \in \mathcal{I}_n\}$. Therefore, for any $m \geqslant n$, we have $U_{t^n_k} \geqslant V^m_{t^n_k}$. Letting $m \to \infty$, we have $U_{t^n_k} \geqslant V^{\infty}_{t^n_k}$, which completes the proof. □

Proposition 5.3 *Assume that the payoff process $\{X_t\}_{t \in [0,1]}$ satisfies Assumption 5.1. Then, we have*

$$V_0 = V^{\infty}_0.$$

Proof Note that for each n,

$$V^n_0 = \sup_{\tau \in \mathcal{T}^n_{0,1}} \hat{\mathbb{E}}[X_\tau].$$

Consequently, we have $V_0 \geqslant V^n_0$, $n \in \mathbb{N}$. Letting n tends to infinity, we have $V_0 \geqslant V^{\infty}_0$. Next, we prove the inverse inequality. For each $\tau \in \mathcal{T}_{0,1}$ and $n \in \mathbb{N}$, set

$$\tau^n = \frac{1}{2^n} I_{\{0 \leqslant \tau \leqslant \frac{1}{2^n}\}} + \sum_{k=2}^{2^n} \frac{k}{2^n} I_{\{\frac{k-1}{2^n} < \tau \leqslant \frac{k}{2^n}\}}.$$

It is easy to check that $\tau^n \in \mathcal{T}^n_{0,1}$. By applying the continuity property of X (see Lemma 2.9), we have

$$\lim_{n \to \infty} \hat{\mathbb{E}}[|X_\tau - X_{\tau^n}|] = 0.$$

It follows that

$$\hat{\mathbb{E}}[X_\tau] = \lim_{n \to \infty} \hat{\mathbb{E}}[X_{\tau^n}] \leqslant \lim_{n \to \infty} V^n_0 = V^{\infty}_0.$$

Since τ is arbitrarily chosen, we deduce that $V_0 \leqslant V^{\infty}_0$. □

According to [5], the value function of the optimal stopping problem defined by g-expectation coincides with the solution of the reflected BSDE with a lower obstacle. The following theorem indicates that our value function (5.1) defined by G-expectation corresponds to the solution of the reflected BSDE driven by G-Brownian motion.

Theorem 5.4 *Let X satisfy (H3) in Subsection 2.2. Let (Y, Z, L) be the solution of the reflected G-BSDE with parameters $(X_1, 0, 0, X)$. Then, for any $t \in \mathcal{I}$, we have*

$$Y_t = V^{\infty}_t = ess\sup_{\tau \in \mathcal{T}^{\infty}_{t,1}} \hat{\mathbb{E}}_t[X_\tau].$$

Proof Since $t \in \mathcal{I}$, we assume that $t = \frac{k}{2^m}$ for some $k = 0, 1, \ldots, 2^m$, $m \in \mathbb{N}$.

Step 1 We first prove $Y_t = V_t^{\infty}$. By Proposition 2.11, Y is a G-supermartingale that dominates the process X. By Theorem 4.2, for all $n \geqslant m$, we have $Y_t \geqslant V_t^n$. It follows that $Y_t \geqslant \lim_{n\to\infty} V_t^n = V_t^{\infty}$. We then show the inverse inequality. For each fixed $n \geqslant m$ and $\varepsilon > 0$, set

$$\tau_\varepsilon^n = \inf\{s \in \mathcal{I}_n, s \geqslant t : A_s \leqslant \varepsilon\},$$

where $A_s = Y_s - X_s$. It is easy to check that τ_ε^n is a G-stopping time taking values in $\mathcal{I}_n$ and

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \tau_\varepsilon^n = \tau,$$

where

$$\tau = \inf\{s \geqslant t : A_s = 0\}.$$

By a similar analysis as the proof of Proposition 7.7 in [16], we have $Y_t = \hat{\mathbb{E}}_t[X_\tau]$. By the continuity property of X (see Lemma 2.9), we have

$$Y_t = \hat{\mathbb{E}}_t[X_\tau] = \lim_{\varepsilon\to 0}\lim_{n\to\infty} \hat{\mathbb{E}}_t[X_{\tau_\varepsilon^n}] \leqslant \lim_{\varepsilon\to 0}\lim_{n\to\infty} V_t^n = V_t^{\infty}.$$

Step 2 We show that $Y_t = V_t^{\infty} = ess\sup_{\tau\in\mathcal{T}_{t,1}^{\infty}} \hat{\mathbb{E}}_t[X_\tau]$. For any $\tau \in \mathcal{T}_{t,1}^{\infty}$ and $n \geqslant m$, set

$$\tau^n = \frac{2^{n-m}k+1}{2^n} I_{\{\frac{k}{2^m}\leqslant\tau\leqslant\frac{2^{n-m}k+1}{2^n}\}} + \sum_{i=2}^{2^n-2^{n-m}k} \frac{2^{n-m}k+i}{2^n} I_{\{\frac{2^{n-m}k+i-1}{2^n}<\tau\leqslant\frac{2^{n-m}k+i}{2^n}\}}.$$

We can check that $\tau^n \in \mathcal{T}_{t,1}^n$. By the continuity property of X, it follows that

$$\hat{\mathbb{E}}_t[X_\tau] = \lim_{n\to\infty} \hat{\mathbb{E}}_t[X_{\tau^n}] \leqslant \lim_{n\to\infty} V_t^n = V_t^{\infty}.$$

It remains to show that if $\eta \geqslant \hat{\mathbb{E}}_t[X_\tau]$, for any $\tau \in \mathcal{T}_{t,1}^{\infty}$, then $\eta \geqslant Y_t$. By the analysis in Step 1, noting that $\tau_\varepsilon^n \in \mathcal{T}_{t,1}^n \subset \mathcal{T}_{t,1}^{\infty}$, we have

$$Y_t = \hat{\mathbb{E}}_t[X_\tau] = \lim_{\varepsilon\to 0}\lim_{n\to\infty} \hat{\mathbb{E}}_t[X_{\tau_\varepsilon^n}] \leqslant \eta.$$

The proof is complete. □

5.2 Infinite-time horizon case

This subsection is focused on the infinite-time horizon case. Since we need to use the Fatou lemma (see Lemma 3.9) in the proof of Proposition 5.6, the payoff process is assumed to be bounded from below.

