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Mean-field stochastic differential equations driven by G-Brownian motion

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Abstract This paper introduces a novel formulation of mean-field stochastic differential equations driven by G-Brownian motion. The proposed formulation generalises existing approaches within the G-framework and enables the study of Fréchet differentiability. Under non-Lipschitz conditions on the coefficients, we establish the existence and uniqueness of a solution for square-integrable stochastic initial data.

Keywords Mean-field stochastic differential equations, G-Brownian motion, G-framework, Fréchet differentiability, Existence and uniqueness of a solution

2020 Mathematics Subject Classification 60H10, 60G65

1. Introduction

A mean-field stochastic differential equation (SDE) driven by Brownian motion is an equation of the form

$$\mathrm{d}X_t = b(X_t, P_{X_t})\mathrm{d}t + \sigma(X_t, P_{X_t})\mathrm{d}B_t,$$

where B denotes Brownian motion, and $P_{X_t} = P \circ (X_t)^{-1}$ denotes the push-forward measure of X_t for $t \geqslant 0$. Mean-field theory can be traced back to the pioneering work of McKean [17] and Vlasov [29], who studied the mean-field limit for interacting particle systems, revealing the connection between microscopic and macroscopic descriptions. The underlying concepts, deeply rooted in statistical physics, have been applied across various disciplines, including physics, economics, and biology, offering fundamental insights into macroscopic system behavior. Mean-field models are widely used to study phenomena such as animal and cell population dynamics, infectious disease spread, systemic risk, and fluid mechanics. In 2006, Lasry and Lions brought a game-theoretic perspective to mean-field interactions and introduced so called mean-field games, cf. [11–13]. The theory of mean-field games has provided a foundation for analyzing dynamic interactions among weakly dependent rational agents in strategic decision-making scenarios. This framework offers valuable insights into strategic behavior in large populations and serves as a powerful tool for studying systems with decentralised decision-making. Further contributions,

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such as those in [6, 7], have extended the scope of mean-field games. Over the past few decades, research in mean-field and mean-field game theory has flourished, as evidenced by the substantial volume of literature on these topics.

Another rapidly growing area of research is Knightian or model uncertainty. Model uncertainty arises in situations where the underlying probability measure is unknown, meaning that any conclusions drawn from such models are inherently affected by this uncertainty. As a result, the need for robust modeling approaches become essential. However, classical probability theory is not inherently designed to accommodate such uncertainty. In the last decades, various approaches to quantify uncertainty have been developed. Notably, significant contributions have emerged in the literature on robust finance, e.g., [14, 16, 19, 20, 25], as well as in the study of imprecise probabilities, e.g., [1, 2, 26]. This study aims to unify mean-field theory and uncertainty quantification and to contribute to the development of a mean-field theory under model uncertainty.

One main approach to study stochastic processes under model uncertainty is to consider the so called G-Brownian motion as underlying driving process. Peng introduced G-Brownian motion in his seminal works on sublinear expectation spaces in the early 2000s, cf. [22–24]. They replaced the classical probability space $(\Omega, \mathcal{F}, P)$ by a sublinear expectation space $(\Omega, \mathcal{F}, \hat{\mathbb{E}})$. In this setting, G-Brownian motion can be regarded as Brownian motion with volatility uncertainty.

A second and more recent approach to stochastic processes under uncertainty is to consider a possibly uncountable set of probability measures $\mathcal{P}$ such that the canonical process X has the desired properties. This is achieved by specifying a set $\Theta(t, \omega)$ of possible local semimartingale characteristics, i.e., each $P \in \mathcal{P}$ is such that

$$\left(b_t^P(\omega), a_t^P(\omega), k_t^P(\omega)\right) \in \Theta(t, \omega) \qquad \mathrm{d}t \otimes \mathrm{d}P\text{-a.e.}, \tag{1.1}$$

where (b^P, a^P, k^P) denote the local semimartingale characteristics of the canonical process under P. In particular, the case where $\mathcal{P}$ is non-dominated and there is no preference among the measures in $\mathcal{P}$ is of great interest; however, it also presents major technical challenges. Within this framework, G-Brownian motion can be associated with a certain non-empty, convex set of probability measures $\mathcal{P}$, representing uncertainty about the volatility. Additionally, this approach allows for more general uncertainty modeling, such as Lévy processes ([18]) and affine processes ([3, 10]) under uncertainty.

For mean-field type processes, the local semimartingale characteristics depend not only on the current state of the process but also on its distribution. Thus, the set of possible characteristics Θ in (1.1) would depend not only on (t, ω) but also on P. This dependence makes the uncertainty set approach significantly more complex for mean-filed processes than for Lévy or affine models. In particular, as part of ongoing research project, we are investigating whether such a set Θ could satisfy Assumption 2.1 in [21] that implies the crucial dynamic programming principle.

In this paper, we introduce mean-field SDEs under volatility uncertainty. By considering G-Brownian motion as the underlying driving process, we leverage the comparatively rich mathematical structure developed within the G-framework. A foundational step toward defining mean-field SDEs in the G-setting was made in [28]. They considered dynamics of the form

$$\mathrm{d}X_s = \hat{\mathbb{E}}[b(s, x, X_s)]\Big|_{x=X_s} \mathrm{d}s + \hat{\mathbb{E}}[h(s, x, X_s)]\Big|_{x=X_s} \mathrm{d}\langle B\rangle_s + \hat{\mathbb{E}}[g(s, x, X_s)]\Big|_{x=X_s} \mathrm{d}B_s, \tag{1.2}$$

where B denotes the G-Brownian motion, and $\hat{\mathbb{E}}$ denotes the corresponding G-expectation.

They proved the existence of a unique solution of (1.2) under a Lipschitz condition on the coefficients. In their recent work [27], they generalised these results to Lipschitz coefficients that depend on the sublinear distribution of the solution. In other words, they studied dynamics of the form

$$\mathrm{d}X_s = b(s,x,F_{X_s})\Big|_{x=X_s}\mathrm{d}s + h(s,x,F_{X_s})\Big|_{x=X_s}\mathrm{d}\langle B\rangle_s + g(s,x,F_{X_s})\Big|_{x=X_s}\mathrm{d}B_s, \tag{1.3}$$

where, for a random variable ξ, the functional F_ξ is given by $F_\xi(\varphi) := \hat{\mathbb{E}}[\varphi(\xi)]$, where $\varphi : \mathbb{R}^d \to \mathbb{R}$ lies in a suitable function space. This corresponds to the sublinear distribution of ξ, cf. [24]. For this purpose, they constructed a space containing all sublinear distribution functions endowed with a metric, which can be regarded as a generalisation of the Kantorovich–Rubinstein metric.

In this paper, we introduce a different, more general formulation of mean-field G-SDEs that has several advantages as discussed below. Instead of letting the coefficients depend on the sublinear distribution F_{X_s}, we allow the coefficients to depend directly on the random variable X_s. To be specific, we consider the generalised G-setting introduced in Chapter 8 in [24] and consider G-SDEs of the form

$$\begin{aligned}\mathrm{d}X_s(\omega) = {}& b(s,x,X_s,\omega)\Big|_{x=X_s(\omega)}\mathrm{d}s + h(s,x,X_s,\omega)\Big|_{x=X_s(\omega)}\mathrm{d}\langle B\rangle_s(\omega)\\ &+ g(s,x,X_s,\omega)\Big|_{x=X_s(\omega)}\mathrm{d}B_s(\omega),\end{aligned} \tag{1.4}$$

where the coefficients are defined on $[0,T]\times\mathbb{R}^d\times \mathrm{L}^{2,d}_*\times\Omega$. Here, $\mathrm{L}^{2,d}_*$ denotes the space of d-dimensional random vectors with finite second moment, cf. Section 2.

For simplicity and conciseness, we use the following notation.

Notation 1.1 *For a function f on $[0,T]\times\mathbb{R}^d\times\mathrm{L}^{2,d}_*\times\Omega$ and random vector $\eta\in\mathrm{L}^{1,d}_*$, define*

$$f(s,\eta,\xi,\omega) := f(s,\eta(\omega),\xi,\omega) = f(s,x,\xi,\omega)\Big|_{x=\eta(\omega)}$$

for any $0\leqslant s\leqslant T$, $\xi\in\mathrm{L}^{2,d}_*$ and $\omega\in\Omega$. Often, we suppress the explicit dependence on ω and write $f(s,\eta,\xi)$ instead of $f(s,\eta,\xi,\omega)$.

Hence, the G-SDE (1.4) can be written as

$$\mathrm{d}X_s = b(s,X_s,X_s)\,\mathrm{d}s + h(s,X_s,X_s)\,\mathrm{d}\langle B\rangle_s + g(s,X_s,X_s)\,\mathrm{d}B_s. \tag{1.5}$$

Our approach, which allows the coefficients to depend on the random variable X_s rather than its distribution, is inspired by [5, 7, 15]. In these studies, the authors lifted functions on probability measures to functions on random variables which allowed the definition of the Lions derivative via the Fréchet derivative of the lifted function. Analogously, our formulation enables the study of the Fréchet differentiability of solutions to (1.5). This will be used in the companion paper [4] to associate (1.5) with a nonlocal nonlinear partial differential equation (PDE). This Feynman-Kac type result allows the computation of functionals of the solution of (1.5) by solving the associated PDE. Our formulation also enables the introduction of a corresponding finite interacting particle system such that mean-field G-SDE (1.5) can be regarded as the asymptotic limit of this system. This result, which is work in progress, corresponds to the propagation of chaos in classical mean-field theory, see also [8] for a set valued approach.

The main contribution of this paper is to study existence and uniqueness of solutions of the mean-field G-SDE (1.5). We remark that coefficients of the form in (1.2) and (1.3) are special cases of the coefficients of the form (1.5). In contrast to [28] and [27], we allow the coefficients to be non-deterministic and non-Lipschitz and consider square integrable instead of deterministic initial

conditions. More precisely, we mainly require the coefficients to satisfy an Osgood type continuity and sublinear growth condition, cf. Assumption 3.1. Under these assumptions, we derive the existence and uniqueness result for (1.5) as formalised in Theorem 3.12. To be specific, for a given square integrable initial condition $X_t = \xi \in \mathrm{L}_*^{2,d}(t)$, $0 \leqslant t \leqslant T$, we construct the solution of (1.4) as the limit of a Picard sequence. Furthermore, we show that this solution is unique and satisfies

$$\hat{\mathbb{E}}\left[\sup_{t \leqslant s \leqslant T} \|X_s\|^2\right] < \infty.$$

This is consistent with the results for classical mean-field SDEs. To the best of our knowledge, this study is the first results for G-SDEs under weaker assumptions on the coefficients. In particular, since coefficients, such as in [27, 28], are a special case of the coefficients considered in this paper, the existence and uniqueness results in [27, 28] follow immediately from Theorem 3.12. The comparison of our results with the existing studies will be illustrated in more detail in Section 4.

Similar to most studies on G-SDEs, [27, 28] work in the G-setting as constructed in [23], which is restricted to quasi-continuous random variables, cf. [9]. In this study, we consider the generalised G-setting from Chapter 8 in [24] and extend many known results from the quasi-continuous to this more general setting. For the sake of completeness, we include the proofs of these results in Appendix B.

Our paper is structured as follows. In Section 2, we recall the sublinear expectation setting from [24]. In Section 3, we introduce our mean-field G-SDE and prove existence and uniqueness of its solution. Finally, we compare our results with existing results in Section 4.

2. The G-setting

In this section, we recall the sublinear expectation setting from Chapter 8 in [24]. This setting builds on the results in quasi-sure analysis related to the G-setting, cf. [9]. Unless stated otherwise, the proofs can be found in Appendix B.

For $n \geqslant 1$, let $\Omega := \mathrm{C}_0(\mathbb{R}_+, \mathbb{R}^n)$ be the space of all continuous $\mathbb{R}^n$-valued paths starting at the origin equipped with the topology of uniform convergence. Let $\mathcal{F}$ denote the corresponding Borel σ-algebra. Moreover, let $\mathbb{F} = (\mathcal{F}_t)_{t \geqslant 0}$ denote the natural filtration generated by the coordinate mapping process B. For $t \geqslant 0$, let $\mathrm{B}_b(\Omega_t)$ denote be the space of all bounded $\mathcal{F}_t$-measurable functions $\xi : \Omega \to \mathbb{R}$.

Fix a convex and closed subset $\Sigma \subseteq \mathbb{S}_+^n$ of symmetric nonnegative definite $n \times n$-matrices and set

$$\mathcal{A}^\Sigma := \left\{\vartheta = (\vartheta_t)_{t \geqslant 0} \,:\, \vartheta \text{ is } \Sigma\text{-valued and } \mathbb{F}\text{-progressively measurable}\right\}.$$

Let P_0 denote the Wiener measure on $(\Omega, \mathcal{F})$, and define

$$\mathcal{P} := \left\{P_0 \circ (\vartheta \bullet B)^{-1} \,:\, \vartheta \in \mathcal{A}^\Sigma\right\},$$

where $\vartheta \bullet B := \int_0^\cdot \vartheta_s \mathrm{d}B_s$ denotes the itô integral with respect to the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, P_0)$. The set of probability measures $\mathcal{P}$ induces an upper expectation on $\mathrm{B}_b(\Omega)$, namely

$$\hat{\mathbb{E}} : \mathrm{B}_b(\Omega) \to \mathbb{R}, \quad \xi \mapsto \hat{\mathbb{E}}[\xi] := \sup_{P \in \mathcal{P}} E_P[\xi],$$

where E_P denotes the linear expectation with respect to P. The process B is a G-Brownian motion with respect to $\hat{\mathbb{E}}$ and $(\Omega, \mathrm{B}_b(\Omega), \hat{\mathbb{E}})$ is a sublinear expectation space.

For $p \geqslant 1$, define the norm

$$\|\cdot\|_{\mathrm{L}_*^p} : \mathrm{B}_b(\Omega) \to \mathbb{R}_+, \quad \xi \mapsto \|\xi\|_{\mathrm{L}_*^p} := \hat{\mathbb{E}}[|\xi|^p]^{\frac{1}{p}}.$$

For any $t \geqslant 0$, let L_*^p and $\mathrm{L}_*^p(t)$ denote the completion of $\mathrm{B}_b(\Omega)$ and $\mathrm{B}_b(\Omega_t)$ with respect to $\|\cdot\|_{\mathrm{L}_*^p}$ respectively.

