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Averaging principles for forward-backward multivalued stochastic systems and applications to systems of nonlinear parabolic partial differential equations

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Abstract This work concerns a type of stochastic systems in which the forward equations are general stochastic differential equations and the backward equations are stochastic variational inequalities. We first prove an averaging principle for general stochastic differential equations in the L^{2p} $(p \geqslant 1)$ sense. In addition, a convergence rate for $p = 1$ is presented. Combining general stochastic differential equations with backward stochastic variational inequalities, we then establish another averaging principle for backward stochastic variational inequalities in the L^2 sense using a time discretization method. Finally, we apply our result to nonlinear parabolic partial differential equations to obtain their averaging principles.

Keywords Averaging principles, Backward stochastic variational inequalities, Averaging principles for nonlinear parabolic partial differential equations

2020 Mathematics Subject Classification 60H10, 35B27

1. Introduction

We take a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ on which l-dimensional standard Brownian motion W is defined, and we assume that $\mathbb{F} := (\mathscr{F}_s)_{s\geqslant 0}$ is the $\mathbb{P}$-augmentation for the natural filtration of W. We set $t \geqslant 0$ and consider the following stochastic differential equation (SDE): for any $s \geqslant t$,

$$\left\{\begin{array}{l} \mathrm{d}X_s^{\varepsilon,t,\zeta} = b(\frac{s}{\varepsilon}, X_s^{\varepsilon,t,\zeta})\mathrm{d}s + \sigma(\frac{s}{\varepsilon}, X_s^{\varepsilon,t,\zeta})\mathrm{d}W_s, \\ X_t^{\varepsilon,t,\zeta} = \zeta, \end{array}\right. \tag{1}$$

where the mappings $b : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}^m$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}^{m\times l}$ are both Borel measurable. ζ is a $\mathscr{F}_t$ -measurable random variable, and $0 < \varepsilon < 1$ is a small parameter. Eq.(1) is usually called a multiscale SDE. People care about the limit of its solution as $\varepsilon \to 0$, that is, the averaging principle. There have been many studies related to the averaging principles for Eq.(1). For example, in [21], N'Goran and N'Zi studied the averaging principle for multivalued SDEs in the probability sense. Later, Xu and Liu [30] improved the convergence result in [21] in the L^2 sense.

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Recently, Guo et al. [15] established an averaging principle for Eq.(1) in the L^2 sense when b and σ satisfy the local Lipschitz and monotone conditions. Shen et al. [29] also proved an averaging principle in the L^2 sense when b and σ depend on the distribution of $X_s^{\varepsilon,t,\zeta}$. Herein, we present an averaging principle for Eq.(1) in the L^{2p} sense. In addition, a convergence rate for $p=1$ is presented.

Next, we couple Eq.(1) with a backward stochastic variational inequality (SVI), and we investigate the limit for the forward-backward multivalued stochastic system as $\varepsilon \to 0$. Concretely speaking, we set $T > t$ and consider the following backward SVI: for any $t \leqslant s \leqslant T$,

$$\left\{\begin{array}{l} \mathrm{d}Y_s^{\varepsilon,t,\zeta} \in \partial\varphi(Y_s^{\varepsilon,t,\zeta})\mathrm{d}s - [f_1(\frac{s}{\varepsilon}, X_s^{\varepsilon,t,\zeta}, Y_s^{\varepsilon,t,\zeta}) + f_2(Z_s^{\varepsilon,t,\zeta})]\mathrm{d}s + Z_s^{\varepsilon,t,\zeta}\mathrm{d}W_s, \\ Y_T^{\varepsilon,t,\zeta} = g(X_T^{\varepsilon,t,\zeta}) \in \overline{\mathcal{D}(\partial\varphi)}, \end{array}\right. \tag{2}$$

where φ is a proper convex lower semicontinuous function, $\partial\varphi$ is its subdifferential operator, and the mappings $f_1 : \mathbb{R}_+ \times \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}^d$, $f_2 : \mathbb{R}^{d\times l} \to \mathbb{R}^d$, and $g : \mathbb{R}^m \to \mathbb{R}^d$ are all Borel measurable. If we take $\varphi = I_{\mathcal{O}}$, where $\mathcal{O}$ is a closed convex subset of $\mathbb{R}^d$ and $\mathrm{Int}(\mathcal{O}) \neq \emptyset$ (see example 2.1), Eq.(2) becomes a reflected backward SDE. We note that our reflected backward SDEs are different from reflected backward SDEs in [9, 12] for the 1-dimensional case, and from those in [11, 14, 19] for the multidimensional case. The difference for the 1-dimensional case is that our boundary is fixed, whereas those results allow randomly moving boundaries. Furthermore, the difference for the multidimensional case is that the conditions that the solution processes satisfy in the reflected backward SDEs are more than the conditions for the solution processes in ours. That is, those reflected backward SDEs cover ours. In addition, it is worth noting that in [8], Chassagneux, Nadtochiy, and Richou proved the well-posedness of our reflected backward SDEs when $\mathcal{O}$ is non-convex and has a weak star-shaped property.