Assumption 5.5 *$\{X_t, t \geqslant 0\}$ is bounded from below and for any $n \in \mathbb{N}$, $X \in S_G^{\beta}(0,n)$, where $\beta > 1$. Furthermore, $\hat{\mathbb{E}}[\sup_{t\geqslant 0}|X_t|^{\beta}] < \infty$.*

Set $t_k^n = \frac{k}{2^n}$ and $\mathcal{I}_n^{\infty} = \{t_k^n : k = 0, 1, \cdots\}$. For each fixed $n, N \in \mathbb{N}$, define the following sequence $\{V_{t_k^n}^{n,N} : k = 0, 1, \cdots, N\}$ backward:

$$V_{t_N^n}^{n,N} = X_{t_N^n}, \quad V_{t_k^n}^{n,N} = \max\{X_{t_k^n}, \hat{\mathbb{E}}_{t_k^n}[V_{t_{k+1}^n}^{n,N}]\}, \quad k \leqslant N-1.$$

It is easy to see that for any $k \leqslant N \leqslant M$, $V_{t_k^n}^{n,N} \leqslant V_{t_k^n}^{n,M}$. We define

$$V_{t_k^n}^n = \lim_{N\geqslant k, N\to\infty} V_{t_k^n}^{n,N}.$$

Then, $V_{t_k^n}^n \in L_G^{*1}(\Omega_{t_k^n})$. If $n < m$, the for any $N \in \mathbb{N}$, we can easily check that

$$V_{t_k^n}^{n,N} \leqslant V_{t_{k2^{m-n}}^m}^{m,N2^{m-n}}, \quad k \leqslant N,$$

which yields that $V_{t_k^n}^n \leqslant V_{t_{k2^{m-n}}^m}^m = V_{t_k^n}^m$. We define

$$V_{t_k^n} = \lim_{m \geqslant n, m \to \infty} V_{t_{k2^{m-n}}^m}^m = \lim_{m \geqslant n, m \to \infty} V_{t_k^n}^m.$$

Then $V_{t_k^n} \in L_G^{*1}(\Omega_{t_k^n})$ and V satisfies the following property.

Proposition 5.6 *The sequence* $\{V_t, t \in \mathcal{I}^\infty\}$ *is the smallest G-supermartingale that dominates* $\{X_t, t \in \mathcal{I}^\infty\}$ *, where* $\mathcal{I}^\infty = \cup_{n=1}^\infty \mathcal{I}_n^\infty$*. Moreover, we have*

$$V_0 = \sup_{\tau \in \mathcal{T}_0^\infty} \hat{\mathbb{E}}[X_\tau], \tag{5.3}$$

where $\mathcal{T}_t^\infty$ *is the collection of all G-stopping time taking values in* $[t, \infty)$ *and satisfying equation* (4.7).

Proof By Proposition 4.5 and 4.7, we derive that $\{V_{t_k^n}^n, k \in \mathbb{N}\}$ is the smallest G-supermartingale dominating $\{X_{t_k^n}, k \in \mathbb{N}\}$ and

$$V_0^n = \sup_{\tau \in \mathcal{T}_0^n} \hat{\mathbb{E}}[X_\tau],$$

where $\mathcal{T}_t^n$ is the collection of all G-stopping time taking values in $\mathcal{I}_n^\infty$, no less than t and satisfying Equation (4.7). It is easy to check that for any $t_k^n, t_l^m \in \mathcal{I}^\infty$ with $t_k^n < t_l^m$, we have

$$\begin{aligned}
\hat{\mathbb{E}}_{t_k^n}[V_{t_l^m}] &= \hat{\mathbb{E}}_{t_k^n}\left[\lim_{M \geqslant m, M \to \infty} V_{t_{l2^{M-m}}^M}^M\right] \\
&= \lim_{M \geqslant (m \vee n), M \to \infty} \hat{\mathbb{E}}_{t_k^n}\left[V_{t_{l2^{M-m}}^M}^M\right] \\
&\leqslant \lim_{M \geqslant (m \vee n), M \to \infty} V_{t_{k2^{M-n}}^M}^M = V_{t_k^n},
\end{aligned}$$

which yields that $\{V_t, t \in \mathcal{I}^\infty\}$ is a G-supermartingale. If $\{U_t, t \in \mathcal{I}^\infty\}$ is another G-supermartingale dominating $\{X_t, t \in \mathcal{I}^\infty\}$, then for $t = t_k^n \in \mathcal{I}^\infty$ and $m \geqslant n$, it is easy to check that $U_{t_k^n} \geqslant V_{t_k^n}^m = V_{t_{k2^{m-n}}^m}^m$. It follows that

$$U_{t_k^n} \geqslant \lim_{m \geqslant n, m \to \infty} V_{t_k^n}^m = V_{t_k^n},$$

which implies that $\{V_t, t \in \mathcal{I}^\infty\}$ is the smallest G-supermartingale dominating $\{X_t, t \in \mathcal{I}^\infty\}$. To prove Equation (5.3), first note that $V_0 = \lim_{n \to \infty} V_0^n \leqslant \sup_{\tau \in \mathcal{T}_0^\infty} \hat{\mathbb{E}}[X_\tau]$. In contrast, for any $\tau \in \mathcal{T}_0^\infty$, there exists $\tau^n \in \mathcal{T}_0^n$ such that $\tau^n \to \tau$. Noting that X is continuous and applying Lemma 3.9, we obtain

$$\hat{\mathbb{E}}[X_\tau] = \hat{\mathbb{E}}[\liminf_{n \to \infty} X_{\tau^n}] \leqslant \liminf_{n \to \infty} \hat{\mathbb{E}}[X_{\tau^n}] \leqslant \liminf_{n \to \infty} V_0^n = V_0.$$

Since τ is chosen arbitrarily, the proof is complete. □

Remark 5.7 $\{V_t, t \in \mathcal{I}^\infty\}$ *can be defined by the following procedure. By Equation* (5.2) *and Proposition 5.2, we can construct a sequence* $\{V_t^{\infty,N}, t \in \mathcal{I}^\infty, t \leqslant N\}$ *such that it is the smallest* G*-supermartingale dominating the process* $\{X_t, t \in \mathcal{I}^\infty, t \leqslant N\}$*. Moreover, by Theorem 5.4, for any* $t \in \mathcal{I}^\infty$ *with* $t \leqslant N$ *, we have*

$$V_t^{\infty,N} = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t^\infty, \tau \leqslant N} \hat{\mathbb{E}}_t[X_\tau].$$