Let $\mathrm{M}_b(0,T)$ denote the space of all processes $X : [0,T] \times \Omega \to \mathbb{R}$ of the form

$$X_s(\omega) = \sum_{k=0}^{m-1} \xi_k(\omega) \mathbf{1}_{[t_k,t_{k+1})}(s), \quad (s,\omega) \in [0,T] \times \Omega,$$

with $m \in \mathbb{N}$, $0 = t_0 < \ldots < t_m = T$, and $\xi_k \in \mathrm{B}_b(\Omega_{t_k})$ for all $0 \leqslant k \leqslant m-1$. For $p \geqslant 1$, define the norm

$$\|\cdot\|_{\mathrm{M}_*^p} : \mathrm{M}_b(0,T) \to \mathbb{R}_+, \quad \|X\|_{\mathrm{M}_*^p} := \left(\int_0^T \hat{\mathbb{E}}[|X_s|^p] \, \mathrm{d}s \right)^{\frac{1}{p}},$$

and let $\mathrm{M}_*^p(0,T)$ denote the completion of $\mathrm{M}_b(0,T)$ with respect to $\|\cdot\|_{\mathrm{M}_*^p}$.

Remark 2.1 *Sometimes, it makes sense to consider the norm*

$$\|\cdot\|_{\overline{\mathrm{M}}_*^p} : \mathrm{M}_b(0,T) \to \mathbb{R}_+, \quad \|X\|_{\overline{\mathrm{M}}_*^p} := \hat{\mathbb{E}} \left[\int_0^T |X_s|^p \, \mathrm{d}s \right]^{\frac{1}{p}},$$

which is weaker than $\|\cdot\|_{\mathrm{M}_*^p}$*, and thus,* $\mathrm{M}_*^p(0,T) \subseteq \overline{\mathrm{M}}_*^p(0,T)$*, where* $\overline{\mathrm{M}}_*^p(0,T)$ *denotes the completion of* $\mathrm{M}_b(0,T)$ *with respect to* $\|\cdot\|_{\overline{\mathrm{M}}_*^p}$*. For the study of* G-SDEs*, the spaces* $\mathrm{M}_*^p(0,T)$ *are more convenient and, thus, we focus on these in this paper.*

Lemma 2.2 *Let* $p \geqslant 1$ *,* $N \in \mathbb{N}$ *,* $0 = t_0 < \ldots < t_N = T$ *, and* $\xi_k \in \mathrm{L}_*^p(t_k)$*,* $0 \leqslant k \leqslant N-1$*. Then*

$$X := \sum_{k=0}^{N-1} \xi_k \mathbf{1}_{[t_k,t_{k+1})} \in \mathrm{M}_*^p(0,T).$$

Lemma 2.3 *Let* $a \in \mathbb{R}^n$*. Then,* $B^a := a^T B \in \mathrm{M}_*^2(0,T)$*.*

For $a \in \mathbb{R}^n$, define the map $\mathcal{I}_a : \mathrm{M}_b(0,T) \to \mathrm{L}_*^2(T)$ by

$$\mathcal{I}_a(X) := \int_0^T X_t \mathrm{d}B_t^a := \sum_{k=0}^{m-1} \xi_k \left(B_{t_{k+1}}^a - B_{t_k}^a \right)$$

for each

$$X = \sum_{k=0}^{m-1} \xi_k \mathbf{1}_{[t_k,t_{k+1})} \in \mathrm{M}_b(0,T).$$

Lemma 2.4 *For* $a \in \mathbb{R}^n$ *, the map* $\mathcal{I}_a : \mathrm{M}_b(0,T) \to \mathrm{L}_*^2(T)$ *is continuous and linearand, thus, it can be continuously extended to* $\mathcal{I}_a : \mathrm{M}_*^2(0,T) \to \mathrm{L}_*^2(T)$*. Moreover, we have for all* $X \in \mathrm{M}_*^2(0,T)$ *that*

$$\hat{\mathbb{E}}\left[I_a(X)\right] = \hat{\mathbb{E}} \left[\int_0^T X_t \mathrm{d}B_t^a \right] = 0,$$

$$\|I_a(X)\|_{\mathrm{L}_*^2}^2 = \hat{\mathbb{E}} \left[\left(\int_0^T X_t \mathrm{d}B_t^a \right)^2 \right] \leqslant \bar{\sigma}_{aa}^2 \hat{\mathbb{E}} \left[\int_0^T |X_t|^2 \mathrm{d}t \right] \leqslant \bar{\sigma}_{aa}^2 \|X\|_{\mathrm{M}_*^2}^2,$$

where

$$\overline{\sigma}_{aa} := \sup_{\sigma\in\Sigma} \sqrt{a^T\sigma\sigma a}.$$

Proof Cf. [24, Lemma 8.1.6]. □

For $0 \leqslant t \leqslant s \leqslant T$ and $X \in \mathrm{M}^2_*(0,T)$, set

$$\int_t^s X_u \mathrm{d}B^a_u := \int_0^T X_u \mathbf{1}_{[t,s)}(u)\mathrm{d}B^a_u = \mathcal{I}_a(X\mathbf{1}_{[t,s)}).$$

By Lemma 2.3, we define the quadratic variation of B^a as process $\langle B^a\rangle$ given by

$$\langle B^a\rangle_t := (B^a_t)^2 - 2\int_0^t B^a_s \mathrm{d}B^a_s \in \mathrm{L}^1_*(t), \quad 0 \leqslant t \leqslant T.$$

Now, define the map $\mathcal{Q}_a : \mathrm{M}_b(0,T) \to \mathrm{L}^1_*(T)$ by

$$\mathcal{Q}_a(X) = \int_0^T X_t \mathrm{d}\langle B^a\rangle_t := \sum_{k=0}^{m-1} \xi_k \left(\langle B^a\rangle_{t_{k+1}} - \langle B^a\rangle_{t_k}\right)$$

for each

$$X = \sum_{k=0}^{m-1} \xi_k \mathbf{1}_{[t_k,t_{k+1})} \in \mathrm{M}_b(0,T).$$

Lemma 2.5 *For $a \in \mathbb{R}^n$, the map $\mathcal{Q}_a : \mathrm{M}_b(0,T) \to \mathrm{L}^2_*(T)$ is continuous and linearand, thus, it can be continuously extended to $\mathcal{Q}_a : \mathrm{M}^1_*(0,T) \to \mathrm{L}^2_*(T)$. Moreover, we have for all $X \in \mathrm{M}^1_*(0,T)$ that*

$$\|\mathcal{Q}_a(X)\|_{\mathrm{L}^1_*} = \hat{\mathbb{E}}\left[\left|\int_0^T X_t \mathrm{d}\langle B^a\rangle_t\right|\right] \leqslant \overline{\sigma}^2_{aa}\hat{\mathbb{E}}\left[\int_0^T |X_t|\,\mathrm{d}t\right] \leqslant \overline{\sigma}^2_{aa}\|X\|_{\mathrm{M}^1_*}.$$

Proof Cf., [24, Proposition 8.1.10]. □

For $0 \leqslant t \leqslant s \leqslant T$ and $X \in \mathrm{M}^1_*(0,T)$, set

$$\int_t^s X_u \mathrm{d}\langle B^a\rangle_u := \int_0^T X_u \mathbf{1}_{[t,s)}(u)\mathrm{d}\langle B^a\rangle_u = \mathcal{Q}_a(X\mathbf{1}_{[t,s)}).$$

Lemma 2.6 *Let $a \in \mathbb{R}^n$ and $X \in \mathrm{M}^2_*(0,T)$. Define Z by*

$$Z_s := \int_0^s X_u \mathrm{d}B^a_u = \mathcal{I}_a\left(X\mathbf{1}_{[0,s)}\right), \quad 0 \leqslant s \leqslant T.$$

Then $Z \in \mathrm{M}^2_*(0,T)$.

Lemma 2.7 *For $a \in \mathbb{R}^n$, we have $\langle B^a\rangle \in \mathrm{M}^1_*(0,T)$.*

Proof Since $B^a \in \mathrm{M}^2_*(0,T)$, we have $(B^a)^2 \in \mathrm{M}^1_*(0,T)$ and $\int_0^\cdot B^a_s \mathrm{d}B^a_s \in \mathrm{M}^2_*(0,T) \subseteq \mathrm{M}^1_*(0,T)$ due to Lemma 2.6. Hence, $\langle B^a\rangle \in \mathrm{M}^1_*(0,T)$ as sum of elements in $\mathrm{M}^1_*(0,T)$. □

Analogous to Lemma 2.6, we have the following result.

Lemma 2.8 *Let $a \in \mathbb{R}^n$ and $X \in \mathrm{M}^1_*(0,T)$. Define Z by*

$$Z_s := \int_0^s X_u \mathrm{d}\langle B^a\rangle_u = \mathcal{Q}_a\left(X\mathbf{1}_{[0,s)}\right), \quad 0 \leqslant s \leqslant T.$$

Then, $Z \in \mathrm{M}^1_*(0,T)$.

Lemma 2.9 *Let $p \geqslant 1$, then $\mathrm{id}_{[0,T]} \in \mathrm{M}_*^p(0,T)$ defined by*

$$\mathrm{id}_{[0,T]} : [0,T] \times \Omega \to \mathbb{R}, \quad (t,\omega) \mapsto t.$$

Proof For $m > 0$, set $t_k^m := \frac{kT}{m}$ and define

$$f^m := \sum_{k=0}^{m-1} t_k \mathbf{1}_{[t_k,t_{k+1})} \in \mathrm{M}_b(0,T).$$

Then

$$\int_0^T E\left[|f^m(s) - s|^p\right] \mathrm{d}s = \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} |t_k - s|^p \,\mathrm{d}s \leqslant \sum_{k=0}^{m-1} \sup_{t_k \leqslant s \leqslant t_{k+1}} |t_k - s|^p \frac{T}{m} = \frac{T^{p+1}}{m^p}.$$

The right-hand side vanishes when $m \to \infty$. Hence, we deduce that $\mathrm{id}_{[0,T]} \in \mathrm{M}_*^p(0,T)$. □

Lemma 2.10 *Let $X \in \mathrm{M}_*^1(0,T)$, and define Z by*

$$Z_s := \int_0^s X_u \mathrm{d}u, \quad 0 \leqslant s \leqslant T.$$

Then $Z \in \mathrm{M}_*^1(0,T)$.

For $a, b \in \mathbb{R}^n$, we define the quadratic covariation of B^a and B^b by

$$\left\langle B^a, B^b \right\rangle := \frac{1}{4}\left(\left\langle B^{a+b}\right\rangle - \left\langle B^{a-b}\right\rangle\right),$$

and, for $X \in \mathrm{M}_*^1(0,T)$, we define

$$\int_t^s X_u \mathrm{d}\left\langle B^a, B^b\right\rangle_u := \frac{1}{4}\left(\mathcal{Q}_{a+b}(X\mathbf{1}_{[t,s)}) - \mathcal{Q}_{a-b}(X\mathbf{1}_{[t,s)})\right).$$

Lemma 2.11 *Let $a, b \in \mathbb{R}^n$, $p \geqslant 1$, $X \in \mathrm{M}_*^p(0,T)$ and $0 \leqslant t \leqslant s \leqslant T$. Then*

$$\hat{\mathbb{E}}\left[\sup_{t \leqslant w \leqslant s}\left|\int_t^w X_u \mathrm{d}\left\langle B^a, B^b\right\rangle_u\right|^p\right] \leqslant \overline{\sigma}_{ab}^{2p}\,(s-t)^{p-1} \int_t^s \hat{\mathbb{E}}\left[|X_u|^p\right] \mathrm{d}u,$$

where

$$\overline{\sigma}_{ab} := \sup_{\sigma \in \Sigma} \sqrt{|a\sigma\sigma b|}.$$

Lemma 2.12 *Let $a \in \mathbb{R}^n$, $p \geqslant 2$, $X \in \mathrm{M}_*^p(0,T)$ and $0 \leqslant t \leqslant s \leqslant T$. Then,*

$$\hat{\mathbb{E}}\left[\sup_{t \leqslant w \leqslant s}\left|\int_t^w X_u \mathrm{d}B_u^a\right|^p\right] \leqslant C_p\,\overline{\sigma}_{aa}^p\,(s-t)^{\frac{p-2}{2}} \int_t^s \hat{\mathbb{E}}\left[|X_u|^p\right] \mathrm{d}u,$$

where $C_p > 0$ is the constant from the Burkholder-Davis-Gundy inequality.

3. Existence and uniqueness results

In this section, we establish existence and uniqueness results for mean-field G-SDEs as introduced in Section 1. To be precise, we fix a finite time horizon $0 < T < \infty$ and a dimension $d \geqslant 1$, and consider coefficients which are defined on $[0,T] \times \mathbb{R}^d \times \mathrm{L}_*^{2,d} \times \Omega$, where $\mathrm{L}_*^{2,d}$ denotes the space of d-dimensional random vectors with components in L_*^2. That is, $\mathrm{L}_*^{2,d} := (\mathrm{L}_*^2)^d$ and $\mathrm{L}_*^{2,d}(t) := (\mathrm{L}_*^2(t))^d$ for $0 \leqslant t \leqslant T$.

We fix an initial time $0 \leqslant t \leqslant T$, and consider the G-SDE

$$\mathrm{d}X_s = b(s, X_s, X_s)\,\mathrm{d}s + h(s, X_s, X_s)\,\mathrm{d}\left\langle B\right\rangle_s + g(s, X_s, X_s)\,\mathrm{d}B_s, \quad t \leqslant s \leqslant T, \tag{3.1}$$

where $b\colon [0,T]\times\mathbb{R}^d\times \mathrm{L}^{2,d}_*\times\Omega\to\mathbb{R}^d$, $h\colon [0,T]\times\mathbb{R}^d\times \mathrm{L}^{2,d}_*\times\Omega\to\mathbb{R}^{d\times n\times n}$ and $g\colon [0,T]\times\mathbb{R}^d\times \mathrm{L}^{2,d}_*\times\Omega\to\mathbb{R}^{d\times n}$ are given.