Essaky and Ouknine [13] concluded that in the case where $f_1(s,x,y)$ is independent of s and $f_2(z) = 0$, if the solution $X^{\varepsilon,t,\zeta}$ of Eq.(1) converges in law to the solution of the corresponding averaging equation, the solution $Y^{\varepsilon,t,\zeta}$ of Eq.(2) also converges in law to the solution of the corresponding averaging equation. Recently, Hu et al. [16] studied Eq.(2) with $\varphi = 0$. However, they did not explicitly express the convergence of $X^{\varepsilon,t,\zeta}$ or $Y^{\varepsilon,t,\zeta}$. Herein, we prove that $Y^{\varepsilon,t,\zeta}$ converges to the solution of the corresponding averaging equation in the L^2 sense. In addition, we note that if $f_1(s,x,y)$ is independent of x and if $g(X_T^{\varepsilon,t,\zeta})$ is replaced by a random variable, a convergence rate can be obtained. However, we do not measure the convergence rate because we want to apply the averaging principle for Eq.(2) to the averaging of nonlinear parabolic partial differential equations (PDEs).

In the following, we consider the nonlinear PDE:

$$\left\{\begin{array}{l} \frac{\partial u^\varepsilon(t,x)}{\partial t} + \mathscr{L}^\varepsilon u^\varepsilon(t,x) + f_1(\frac{t}{\varepsilon}, x, u^\varepsilon(t,x)) + f_2(\nabla u^\varepsilon(t,x)\sigma(\frac{t}{\varepsilon}, x)) \in \partial\varphi(u^\varepsilon(t,x)), \quad t \in [0,T], \\ u^\varepsilon(T,x) = g(x), \quad u^\varepsilon(t,x) \in \overline{\mathrm{Dom}(\varphi)}, \quad x \in \mathbb{R}^m, \end{array}\right. \tag{3}$$

where

$$\mathscr{L}^\varepsilon := \frac{1}{2}\sum_{i,j=1}^m (\sigma\sigma^*)_{ij}\left(\frac{t}{\varepsilon}, x\right)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^m b_i\left(\frac{t}{\varepsilon}, x\right)\frac{\partial}{\partial x_i}.$$

Eq.(3) is often used to model fluctuation problems that involve obstacles and constraints in engineering, biology, and other fields (cf. [1, 13, 26]). As ε tends to 0, the limit problem for Eq.(3) is often closely related to engineering robustness, or to changes in biological populations, and so on. Therefore, it is important to study the limit problem of Eq.(3).

There have been many results for the limit problem for Eq.(3) (cf. [1, 3–5, 10, 23]) where

$\varphi = 0$ and b, σ, f_1 are independent of s. Let us recall some examples. In [1], Bensoussan et al. systematically elaborated the limit problem for PDEs. Later, Buckdahn and Ichihara [5] studied, using a probability approach, the homogenization of Hamilton-Jacobi-Bellman equations with periodic structures. Hu et al. [16] proved, using a probability approach, the averaging for Eq.(3) where $\varphi = 0$ and b, σ, and f_1 depend on s. Essaky and Ouknine [13] observed, using a probability approach, the averaging for Eq.(3) where $\varphi \neq 0$, $f_1(s,x,y)$ is independent of s and $f_2(z) = 0$. However, until now, there have been few studies on the averaging of Eq.(3) with $\varphi \neq 0$ and $f_2(z) \neq 0$. Here, we investigate the averaging of Eq.(3) with $\varphi \neq 0$ and $f_2(z) \neq 0$ and two other types of PDEs.

The novelty of this study is threefold. The first point of novelty is that we prove one averaging principle for general SDEs in the L^{2p} $(p \geqslant 1)$ sense. Moreover, a convergence rate is presented for $p = 1$, which is important for numerical simulation. The second point is that we establish the averaging principle for backward stochastic variational inequalities in the L^2 sense through a time discretization method. Because the operator $\partial\varphi$ is multivalued, nonlinear, and not smooth, we need new ideas and approaches to reach this goal. The third point is that we obtain the averaging of nonlinear parabolic PDEs.