It is easy to check that for $t \leqslant N \leqslant M$, $V_t^{\infty,N} \leqslant V_t^{\infty,M}$*. For each* $t = t_k^n \in \mathcal{I}^\infty$ *, We define*

$$\tilde{V}_t = \lim_{N \geqslant t, N \to \infty} V_t^{\infty,N}.$$

We claim that $V_t = \tilde{V}_t$ *for any* $t \in \mathcal{I}^\infty$*. It suffices to prove that* $\{\tilde{V}_t, t \in \mathcal{I}^\infty\}$ *is the smallest* G*-supermartingale dominating* $\{X_t, t \in \mathcal{I}^\infty\}$*. For any* $s, t \in \mathcal{I}^\infty$ *with* $s \leqslant t$ *, we have*

$$\hat{\mathbb{E}}_s[\tilde{V}_t] = \hat{\mathbb{E}}_s\left[\lim_{N \geqslant t, N \to \infty} V_t^{\infty,N}\right] = \lim_{N \geqslant t, N \to \infty} \hat{\mathbb{E}}_s[V_t^{\infty,N}] \leqslant \lim_{N \geqslant t, N \to \infty} V_s^{\infty,N} = \tilde{V}_s.$$

Now suppose that $\{U_t, t \in \mathcal{I}^\infty\}$ *is the a* G*-supermartingale dominating* $\{X_t, t \in \mathcal{I}^\infty\}$ *, then we have* $U_t \geqslant V_t^{\infty,N}$ *for any* $t \leqslant N$*. Letting* $N \to \infty$ *yields that* $U_t \geqslant \tilde{V}_t$*.*

By a similar analysis as the proof of Proposition 4.8, we obtain that for any $\tau \in \mathcal{T}_t^\infty$ *and* $t \in \mathcal{I}^\infty$

$$\hat{\mathbb{E}}_t[X_\tau] = \lim_{N \geqslant t, N \to \infty} \hat{\mathbb{E}}_t[X_{\tau \wedge N}] \leqslant \lim_{N \geqslant t, N \to \infty} V_t^{\infty,N} = V_t.$$

In contrast, if there exists some $\eta \in \mathcal{L}(\Omega_t)$ *, such that* $\eta \geqslant \hat{\mathbb{E}}_t[X_\tau]$ *for any* $\tau \in \mathcal{T}_t^\infty$ *with some* $t \in \mathcal{I}^\infty$ *, then* $\eta \geqslant V_t^{\infty,N}$ *for any* $N \geqslant t$ *, which implies that* $\eta \geqslant V_t$*. By the definition of essential supremum, the above analysis shows that*

$$V_t = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t^\infty} \hat{\mathbb{E}}_t[X_\tau]$$

for each $t \in \mathcal{I}^\infty$*.*

6. Markovian case

In this section, we present some results of optimal stopping under G-expectation when the payoff process is Markovian. More precisely, consider the payoff process $\{X^{t,\xi}\}$ generated by the following G-SDE:

$$X_s^{t,\xi} = \xi + \int_t^s b(X_r^{t,\xi})\mathrm{d}r + \int_t^s h(X_r^{t,\xi})d\langle B\rangle_r + \int_t^s \sigma(X_r^{t,\xi})\mathrm{d}B_r, \tag{6.1}$$

where $\xi \in L_G^p(\Omega_t)$, $p \geqslant 2$ and b, h, $\sigma : \mathbb{R} \to \mathbb{R}$ are deterministic functions satisfying the following:

(H1) There exists a constant $L > 0$, such that for any $x, y \in \mathbb{R}$,

$$|b(x) - b(y)| + |h(x) - h(y)| + |\sigma(x) - \sigma(y)| \leqslant L|x - y|.$$

Then, we have the following estimates, which can be found in [16, 23].

Proposition 6.1 *Let* $\xi, \xi' \in L_G^p(\Omega_t)$ *and* $p \geqslant 2$*. Then, we have, for each* $\delta \in [0, T-t]$*,*

$$\hat{\mathbb{E}}_t\left[\sup_{s \in [t, t+\delta]} |X_s^{t,\xi} - X_s^{t,\xi'}|^p\right] \leqslant C|\xi - \xi'|^p,$$

$$\hat{\mathbb{E}}_t\left[|X_{t+\delta}^{t,\xi}|^p\right] \leqslant C(1 + |\xi|^p),$$

$$\hat{\mathbb{E}}_t\left[\sup_{s \in [t, t+\delta]} |X_s^{t,\xi} - \xi|^p\right] \leqslant C(1 + |\xi|^p)\delta^{p/2},$$

where the constant C *depends on* L, G, p *and* T*.*

For simplicity, set $X_s^x := X_s^{0,x}$. By Lemma 4.1 in [9], we have the following Markov property.

Lemma 6.2 *For each given* $\varphi \in C_{b,Lip}(\mathbb{R})$ *and* $s, t \geqslant 0$ *, we have*

$$\hat{\mathbb{E}}_t[\varphi(X_{t+s}^x)] = \hat{\mathbb{E}}[\varphi(X_s^y)]_{y = X_t^x}.$$

6.1 Discrete time case

In this subsection, we first investigate the discrete-time case. For a given function $f \in C_{b,Lip}(\mathbb{R})$ and $x \in \mathbb{R}$, consider the following optimal stopping problem:

$$F^N(x) := \sup_{\tau \in \mathcal{T}_{0,N}} \hat{\mathbb{E}}[f(X^x_\tau)], \tag{6.2}$$

where $\mathcal{T}_{0,N}$ denotes the set of all G-stopping times taking values in $\{0, 1, \cdots, N\}$.

Lemma 6.3 *For each $N \in \mathbb{N}$, the function F^N defined by Equation (6.2) is bounded and Lipschitz.*

Proof Since $f \in C_{b,Lip}(\mathbb{R})$, F^N is bounded. Moreover, by Proposition 6.1, we have

$$|F^N(x) - F^N(y)| \leqslant \sup_{\tau \in \mathcal{T}_{0,N}} \hat{\mathbb{E}}[|f(X^x_\tau) - f(X^y_\tau)|] \leqslant C\hat{\mathbb{E}}[\sup_{t \in [0,N]} |X^x_t - X^y_t|] \leqslant C|x-y|.$$

□

Set $\tilde{X}^x_n = f(X^x_n)$. It is easy to check that $\{\tilde{X}^x_n, n \in \mathbb{N}\}$ satisfies Assumption 4.4. Similar to Section 4, we define the following sequence $\{V^N_n, n = 0, 1, \cdots, N\}$ backward. Let $V^N_N(x) = \tilde{X}^x_N$ and

$$V^N_n(x) = \max\{\tilde{X}^x_n, \hat{\mathbb{E}}_n[V^N_{n+1}(x)]\}, \quad n \leqslant N-1.$$