Let $\mathrm{M}^{p,d}_*(t,T)$ denote the space of all d-dimensional processes $X\colon [t,T]\times\Omega\to\mathbb{R}^d$ such that $X=(X^1,\dots,X^d)^T$ and $X^k\mathbf{1}_{[t,T)}\in \mathrm{M}^p_*(0,T)$ for $1\leqslant k\leqslant d$. A solution of (3.1) is a process $X\in \mathrm{M}^{2,d}_*(t,T)$ such that its components X^k, $1\leqslant k\leqslant d$ satisfy

$$\mathrm{d}X^k_s = b_k(s,X_s,X_s)\,\mathrm{d}s+\sum_{i,j=1}^n h_{kij}(s,X_s,X_s)\,\mathrm{d}\left\langle B^i,B^j\right\rangle_s+\sum_{i=1}^n g_{ki}(s,X_s,X_s)\,\mathrm{d}B_s,\quad t\leqslant s\leqslant T,$$

where b_k,h_{kij},g_{ki}, $1\leqslant i,j\leqslant n$, $1\leqslant k\leqslant d$ denote the components of the coefficients.

Assumption 3.1 *The coefficients $b\colon [0,T]\times\mathbb{R}^d\times \mathrm{L}^{2,d}_*\times\Omega\to\mathbb{R}^d$, $h\colon [0,T]\times\mathbb{R}^d\times \mathrm{L}^{2,d}_*\times\Omega\to\mathbb{R}^{d\times n\times n}$, and $g\colon [0,T]\times\mathbb{R}^d\times \mathrm{L}^{2,d}_*\times\Omega\to\mathbb{R}^{d\times n}$ are such that the following holds for all components $f=b_k,h_{kij},g_{ki}$, $1\leqslant i,j\leqslant n$, $1\leqslant k\leqslant d$.*

(1) *$f(\cdot,x,\xi)\mathbf{1}_{[s,T]}\in \mathrm{M}^1_*(0,T)$ for all $x\in\mathbb{R}^d$, $\xi\in \mathrm{B}_b(\Omega_s)^d$ and $t\leqslant s\leqslant T$.*

(2) *There exist an integrable function $\kappa\colon [0,T]\to\mathbb{R}_+$, a process $K\in \mathrm{M}^1_*(0,T)$, and continuous, increasing and concave functions $\rho_1,\rho_2\colon \mathbb{R}_+\to\mathbb{R}_+$ with $\rho_1(0)=\rho_2(0)=0$ and*

$$\int_0^1 \frac{1}{\rho_1(r)+\rho_2(r)}\mathrm{d}r=+\infty, \tag{3.2}$$

such that

$$|f(s,x,\xi,\omega)-f(s,y,\eta,\omega)|^2\leqslant \kappa(s)\,\rho_1\Big(\|x-y\|^2\Big)+K_s(\omega)\,\rho_2\Big(\|\xi-\eta\|^2_{\mathrm{L}^2_*}\Big), \tag{3.3}$$

$$|f(s,x,\xi,\omega)|^2\leqslant \kappa(s)\,\|x\|^2+K_s(\omega)\left(1+\|\xi\|^2_{\mathrm{L}^2_*}\right) \tag{3.4}$$

for all $\omega\in\Omega$, $t\leqslant s\leqslant T$, $x,y\in\mathbb{R}^d$, and $\xi,\eta\in \mathrm{L}^{2,d}_*(T)$.

For convenience, define the constant $C:=\int_0^T\kappa(s)\mathrm{d}s+\|K\|_{\mathrm{M}^1_*}$.

Remark 3.2 *In this paper, we consider the generalised G-setting from Section 8 in [24]. In particular, the processes in $\mathrm{M}^1_*(0,T)$ need not be quasi-continuous in the sense of [9]. However, if the initial solution and the coefficients are quasi-continuous, then the solution process of (3.1) is also quasi-continuous. For instance, if Assumption 3.1 holds, the stochastic initial condition ξ in (3.1) is quasi-continuous, and the coefficients $f=b_k,h_{kij},g_{ki}$, $1\leqslant i,j\leqslant n$, $1\leqslant k\leqslant d$ are such that $f(\cdot,x,\xi)\mathbf{1}_{[s,T]}$ is quasi-continuous for all $x\in\mathbb{R}^d$, $\xi\in \mathrm{B}_b(\Omega_s)$ and $t\leqslant s\leqslant T$, then the solution of (3.1) is quasi-continuous.*

Lemma 3.3 *Let $X,Y\in \mathrm{M}_b(0,T)^d$. If Assumption 3.1 is satisfied, then $f(\cdot,X,Y)\in \mathrm{M}^2_*(0,T)$ for all components $f=b_k,h_{kij},g_{ki}$, $1\leqslant i,j\leqslant n$, $1\leqslant k\leqslant d$.*

Proof Since $X,Y\in \mathrm{M}_b(0,T)$, there exist $N\in\mathbb{N}$, $0=t_0<\ldots<t_N=T$ and $\xi_k,\eta_k\in \mathrm{B}_b(\Omega_{t_k})$ for $0\leqslant k\leqslant N-1$ such that

$$X=\sum_{k=0}^{N-1}\xi_k\mathbf{1}_{[t_k,t_{k+1})},\quad Y=\sum_{k=0}^{N-1}\eta_k\mathbf{1}_{[t_k,t_{k+1})}.$$

Then we have

$$f(\cdot,X,Y)=\sum_{k=0}^{N-1}f(\cdot,\xi_k,\eta_k)\mathbf{1}_{[t_k,t_{k+1})}.$$

For $0 \leqslant k \leqslant N-1$, define $f_k := f(\cdot, \eta_k)\mathbf{1}_{[t_k,t_{k+1})} : [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$. By Assumption 3.1, $f_k(\cdot, x) = f(\cdot, x, \eta_k)\mathbf{1}_{[t_k,t_{k+1})} \in \mathrm{M}^2_*(0,T)$ for all $x \in \mathbb{R}$. The continuity assumption (3.3) implies (A.1) with $\rho(r) := \rho_1(r^2)$. Thus, we can apply Lemma A.4 and obtain that $f_k(\cdot, \xi_k)\mathbf{1}_{[t_k,T]} = f(\cdot, \xi_k, \eta_k)\mathbf{1}_{[t_k,t_{k+1}]} \in \mathrm{M}^2_*(0,T)$ since $\xi_k \in \mathrm{B}_b(\Omega_{t_k})$ is $\mathcal{F}_t$-measurable and bounded. Thus, $f(\cdot, X, Y) \in \mathrm{M}^2_*(0,T)$ as finite sum of elements in $\mathrm{M}^2_*(0,T)$. □

Corollary 3.4 *Let $X, Y \in \mathrm{M}^{2,d}_*(0,T)$ be such that*

$$\sup_{0\leqslant s\leqslant T} \hat{\mathbb{E}}\Big[|X_s|^2\Big] + \sup_{0\leqslant s\leqslant T} \hat{\mathbb{E}}\Big[|Y_s|^2\Big] < \infty. \tag{3.5}$$

If Assumption 3.1 is satisfied, then $f(\cdot, X, Y) \in \mathrm{M}^2_*(0,T)$ for all components $f = b_k, h_{kij}, g_{ki}$, $1 \leqslant i, j \leqslant n$, $1 \leqslant k \leqslant d$.

Proof Since $X, Y \in \mathrm{M}^2_*(0,T)$ satisfy (3.5), there exist sequences $(X^m)_{m\in\mathbb{N}}, (Y^m)_{m\in\mathbb{N}}$ in $\mathrm{M}_b(0,T)$ such that

$$\lim_{m\to\infty} \sup_{0\leqslant s\leqslant T} \hat{\mathbb{E}}\Big[|X^m_s - X_s|^2\Big] = 0, \quad \lim_{m\to\infty} \sup_{0\leqslant s\leqslant T} \hat{\mathbb{E}}\Big[|Y^m_s - Y_s|^2\Big] = 0. \tag{3.6}$$

Lemma 3.3 implies $f(\cdot, X^m, Y^m) \in \mathrm{M}^2_*(0,T)$ for all $m \in \mathbb{N}$. By the continuity assumption (3.3), we have

$$\begin{aligned}
&\int_0^T \hat{\mathbb{E}}\left[|f(s, X^m_s, Y^m_s) - f(s, X_s, Y_s)|^2\right] \mathrm{d}s \\
&\leqslant \int_0^T \hat{\mathbb{E}}\left[\kappa(s)\rho_1\left(|X^m_s - X_s|^2\right) + K_s\rho_2\left(\|Y^m_s - Y_s\|^2_{\mathrm{L}^2_*}\right)\right] \mathrm{d}s \\
&\leqslant \int_0^T \kappa(s)\rho_1\left(\hat{\mathbb{E}}\left[|X^m_s - X_s|^2\right]\right) + \hat{\mathbb{E}}\left[K_s\right]\rho_2\left(\|Y^m_s - Y_s\|^2_{\mathrm{L}^2_*}\right) \mathrm{d}s \\
&\leqslant \int_0^T \kappa(s)\mathrm{d}s\rho_1\left(\sup_{0\leqslant w\leqslant T} \hat{\mathbb{E}}\left[|X^m_w - X_w|^2\right]\right) + \int_0^T \hat{\mathbb{E}}\left[K_s\right]\mathrm{d}s\rho_2\left(\sup_{0\leqslant w\leqslant T} \hat{\mathbb{E}}\left[|Y^m_w - Y_w|^2\right]\right) \\
&\leqslant C\rho_1\left(\sup_{0\leqslant w\leqslant T} \hat{\mathbb{E}}\left[|X^m_w - X_w|^2\right]\right) + C\rho_2\left(\sup_{0\leqslant w\leqslant T} \hat{\mathbb{E}}\left[|Y^m_w - Y_w|^2\right]\right),
\end{aligned}$$

which tends to 0 as $m \to \infty$ due to (3.6). Thus, $f(\cdot, X, Y) \in \mathrm{M}^2_*(0,T)$ since $\mathrm{M}^2_*(0,T)$ is complete. □

For $\xi \in \mathrm{L}^{2,d}_*(t)$ and $X, Y \in \mathrm{M}^{2,d}_*(t,T)$, consider the process $\Phi^{t,\xi}(X,Y)$ given by

$$\begin{aligned}
\Phi^{t,\xi}(X,Y)_s := \xi &+ \int_t^s b(u, X_u, Y_u)\,\mathrm{d}u + \sum_{i,j=1}^n \int_t^s h_{ij}(u, X_u, Y_u)\,\mathrm{d}\left\langle B^i, B^j\right\rangle_u \\
&+ \sum_{i=1}^n \int_t^s g_i(u, X_u, Y_u)\,\mathrm{d}B^i_u, \quad t \leqslant s \leqslant T,
\end{aligned} \tag{3.7}$$

if it is well-defined.

Lemma 3.5 *Let $X, Y \in \mathrm{M}^{2,d}_*(t,T)$ be such that $\Phi(X,Y) \in \mathrm{M}^{1,d}_*(t,T)$. If Assumption 3.1 is satisfied, then there exists a constant $\mathcal{K} > 0$ such that*

$$\hat{\mathbb{E}}\Big[\sup_{t\leqslant w\leqslant s} \left\|\Phi^{t,\xi}(X,Y)\right\|^2\Big] \leqslant \mathcal{K}\left(\|\xi\|^2_{\mathrm{L}^2_*} + \hat{\mathbb{E}}\Big[\int_t^s \kappa(u)\,|X_u|^2 + K_u\left(1 + \|Y_u\|^2_{\mathrm{L}^2_*}\right)\mathrm{d}u\Big]\right)$$

for all $t \leqslant s \leqslant T$ and $\xi \in \mathrm{L}^{2,d}_*(t)$. Moreover, $\mathcal{K}$ is independent of X, Y.

Proof By Jensen's inequality, we have

$$\begin{aligned}\hat{\mathbb{E}}\left[\sup_{t\leqslant w\leqslant s}\left|\Phi^{t,\xi}(X,Y)_w\right|^2\right] \leqslant \left(2+n+n^2\right)\Bigg(&\hat{\mathbb{E}}\left[|\xi|^2\right]+\hat{\mathbb{E}}\left[\sup_{t\leqslant w\leqslant s}\left|\int_t^w b(u,X_u,Y_u)\,\mathrm{d}u\right|^2\right]\\ &+\sum_{i,j=1}^n\hat{\mathbb{E}}\left[\sup_{t\leqslant w\leqslant s}\left|\int_t^w h_{ij}(u,X_u,Y_u)\,\mathrm{d}\left\langle B^i,B^j\right\rangle_u\right|^2\right]\\ &+\sum_{i=1}^n\hat{\mathbb{E}}\left[\sup_{t\leqslant w\leqslant s}\left|\int_t^w g_i(u,X_u,Y_u)\,\mathrm{d}B_u^i\right|^2\right]\Bigg).\end{aligned}\tag{3.8}$$

By the growth assumption (3.4), we have for the integral with respect to u that

$$\begin{aligned}\hat{\mathbb{E}}\left[\sup_{t\leqslant w\leqslant s}\left|\int_t^w b(u,X_u,Y_u)\,\mathrm{d}u\right|^2\right] &\leqslant (s-t)\,\hat{\mathbb{E}}\left[\int_t^s |b(u,X_u,Y_u)|^2\,\mathrm{d}u\right]\\ &\leqslant (s-t)\,\hat{\mathbb{E}}\left[\int_t^s \kappa(u)\,|X_u|^2+K_u\left(1+\|Y_u\|_{\mathrm{L}^2_*}^2\right)\mathrm{d}u\right].\end{aligned}\tag{3.9}$$