Finally, this paper is arranged as follows. In the next section, we introduce notation and concepts, and we recall some results that are used in the sequel. In Section 3, two main results are formulated. Next, the proofs of the main results are presented in Sections 4 and 5. In Section 6, we apply our result to nonlinear parabolic PDEs. Finally, we provide an example to explain our results in Section 7.

The following convention will be used throughout the paper: C, with or without indices, denotes different positive constants whose values may change from one context to another.

2. Preliminaries

In this section, we introduce notation and concepts, and we recall some results that are used in the sequel.

2.1 Notation

In this subsection, we introduce some notation.

For convenience, we will use $|\cdot|$ and $\|\cdot\|$ as the norms of vectors and matrices, respectively. Furthermore, we let $\langle \cdot, \cdot \rangle$ denote the scalar product in $\mathbb{R}^d$, and we let B^* denote the transpose of a matrix B.

Let $C(\mathbb{R}^d)$ be the collection of continuous functions on $\mathbb{R}^d$, and let $C^2(\mathbb{R}^d)$ be the space of continuous functions on $\mathbb{R}^d$ that have continuous partial derivatives of up to order 2.

2.2 Maximal monotone operators

In this subsection, we introduce maximal monotone operators.

For a multivalued operator $A : \mathbb{R}^d \mapsto 2^{\mathbb{R}^d}$, where $2^{\mathbb{R}^d}$ stands for all the subsets of $\mathbb{R}^d$, we set

$$\mathcal{D}(A) := \left\{ x \in \mathbb{R}^d : A(x) \neq \emptyset \right\},$$

$$Gr(A) := \left\{ (x,y) \in \mathbb{R}^{2d} : x \in \mathcal{D}(A),\ y \in A(x) \right\}.$$

We say that A is monotone if $\big\langle x_1 - x_2, y_1 - y_2 \big\rangle \geqslant 0$ for any $(x_1,y_1), (x_2,y_2) \in Gr(A)$, and A is maximal monotone if

$$(x_1,y_1) \in Gr(A) \iff \big\langle x_1 - x_2, y_1 - y_2 \big\rangle \geqslant 0, \quad \forall (x_2,y_2) \in Gr(A).$$

We next give an example to explain maximal monotone operators.

Example 2.1 *For a lower semicontinuous convex function* $\varphi:\mathbb{R}^d\mapsto(-\infty,+\infty]$ *, we assume* $\mathrm{Int}(Dom(\varphi))\neq\emptyset$ *, where* $Dom(\varphi)\equiv\{x\in\mathbb{R}^d;\varphi(x)<\infty\}$ *and* $\mathrm{Int}(Dom(\varphi))$ *is the interior of* $Dom(\varphi)$*. We define the subdifferential operator of the function* φ *as*

$$\partial\varphi(x):=\{y\in\mathbb{R}^d:\langle y,z-x\rangle+\varphi(x)\leqslant\varphi(z),\forall z\in\mathbb{R}^d\}.$$

Then, $\partial\varphi$ *is a maximal monotone operator.*

If we take a closed convex set $\mathcal{O}\subset\mathbb{R}^d$ *with* $\mathrm{Int}(\mathcal{O})\neq\emptyset$ *and define* $I_{\mathcal{O}}$ *as follows:*

$$I_{\mathcal{O}}(x):=\begin{cases}0, & \text{if } x\in\mathcal{O},\\ +\infty, & \text{if } x\notin\mathcal{O}.\end{cases}$$

$I_{\mathcal{O}}$ *is a lower semicontinuous convex function with* $\mathrm{Int}(Dom(I_{\mathcal{O}}))=\mathrm{Int}(\mathcal{O})\neq\emptyset$.

Take any $T>0$ and set it to a constant value. Let $\mathscr{V}_0$ be the set of all continuous functions $K:[0,T]\mapsto\mathbb{R}^d$ with finite variations, and let $K_0=0$. For $K\in\mathscr{V}_0$ and $s\in[0,T]$, we will use $|K|_0^s$ to denote the variation of K on $[0,s]$, and we write $|K|_{TV}:=|K|_0^T$. Then, we set

$$\begin{aligned}\mathscr{A}:=\Big\{&(X,K):X\in C([0,T],\overline{\mathcal{D}(\mathcal{A})}),K\in\mathscr{V}_0,\\ &\text{and }\langle X_t-x,\mathrm{d}K_t-y\mathrm{d}t\rangle\geqslant0\text{ for any }(x,y)\in Gr(A)\Big\},\end{aligned}$$

where $\overline{\mathcal{D}(\mathcal{A})}$ stands for the closure of $\mathcal{D}(A)$ in $\mathbb{R}^d$ and $\langle X_t-x,\mathrm{d}K_t-y\mathrm{d}t\rangle$ is the scalar product in $\mathbb{R}^d$. For $\mathscr{A}$, we obtain the two following results (cf.[7, 31]).