It is worth noting that $V^N_0(x) = F^N(x)$. Moreover, we have the following identity

$$V^N_n(x) = F^{N-n}(X^x_n), \quad \text{for } 0 \leqslant n \leqslant N. \tag{6.3}$$

This will be shown in the proof of the next theorem. Next, we set

$$C_n = \{x \in \mathbb{R} : F^{N-n}(x) > f(x)\},$$
$$D_n = \{x \in \mathbb{R} : F^{N-n}(x) = f(x)\},$$

for any $n = 0, 1, \cdots, N$. Then, we define

$$\tau^{N,x}_D = \inf\{0 \leqslant n \leqslant N : X^x_n \in D_n\}.$$

Since both F^{N-n} and f are Lipschitz continuous, then D_n is a closed set, which implies that $I_{\{X^x_n \in D_n\}} \in L^{1^*}_G(\Omega_n)$. Therefore, we conclude that $\tau^{N,x}_D$ is a G-stopping time. Finally, for any $f \in C_{b,Lip}(\mathbb{R})$, define the following transition operator T:

$$Tf(x) = \hat{\mathbb{E}}[f(X^x_1)].$$

Theorem 6.4 *Consider the optimal stopping time problem* (6.2)*. Then, for any $n = 1, 2, \cdots, N$, the value function F^n satisfies the Wald-Bellman equations*

$$F^n(x) = \max\{f(x), TF^{n-1}(x)\}, \tag{6.4}$$

where $F^0(x) = f(x)$. Furthermore, we have

(i) *$\tau^{N,x}_D$ is a G-stopping time and optimal for the problem given in* (6.2)*;*

(ii) *The sequence $\{F^{N-n}(X^x_n), n = 0, 1, \cdots, N\}$ is the smallest G-supermartingale that dominates $\{f(X^x_n), n = 0, 1, \cdots, N\}$ for each $x \in \mathbb{R}$;*

(iii) *The stopped process $\{F^{N-n\wedge\tau^{N,x}_D}(X^x_{n\wedge\tau^{N,x}_D}), n = 0, 1, \cdots, N\}$ is a G-martingale for each $x \in \mathbb{R}$.*

Proof We claim that $\{V^n_0\}$ satisfies the Wald-Bellman equations. Indeed, it is easy to check that $V^0_0(x) = \tilde{X}^x_0 = f(x)$ and

$$V_0^1(x) = \max\{\tilde{X}_0^x, \hat{\mathbb{E}}[V_1^1(x)]\} = \max\{f(x), \hat{\mathbb{E}}[f(X_1^x)]\} = \max\{f(x), TV_0^0(x)\}.$$

We assume that for any $n \leqslant k$, $V_0^n(x) = \max\{f(x), \hat{\mathbb{E}}[V_0^{n-1}(X_1^x)]\}$. We then obtain that

$$V_k^{k+1}(x) = \max\{f(X_k^x), \hat{\mathbb{E}}_k[f(X_{k+1}^x)]\} = \max\{f(X_k^x), \hat{\mathbb{E}}[f(X_1^y)]_{y=X_k^x}\} = V_0^1(X_k^x),$$

and

$$\begin{aligned} V_{k-1}^{k+1}(x) &= \max\{f(X_{k-1}^x), \hat{\mathbb{E}}_{k-1}[V_k^{k+1}(x)]\} = \max\{f(X_{k-1}^x), \hat{\mathbb{E}}_{k-1}[V_0^1(X_k^x)]\} \\ &= \max\{f(X_{k-1}^x), \hat{\mathbb{E}}[V_0^1(X_1^y)]_{y=X_{k-1}^x}\} = V_0^2(X_{k-1}^x). \end{aligned}$$

By the above procedure, we have

$$\begin{aligned} V_1^{k+1}(x) &= \max\{f(X_1^x), \hat{\mathbb{E}}_1[V_2^{k+1}(x)]\} = \max\{f(X_1^x), \hat{\mathbb{E}}_1[V_0^{k-1}(X_2^x)]\} \\ &= \max\{f(X_1^x), \hat{\mathbb{E}}[V_0^{k-1}(X_1^y)]_{y=X_1^x}\} = V_0^k(X_1^x), \end{aligned}$$

which yields that

$$V_0^{k+1}(x) = \max\{f(x), \hat{\mathbb{E}}[V_1^{k+1}(x)]\} = \max\{f(x), \hat{\mathbb{E}}[V_0^k(X_1^x)]\}.$$

Note that the above analysis also establishes that for any $0 \leqslant j \leqslant n \leqslant N$, $V_j^n(x) = V_0^{n-j}(X_j^x)$. Recall that $F^n(x) = V_0^n(x)$, $n = 0, 1, \cdots, N$, which implies that (6.3) holds and F^n satisfies the Wald-Bellman equation. Applying Theorem 4.2, the conclusions (i)−(iii) hold. □

For the infinite-time case, the value function is defined by

$$F(x) = \sup_{\tau \in \mathcal{T}_0} \hat{\mathbb{E}}[f(X_\tau^x)], \tag{6.5}$$

where $f \in C_{b,Lip}(\mathbb{R})$. Let $V_n^\infty(x) = \lim_{N\to\infty} V_n^N(x)$. By Proposition 4.7, we have

$$F(x) = V_0^\infty(x) = \lim_{N\to\infty} F^N(x), \tag{6.6}$$

which implies that F is a bounded lower semicontinuous function. Then, letting $N \to \infty$ in Equation (6.3), it follows that

$$V_n^\infty(x) = F(X_n^x), \quad \text{for } n \in \mathbb{N}.$$

Set

$$\begin{aligned} C &= \{x \in \mathbb{R} : F(x) > f(x)\}, \\ D &= \{x \in \mathbb{R} : F(x) = f(x)\}. \end{aligned}$$

Since F is lower semicontinuous, D is a closed subset of $\mathbb{R}$. Then, we define

$$\tau_D^x = \inf\{n \geqslant 0 : X_n^x \in D\}.$$

Similar to the finite-time case, τ_D^x is a G-stopping time.