Since $\Phi(X,Y)\in\mathrm{M}^1_*(t,T)$, we know that $h_{ij}(\cdot,X,Y)\in\mathrm{M}^1_*(t,T)$ and $g_i(\cdot,X,Y)\in\mathrm{M}^2_*(t,T)$ for $1\leqslant i,j\leqslant n$, and we can apply Lemmas 2.11 and 2.12. Thus, the growth condition (3.4) yields for the integral with respect to $\left\langle B^i,B^j\right\rangle$ with $1\leqslant i,j\leqslant n$ that

$$\begin{aligned}\hat{\mathbb{E}}\left[\sup_{t\leqslant w\leqslant s}\left|\int_t^w h_{ij}(u,X_u,Y_u)\,\mathrm{d}\left\langle B^i,B^j\right\rangle_u\right|^2\right] &\leqslant \overline{\sigma}_{ij}^4\,(s-t)\,\hat{\mathbb{E}}\left[\int_t^s |h_{ij}(u,X_u,Y_u)|^2\,\mathrm{d}u\right]\\ &\leqslant \overline{\sigma}_{ij}^4\,(s-t)\,\hat{\mathbb{E}}\left[\int_t^s \kappa(u)\,|X_u|^2+K_u\left(1+\|Y_u\|_{\mathrm{L}^2_*}^2\right)\mathrm{d}u\right],\end{aligned}\tag{3.10}$$

and for the integral with respect to B^i with $1\leqslant i\leqslant n$ that

$$\begin{aligned}\hat{\mathbb{E}}\left[\sup_{t\leqslant w\leqslant s}\left|\int_t^w g_i(u,X_u,X_u)\,\mathrm{d}B_u^i\right|^2\right] &\leqslant C_2\,\overline{\sigma}_{ii}^2\hat{\mathbb{E}}\left[\int_t^s |g_i(u,X_u,Y_u)|^2\,\mathrm{d}u\right]\\ &\leqslant C_2\,\overline{\sigma}_{ii}^2\hat{\mathbb{E}}\left[\int_t^s \kappa(u)\,|X_u|^2+K_u\left(1+\|Y_u\|_{\mathrm{L}^2_*}^2\right)\mathrm{d}u\right].\end{aligned}\tag{3.11}$$

Combining (3.8), (3.9), (3.10) and (3.11), we obtain

$$\hat{\mathbb{E}}\left[\sup_{t\leqslant w\leqslant s}\left|\Phi^{t,\xi}(X,Y)_w\right|^2\right]\leqslant\mathcal{K}\left(\|\xi\|_{\mathrm{L}^2_*}^2+\hat{\mathbb{E}}\left[\int_t^s\kappa(u)\,|X_u|^2+K_u\left(1+\|Y_u\|_{\mathrm{L}^2_*}^2\right)\mathrm{d}u\right]\right),$$

where the constant $\mathcal{K}$ is given by

$$\mathcal{K}:=\left(2+n+n^2\right)\left(1+T+\sum_{i,j=1}^n\overline{\sigma}_{ij}^4T+C_2\sum_{i=1}^n\overline{\sigma}_{ii}^2\right),\tag{3.12}$$

which is independent of X,Y. □

For $0\leqslant s\leqslant T$ and $d\geqslant 1$, let us introduce the space

$$\mathrm{H}^{2,d}_*(t,T):=\left\{X\in\mathrm{M}^{2,d}_*(s,T)\,:\,\hat{\mathbb{E}}\left[\sup_{w\leqslant w\leqslant T}\|X_w\|^2\right]<\infty\right\}.$$

Clearly, $\mathrm{H}^2_*(0,T):=\mathrm{H}^{2,1}_*(0,T)\subseteq\mathrm{M}^2_*(0,T)$ is the completion of $\mathrm{M}_b(0,T)$ with respect to the norm

$$\|\cdot\|_{\mathrm{H}^2_*}:\ \mathrm{M}_b(0,T)\to\mathbb{R}_+,\quad \|X\|_{\mathrm{H}^2_*}:=\hat{\mathbb{E}}\left[\sup_{0\leqslant s\leqslant T}|X_s|^2\right]^{\frac{1}{2}}.$$

Corollary 3.6 *Let* $\xi \in \mathrm{L}_*^{2,d}(t)$ *and* $X, Y \in \mathrm{H}_*^{2,d}(t,T)$. *If Assumption 3.1 is satisfied, then* $\Phi^{t,\xi}(X,Y) \in \mathrm{M}_*^{1,d}(t,T)$.

Proof Corollary 3.4 implies that $Z := f\big(\cdot, X\mathbf{1}_{[t,T]}, Y\mathbf{1}_{[t,T]}\big) \in \mathrm{M}_*^2(0,T)$ for all components $f = b, h_{ij}, g_i$, $1 \leqslant i,j \leqslant n$. In particular, $Z\mathbf{1}_{[t,s]} = f(\cdot, X, Y)\,\mathbf{1}_{[t,s]} \in \mathrm{M}_*^2(0,T)$ for all $t \leqslant s \leqslant T$. Hence, $f(\cdot, X, Y) \in \mathrm{M}_*^2(t,T)$ and all integrals in (3.7) are well-defined. Thus, by Lemmas 2.6, 2.8, and 2.10, we obtain $\Phi^{t,\xi}(X,Y) \in \mathrm{M}_*^1(t,T)$. □

Corollary 3.7 *Let* $\xi \in \mathrm{L}_*^{2,d}(t)$ *and* $X, Y \in \mathrm{H}_*^{2,d}(t,T)$. *If Assumption 3.1 is satisfied, then* $\Phi^{t,\xi}(X,Y) \in \mathrm{H}_*^{2,d}(t,T)$.

Proof Lemma 3.5 implies that

$$\begin{aligned}
\big\|\Phi^{t,\xi}(X,Y)\big\|_{\mathrm{H}_*^2}^2 &= \hat{\mathbb{E}}\Big[\sup_{t\leqslant s\leqslant T}\big|\Phi^{t,\xi}(X,Y)_s\big|^2\Big] \\
&\leqslant \mathcal{K}\left(\|\xi\|_{\mathrm{L}_*^2}^2 + \hat{\mathbb{E}}\left[\int_t^T \kappa(u)\,|X_u|^2 + K_u\left(1 + \|Y_u\|_{\mathrm{L}_*^2}^2\right)\mathrm{d}u\right]\right) \\
&\leqslant \mathcal{K}\left(\|\xi\|_{\mathrm{L}_*^2}^2 + \int_t^T \kappa(u)\hat{\mathbb{E}}\Big[|X_u|^2\Big] + \hat{\mathbb{E}}[K_u]\left(1 + \|Y_u\|_{\mathrm{L}_*^2}^2\right)\mathrm{d}u\right) \\
&\leqslant \mathcal{K}\left(\|\xi\|_{\mathrm{L}_*^2}^2 + \int_t^T \kappa(u)\hat{\mathbb{E}}\Big[\sup_{t\leqslant w\leqslant T}|X_w|^2\Big] + \hat{\mathbb{E}}[K_u]\left(1 + \hat{\mathbb{E}}\Big[\sup_{t\leqslant w\leqslant T}|Y_w|^2\Big]\right)\mathrm{d}u\right) \\
&= \mathcal{K}\left(\|\xi\|_{\mathrm{L}_*^2}^2 + \int_t^T \kappa(u)\,\|X\|_{\mathrm{H}_*^2}^2 + \hat{\mathbb{E}}[K_u]\left(1 + \|Y\|_{\mathrm{H}_*^2}^2\right)\mathrm{d}u\right) \\
&\leqslant \mathcal{K}\left(\|\xi\|_{\mathrm{L}_*^2}^2 + C\left(1 + \|X\|_{\mathrm{H}_*^2}^2 + \|Y\|_{\mathrm{H}_*^2}^2\right)\right) < \infty,
\end{aligned}$$

i.e., $\Phi^{t,\xi}(X,Y) \in \mathrm{H}_*^2(t,T)$. □

Lemma 3.8 *If Assumption 3.1 is satisfied, then there exists a constant* $\mathcal{K} > 0$ *such that*

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{t\leqslant w\leqslant s}\big\|\Phi^{t,\xi}(X,Y)_w - \Phi^{t,\eta}(X',Y')_w\big\|^2\right] \\
&\leqslant \mathcal{K}\left(\|\xi-\eta\|_{\mathrm{L}_*^2}^2 + \hat{\mathbb{E}}\left[\int_t^s \kappa(u)\,\rho_1\Big(|X_u - X_u'|^2\Big) + K_u\,\rho_2\Big(\|Y_u - Y_u'\|_{\mathrm{L}_*^2}^2\Big)\mathrm{d}u\right]\right)
\end{aligned}$$

for all $t \leqslant s \leqslant T$, $\xi, \eta \in \mathrm{L}_*^{2,d}(t)$, *and* $X, X', Y, Y' \in \mathrm{H}_*^{2,d}(t,T)$.

Proof Jensen's inequality yields

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{t\leqslant w\leqslant s}\big|\Phi^{t,\xi}(X,Y)_w - \Phi^{t,\eta}(X',Y')_w\big|^2\right] \\
\leqslant\ & \left(2 + n + n^2\right)\Bigg(\hat{\mathbb{E}}\Big[|\xi-\eta|^2\Big] + \hat{\mathbb{E}}\left[\sup_{t\leqslant w\leqslant s}\left|\int_t^w b(u,X_u,Y_u) - b(u,X_u',Y_u')\,\mathrm{d}u\right|^2\right] \\
&+ \sum_{i,j=1}^n \hat{\mathbb{E}}\left[\sup_{t\leqslant w\leqslant s}\left|\int_t^w h_{ij}(u,X_u,Y_u) - h_{ij}(u,X_u',Y_u')\,\mathrm{d}\left\langle B^i, B^j\right\rangle_u\right|^2\right] \\
&+ \sum_{i=1}^n \hat{\mathbb{E}}\left[\sup_{t\leqslant w\leqslant s}\left|\int_t^w g_i(u,X_u,Y_u) - g_i(u,X_u',Y_u')\,\mathrm{d}B_u^i\right|^2\right]\Bigg).
\end{aligned} \tag{3.13}$$

Due to the continuity assumption (3.3), we have for the integral with respect to u that

$$
\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{t\leqslant w\leqslant s}\left|\int_t^w b(u,X_u,Y_u)-b(u,X'_u,Y'_u)\,\mathrm{d}u\right|^2\right]\\
&\leqslant (s-t)\,\hat{\mathbb{E}}\left[\int_t^s |b(u,X_u,Y_u)-b(u,X'_u,Y'_u)|^2\,\mathrm{d}u\right]\\
&\leqslant (s-t)\,\hat{\mathbb{E}}\left[\int_t^s \kappa(u)\,\rho_1\Big(|X_u-X'_u|^2\Big)+K_u\,\rho_2\Big(\|Y_u-Y'_u\|^2_{\mathrm{L}^2_*}\Big)\,\mathrm{d}u\right].
\end{aligned}
\tag{3.14}
$$

Analogously, by Lemmas 2.11 and 2.12, we have for the integral with respect to $\langle B^i,B^j\rangle$ with $1\leqslant i,j\leqslant n$ that

$$
\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{t\leqslant w\leqslant s}\left|\int_t^w h_{ij}(u,X_u,Y_u)-h_{ij}(u,X'_u,Y'_u)\,\mathrm{d}\left\langle B^i,B^j\right\rangle_u\right|^2\right]\\
&\leqslant \overline{\sigma}^4_{ij}\,(s-t)\,\hat{\mathbb{E}}\left[\int_t^s |h_{ij}(u,X_u,Y_u)-h_{ij}(u,X'_u,Y'_u)|^2\,\mathrm{d}u\right]\\
&\leqslant \overline{\sigma}^4_{ij}\,(s-t)\,\hat{\mathbb{E}}\left[\int_t^s \kappa(u)\,\rho_1\Big(|X_u-X'_u|^2\Big)+K_u\,\rho_2\Big(\|Y_u-Y'_u\|^2_{\mathrm{L}^2_*}\Big)\,\mathrm{d}u\right],
\end{aligned}
\tag{3.15}
$$

and for the integral with respect to B^i with $1\leqslant i\leqslant n$ that

$$
\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{t\leqslant w\leqslant s}\left|\int_t^w g_i(u,X_u,Y_u)-g_i(u,X'_u,Y'_u)\,\mathrm{d}B^i_u\right|^2\right]\\
&\leqslant C_2\,\overline{\sigma}^2_{ii}\hat{\mathbb{E}}\left[\int_t^s |g_i(u,X_u,Y_u)-g_i(u,X'_u,Y'_u)|^2\,\mathrm{d}u\right]\\
&\leqslant C_2\,\overline{\sigma}^2_{ii}\hat{\mathbb{E}}\left[\int_t^s \kappa(u)\,\rho_1\Big(|X_u-X'_u|^2\Big)+K_u\,\rho_2\Big(\|Y_u-Y'_u\|^2_{\mathrm{L}^2_*}\Big)\,\mathrm{d}u\right].
\end{aligned}
\tag{3.16}
$$

Combining (3.13), (3.14), (3.15), and (3.16), we obtain

$$
\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{t\leqslant w\leqslant s}\left|\Phi^{t,\xi}(X,Y)_w-\Phi^{t,\eta}(X',Y')_w\right|^2\right]\\
&\leqslant \mathcal{K}\left(\|\xi-\eta\|^2_{\mathrm{L}^2_*}+\hat{\mathbb{E}}\left[\int_t^s \kappa(u)\,\rho_1\Big(|X_u-X'_u|^2\Big)+K_u\,\rho_2\Big(\|Y_u-Y'_u\|^2_{\mathrm{L}^2_*}\Big)\,\mathrm{d}u\right]\right),
\end{aligned}
$$

where the constant $\mathcal{K}$ can be chosen as in (3.12). □

Corollary 3.9 *If Assumption 3.1 is satisfied, then the map*

$$
\mathrm{L}^{2,d}_*(t)\times\mathrm{H}^{2,d}_*(t,T)\times\mathrm{H}^{2,d}_*(t,T)\to\mathrm{H}^{2,d}_*(t,T),\quad (\xi,X,Y)\mapsto\Phi^{t,\xi}(X,Y)
$$

is continuous.