Lemma 2.2 *For* $X\in C([0,T],\overline{\mathcal{D}(\mathcal{A})})$ *and* $K\in\mathscr{V}_0$ *, the following statements are equivalent:*

(i) $(X,K)\in\mathscr{A}$;

(ii) *For any* $x,y\in C([0,T],\mathbb{R}^d)$ *with* $(x_s,y_s)\in Gr(A)$ *for any* $s\in[0,T]$ *, it holds that*

$$\langle X_s-x_s,\mathrm{d}K_s-y_s\mathrm{d}s\rangle\geqslant0;$$

(iii) *For any* $(X',K')\in\mathscr{A}$ *, it holds that*

$$\Big\langle X_s-X_s',\mathrm{d}K_s-\mathrm{d}K_s'\Big\rangle\geqslant0.$$

Lemma 2.3 *Assume that* $\mathrm{Int}(\mathcal{D}(A))\neq\emptyset$*. For any* $a\in\mathrm{Int}(\mathcal{D}(A))$ *, there exists* $M_1>0$ *, and* $M_2,M_3\geqslant0$ *such that for any* $(X,K)\in\mathscr{A}$ *and* $0\leqslant t<s\leqslant T$,

$$\int_t^s\langle X_r-a,\mathrm{d}K_r\rangle\geqslant M_1|K|_t^s-M_2\int_t^s|X_r-a|\mathrm{d}r-M_3(s-t).$$

2.3 Backward stochastic variational inequalities

In this subsection, we introduce backward SVIs.

Consider the following backward SVI on $\mathbb{R}^d$: for any $0\leqslant s\leqslant T$,

$$\begin{cases}\mathrm{d}Y_s\in\partial\varphi(Y_s)\mathrm{d}s-F(s,Y_s,Z_s)\mathrm{d}s+Z_s\mathrm{d}W_s,\\ Y_T=\xi,\end{cases}\tag{4}$$

where $\varphi:\mathbb{R}^d\to(-\infty,+\infty]$ is a proper ($\varphi\not\equiv+\infty$) convex and lower semicontinuous function, $\varphi(y)\geqslant\varphi(0)=0$, the coefficient $F:\Omega\times[0,T]\times\mathbb{R}^d\times\mathbb{R}^{d\times l}\mapsto\mathbb{R}^d$ is Borel measurable, $\forall(y,z)\in\mathbb{R}^d\times\mathbb{R}^{d\times l}$, $F(\cdot,y,z)$ is $(\mathscr{F}_t)_{t\in[0,T]}$-progressively measurable, and ξ is a $\mathscr{F}_T$-measurable random variable with values in $\overline{\mathcal{D}(\partial\varphi)}$ and $\mathbb{E}[|\xi|^2+\varphi(\xi)]<\infty$. We define solutions for Eq.(4).

Definition 2.4 *We say that Eq. (4) admits a solution having the terminal value* ξ *if there exists a triple* $\{(Y_t, K_t, Z_t) : t \in [0,T]\}$ *that is a* $(\mathscr{F}_t)_{t\in[0,T]}$*-progressively measurable process and that satisfies the following:*

(i) $(Y_\cdot, K_\cdot) \in \mathscr{A}$, $\mathrm{d}\mathbb{P} \times \mathrm{d}t$ *-a.e. on* $\Omega \times [0,T]$ *, where* A *in* $\mathscr{A}$ *is replaced by* $\partial\varphi$*;*

(ii)

$$\mathbb{E}\left(\sup_{0\leqslant t\leqslant T}|Y_t|^2 + \int_0^T \|Z_s\|^2\mathrm{d}s + |K|_0^T\right) < \infty;$$

(iii)

$$Y_t = \xi - (K_T - K_t) + \int_t^T F(s, Y_s, Z_s)\,\mathrm{d}s - \int_t^T Z_s\mathrm{d}W_s.$$

3. Main results

In this section, we formulate the main results.

3.1 Averaging principle for SDEs

In this subsection, we present an averaging principle for Eq.(1).