Definition 6.5 *A measurable function $F : \mathbb{R} \to \mathbb{R}$ is said to be superharmonic, if for all $x \in \mathbb{R}$,*

$$TF(x) \leqslant F(x).$$

Remark 6.6 *It is worth noting that there is an implicit assumption in the above definition that $F(X_1^x) \in \mathbb{L}^1(\Omega)$ for each $x \in \mathbb{R}$.*

Lemma 6.7 *Suppose that F is lower semicontinuous and bounded from below (resp. upper semicontinuous and bounded from above). Then, F is superharmonic if and only if $\{F(X_n^x), n \in \mathbb{N}\}$ is a G-supermartingale for any $x \in \mathbb{R}$.*

Proof Since F is lower semicontinuous and bounded from below, there exists a sequence $\{F^m, m \in \mathbb{N}\} \subset C_{b,Lip}(\mathbb{R})$ such that $F^m \uparrow F$. For the "if " part, suppose that F is superharmonic. Note that $F^m(X_n^x) \in L_G^1(\Omega_n)$ and $F^m(X_n^x) \uparrow F(X_n^x)$ for any $m, n \in \mathbb{N}$ and $x \in \mathbb{R}$. We obtain that

$$\begin{aligned} F(X_n^x) \geqslant TF(X_n^x) &= \hat{\mathbb{E}}[F(X_1^y)]_{y=X_n^x} = \lim_{m\to\infty} \hat{\mathbb{E}}[F^m(X_1^y)]_{y=X_n^x} \\ &= \lim_{m\to\infty} \hat{\mathbb{E}}_n[F^m(X_{n+1}^x)] = \hat{\mathbb{E}}_n[F(X_{n+1}^x)], \end{aligned} \tag{6.7}$$

for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$, which implies $\{F(X_n^x), n \in \mathbb{N}\}$ is a G-supermartingale. For the "only if " part, note that $F(X_n^x) \geqslant \hat{\mathbb{E}}_n[F(X_{n+1}^x)]$ holds for any $n \in \mathbb{N}$. Letting $n = 0$ yields that F is superharmonic. □

Using the relationship between F and F^N (see (6.6)) and Theorem 6.4, we obtain the following result.

Theorem 6.8 *Consider the optimal stopping time problem* (6.5). *Then, the value function F satisfies the Wald-Bellman equation*

$$F(x) = \max\{f(x), TF(x)\}. \tag{6.8}$$

Furthermore, assume that for any $x \in \mathbb{R}$, τ_D^x satisfies (4.7). *Then, we have*

(i) *τ_D^x is a G-stopping time and optimal in Equation* (6.5)*;*
(ii) *The value function F is the smallest superharmonic function that dominates f on* $\mathbb{R}$;
(iii) *The stopped process $\{F(X_{n\wedge\tau_D^x}^x), n = 0, 1, \cdots, N\}$ is a G-martingale for each $x \in \mathbb{R}$.*

It is easy to check that for any $n = 1, 2, \cdots, N$, there exists a unique solution to the Wald-Bellman Equation (6.4). However, when the time horizon is infinite, there may be many solutions to the Wald-Bellman Equation (6.8). For example, if $f(x) \equiv c$, any $F(x) = C \geqslant c$ solves this equation. In the following, we give a sufficient condition under which the solution to Equation (6.8) is unique.

Theorem 6.9 *Suppose that $G : \mathbb{R} \to \mathbb{R}$ is lower semicontinuous and bounded from below satisfying the Wald-Bellman equation*

$$G(x) = \max\{f(x), TG(x)\}$$

for $x \in \mathbb{R}$. Furthermore, we assume that, for some $p > 1$,

$$\hat{\mathbb{E}}\left[\sup_{n\in\mathbb{N}} |G(X_n^x)|^p\right] < \infty$$

for any $x \in \mathbb{R}$. If the following "boundary condition at infinity" holds,

$$\limsup_{n\to\infty} G(X_n^x) = \limsup_{n\to\infty} f(X_n^x), \quad q.s. \tag{6.9}$$

for any $x \in \mathbb{R}$, then G equals to the value function F.

Proof Without loss of generality, we assume that $G \geqslant 0$. Since G satisfy the Wald-Bellman equation, it is superharmonic and $G \geqslant f$. By Theorem 6.8, we have $G \geqslant F$. In the following, we show the converse inequality. Define the random time

$$\tau_\varepsilon^x = \inf\{n \geqslant 0 : G(X_n^x) \leqslant f(X_n^x) + \varepsilon\} = \inf\{n \geqslant 0 : X_n^x \in D_\varepsilon\},$$

where $\varepsilon > 0$ and

$$D_\varepsilon = \{x \in \mathbb{R} : G(x) \leqslant f(x) + \varepsilon\}.$$

It is a closed subset of $\mathbb{R}$ due to the lower semicontinuity of G. Therefore, τ_ε^x is a G-stopping time for any $\varepsilon > 0$ and $x \in \mathbb{R}$. Additionally, by (6.9), τ_ε^x satisfy condition (4.7). We claim that $\{G(X^x_{\tau_\varepsilon^x \wedge n}), n \in \mathbb{N}\}$ is a G-martingale for all $x \in \mathbb{R}$. Following a similar analysis in the proof of Theorem 4.8, we have $G(X^x_{\tau_\varepsilon^x \wedge n}) \in L_G^{1^*}(\Omega_n)$, for each $n \in \mathbb{N}$. Note that $I_{\{\tau_\varepsilon^x \leqslant n-1\}} \in L_G^{1^*}(\Omega_{n-1})$. It follows that $I_{\{\tau_\varepsilon^x \geqslant n\}} \in L_G^{1^*}(\Omega_{n-1})$ and $G(X^x_{\tau_\varepsilon^x \wedge (n-1)}) I_{\{\tau_\varepsilon^x \leqslant n-1\}} \in L_G^{1^*}(\Omega_{n-1})$. We obtain that for each $n \geqslant 1$ and $x \in \mathbb{R}$,

$$\begin{aligned}
\hat{\mathbb{E}}_{n-1}[G(X^x_{\tau_\varepsilon^x \wedge n})] &= \hat{\mathbb{E}}_{n-1}[G(X^x_n) I_{\{\tau_\varepsilon^x \geqslant n\}}] + G(X^x_{\tau_\varepsilon^x \wedge (n-1)}) I_{\{\tau_\varepsilon^x \leqslant n-1\}} \\
&= \hat{\mathbb{E}}_{n-1}[G(X^x_n)] I_{\{\tau_\varepsilon^x \geqslant n\}} + G(X^x_{\tau_\varepsilon^x}) I_{\{\tau_\varepsilon^x \leqslant n-1\}} \\
&= TG(X^x_{n-1}) I_{\{\tau_\varepsilon^x \geqslant n\}} + G(X^x_{\tau_\varepsilon^x}) I_{\{\tau_\varepsilon^x \leqslant n-1\}} \\
&= G(X^x_{n-1}) I_{\{\tau_\varepsilon^x \geqslant n\}} + G(X^x_{\tau_\varepsilon^x}) I_{\{\tau_\varepsilon^x \leqslant n-1\}} \\
&= G(X^x_{\tau_\varepsilon^x \wedge (n-1)}),
\end{aligned}$$