Proof Let $\xi,\eta\in\mathrm{L}^2_*(t)$, and $X,X',Y,Y'\in\mathrm{H}^2_*(t,T)$. By Lemma 3.8, we have

$$
\begin{aligned}
&\|\Phi^{t,\xi}(X,Y)-\Phi^{t,\eta}(X',Y')\|^2_{\mathrm{H}^2_*}\\
&=\hat{\mathbb{E}}\left[\sup_{t\leqslant w\leqslant T}\left\|\Phi^{t,\xi}(X,Y)_w-\Phi^{t,\eta}(X',Y')_w\right\|^2\right]\\
&\leqslant \mathcal{K}\left(\|\xi-\eta\|^2_{\mathrm{L}^2_*}+\hat{\mathbb{E}}\left[\int_t^T \kappa(u)\,\rho_1\Big(|X_u-X'_u|^2\Big)+K_u\,\rho_2\Big(\|Y_u-Y'_u\|^2_{\mathrm{L}^2_*}\Big)\,\mathrm{d}u\right]\right)\\
&\leqslant \mathcal{K}\left(\|\xi-\eta\|^2_{\mathrm{L}^2_*}+\int_t^T \kappa(u)\,\rho_1\Big(\hat{\mathbb{E}}\Big[|X_u-X'_u|^2\Big]\Big)+\hat{\mathbb{E}}[K_u]\,\rho_2\Big(\hat{\mathbb{E}}\Big[|Y_u-Y'_u|^2\Big]\Big)\,\mathrm{d}u\right)\\
&\leqslant \mathcal{K}\left(\|\xi-\eta\|^2_{\mathrm{L}^2_*}+\int_t^T \kappa(u)\,\rho_1\Big(\|X-X'\|^2_{\mathrm{H}^2_*}\Big)\,\mathrm{d}u+\int_t^T \hat{\mathbb{E}}[K_u]\,\rho_2\Big(\|Y-Y'\|^2_{\mathrm{H}^2_*}\Big)\,\mathrm{d}u\right).
\end{aligned}
$$

Since ρ_1 and ρ_2 are continuous, the right-hand side vanishes when $(\xi, X, Y) \to (\eta, X', Y')$ in $\mathrm{L}^2_* \times \mathrm{H}^2_*(t,T) \times \mathrm{H}^2_*(t,T)$. □

Proposition 3.10 *Let $\xi \in \mathrm{L}^{2,d}_*(t)$. If Assumption 3.1 is satisfied, then the map*

$$\mathrm{H}^{2,d}_*(t,T) \to \mathrm{H}^{2,d}_*(t,T), \quad X \mapsto \Phi^{t,\xi}(X) := \Phi^{t,\xi}(X,X)$$

has a unique fixed-point.

Proof *Existence:* Fix $\xi \in \mathrm{L}^2_*(t)$, and define the Picard sequence $(X^m)_{m\in\mathbb{N}}$ recursively by

$$X^0 := \xi \mathbf{1}_{[t,T)} \quad \text{and} \quad X^{m+1} := \Phi^{t,\xi}(X^m), \quad m \in \mathbb{N}.$$

Since $\left\|X^0\right\|_{\mathrm{H}^2_*} = \|\xi\|_{\mathrm{L}^2_*} < \infty$ and thus $X^0 \in \mathrm{H}^2_*(t,T)$, the sequence is well-defined in $\mathrm{H}^2_*(t,T)$ due to Corollary 3.7.

Let $\mathcal{K}$ be the constant given in (3.12), and consider the function $q : [t,T] \to \mathbb{R}$ defined by

$$q(s) := \mathcal{K} e^{\mathcal{K}\int_t^s \gamma(u)\mathrm{d}u} \left(\|\xi\|^2_{\mathrm{L}^2_*} + \int_t^s \hat{\mathbb{E}}[K_u]\, e^{-\mathcal{K}\int_t^u \gamma(w)\mathrm{d}w} \mathrm{d}u \right),$$

where $\gamma(u) := \hat{\mathbb{E}}\left[K_u\right] + \kappa(u) \geqslant 0$ for $t \leqslant u \leqslant T$. Then, q is continuous, increasing and bounded, and the solution of the ordinary differential equation

$$\begin{aligned} &\mathrm{d}q(s) = \mathcal{K}\left(\hat{\mathbb{E}}[K_s] + \gamma(s)\, q(s)\right) \mathrm{d}s, \quad t \leqslant s \leqslant T, \\ &q(t) = \mathcal{K}\, \|\xi\|^2_{\mathrm{L}^2_*}. \end{aligned}$$

By induction over $m \in \mathbb{N}$, we show that

$$\hat{\mathbb{E}}\left[\sup_{t\leqslant w\leqslant s} |X^m_w|^2\right] \leqslant q(s)$$

for all $t \leqslant s \leqslant T$ and $m \in \mathbb{N}$. Since $\mathcal{K} > 1$, we have for all $t \leqslant s \leqslant T$ that

$$\hat{\mathbb{E}}\left[\sup_{t\leqslant w\leqslant s} \left|X^0_w\right|^2\right] = \|\xi\|^2_{\mathrm{L}^2_*} \leqslant q(t) \leqslant q(s).$$

Now, we consider the induction method. Suppose $\hat{\mathbb{E}}\left[\sup_{t\leqslant w\leqslant s} |X^m_w|^2\right] \leqslant q(s)$ for all $t \leqslant s \leqslant T$ for some $m \in \mathbb{N}$. By Lemma 3.5, we have for all $t \leqslant s \leqslant T$ that

$$\begin{aligned} \hat{\mathbb{E}}\left[\sup_{t\leqslant w\leqslant s} \left|X^{m+1}_w\right|^2\right] &= \hat{\mathbb{E}}\left[\sup_{t\leqslant w\leqslant s} \left|\Phi^{t,\xi}(X^m)_w\right|^2\right] \\ &\leqslant \mathcal{K}\left(\|\xi\|^2_{\mathrm{L}^2_*} + \int_t^s \hat{\mathbb{E}}[K_u] + \gamma(s)\hat{\mathbb{E}}\left[|X^m_u|^2\right] \mathrm{d}u \right) \\ &\leqslant \mathcal{K}\left(\|\xi\|^2_{\mathrm{L}^2_*} + \int_t^s \hat{\mathbb{E}}[K_u] + \gamma(s)\hat{\mathbb{E}}\left[\sup_{t\leqslant w\leqslant u} |X^m_w|^2\right] \mathrm{d}u \right) \\ &\leqslant \mathcal{K}\left(\|\xi\|^2_{\mathrm{L}^2_*} + \int_t^s \hat{\mathbb{E}}[K_u] + \gamma(s) q(u) \mathrm{d}u \right) \\ &= q(s), \end{aligned}$$

which completes the induction. In particular, we established the uniform bound

$$\sup_{m\in\mathbb{N}} \hat{\mathbb{E}}\left[\sup_{t\leqslant s\leqslant T} |X^m_s|^2\right] \leqslant q(T) =: \hat{q}.$$

Next, we show that the sequence $(X^m)_{m\in\mathbb{N}}$ is Cauchy. Let $m, k \in \mathbb{N}$ and $t \leqslant s \leqslant T$. By Lemma 3.8, we have

$$
\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{t\leqslant w\leqslant s}\left|X_w^{m+k+1}-X_w^{m+1}\right|^2\right]\\
&=\hat{\mathbb{E}}\left[\sup_{t\leqslant w\leqslant s}\left|\Phi^{t,\xi}(X^{m+k})_w-\Phi^{t,\xi}(X^m)_w\right|^2\right]\\
&\leqslant \mathcal{K}\int_t^s \gamma(u)\rho\Big(\hat{\mathbb{E}}\Big[\left|X_u^{m+k}-X_u^m\right|^2\Big]\Big)\,\mathrm{d}u\\
&\leqslant \mathcal{K}\int_t^s \gamma(u)\rho\left(\hat{\mathbb{E}}\left[\sup_{t\leqslant w\leqslant u}\left|X_w^{m+k}-X_w^m\right|^2\right]\right)\mathrm{d}u. \qquad (3.17)
\end{aligned}
$$

For $k,m\in\mathbb{N}$, define the function $u_m^k:[t,T]\to\mathbb{R}_+$ by

$$
u_m^k(s):=\hat{\mathbb{E}}\left[\sup_{t\leqslant w\leqslant s}\left|X_w^{m+k}-X_w^m\right|^2\right],
$$

then plugging into (3.17) yields

$$
u_{m+1}^k(s)\leqslant \mathcal{K}\int_t^s \gamma(w)\rho\big(u_m^k(w)\big)\,\mathrm{d}w. \qquad (3.18)
$$

Taking the point-wise supremum of (3.18) over $k\in\mathbb{N}$, we obtain

$$
\begin{aligned}
\sup_{k\in\mathbb{N}} u_{m+1}^k(s)&\leqslant \sup_{k\in\mathbb{N}}\mathcal{K}\int_t^s \gamma(w)\rho\big(u_m^k(w)\big)\,\mathrm{d}w\\
&\leqslant \mathcal{K}\int_t^s \gamma(w)\rho\left(\sup_{k\in\mathbb{N}} u_m^k(w)\right)\mathrm{d}w. \qquad (3.19)
\end{aligned}
$$

Define the point-wise limit superior

$$
u(s):=\limsup_{m\to\infty}\sup_{k\in\mathbb{N}} u_m^k(s),\quad t\leqslant s\leqslant T.
$$

Since for all $k,m\in\mathbb{N}$ and $t\leqslant s\leqslant T$, we have

$$
0\leqslant u_m^k(s)=\sup_{t\leqslant w\leqslant s}\hat{\mathbb{E}}\Big[\left|X_w^{m+k}-X_w^m\right|^2\Big]\leqslant \sup_{t\leqslant w\leqslant s}\left(\hat{\mathbb{E}}\Big[\left|X_w^{m+k}\right|^2\Big]+\hat{\mathbb{E}}\Big[\left|X_w^m\right|^2\Big]\right)\leqslant 2\,\hat{q},
$$

we also have $0\leqslant u(s)\leqslant 2\,\hat{q}$ for all $t\leqslant s\leqslant T$. In particular, u and u_m^k are Lebesgue integrable for any $k,m\in\mathbb{N}$. Applying the Fatou-Lebesgue Theorem to (3.19), we obtain

$$
0\leqslant u(s)\leqslant \mathcal{K}\int_t^s \gamma(w)\rho(u(w))\,\mathrm{d}w
$$

for all $t\leqslant s\leqslant T$. Bihari's inequality implies that

$$
u(s)\leqslant G_z^{-1}\left(\mathcal{K}\int_t^s \gamma(w)\,\mathrm{d}w\right) \qquad (3.20)
$$

for all $t\leqslant s\leqslant T$ and $z>0$, where G_z^{-1} denotes the inverse of

$$
G_z(y):=\int_z^y \frac{1}{\rho(r)}\mathrm{d}r,\quad y>0.
$$

Assumption 3.1 implies that

$$
\lim_{z\downarrow 0} G_z(y)=\int_{0+}^y \frac{1}{\rho(r)}\mathrm{d}r=\infty
$$

for all $y>0$ since ρ is increasing. Thus, by letting z tend to 0 in (3.20), we obtain $u(s)=0$ for all $t\leqslant s\leqslant T$ since $\mathcal{K}\int_t^T \gamma(w)\mathrm{d}w<\infty$.

Hence, $(X^m)_{m\in\mathbb{N}}$ is a Cauchy sequence with respect to the norm $\|\cdot\|_{\mathrm{H}^2_*}$. Since $\mathrm{H}^2_*(t,T)$ is complete, we have

$$\lim_{m\to\infty} X^m =: X \in \mathrm{H}^2_*(t,T).$$

By Corollary 3.9, the map $X \mapsto \Phi^{t,\xi}(X)$ is continuous. Thus, we have

$$X = \lim_{m\to\infty} X^{m+1} = \lim_{m\to\infty} \Phi^{t,\xi}(X^m) = \Phi^{t,\xi}\Big(\lim_{m\to\infty} X^m\Big) = \Phi^{t,\xi}(X)$$

in $\mathrm{H}^2_*(t,T)$. That is, X is a fixed point of $\Phi^{t,\xi}$.

Uniqueness. Suppose $X, Y \in \mathrm{H}^2_*(t,T)$ satisfy $X = \Phi^{t,\xi}(X)$ and $Y = \Phi^{t,\xi}(Y)$. By Lemma 3.8, we have for all $t \leqslant s \leqslant T$ that

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{t\leqslant w\leqslant s} |X_w - Y_w|^2\right] \\
&= \hat{\mathbb{E}}\left[\sup_{t\leqslant w\leqslant s} \left|\Phi^{t,\xi}(X)_w - \Phi^{t,\xi}(Y)_w\right|^2\right] \\
&\leqslant \mathcal{K}\int_t^s \gamma(u)\rho\left(\hat{\mathbb{E}}\left[\sup_{t\leqslant w\leqslant u} |X_w - Y_w|^2\right]\right) \mathrm{d}u.
\end{aligned} \tag{3.21}$$

Define the function $u : [t,T] \to \mathbb{R}_+$ by

$$u(s) := \hat{\mathbb{E}}\left[\sup_{t\leqslant w\leqslant s} |X_w - Y_w|^2\right].$$

By substituting u into (3.21), we obtain

$$u(s) \leqslant \mathcal{K}\int_t^s \gamma(w)\rho(u(w))\,\mathrm{d}w,$$

and Bihari's inequality yields $u \equiv 0$. Thus, $\Phi^{t,\xi}$ has a unique fixed-point. □

Consequently, we deduce the following result.

Corollary 3.11 *Let $\xi \in \mathrm{L}^{2,d}_*(t)$ and $Y \in \mathrm{H}^{2,d}_*(t,T)$. If Assumption 3.1 is satisfied, then the map*

$$\mathrm{H}^{2,d}_*(t,T) \to \mathrm{H}^{2,d}_*(t,T), \quad X \mapsto \Phi^{t,\xi}(X,Y)$$

has a unique fixed-point.