Consider Eq.(1), i.e.,

$$\left\{\begin{array}{l} \mathrm{d}X_s^{\varepsilon,t,\zeta} = b(\frac{s}{\varepsilon}, X_s^{\varepsilon,t,\zeta})\mathrm{d}s + \sigma(\frac{s}{\varepsilon}, X_s^{\varepsilon,t,\zeta})\mathrm{d}W_s, \\ X_t^{\varepsilon,t,\zeta} = \zeta. \end{array}\right.$$

We assume:

$(\mathbf{H}^1_{b,\sigma})$ There exists a constant $L_1 > 0$ such that for any $s \in \mathbb{R}_+, x_i \in \mathbb{R}^m, i = 1,2,$

$$|b(s,x_1) - b(s,x_2)| + \|\sigma(s,x_1) - \sigma(s,x_2)\| \leqslant L_1|x_1 - x_2|,$$
$$|b(s,0)| + \|\sigma(s,0)\| \leqslant L_1;$$

$(\mathbf{H}^2_{b,\sigma})$ There exist $\bar{b} : \mathbb{R}^m \to \mathbb{R}^m$, $\bar{\sigma} : \mathbb{R}^m \to \mathbb{R}^{m\times l}$ such that for any $\hat{T} \in \mathbb{R}_+, x \in \mathbb{R}^m$,

$$\left|\frac{1}{\hat{T}}\int_0^{\hat{T}} b(s,x)\mathrm{d}s - \bar{b}(x)\right|^2 \leqslant \kappa_1(\hat{T})(1 + |x|^2),$$
$$\frac{1}{\hat{T}}\int_0^{\hat{T}} \|\sigma(s,x) - \bar{\sigma}(x)\|^2\mathrm{d}s \leqslant \kappa_2(\hat{T})(1 + |x|^2),$$

where $\kappa_i(\cdot)$ is a continuous and positive bounded function with $\lim\limits_{\hat{T}\to\infty} \kappa_i(\hat{T}) = 0, i = 1,2.$

Remark 3.1 $\bar{b}$ *and* $\bar{\sigma}$ *are Lipschitz continuous in* x*. Indeed, for* $x_1, x_2 \in \mathbb{R}^m$,

$$\begin{aligned}
&|\bar{b}(x_1) - \bar{b}(x_2)|^2 \\
\leqslant\ & 3\left|\bar{b}(x_1) - \frac{1}{\hat{T}}\int_0^{\hat{T}} b(s,x_1)\mathrm{d}s\right|^2 + 3\left|\frac{1}{\hat{T}}\int_0^{\hat{T}} b(s,x_1)\mathrm{d}s - \frac{1}{\hat{T}}\int_0^{\hat{T}} b(s,x_2)\mathrm{d}s\right|^2 + 3\left|\frac{1}{\hat{T}}\int_0^{\hat{T}} b(s,x_2)\mathrm{d}s - \bar{b}(x_2)\right|^2 \\
\leqslant\ & 3\left|\bar{b}(x_1) - \frac{1}{\hat{T}}\int_0^{\hat{T}} b(s,x_1)\mathrm{d}s\right|^2 + 3\frac{1}{\hat{T}}\int_0^{\hat{T}} |b(s,x_1) - b(s,x_2)|^2\mathrm{d}s + 3\left|\frac{1}{\hat{T}}\int_0^{\hat{T}} b(s,x_2)\mathrm{d}s - \bar{b}(x_2)\right|^2 \\
\leqslant\ & C\kappa_1(\hat{T})(1 + |x_1|^2 + |x_2|^2) + 3L_1^2|x_1 - x_2|^2.
\end{aligned}$$

Let $\hat{T}\to\infty$. We obtain that $|\bar{b}(x_1)-\bar{b}(x_2)|^2\leqslant 3L_1^2|x_1-x_2|^2$. By the same deduction, one can verify that $\bar{\sigma}$ is Lipschitz continuous in x.

If $\mathbb{E}|\zeta|^{2p+2}<\infty$ for any $p\geqslant 0$, then under $(\mathbf{H}^1_{b,\sigma})$ and by [18, Theorem 19.3], Eq.(1) has a unique solution $X^{\varepsilon,t,\zeta}$ with $\mathbb{E}|X^{\varepsilon,t,\zeta}_s|^{2p+2}<\infty$ for any $s\geqslant t$. We can then construct the following SDE:

$$\left\{\begin{array}{l} \mathrm{d}\bar{X}^{t,\zeta}_s=\bar{b}(\bar{X}^{t,\zeta}_s)\mathrm{d}s+\bar{\sigma}(\bar{X}^{t,\zeta}_s)\mathrm{d}W_s,\\ \bar{X}^{t,\zeta}_t=\zeta. \end{array}\right. \tag{5}$$

By Remark 3.1 and [18, Theorem 19.3], we obtain that Eq.(5) also has a unique solution $\bar{X}^{t,\zeta}$ with $\mathbb{E}|\bar{X}^{t,\zeta}_s|^{2p+2}<\infty$ for any $s\geqslant t$. The following theorem indicates the relationship between $X^{\varepsilon,t,\zeta}$ and $\bar{X}^{t,\zeta}$.