where in the third equality we use Equation (6.7). Therefore, we have

$$\hat{\mathbb{E}}[G(X^x_{\tau_\varepsilon^x \wedge n})] = G(x)$$

for all $n \geqslant 0$ and $x \in \mathbb{R}$. A similar analysis as (4.8) shows that

$$\hat{\mathbb{E}}[G(X^x_{\tau_\varepsilon^x})] = \lim_{n \to \infty} \hat{\mathbb{E}}[G(X^x_{\tau_\varepsilon^x \wedge n})] = G(x)$$

for all $x \in \mathbb{R}$. Recalling (6.5) and the definition of τ_ε^x, we obtain that

$$F(x) \geqslant \hat{\mathbb{E}}[f(X^x_{\tau_\varepsilon^x})] \geqslant \hat{\mathbb{E}}[G(X^x_{\tau_\varepsilon^x})] - \varepsilon = G(x) - \varepsilon.$$

Since ε can be arbitrarily small, we obtain $F \geqslant G$, which completes the proof. □

Remark 6.10 *The results in this subsection extends those in Subsection 1.2 in [24] to the G-expectation framework (Theorem 6.4, Theorem 6.8 and Theorem 6.9 generalizes Theorem 1.7, Theorem 1.11 and Theorem 1.13 in [24], respectively).*

6.2 Continuous-time case

In this section, we investigate the optimal stopping problem in the continuous-time case when the payoff process is Markovian satisfying Equation (6.1). Similar to Definition 6.5, a basic concept is given as follows:

Definition 6.11 *A measurable function $f : \mathbb{R} \to \mathbb{R}$ is called excessive (w.r.t X_t), if*

$$f(x) \geqslant \hat{\mathbb{E}}[f(X^x_t)] \text{ for all } t \geqslant 0, \quad x \in \mathbb{R}.$$

Definition 6.12 *A measurable function $f : \mathbb{R} \to \mathbb{R}$ is called superharmonic (w.r.t X_t), if*

$$f(x) \geqslant \hat{\mathbb{E}}[f(X^x_\tau)]$$

for all G-stopping time τ satisfying Equation (4.7) and all $x \in \mathbb{R}$.

Remark 6.13 *It is worth noting that in the above definitions, there is an implicit assumption that $f(X^x_\tau) \in \mathbb{L}^1(\Omega)$ for each G-stopping time τ and $x \in \mathbb{R}$.*

It is easy to check that a superharmonic function is excessive. The following proposition shows that the converse is true for some typical f.

Proposition 6.14 *Suppose that f is bounded and lower semicontinuous. If f is excessive, then it is also superharmonic.*

Proof We first prove this result for $f \in C_{b,Lip}(\mathbb{R})$. Without loss of generality, we assume that $f \geqslant 0$.

Step 1 Suppose that τ is a discrete G-stopping time of the following form:

$$\tau = \sum_{i=0}^{n} t_i I_{\{\tau = t_i\}}.$$

By a similar analysis as the proof of Theorem 4.2, we have $f(X_\tau^x) \in L_G^{*1}(\Omega_{t_n})$ and

$$\begin{aligned}
\hat{\mathbb{E}}\left[f(X_\tau^x)\right] &= \hat{\mathbb{E}}\left[\sum_{i=0}^{n} f(X_{t_i}^x) I_{\{\tau=t_i\}}\right] \\
&= \hat{\mathbb{E}}\left[\sum_{i=0}^{n-1} f(X_{t_i}^x) I_{\{\tau=t_i\}} + \hat{\mathbb{E}}_{t_{n-1}}\left[f(X_{t_n}^x) I_{\{\tau=t_n\}}\right]\right] \\
&= \hat{\mathbb{E}}\left[\sum_{i=0}^{n-1} f(X_{t_i}^x) I_{\{\tau=t_i\}} + \hat{\mathbb{E}}_{t_{n-1}}\left[f(X_{t_n}^x)\right] I_{\{\tau=t_n\}}\right] \\
&= \hat{\mathbb{E}}\left[\sum_{i=0}^{n-1} f(X_{t_i}^x) I_{\{\tau=t_i\}} + \hat{\mathbb{E}}\left[f(X_{t_n - t_{n-1}}^y)\right]_{y = X_{t_{n-1}}^x} I_{\{\tau=t_n\}}\right] \\
&\leqslant \hat{\mathbb{E}}\left[\sum_{i=0}^{n-2} f(X_{t_i}^x) I_{\{\tau=t_i\}} + f(X_{t_{n-1}}^x) I_{\{\tau \geqslant t_{n-1}\}}\right] \leqslant \cdots \leqslant f(x),
\end{aligned}$$

where we use the Markov property in the fourth equality.

Step 2 If τ is bounded and continuous, there exists a sequence of discrete G-stopping time $\{\tau^n\}_{n=1}^\infty$ such that $|\tau - \tau^n| \leqslant 1/2^n$. By applying the continuity of $f(X^x)$, similar to the proof of Proposition 5.3, we have

$$\hat{\mathbb{E}}[f(X_\tau^x)] = \lim_{n\to\infty} \hat{\mathbb{E}}[f(X_{\tau^n}^x)] \leqslant f(x).$$

Step 3 If τ satisfy Equation (4.7), by a similar analysis as the proof of Proposition 4.7, it follows that

$$\hat{\mathbb{E}}[f(X_\tau^x)] = \lim_{N\to\infty} \hat{\mathbb{E}}[f(X_{\tau\wedge N}^x)] \leqslant f(x).$$

Next, we show that the result still hold for f, which is bounded and lower semicontinuous. We can choose a sequence $\{f_n\}_{n=1}^\infty \subset C_{b,Lip}(\mathbb{R})$ such that $f_n \uparrow f$. By the proof of Theorem 3.11, we obtain that $f(X_\tau^x) \in L_G^{1^*_*}(\Omega_{t_n})$ for each discrete G-stopping time τ with the form in Step 1. Moreover, since $f_n(X_t^x) \uparrow f(X_t^x)$ for any $t \geqslant 0$ and $x \in \mathbb{R}$, we have

$$\hat{\mathbb{E}}_t[f(X_{t+s}^x)] = \lim_{n\to\infty} \hat{\mathbb{E}}_t[f_n(X_{t+s}^x)] = \lim_{n\to\infty} \hat{\mathbb{E}}[f_n(X_s^y)]_{y=X_t^x} = \hat{\mathbb{E}}[f(X_s^y)]_{y=X_t^x}. \tag{6.10}$$