Theorem 3.12 *Let $\xi \in \mathrm{L}^{2,d}_*(t)$. If Assumption 3.1 is satisfied, then there exists a unique $X \in \mathrm{H}^{2,d}_*(t,T)$ which solves* (3.1) *with $X_t = \xi$.Moreover, if*

$$\gamma(s) = \hat{\mathbb{E}}\left[K_s\right] + \kappa(s), \quad t \leqslant s \leqslant T$$

is bounded, then the solution X is unique in $\mathrm{M}^{2,d}_*(t,T)$ and

$$\hat{\mathbb{E}}\left[\sup_{t\leqslant s\leqslant T} \|X_s\|^2\right] < \infty.$$

Proof From the definition of $\Phi^{t,\xi}(X,Y)$ in (3.7), we deduce that $X \in \mathrm{M}^2_*(t,T)$ is a solution of (3.1) with $X_t = \xi$ if and only if $X = \Phi^{t,\xi}(X,X)$.

Suppose $X \in \mathrm{M}^2_*(t,T)$ satisfies $X = \Phi^{t,\xi}(X,X)$, then $\Phi^{t,\xi}(X,X) = X \in \mathrm{M}^2_*(t,T) \subseteq \mathrm{M}^1_*(t,T)$. Set $M := \sup_{t\leqslant s\leqslant T} \gamma(s)$. By Lemma 3.5, we obtain

$$\begin{aligned}\hat{\mathbb{E}}\left[\sup_{t\leqslant s\leqslant T}|X_s|^2\right] &= \hat{\mathbb{E}}\left[\sup_{t\leqslant s\leqslant T}\left|\Phi^{t,\xi}(X,X)_s\right|^2\right]\\ &\leqslant \mathcal{K}\left(\|\xi\|^2_{\mathrm{L}^2_*}+\|K\|_{\mathrm{M}^1_*}+\int_t^T\gamma(u)\hat{\mathbb{E}}\left[|X_u|^2\right]\mathrm{d}u\right)\\ &\leqslant \mathcal{K}\left(\|\xi\|^2_{\mathrm{L}^2_*}+\|K\|_{\mathrm{M}^1_*}+M\,\|X\|^2_{\mathrm{M}^2_*}\right)<\infty.\end{aligned}$$

That is, $X\in \mathrm{H}^2_*(t,T)$. By Proposition 3.10, there exists a unique $X\in \mathrm{H}^2_*(t,T)$ with $X=\Phi^{t,\xi}(X)=\Phi^{t,\xi}(X,X)$, which implies the desired result. □

4. Comparison to existing literature

In this section, we discuss how our results relate to the existing results on mean-field G-SDEs. In particular, we show that Theorem 3.12 generalises the existing results in the literature.

First, we show that the G-SDE considered in [27] is a special case of the mean-field G-SDE (3.1). For the sake of convenience, we recall the definitions from Section 3 in [27]. Let $\mathcal{D}$ denote the space of all functionals $F:\mathrm{Lip}(\mathbb{R}^d)\to\mathbb{R}$, which are monotonous, positive homogeneous, subadditive, constant-invariant and such that

$$\sup_{L_\varphi\leqslant 1}|F_1(\varphi)-\varphi(0)|<\infty,$$

where the supremum ranges over all Lipschitz continuous $\varphi:\mathbb{R}^d\to\mathbb{R}$ with Lipschitz constant $L_\varphi\leqslant 1$. Moreover, define the metric d_1 on $\mathcal{D}$ by

$$d_1(F_1,F_2):=\sup_{L_\varphi\leqslant 1}|F_1(\varphi)-F_2(\varphi)|\,,\quad f,g\in\mathcal{D}.$$

For any $\xi\in\mathrm{L}^{1,d}_*$, define the functional

$$F_\xi:\ \mathrm{Lip}(\mathbb{R}^d)\to\mathbb{R},\quad \varphi\mapsto\hat{\mathbb{E}}[\varphi(\xi)]\,.$$

Clearly, $F_\xi\in\mathcal{D}$, cf. also Remark 3.2 in [27]. Moreover, we have for any $\xi,\eta\in\mathrm{L}^{1,d}_*$ that

$$\begin{aligned}d_1(F_\xi,F_\eta) &= \sup_{L_\varphi\leqslant 1}|F_\xi(\varphi)-F_\eta(\varphi)|\\ &= \sup_{L_\varphi\leqslant 1}\left|\hat{\mathbb{E}}[\varphi(\xi)]-\hat{\mathbb{E}}[\varphi(\eta)]\right|\\ &\leqslant \sup_{L_\varphi\leqslant 1}\hat{\mathbb{E}}[|\varphi(\xi)-\varphi(\eta)|]\\ &\leqslant \sup_{L_\varphi\leqslant 1}L_\varphi\,\hat{\mathbb{E}}[|\xi-\eta|]\\ &= \|\xi-\eta\|_{\mathrm{L}^1_*}\,.\end{aligned}\tag{4.1}$$

In [27], the authors consider dynamics of the form

$$\mathrm{d}X_s=b(s,X_s,F_{X_s})\,\mathrm{d}s+h(s,X_s,F_{X_s})\,\mathrm{d}\langle B\rangle_s+g(s,X_s,F_{X_s})\,\mathrm{d}B_s.\tag{4.2}$$

The coefficients $\tilde{b},\tilde{h},\tilde{g}$ are defined on $[0,T]\times\mathbb{R}^d\times\mathcal{D}$ such that the components $\tilde{f}=\tilde{b}_k,\tilde{h}_{kij},\tilde{g}_{ki}$ with $1\leqslant i,j\leqslant n$, $1\leqslant k\leqslant d$ satisfy

$$|\tilde{f}(t,x,F_1)-\tilde{f}(t,y,F_2)|\leqslant K\left(\|x-y\|+d_1(F_1,F_2)\right)$$

for some constant $K>0$, cf. (H1) in [27].

Now, we define b,g,h on $[0,T]\times\mathbb{R}^d\times\mathrm{L}^{2,d}_*\times\Omega$ by

$$\begin{aligned} b(s,x,\xi,\omega) &:= \tilde{b}(s,x,F_\xi), \\ h(s,x,\xi,\omega) &:= \tilde{h}(s,x,F_\xi), \\ g(s,x,\xi,\omega) &:= \tilde{g}(s,x,F_\xi), \end{aligned}$$

for all $0 \leqslant s \leqslant T$, $x \in \mathbb{R}^d$, $\xi \in \mathrm{L}_*^{2,d}$, $\omega \in \Omega$.

Note that the coefficients b, h, g are deterministic. For the components $f = b_k, h_{kij}, g_{ki}$, $1 \leqslant i,j \leqslant n$, $1 \leqslant k \leqslant d$, we have that

$$\begin{aligned} |f(t,x,\xi,\omega) - f(t,y,\eta,\omega)| &= |\tilde{f}(t,x,F_\xi) - \tilde{f}(t,y,F_\eta)| \\ &\leqslant K\left(\|x-y\| + d_1(F_\xi,F_\eta)\right) \\ &\leqslant K\left(\|x-y\| + \hat{\mathbb{E}}[|\xi-\eta|]\right) \end{aligned}$$

for all $\omega \in \Omega$, $0 \leqslant s \leqslant T$, $x, y \in \mathbb{R}^d$ and $\xi, \eta \in \mathrm{L}_*^{2,d}$ due to (4.1). In particular, by Jensen's inequality, we obtain

$$\begin{aligned} |f(t,x,\xi,\omega) - f(t,y,\eta,\omega)|^2 &\leqslant K^2\left(\|x-y\| + \hat{\mathbb{E}}[|\xi-\eta|]\right)^2 \\ &\leqslant 2\,K^2\left(\|x-y\|^2 + \hat{\mathbb{E}}\Big[|\xi-\eta|^2\Big]\right) \\ &= 2\,K^2\left(\|x-y\|^2 + \|\xi-\eta\|_{\mathrm{L}_*^2}^2\right), \end{aligned}$$

i.e., the coefficients b, h, g satisfy (3.3) from Assumption 3.1. Thus, Theorem 3.12 implies Theorem 4.1 in [27].

In [28], the author considers the G-SDE

$$\begin{aligned} \mathrm{d}X_s = \hat{\mathbb{E}}\Big[\tilde{b}\left(s,x,X_s\right)\Big]\Big|_{x=X_s}\mathrm{d}s + \sum_{i,j=1}^n \hat{\mathbb{E}}\Big[\tilde{h}_{ij}(s,x,X_s)\Big]\Big|_{x=X_s}\mathrm{d}\left\langle B^i, B^j\right\rangle_s \\ + \sum_{i=1}^n \hat{\mathbb{E}}\Big[\tilde{g}_i(s,x,X_s)\Big]\Big|_{x=X_s}\mathrm{d}B_s^i, \end{aligned}$$

where the functions $\tilde{b}, \tilde{h}_{ij}, \tilde{g}_i$, $1 \leqslant i,j \leqslant n$ are defined on $[0,T] \times \mathbb{R} \times \mathbb{R}$. That is,

$$\hat{\mathbb{E}}\left[\tilde{f}(s,x,X_s)\right]\Big|_{x=X_s} = F_{X_s}\tilde{f}(s,x,\cdot)\Big|_{x=X_s} =: f(s,X_s,F_{X_s})$$

for $\tilde{f} = \tilde{b}, \tilde{h}_{ij}, \tilde{g}_i$, $1 \leqslant i,j \leqslant n$. That is, it is a special case of (4.2) and, thus, also of (3.1) with $d = 1$.

Appendix

A Auxiliary results

Lemma A.1 *Let $0 \leqslant t \leqslant T$, $\eta \in \mathrm{B}_b(\Omega_t)$, and $X \in \mathrm{M}_b(0,T)$. Then, $X\eta\mathbf{1}_{[t,T]} \in \mathrm{M}_b(0,T)$.*

Proof Since $X \in \mathrm{M}_b(0,T)$, it is of the form

$$X = \sum_{k=0}^{N-1} \xi_k \mathbf{1}_{[t_k,t_{k+1})}$$

for some $N \in \mathbb{N}$, $0 = t_0 < \ldots < t_N = T$ and $\xi_k \in \mathrm{B}_b(\Omega_{t_k})$, $0 \leqslant k \leqslant N-1$. Fix $0 \leqslant l \leqslant N$ such that $t_l \leqslant t < t_{l+1}$. For each $l \leqslant k \leqslant N-1$, we have $\mathrm{B}_b(\Omega_t) \subseteq \mathrm{B}_b(\Omega_{t_k})$ and $\mathrm{B}_b(\Omega_{t_k})$ is closed under multiplication, i.e., $\xi_k\eta \in \mathrm{B}_b(\Omega_{t_k})$. Hence,

$$X\eta\mathbf{1}_{[t,T]} = \xi_l\eta\mathbf{1}_{[t,t_{l+1})} + \sum_{k=l+1}^{N-1} \xi_k\eta\mathbf{1}_{[t_k,t_{k+1})} \in \mathrm{M}_b(0,T),$$

as desired. □

Corollary A.2 *Let* $0 \leqslant t \leqslant T$, $\eta \in \mathrm{B}_b(\Omega_t)$, *and* $X \in \mathrm{M}_*^p(0,T)$, $p \geqslant 1$. *Then,* $X\eta\mathbf{1}_{[t,T]} \in \mathrm{M}_*^p(0,T)$.

Proof Since $X \in \mathrm{M}_*^1(0,T)$, there exists a sequence $(X^m)_{m\in\mathbb{N}}$ in $\mathrm{M}_b(0,T)$ such that

$$\lim_{m\to\infty} \|X^m - X\|_{\mathrm{M}_*^p} = 0.$$

By Lemma A.1, we have $X^m\eta\mathbf{1}_{[t,T]} \in \mathrm{M}_b(0,T)$ for all $m \in \mathbb{N}$. Since $\eta \in \mathrm{B}_b(\Omega_t)$, there exists a constant $M > 0$ such that $|\eta| < M$. Thus,

$$\lim_{m\to\infty} \left\|X^m\eta\mathbf{1}_{[t,T]} - X\eta\mathbf{1}_{[t,T]}\right\|_{\mathrm{M}_*^p} \leqslant \lim_{m\to\infty} M\,\|X^m - X\|_{\mathrm{M}_*^p} = 0.$$

Since $\mathrm{M}_*^p(0,T)$ is the completion of $\mathrm{M}_b(0,T)$ with respect to the M_*^p-norm, we obtain that $X\eta\mathbf{1}_{[t,T]} \in \mathrm{M}_*^p(0,T)$. □

Corollary A.3 *Let* $X \in \mathrm{M}_b(0,T)$ *and* $Y \in \mathrm{M}_*^p(0,T)$. *Then,* $XY \in \mathrm{M}_*^p(0,T)$.

Proof Since $X \in \mathrm{M}_b(0,T)$, it is of the form

$$X = \sum_{k=0}^{N-1} \xi_k\mathbf{1}_{[t_k,t_{k+1})}$$

for some $N \in \mathbb{N}$, $0 = t_0 < \ldots < t_N = T$ and $\xi_k \in \mathrm{B}_b(\Omega_{t_k})$, $0 \leqslant k \leqslant N-1$. We have

$$XY = \sum_{k=0}^{N-1} \xi_k Y\mathbf{1}_{[t_k,t_{k+1})}.$$

Corollary A.2 implies $\xi_k Y\mathbf{1}_{[t_k,t_{k+1})} \in \mathrm{M}_*^p(0,T)$ and, thus, $XY \in \mathrm{M}_*^p(0,T)$ as finite sum of elements in $\mathrm{M}_*^p(0,T)$. □

Lemma A.4 *Let* (E,d_E) *be a metric space, and* $\xi : \Omega \to E$ *be* $\mathcal{F}_t$*-measurable and such that* image $\xi \subseteq K$ *for some compact* $K \subseteq E$. *Moreover, let* $p \geqslant 1$ *and* $f : [0,T] \times E \times \Omega \to \mathbb{R}$ *be such that* $f(\cdot,x) \in \mathrm{M}_*^p(0,T)$ for all $x \in E$. *If there exist a constant* $\delta > 0$, *a process* $M \in \mathrm{M}_*^1(t,T)$ *and an increasing function* $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ *with* $\lim_{\varepsilon\downarrow 0} \rho(\varepsilon) = 0$ *such that*

$$|f(s,x,\omega) - f(s,y,\omega)|^p \leqslant M_s(\omega)\,\rho(d_E(x,y)) \tag{A.1}$$

for all $x,y \in E$ *with* $d_E(x,y) < \delta$, *quasi all* $\omega \in \Omega$, *and almost all* $0 \leqslant s \leqslant T$; *then* $f(\cdot,\eta)\mathbf{1}_{[t,T]} \in \mathrm{M}_*^p(0,T)$.