Theorem 3.2 *Suppose that $(\mathbf{H}^1_{b,\sigma})$ and $(\mathbf{H}^2_{b,\sigma})$ hold, and $\mathbb{E}|\zeta|^{2p+2}<\infty$ for any $p\geqslant 1$. Then, it holds that*

$$\lim_{\varepsilon\to 0}\mathbb{E}\left(\sup_{s\in[t,T]}|X^{\varepsilon,t,\zeta}_s-\bar{X}^{t,\zeta}_s|^{2p}\right)=0. \tag{6}$$

In particular, we have that for $0<\gamma<1$,

$$\mathbb{E}\sup_{s\in[t,T]}|X^{\varepsilon,t,\zeta}_s-\bar{X}^{t,\zeta}_s|^2\leqslant C(\varepsilon^\gamma+\varepsilon^{2\gamma}+\kappa_1(\varepsilon^{\gamma-1})+\kappa_2(\varepsilon^{\gamma-1})), \tag{7}$$

where the constant $C>0$ is independent of ε.

The proof of the above theorem is presented in Section 4.

Remark 3.3 *In [15], Guo et al. also studied Eq.(1) and obtained only that*

$$\lim_{\varepsilon\to 0}\mathbb{E}\left(\sup_{s\in[t,T]}|X^{\varepsilon,t,\zeta}_s-\bar{X}^{t,\zeta}_s|^2\right)=0$$

under the local Lipschitz and monotone conditions. Here, we are able to give the convergence rate.

3.2 Averaging principle for backward SVIs

In this subsection, we present an averaging principle for Eq.(2).

Consider Eq.(2), i.e.,

$$\left\{\begin{array}{l} \mathrm{d}Y^{\varepsilon,t,\zeta}_s\in\partial\varphi(Y^{\varepsilon,t,\zeta}_s)\mathrm{d}s-[f_1(\frac{s}{\varepsilon},X^{\varepsilon,t,\zeta}_s,Y^{\varepsilon,t,\zeta}_s)+f_2(Z^{\varepsilon,t,\zeta}_s)]\mathrm{d}s+Z^{\varepsilon,t,\zeta}_s\mathrm{d}W_s,\\ Y^{\varepsilon,t,\zeta}_T=g(X^{\varepsilon,t,\zeta}_T)\in\overline{\mathcal{D}(\partial\varphi)}, \end{array}\right.$$

where φ is the same as that in Subsection 2.3.

We assume:

$(\mathbf{H}_\varphi)$ There exist $q_1\in\mathbb{N}_+\cup\{0\}$ and constant $L_2>0$ such that

$$|\varphi(g(x))|\leqslant L_2(1+|x|^{q_1}),\quad x\in\mathbb{R}^m;$$

$(\mathbf{H}_g)$ There exist $q_2\in\mathbb{N}_+$ and constant $L_3>0$ such that for any $x_i\in\mathbb{R}^m, i=1,2$,

$$|g(x_1)-g(x_2)|\leqslant L_3(1+|x_1|^{q_2}+|x_2|^{q_2})|x_1-x_2|;$$

$(\mathbf{H}^1_f)$ There exist $q_3\in\mathbb{N}_+$ and two constants $L_4,L_5>0$ such that for any $s\in\mathbb{R}_+, x_i\in\mathbb{R}^m$, $y_i\in\mathbb{R}^d, z_i\in\mathbb{R}^{d\times l}, i=1,2$,

$$
\begin{aligned}
&|f_1(s,x_1,y_1)-f_1(s,x_2,y_2)|\leqslant L_4((1+|x_1|^{q_3}+|x_2|^{q_3})|x_1-x_2|+|y_1-y_2|),\\
&|f_1(s,0,0)|\leqslant L_4,\quad |f_2(z_1)-f_2(z_2)|\leqslant L_5\|z_1-z_2\|;
\end{aligned}
$$

($\mathbf{H}_f^2$) There exists $\bar{f}_1:\mathbb{R}^m\times\mathbb{R}^d\to\mathbb{R}^d$ that, for any $\hat{T}\in\mathbb{R}_+, x\in\mathbb{R}^m, y\in\mathbb{R}^d$, satisfies

$$\left|\frac{1}{\hat{T}}\int_0^{\hat{T}} f_1(s,x,y)\mathrm{d}s-\bar{f}_1(x,y)\right|^2\leqslant\kappa_3(\hat{T})(1+|x|^2+|y|^2),$$

where $\kappa_3(\cdot)$ is a continuous and positive bounded function with $\lim\limits_{\hat{T}\to\infty}\kappa_3(\hat{T})=0$.