Then, the proof of Step 1 can be extended to the case where f is lower semicontinuous and bounded from below. If τ is continuous and bounded and τ^n is chosen as Step 2, noting that f is lower semicontinuous and applying Fatou's Lemma, it is easy to check that

$$\hat{\mathbb{E}}[f(X_\tau^x)] \leqslant \hat{\mathbb{E}}\left[\liminf_{n\to\infty} f(X_{\tau^n}^x)\right] \leqslant \liminf_{n\to\infty} \hat{\mathbb{E}}[f(X_{\tau^n}^x)] \leqslant f(x).$$

Repeating the proof of Step 3, we finally obtain the desired result. □

Proposition 6.15 *Suppose that f is bounded and lower semicontinuous. Then, f is excessive if and only if $\{f(X_t^x)\}$ is a supermartingale for any $x \in \mathbb{R}$.*

Proof If $\{f(X_t^x)\}$ is a supermartingale, it follows that

$$\hat{\mathbb{E}}[f(X_t^x)] \leqslant f(X_0^x) = f(x),$$

which implies that f is excessive. The other direction can be shown easily by using Equation (6.10). The proof is complete. □

For any given bounded and Lipschitz continuous function g, by the following iterative procedure, we can construct the smallest superharmonic function that dominates g.

Proposition 6.16 *Let g be a bounded and Lipschitz continuous and define the following sequence: $g_0(x) = g(x)$,*

$$g_n(x) = \sup_{t \in S_n} \hat{\mathbb{E}}[g_{n-1}(X_t^x)], \quad n = 1, 2, \cdots,$$

where $S_n = \{k \cdot 2^{-n} : 0 \leqslant k \leqslant 4^n\}$. Then, $g_n \uparrow \bar{g}$ and $\bar{g}$ is the smallest superharmonic function that dominates g.

Proof It is obvious that $\{g_n\}$ is bounded and increasing. Moreover, by Proposition 6.1, we have

$$|g_1(x) - g_1(y)| \leqslant \sup_{t \in S_1} \hat{\mathbb{E}}[|g(X_t^x) - g(X_t^y)|] \leqslant C \sup_{t \in S_1} \hat{\mathbb{E}}[|X_t^x - X_t^y|] \leqslant C|x - y|,$$

where C is the Lipschitz constant for g. By induction, we obtain that g_n is continuous. Define $\bar{g}(x) = \lim_{n \to \infty} g_n(x)$. Then, $\bar{g}$ is bounded and lower semicontinuous. We claim that $\bar{g}$ is excessive. Indeed, we can show that

$$\bar{g}(x) \geqslant g_n(x) = \hat{\mathbb{E}}[g_{n-1}(X_t^x)], \quad \text{for any } t \in S_n, \quad n \geqslant 1.$$

Letting $n \to \infty$, it follows that

$$\bar{g} \geqslant \hat{\mathbb{E}}[\bar{g}(X_t^x)], \quad \text{for any } t \in S = \cup_{n=1}^{\infty} S_n.$$

If $t \notin S$, there exists $\{t_n\}_{n=1}^{\infty} \subset S$ such that $t_n \to t$. By Fatou's Lemma and noting that $\bar{g}$ is lower semicontinuous, we have

$$\hat{\mathbb{E}}[\bar{g}(X_t^x)] \leqslant \hat{\mathbb{E}}[\liminf_{n \to \infty} \bar{g}(X_{t_n}^x)] \leqslant \liminf_{n \to \infty} \hat{\mathbb{E}}[\bar{g}(X_{t_n}^x)] \leqslant \bar{g}(x).$$

The above two inequalities imply $\bar{g}$ is excessive. By Proposition 6.14, $\bar{g}$ is superharmonic. If f is a superharmonic function and $f \geqslant g$, by induction, it is easy to check that $f \geqslant g_n$ for any $n = 1, 2, \cdots$. Letting n tend to infinity, we finally obtain the desired result. □

Theorem 6.17 *Let g be a bounded and Lipschitz function. Define*

$$V(x) = \sup_{\tau \in \mathcal{T}} \hat{\mathbb{E}}[g(X_\tau^x)],$$

where $\mathcal{T}$ is the collection of all G-stopping times satisfying Equation (4.7). Then, V is the smallest superharmonic function that dominates g.

Proof Set $\mathcal{I}_n^\infty = \{0, \frac{1}{2^n}, \cdots, \frac{k}{2^n}, \cdots\}$ for any $n \geqslant 1$ and denote by $\mathcal{T}_n^\infty$ the set of all G-stopping times taking values in $\mathcal{I}_n^\infty$ and satisfying Equation (4.7). Consider the following optimal stopping problem:

$$V^n(x) = \sup_{\tau \in \mathcal{T}_n^\infty} \hat{\mathbb{E}}[g(X_\tau^x)].$$

It is easy to see that $V \geqslant V^n$, for any $n \geqslant 1$. Moreover, for any $\tau \in \mathcal{T}$, there exists a sequence of G-stopping time $\{\tau^n\}$ such that $\tau^n \in \mathcal{T}_n^\infty$ and $\tau^n \to \tau$. Applying Fatou's Lemma, we have

$$\hat{\mathbb{E}}[g(X_\tau^x)] = \hat{\mathbb{E}}[\liminf_{n\to\infty} g(X_{\tau^n}^x)] \leqslant \liminf_{n\to\infty} \hat{\mathbb{E}}[g(X_{\tau^n}^x)] \leqslant \liminf_{n\to\infty} V^n(x),$$

which implies that $V^n \uparrow V$ and V is bounded and lower semicontinuous. By Lemma 6.7 and Theorem 6.8, we know that $\{V^n(X_t^x), t \in \mathcal{I}_n^\infty\}$ is a G-supermartingale for each $x \in \mathbb{R}$. If $t \in \mathcal{I}^\infty = \cup_{n=1}^\infty \mathcal{I}_n^\infty$, without loss of generality, we assume that $t = \frac{k}{2^m}$. By simple calculation, we have

$$\hat{\mathbb{E}}[V(X_t^x)] = \lim_{n\to\infty, n\geqslant m} \hat{\mathbb{E}}[V^n(X_t^x)] \leqslant \lim_{n\to\infty, n\geqslant m} V^n(x) = V(x).$$