Proof Let $0 < \varepsilon < \delta$ and $B_\varepsilon(y)$ denote the open ε-ball centered at $y \in E$. Then, $\{B_\varepsilon(y)\}_{y\in K}$ is an open cover of K. Since K is compact, we can fix a finite $I \subseteq K$ such that $\{B_\varepsilon(y)\}_{y\in I}$ is an open cover of K. Moreover, there exists a partition of unity $\{\psi_y^\varepsilon\}_{y\in I}$ subordinate to $\{B_\varepsilon(y)\}_{y\in I}$. That is, $\psi_y^\varepsilon : E \to [0,1]$ is continuous with supp $\psi_y^\varepsilon \subseteq B_\varepsilon(y)$ for each $y \in I$, and $\sum_{y\in I} \psi_y^\varepsilon(x) = 1$ for all $x \in K$.

Define the function $f_\eta^\varepsilon : [0,T] \times \Omega \to \mathbb{R}$ by

$$f_\eta^\varepsilon := \sum_{y\in I} f(\cdot,y)\,\psi_y^\varepsilon(\eta)\,\mathbf{1}_{[t,T]}.$$

For each $y \in I$, the concatenation $\psi_y^\varepsilon(\eta) : \Omega \to [0,1]$ is $\mathcal{F}_t$-measurable and bounded, i.e., $\psi_y^\varepsilon(\eta) \in \mathrm{B}_b(\Omega_t)$. Moreover, $f(\cdot, y) \in \mathrm{M}_*^p(0,T)$ implies $f(\cdot,y)\,\psi_y^\varepsilon(\eta)\,\mathbf{1}_{[t,T]} \in \mathrm{M}_*^p(0,T)$ due to Corollary A.2. Thus, $f_\eta^\varepsilon \in \mathrm{M}_*^p(0,T)$ as finite sum of elements in $\mathrm{M}_*^p(0,T)$.

For $t \leqslant s \leqslant T$, we have

$$\begin{aligned}
\hat{\mathbb{E}}\big[\big|f(s,\eta) - f_\eta^\varepsilon(s)\big|^p\big] &= \hat{\mathbb{E}}\left[\left|f(s,\eta) - \sum_{y\in I} f(s,y)\,\psi_y^\varepsilon(\eta)\right|^p\right] \\
&= \hat{\mathbb{E}}\left[\left|\sum_{y\in I} \big(f(s,\eta) - f(s,y)\big)\,\psi_y^\varepsilon(\eta)\right|^p\right] \\
&\leqslant \hat{\mathbb{E}}\left[\sum_{y\in I} |f(s,\eta) - f(s,y)|^p\,\psi_y^\varepsilon(\eta)\right]. \qquad \text{(A.2)}
\end{aligned}$$

There exists a polar set $N \subseteq \Omega$ such that (A.1) holds for all $\omega \in \Omega \setminus N$, almost every $0 \leqslant s \leqslant T$ and all $x, y \in E$ with $d_E(x,y) < \delta$. By construction, $\psi_y^\varepsilon(\eta) = 0$ on $\{\eta \notin B_\varepsilon(y)\}$. Furthermore,

$$|f(s,\eta) - f(s,y)|^p \leqslant M_s\,\rho(d_E(\eta,y)) < M_s\,\rho(\varepsilon) \quad \text{on } \{\eta \in B_\varepsilon(y)\} \setminus N$$

for almost every $t \leqslant s \leqslant T$. Hence,

$$|f(s,\eta) - f(s,y)|^p\,\psi_y^\varepsilon(\eta) < M_s\,\rho(\varepsilon)\,\psi_y^\varepsilon(\eta) \quad \text{on } \Omega \setminus N$$

for almost every $t \leqslant s \leqslant T$. Since N is polar, we deduce from (A.2) that

$$\begin{aligned}
\big\|f(s,\eta)\mathbf{1}_{[t,T]} - f_\eta^\varepsilon(s)\big\|_{\mathrm{M}_*^p}^p &= \int_t^T \hat{\mathbb{E}}\big[\big|f(s,\eta) - f_\eta^\varepsilon(s)\big|^p\big]\,\mathrm{d}s \\
&\leqslant \int_t^T \hat{\mathbb{E}}\left[\sum_{y\in I} |f(s,\eta) - f(s,y)|^p\,\psi_y^\varepsilon(\eta)\right]\mathrm{d}s \\
&< \int_t^T \hat{\mathbb{E}}\left[\sum_{y\in I} M_s\,\rho(\varepsilon)\,\psi_y^\varepsilon(\eta)\right]\mathrm{d}s \\
&= \int_t^T \hat{\mathbb{E}}[M_s]\,\mathrm{d}s\;\rho(\varepsilon) \\
&\leqslant \|M\|_{\mathrm{M}_*^1}\,\rho(\varepsilon)\,.
\end{aligned}$$

Since $\lim_{\varepsilon\downarrow 0} \rho(\varepsilon) = 0$, we have $f_\eta^\varepsilon \to f(\cdot,\eta)$ with respect to the M_*^p-norm as $\varepsilon \downarrow 0$. Finally, we obtain $f(\cdot,\eta)\mathbf{1}_{[t,T]} \in \mathrm{M}_*^p(0,T)$ because $\mathrm{M}_*^p(0,T)$ is complete. □

B. Proofs to results in Section 2

Proof of Lemma 2.2 For each $0 \leqslant k \leqslant N-1$, there exists a sequence $(\xi_k^m)_{m\in\mathbb{N}}$ in $\mathrm{B}_b(\Omega_{t_k})$ with

$$\lim_{m\to\infty} \hat{\mathbb{E}}\left[|\xi_k - \xi_k^m|^p\right] = 0.$$

For each $m \in \mathbb{N}$, we have

$$X^m := \sum_{k=0}^{N-1} \xi_k^m \mathbf{1}_{[t_k,t_{k+1})} \in \mathrm{M}_b(t,T).$$

Further,

$$\lim_{m\to\infty}\int_t^T \hat{\mathbb{E}}\left[|X_s - X_s^m|^p\right]\mathrm{d}s = \lim_{m\to\infty}\sum_{k=0}^{N-1}\hat{\mathbb{E}}\left[|\xi_k - \xi_k^m|^p\right](t_{k+1}-t_k) = 0.$$

Hence, $X^m \in \mathrm{M}_b(t,T)$ converges to X with respect to $\|\cdot\|_{\mathrm{M}_*^p}$ as $m\to\infty$. Thus, $X \in \mathrm{M}_*^p(t,T)$ since $\mathrm{M}_*^p(t,T)$ the completion of $\mathrm{M}_b(t,T)$. □

Proof of Lemma 2.3 For $m\in\mathbb{N}$, set $t_k^m := \frac{kT}{m}$ and define

$$B^{a,m} := \sum_{k=1}^{m-1} B^a_{t_k^m}\mathbf{1}_{[t_k^m,t_{k+1}^m)}.$$

Clearly, $B^a_{t_k^m} \in \mathrm{L}_*^2(t_k^m)$ for $0\leqslant k\leqslant m-1$; hence, $B^{a,m}\in\mathrm{M}_*^2(0,T)$ due to Lemma 2.2. We have

$$\begin{aligned}\int_0^T \hat{\mathbb{E}}\left[|B_s^a - B_s^{a,m}|^2\right]\mathrm{d}s &= \sum_{k=0}^{m-1}\int_{t_k^m}^{t_{k+1}^m}\hat{\mathbb{E}}\left[\left|B_s^a - B^a_{t_k^m}\right|^2\right]\mathrm{d}s\\ &= \sum_{k=0}^{m-1}\int_{t_k^m}^{t_{k+1}^m}\overline{\sigma}_{aa}^2\,(s-t_k^m)\,\mathrm{d}s\\ &\leqslant \sum_{k=0}^{m-1}\overline{\sigma}_{aa}^2\frac{T^2}{m^2}\\ &= \frac{\overline{\sigma}_{aa}^2\,T^2}{m},\end{aligned}$$

which tends to 0 when $m\to\infty$, i.e., $B^{a,m}\to B^a$ with respect to $\|\cdot\|_{\mathrm{M}_*^2}$. Since $\mathrm{M}_*^2(0,T)$ is complete, we deduce $B^a\in\mathrm{M}_*^2(0,T)$. □

Proof of Lemma 2.6 First, suppose $X\in\mathrm{M}_b(0,T)$. Then, there exist $m\in\mathbb{N}$, $0 = t_0<\ldots<t_m = T$, and $\xi_k\in\mathrm{B}_b(\Omega_{t_k})$, $0\leqslant k\leqslant N-1$ such that

$$X = \sum_{k=0}^{m-1}\xi_k\mathbf{1}_{[t_k,t_{k+1})}.$$

We have

$$\begin{aligned}Z &= \int_0^{\cdot} X_u\mathrm{d}B_u^a\\ &= \sum_{k=0}^{m-1}\xi_k\left(B^a_{t_{k+1}\wedge\cdot} - B^a_{t_k\wedge\cdot}\right)\\ &= \sum_{k=0}^{m-1}\xi_k\left(B^a_{\cdot} - B^a_{t_k}\right)\mathbf{1}_{[t_k,t_{k+1})} + \sum_{k=0}^{m-1}\xi_k\left(B^a_{t_{k+1}} - B^a_{t_k}\right)\mathbf{1}_{[t_{k+1},T)}.\end{aligned}\tag{B.1}$$

For the former sum on the right-hand side of (B.1), note that

$$\xi_k\left(B^a_{\cdot} - B^a_{t_k}\right)\mathbf{1}_{[t_k,t_{k+1})} = \xi_k\mathbf{1}_{[t_k,t_{k+1})}\cdot\left(B^a_{\cdot} - B^a_{t_k}\mathbf{1}_{[t_k,t_{k+1})}\right).$$

Clearly, $B^a_{t_k}\in\mathrm{L}_*^2(t_k)$ and, thus, $B^a_{t_k}\mathbf{1}_{[t_k,t_{k+1})}\in\mathrm{M}_*^2(0,T)$ due to Lemma 2.2. By Lemma 2.3, we have $B^a\in\mathrm{M}_*^2(0,T)$ and, thus, $(B^a - B^a_{t_k}\mathbf{1}_{[t_k,t_{k+1})})\in\mathrm{M}_*^2(0,T)$. Since $\xi_k\mathbf{1}_{[t_{k+1},T)}\in\mathrm{M}_b(0,T)$, we obtain $\xi_k\left(B^a_{\cdot} - B^a_{t_k}\right)\mathbf{1}_{[t_k,t_{k+1})}\in\mathrm{M}_*^2(0,T)$ due to Corollary A.3.

Similarly, for the latter sum on the right-hand side of (B.1),

$$\xi_k\left(B^a_{t_{k+1}} - B^a_{t_k}\right)\mathbf{1}_{[t_{k+1},T)} = \xi_k\mathbf{1}_{[t_{k+1},T)}\cdot\left(B^a_{t_{k+1}} - B^a_{t_k}\right)\mathbf{1}_{[t_{k+1},T)}.$$

We have $B^a_{t_k}, B^a_{t_{k+1}} \in \mathrm{L}^2_*(t_{k+1})$ and thus $(B^a_{t_{k+1}} - B^a_{t_k})\mathbf{1}_{[t_{k+1},T)} \in \mathrm{M}^2_*(0,T)$ due to Lemma 2.2. Corollary A.3 yields $\xi_k(B^a_{t_{k+1}} - B^a_{t_k})\mathbf{1}_{[t_{k+1},T)} \in \mathrm{M}^2_*(0,T)$.

Finally, we deduce that $Z \in \mathrm{M}^2_*(0,T)$ as finite sum of elements in $\mathrm{M}^2_*(0,T)$.

Now, suppose $X \in \mathrm{M}^2_*(0,T)$. Then, there exists a sequence $(X^m)_{m\in\mathbb{N}}$ in $\mathrm{M}_b(0,T)$ with

$$\lim_{m\to\infty} \|X - X^m\|_{\mathrm{M}^2_*} = 0.$$

For each $m \in \mathbb{N}$, define Z^m by

$$Z^m_t := \int_0^t X^m_s \mathrm{d}B^a_s = \mathcal{I}_a\left(X^m \mathbf{1}_{[0,t)}\right), \quad 0 \leqslant t \leqslant T.$$

Then, $Z^m \in \mathrm{M}^2_*(0,T)$. By Lemma 2.4, we have

$$\hat{\mathbb{E}}\left[|Z^m_t - Z_t|^2\right] \leqslant \overline{\sigma}^2_{aa} \left\|(X^m - X)\,\mathbf{1}_{[0,t)}\right\|^2_{\mathrm{M}^2_*} \leqslant \overline{\sigma}^2_{aa} \|X^m - X\|^2_{\mathrm{M}^2_*},$$

and thus

$$\lim_{m\to\infty} \int_0^T \hat{\mathbb{E}}\left[|Z^m_t - Z_t|^2\right] \mathrm{d}t \leqslant \overline{\sigma}^2_{aa}\, T\, \|X^m - X\|^2_{\mathrm{M}^2_*} = 0.$$

Hence, $Z \in \mathrm{M}^2_*(0,T)$ since $\mathrm{M}^2_*(0,T)$ is complete. □

Proof of Lemma 2.8 First, suppose $X \in \mathrm{M}_b(0,T)$. Then, there exist $m \in \mathbb{N}$, $0 = t_0 < \ldots < t_m = T$, and $\xi_k \in \mathrm{B}_b(\Omega_{t_k})$, $0 \leqslant k \leqslant m-1$ such that

$$X = \sum_{k=0}^{N-1} \xi_k \mathbf{1}_{[t_k,t_{k+1})}.$$

Analogous to the proof of Lemma 2.6, we have

$$Z = \sum_{k=0}^{m-1} \xi_k \left(\langle B^a\rangle_\cdot - \langle B^a\rangle_{t_k}\right)\mathbf{1}_{[t_k,t_{k+1})} + \sum_{k=0}^{N-1} \xi_k \left(\langle B^a\rangle_{t_{k+1}} - \langle B^a\rangle_{t_k}\right)\mathbf{1}_{[t_{k+1},T)}. \tag{B.2}$$

For the former sum on the right-hand side of (B.2), note that

$$\xi_k \left(\langle B^a\rangle_\cdot - \langle B^a\rangle_{t_k}\right)\mathbf{1}_{[t_k,t_{k+1})} = \xi_k \mathbf{1}_{[t_k,t_{k+1})} \cdot \left(\langle B^a\rangle_\cdot - \langle B^a\rangle_{t_k}\mathbf{1}_{[t_k,t_{k+1})}\right).$$

Clearly, $\langle B^a\rangle_{t_k} \in \mathrm{L}^1_*(t_k)$. Thus, $\langle B^a\rangle_{t_k}\mathbf{1}_{[t_k,t_{k+1})} \in \mathrm{M}^1_*(0,T)$ due to Lemma 2.2. By Lemma 2.3, we have $\langle B^a\rangle \in \mathrm{M}^1_*(0,T)$ and thus $(\langle B^a\rangle - \langle B^a\rangle_{t_k}\mathbf{1}_{[t_k,t_{k+1})}) \in \mathrm{M}^1_*(0,T)$. Since $\xi_k\mathbf{1}_{[t_{k+1},T)} \in \mathrm{M}_b(0,T)$, we obtain $\xi_k\left(\langle B^a\rangle_\cdot - \langle B^a\rangle_{t_k}\right)\mathbf{1}_{[t_k,t_{k+1})} \in \mathrm{M}^1_*(0,T)$ due to Corollary A.3.