Remark 3.4 (i) *By* ($\mathbf{H}_g$) *, it holds that for* $x\in\mathbb{R}^m$,

$$|g(x)|\leqslant(2L_3+|g(0)|)(1+|x|^{q_2+1}). \tag{8}$$

(ii) ($\mathbf{H}_f^1$) *implies that for* $s\in\mathbb{R}_+, x\in\mathbb{R}^m, y\in\mathbb{R}^d, z\in\mathbb{R}^{d\times l}$,

$$|f_1(s,x,y)|\leqslant 2L_4(1+|x|^{q_3+1}+|y|),\quad |f_2(z)|\leqslant(L_5+|f_2(0)|)(1+\|z\|). \tag{9}$$

(iii) *By a deduction similar to that in Remark 3.1, we know that for any* $x_i\in\mathbb{R}^m, y_i\in\mathbb{R}^d$, $i=1,2$,

$$
\begin{gathered}
|\bar{f}_1(x_1,y_1)-\bar{f}_1(x_2,y_2)|\leqslant C((1+|x_1|^{q_3}+|x_2|^{q_3})|x_1-x_2|+|y_1-y_2|),\\
|\bar{f}_1(0,0)|\leqslant L_4.
\end{gathered}
$$

If $\mathbb{E}|\zeta|^{2p+2}<\infty$ for any $p\geqslant\frac{q_1-2}{2}\vee q_2\vee q_3$, then under ($\mathbf{H}_{b,\sigma}^1$), ($\mathbf{H}_\varphi$), ($\mathbf{H}_g$), and ($\mathbf{H}_f^1$) and by [26, Theorem 1.1], the system (2) has a unique solution $(Y^{\varepsilon,t,\zeta},K^{\varepsilon,t,\zeta},Z^{\varepsilon,t,\zeta})$ that satisfies:

$$\mathbb{E}\sup_{s\in[t,T]}|Y_s^{\varepsilon,t,\zeta}|^2+\int_t^T\mathbb{E}\|Z_r^{\varepsilon,t,\zeta}\|^2\mathrm{d}r+\mathbb{E}|K^{\varepsilon,t,\zeta}|_t^T<\infty.$$

We then construct the following backward SVI:

$$
\begin{cases}
\mathrm{d}\bar{Y}_s^{t,\zeta}\in\partial\varphi(\bar{Y}_s^{t,\zeta})\mathrm{d}s-[\bar{f}_1(\bar{X}_s^{t,\zeta},\bar{Y}_s^{t,\zeta})+f_2(\bar{Z}_s^{t,\zeta})]\mathrm{d}s+\bar{Z}_s^{t,\zeta}\mathrm{d}W_s,\\
\bar{Y}_T^{t,\zeta}=g(\bar{X}_T^{t,\zeta})\in\overline{\mathcal{D}(\partial\varphi)}.
\end{cases} \tag{10}
$$

According to Remark 3.1, [18, Theorem 19.3], Remark 3.4 (iii), and [26, Theorem 1.1], the assumptions ($\mathbf{H}_{b,\sigma}^1$), ($\mathbf{H}_{b,\sigma}^2$), ($\mathbf{H}_\varphi$), ($\mathbf{H}_g$), ($\mathbf{H}_f^1$), and ($\mathbf{H}_f^2$) assure that Eq.(10) has a unique solution $(\bar{Y}^{t,\zeta},\bar{K}^{t,\zeta},\bar{Z}^{t,\zeta})$ with

$$\mathbb{E}\sup_{s\in[t,T]}|\bar{Y}_s^{t,\zeta}|^2+\int_t^T\mathbb{E}\|\bar{Z}_r^{t,\zeta}\|^2\mathrm{d}r+\mathbb{E}|\bar{K}^{t,\zeta}|_t^T<\infty.$$

Now, it is time to state the main result in this subsection.

Theorem 3.5 *Assume that* ($\mathbf{H}_{b,\sigma}^1$) *,* ($\mathbf{H}_{b,\sigma}^2$) *,* ($\mathbf{H}_\varphi$) *,* ($\mathbf{H}_g$) *,* ($\mathbf{H}_f^1$) *, and* ($\mathbf{H}_f^2$) *hold, and* $\mathbb{E}|\zeta|^{2p+2}<\infty$ *for any* $p\geqslant\frac{q_1-2}{2}\vee q_2\vee q_3$. *It then holds that*

$$\lim_{\varepsilon\to0}\mathbb{E}\sup_{s\in[t,T]}|Y_s^{\varepsilon,t,\zeta}-\bar{Y}_s^{t,\zeta}|^2=0.$$

The proof of the above theorem is presented in Section 5.