If $t \notin \mathcal{I}^\infty$, there exists $\{t_n\} \subset \mathcal{I}^\infty$ such that $t_n \to t$. Noting that V is lower semicontinuous, it follows that

$$\hat{\mathbb{E}}[V(X_t^x)] \leqslant \hat{\mathbb{E}}[\liminf_{n\to\infty} V(X_{t_n}^x)] \leqslant \liminf_{n\to\infty} \hat{\mathbb{E}}[V(X_{t_n}^x)] \leqslant V(x).$$

We obtain that V is an excessive function. Applying Proposition 6.14 shows that V is also superharmonic. For any superharmonic function $f \geqslant g$, it is easy to check that

$$V(x) = \sup_{\tau\in\mathcal{T}} \hat{\mathbb{E}}[g(X_\tau^x)] \leqslant \sup_{\tau\in\mathcal{T}} \hat{\mathbb{E}}[f(X_\tau^x)] \leqslant f(x),$$

which yields that V is the smallest superharmonic function that dominates g. □

Remark 6.18 *Proposition 6.14, Proposition 6.16, and Theorem 6.17 generalize Theorem 10.1.6, Theorem 10.1.7, and Theorem 10.1.9 (a) in [20] to the G-expectation framework.*

Acknowledgements

This work was supported by the Natural Science Foundation of Shandong Province for Excellent Young Scientists Fund Program (Overseas) (Grant No. 2023HWYQ-049), the National Natural Science Foundation of China (Grant No. 12301178), the Natural Science Foundation of Shandong Province (Grant No. ZR2022QA022), and the Fundamental Research Funds for the Central Universities.

References

[1] Bayraktar, E. and Yao, S., Optimal stopping for nonlinear expectations: Part I, Stochastic Processes and Their Applications, 2011, 121: 185–211.

[2] Bayraktar, E. and Yao, S., Optimal stopping for nonlinear expectations: Part II, Stochastic Processes and Their Applications, 2011, 121: 212–264.

[3] Bayraktar, E. and Yao, S., On the robust optimal stopping problem, SIAM Journal on Control and Optimization, 2014, 52(5): 3135–3175.

[4] Bayraktar, E. and Yao, S., Optimal stopping with random maturity under nonlinear expectations, Stochastic Processes and Their Applications, 2017, 127: 2586–2629.

[5] Cheng, X. and Riedel, F., Optimal stopping under ambiguity in continuous-time, Mathematics and Financial Economics, 2013, 7: 29–68.

[6] Denis, L., Hu, M. and Peng, S., Function spaces and capacity related to a sublinear expectation: Application to *G*-Brownian motion paths, Potential Analysis, 2011, 34: 139–161.

[7] Ekren, I., Touzi, N. and Zhang, J., Optimal stopping under nonlinear expectation, Stochastic Processes and Their Applications, 2014, 124: 3277–3311.

[8] Grigorova, M., Imkeller, P., Ouknine, Y. and Quenez, M. C., Optimal stopping with *f*-expectation: The irregular

case, Stochastic Processes and Their Applications, 2020, 130(3): 1258–1288.

[9] Hu, M., Ji, X. and Liu, G., On the strong Markov property for stochastic differential equations driven by G-Brownian motion, Stochastic Processes and Their Applications, 2021, 131: 417–453.

[10] Hu, M., Ji, S., Peng, S. and Song, Y., Backward stochastic differential equations driven by G-Brownian motion, Stochastic Processes and Their Applications, 2014, 124: 759–784.

[11] Hu, M., Ji, S., Peng, S. and Song, Y., Comparison theorem, Feynman--Kac formula and Girsanov transformation for BSDEs driven by G-Brownian motion, Stochastic Processes and Their Applications, 2014, 124: 1170–1195.

[12] Hu, M. and Peng, S., Extended conditional G-expectations and related stopping times, Probability, Uncertainty and Quantitative Risk, 2021, 64(4): 369–390.

[13] Huang, Y. and Yu, X., Optimal stopping under model uncertainty ambiguity: A time-consistent equilibrium approach, Mathematical Finance, 2021, 31: 979–1012.

[14] Li, H., Reflected backward stochastic differential equations under nonlinear expectations and their applications, PhD diss., Shandong University, 2018.

[15] Li, H., Peng, S. and Song, Y., Supermartingale decomposition theorem under G-expectation, Electronic Journal of Probability, 2018, 23: 1–20.

[16] Li, H., Peng, S. and Soumana Hima, A., Reflected solutions of backward stochastic differential equations driven by G-Brownian motion, Science China Mathematics, 2018, 61: 1–26.

[17] Li, H. and Song, Y., Backward stochastic differential equations driven by G-Brownian motion with double reflections, Journal of Theoretical Probability, 2021, 34: 2285–2314.

[18] Matoussi, A., Possamai, D. and Zhou, C., Second order backward stochastic differential equations, Annals of Applied Probability, 2013, 23(6): 2420–2457.

[19] Nutz, M. and Zhang, J., Optimal stopping under adverse nonlinear expectation and related games, Annals of Applied Probability, 2015, 25(5): 2503–2534.

[20] Øksendal, B., Stochastic differential equations: An introduction with applications, 6th edn., Springer, Berlin, 2003.

[21] Peng, S., G-expectation, G-Brownian motion and related stochastic calculus of Itô type, In: Benth, F. E., Di Nunno, G., Lindstrøm, T., Øksendal, B. and Zhang, T.(eds.), Stochastic Analysis and Applications, Abel Symposia 2, Springer, Berlin, 2007: 541–567.

[22] Peng, S., Multi-dimensional G-Brownian motion and related stochastic calculus under G-expectation, Stochastic Processes and Their Applications, 2008, 118(12): 2223–2253.

[23] Peng, S., Nonlinear Expectations and Stochastic Calculus under Uncertainty, Springer, Berlin, 2019.

[24] Peskir, G. and Shiryaev, A., Optimal Stopping and Free-Boundary Problems, Birkhäuser, Basel, 2006.

[25] Riedel, F., Optimal stopping with multiple priors, Econometrica, 2009, 77: 857–908.

[26] Song, Y., Some properties on G-evaluation and its applications to G-martingale decomposition, Science China Mathematics, 2011, 54: 287–300.