Similarly, for the latter sum on the right-hand side of (B.2),

$$\xi_k \left(\langle B^a\rangle_{t_{k+1}} - \langle B^a\rangle_{t_k}\right)\mathbf{1}_{[t_{k+1},T)} = \xi_k \mathbf{1}_{[t_{k+1},T)} \cdot \left(\langle B^a\rangle_{t_{k+1}} - \langle B^a\rangle_{t_k}\right)\mathbf{1}_{[t_{k+1},T)}.$$

We have $\langle B^a\rangle_{t_k}, \langle B^a\rangle_{t_{k+1}} \in \mathrm{L}^1_*(t_{k+1})$ and thus $(\langle B^a\rangle_{t_{k+1}} - \langle B^a\rangle_{t_k})\mathbf{1}_{[t_{k+1},T)} \in \mathrm{M}^1_*(0,T)$ due to Lemma 2.2. Corollary A.3 yields $\xi_k(\langle B^a\rangle_{t_{k+1}} - \langle B^a\rangle_{t_k})\mathbf{1}_{[t_{k+1},T)} \in \mathrm{M}^1_*(0,T)$.

Finally, we deduce that $Z \in \mathrm{M}^1_*(0,T)$ as finite sum of elements in $\mathrm{M}^1_*(0,T)$.

Now, suppose $X \in \mathrm{M}^1_*(0,T)$. Then, there exists a sequence $(X^m)_{m\in\mathbb{N}}$ in $\mathrm{M}_b(0,T)$ with

$$\lim_{m\to\infty} \|X - X^m\|_{\mathrm{M}^1_*} = 0.$$

For each $m \in \mathbb{N}$, define Z^m by

$$Z^m_t := \int_0^t X^m_s \mathrm{d}\langle B^a\rangle_s = \mathcal{Q}_a\left(X^m \mathbf{1}_{[0,t)}\right), \quad 0 \leqslant t \leqslant T.$$

Then, $Z^m \in \mathrm{M}^1_*(0,T)$. By Lemma 2.5, we have

$$\hat{\mathbb{E}}\left[|Z^m_t - Z_t|\right] \leqslant \overline{\sigma}^2_{aa} \left\|(X^m - X)\,\mathbf{1}_{[0,t)}\right\|_{\mathrm{M}^1_*} \leqslant \overline{\sigma}^2_{aa} \left\|X^m - X\right\|_{\mathrm{M}^1_*}.$$

Thus,

$$\lim_{m\to\infty} \int_0^T \hat{\mathbb{E}}\left[\left|Z^m_t - Z_t\right|^1\right] \mathrm{d}t \leqslant \lim_{m\to\infty} \overline{\sigma}^2_{aa}\, T \left\|X^m - X\right\|_{\mathrm{M}^1_*} = 0.$$

Hence, $Z \in \mathrm{M}^1_*(0,T)$ since $\mathrm{M}^1_*(0,T)$ is complete. □

Proof of Lemma 2.10 First, suppose $X \in \mathrm{M}_b(t,T)$. Then, there exist $m \in \mathbb{N}$, $t = t_0 < \ldots < t_m = T$, and $\xi_k \in \mathrm{B}_b(\Omega_{t_k})$, $0 \leqslant k \leqslant m-1$ such that

$$X = \sum_{k=0}^{m-1} \xi_k \mathbf{1}_{[t_k,t_{k+1})}.$$

Analogous to the proof of Lemma 2.6, we have

$$Z = \sum_{k=0}^{m-1} \xi_k \left(\mathrm{id}_{[0,T]} - t_k\right) \mathbf{1}_{[t_k,t_{k+1})} + \sum_{k=0}^{m-1} \xi_k \left(t_{k+1} - t_k\right) \mathbf{1}_{[t_{k+1},T)}. \tag{B.3}$$

For the former sum on the right-hand side of (B.3), note that

$$\xi_k \left(\mathrm{id}_{[t,T]} - t_k\right) \mathbf{1}_{[t_k,t_{k+1})} = \xi_k \mathbf{1}_{[t_k,t_{k+1})} \cdot \left(\mathrm{id}_{[t,T]} - t_k \mathbf{1}_{[t_k,t_{k+1})}\right).$$

Clearly, $t_k \mathbf{1}_{[t_k,t_{k+1})} \in \mathrm{M}_b(0,T)$ since t_k is constant. By Lemma 2.9, we have $\mathrm{id}_{[t,T]} \in \mathrm{M}^p_*(t,T)$ and, thus, $(\mathrm{id}_{[t,T]} - t_k \mathbf{1}_{[t_k,t_{k+1})}) \in \mathrm{M}^p_*(t,T)$. Since $\xi_k \mathbf{1}_{[t_{k+1},T)} \in \mathrm{M}_b(t,T)$, we obtain $\xi_k(\mathrm{id}_{[t,T]} - t_k \mathbf{1}_{[t_k,t_{k+1})})\mathbf{1}_{[t_k,t_{k+1})} \in \mathrm{M}^p_*(t,T)$ due to Corollary A.3.

Similarly, for the latter sum on the right-hand side of (B.2),

$$\xi_k \left(\langle B^a\rangle_{t_{k+1}} - \langle B^a\rangle_{t_k}\right) \mathbf{1}_{[t_{k+1},T)} = \xi_k \mathbf{1}_{[t_{k+1},T)} \cdot \left(\langle B^a\rangle_{t_{k+1}} - \langle B^a\rangle_{t_k}\right) \mathbf{1}_{[t_{k+1},T)}.$$

We have $\langle B^a\rangle_{t_k}, \langle B^a\rangle_{t_{k+1}} \in \mathrm{L}^1_*(t_{k+1})$ and thus $(\langle B^a\rangle_{t_{k+1}} - \langle B^a\rangle_{t_k})\mathbf{1}_{[t_{k+1},T)} \in \mathrm{M}^1_*(0,T)$ due to Lemma 2.2. Corollary A.3 yields $\xi_k(\langle B^a\rangle_{t_{k+1}} - \langle B^a\rangle_{t_k})\mathbf{1}_{[t_{k+1},T)} \in \mathrm{M}^1_*(0,T)$.

Finally, we deduce that $Z \in \mathrm{M}^1_*(0,T)$ as finite sum of elements in $\mathrm{M}^1_*(0,T)$.

Now, suppose $X \in \mathrm{M}^1_*(0,T)$. Then, there exists a sequence $(X^m)_{m\in\mathbb{N}}$ in $\mathrm{M}_b(0,T)$ with

$$\lim_{m\to\infty} \left\|X - X^m\right\|_{\mathrm{M}^1_*} = 0.$$

For each $m \in \mathbb{N}$, define Z^m by

$$Z^m_t := \int_0^t X^m_s \mathrm{d}\langle B^a\rangle_s = \mathcal{Q}_a\left(X^m \mathbf{1}_{[0,t)}\right), \quad 0 \leqslant t \leqslant T.$$

Then, $Z^m \in \mathrm{M}^1_*(0,T)$. By Lemma 2.5, we have

$$\hat{\mathbb{E}}\left[|Z^m_t - Z_t|\right] \leqslant \overline{\sigma}^2_{aa} \left\|(X^m - X)\,\mathbf{1}_{[0,t)}\right\|_{\mathrm{M}^1_*} \leqslant \overline{\sigma}^2_{aa} \left\|X^m - X\right\|_{\mathrm{M}^1_*}.$$

Thus,

$$\lim_{m\to\infty} \int_0^T \hat{\mathbb{E}}\left[\left|Z^m_t - Z_t\right|^1\right] \mathrm{d}t \leqslant \lim_{m\to\infty} \overline{\sigma}^2_{aa}\, T \left\|X^m - X\right\|_{\mathrm{M}^1_*} = 0.$$

Hence, $Z \in \mathrm{M}^1_*(0,T)$ since $\mathrm{M}^1_*(0,T)$ is complete. □

Proof of Lemma 2.11 Let $P \in \mathcal{P}$, then B is a continuous martingale under P. By the martingale representation theorem, there exists P-Brownian motion W^P and a progressively

measurable σ^P such that $B = \sigma^P \cdot W^P$ under P. Moreover, we can choose σ^P to be Σ-valued due to the construction of $\mathcal{P}$.

By the definition of the quadratic covariation, we have for all $0 \leqslant t \leqslant T$ under P that

$$\begin{aligned}
\left\langle B^a, B^b \right\rangle_t &= \frac{1}{4}\left(\left\langle B^{a+b}\right\rangle_t - \left\langle B^{a-b}\right\rangle_t\right) \\
&= \frac{1}{4}\left(\left\langle (a+b)^T B\right\rangle_t - \left\langle (a-b)^T B\right\rangle_t\right) \\
&= \frac{1}{4}\left(\left\langle (a+b)^T \sigma^P \cdot W^P\right\rangle_t - \left\langle (a-b)^T \sigma^P \cdot W^P\right\rangle_t\right) \\
&= \frac{1}{4}\left(\int_0^t (a+b)^T \sigma_s^P \mathrm{d}\left\langle W^P\right\rangle_s \sigma_s^P (a+b) - \int_0^t (a-b)^T \sigma_s^P \mathrm{d}\left\langle W^P\right\rangle_s \sigma_s^P (a-b)\right) \\
&= \frac{1}{4}\int_0^t (a+b)^T \sigma_s^P \sigma_s^P (a+b) - (a-b)^T \sigma_s^P \sigma_s^P (a-b)\,\mathrm{d}s \\
&= \int_0^t a^T \sigma_s^P \sigma_s^P b \mathrm{d}s.
\end{aligned}$$

Using the representation of $\hat{\mathbb{E}}$ as upper expectation, we obtain

$$\begin{aligned}
\hat{\mathbb{E}}\left[\sup_{t\leqslant w\leqslant s}\left|\int_t^w X_u \mathrm{d}\left\langle B^a, B^b\right\rangle_u\right|^p\right] &= \sup_{P\in\mathcal{P}} E_P\left[\sup_{t\leqslant w\leqslant s}\left|\int_t^w X_u a^T \sigma_u^P \sigma_u^P b \mathrm{d}u\right|^p\right] \\
&\leqslant (s-t)^{p-1} \sup_{P\in\mathcal{P}} E_P\left[\int_t^s \left|X_u a^T \sigma_u^P \sigma_u^P b\right|^p \mathrm{d}u\right] \\
&\leqslant \overline{\sigma}_{ab}^{2p} (s-t)^{p-1} \sup_{P\in\mathcal{P}} E_P\left[\int_t^s |X_u|^p \mathrm{d}u\right] \\
&= \overline{\sigma}_{ab}^{2p} (s-t)^{p-1} \hat{\mathbb{E}}\left[\int_t^s |X_u|^p \mathrm{d}u\right] \\
&\leqslant \overline{\sigma}_{ab}^{2p} (s-t)^{p-1} \int_t^s \hat{\mathbb{E}}\left[|X_u|^p\right] \mathrm{d}u.
\end{aligned}$$

□

Proof of Lemma 2.12 Let $P \in \mathcal{P}$, then $X \cdot B$ is a continuous local martingale. By the Burkholder-Davis-Gundy inequality, there exists a constant $C_p > 0$ such that

$$\begin{aligned}
\hat{\mathbb{E}}\left[\sup_{t\leqslant w\leqslant s}\left|\int_t^w X_u \mathrm{d}B_u^a\right|^p\right] &= \sup_{P\in\mathcal{P}} E_P\left[\sup_{t\leqslant w\leqslant s}\left|\int_t^w X_u \mathrm{d}B_u^a\right|^p\right] \\
&\leqslant C_p \sup_{P\in\mathcal{P}} E_P\left[\sup_{t\leqslant w\leqslant s}\left|\int_t^w X_u^2 \mathrm{d}\left\langle B^a\right\rangle_u\right|^{\frac{p}{2}}\right] \\
&= C_p \hat{\mathbb{E}}\left[\sup_{t\leqslant w\leqslant s}\left|\int_t^w X_u^2 \mathrm{d}\left\langle B^a\right\rangle_u\right|^{\frac{p}{2}}\right] \\
&\leqslant C_p\, \overline{\sigma}_{aa}^p (s-t)^{\frac{p-2}{2}} \int_t^s \hat{\mathbb{E}}\left[|X_u|^p\right] \mathrm{d}u,
\end{aligned}$$

where the last step follows from Lemma 2.11 since $X^2 \in \mathrm{M}_*^q(0,T)$ with $q = \frac{p}{2} \geqslant 1$. □

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