Remark 3.6 *We note that if* $f_1(s,x,y)$ *is independent of* s *and if* $f_2(z)=0$ *, the backward equation in system (2) is the same as equation (2.6) in [13]. Under similar assumptions, Essaky and Ouknine proved that if* $X^{\varepsilon,t,\zeta}$ *converges in law to* $\bar{X}^{t,\zeta}$ *, then* $Y^{\varepsilon,t,\zeta}$ *also converges in law to* $\bar{Y}^{t,\zeta}$ *(cf. [13, Theorem 3.1]). Because the mean square convergence implies convergence in law, our result is stronger.*

4. Proof of Theorem 3.2

In this section, we present Theorem 3.2. We begin with some key estimates.

Lemma 4.1 *Under* $(\mathbf{H}^1_{b,\sigma})$ *and* $\mathbb{E}|\zeta|^{2p+2}<\infty$ *for any* $p\geqslant 0$ *, it holds that*

$$\mathbb{E}\sup_{s\in[t,T]}|X^{\varepsilon,t,\zeta}_s|^{2p+2}\leqslant C(1+\mathbb{E}|\zeta|^{2p+2}), \tag{11}$$

$$\mathbb{E}|X^{\varepsilon,t,\zeta}_{s+h}-X^{\varepsilon,t,\zeta}_s|^{2p+2}\leqslant C(h^{p+1}+h^{2p+2}), \quad t\leqslant s\leqslant s+h\leqslant T, \tag{12}$$

where C *is independent of* ε*.*

Lemma 4.2 *Suppose that* $(\mathbf{H}^1_{b,\sigma})$ *and* $(\mathbf{H}^2_{b,\sigma})$ *hold, and* $\mathbb{E}|\zeta|^{2p+2}<\infty$ *for any* $p\geqslant 0$*. It then holds that*

$$\mathbb{E}\sup_{s\in[t,T]}|\bar{X}^{t,\zeta}_s|^{2p+2}\leqslant C(1+\mathbb{E}|\zeta|^{2p+2}), \tag{13}$$

where C *is independent of* ε*.*

Because the proofs for the two lemmas above are standard, we omit them (cf. [18, Lemma 21.2]).

Proof of Theorem 3.2 First, by (1) and (5), it holds that

$$\begin{aligned}
&\mathbb{E}\sup_{s\in[t,T]}|X^{\varepsilon,t,\zeta}_s-\bar{X}^{t,\zeta}_s|^2\\
&\leqslant 2\mathbb{E}\sup_{s\in[t,T]}\left|\int_t^s\left(b\left(\frac{r}{\varepsilon},X^{\varepsilon,t,\zeta}_r\right)-\bar{b}(\bar{X}^{t,\zeta}_r)\right)\mathrm{d}r\right|^2+8\mathbb{E}\int_t^T\|\sigma\left(\frac{r}{\varepsilon},X^{\varepsilon,t,\zeta}_r\right)-\bar{\sigma}(\bar{X}^{t,\zeta}_r)\|^2\mathrm{d}r\\
&\leqslant 4\mathbb{E}\sup_{s\in[t,T]}\left|\int_t^s\left(b\left(\frac{r}{\varepsilon},X^{\varepsilon,t,\zeta}_r\right)-\bar{b}(X^{\varepsilon,t,\zeta}_r)\right)\mathrm{d}r\right|^2+4\mathbb{E}\sup_{s\in[t,T]}\left|\int_t^s\left(\bar{b}(X^{\varepsilon,t,\zeta}_r)-\bar{b}(\bar{X}^{t,\zeta}_r)\right)\mathrm{d}r\right|^2\\
&\quad+16\mathbb{E}\int_t^T\|\sigma\left(\frac{r}{\varepsilon},X^{\varepsilon,t,\zeta}_r\right)-\bar{\sigma}(X^{\varepsilon,t,\zeta}_r)\|^2\mathrm{d}r+16\mathbb{E}\int_t^T\|\bar{\sigma}(X^{\varepsilon,t,\zeta}_r)-\bar{\sigma}(\bar{X}^{t,\zeta}_r)\|^2\mathrm{d}r\\
&\leqslant C\int_t^T\mathbb{E}\sup_{s\in[t,r]}|X^{\varepsilon,t,\zeta}_s-\bar{X}^{t,\zeta}_s|^2\mathrm{d}r+4\mathbb{E}\sup_{s\in[t,T]}\left|\int_t^s\left(b\left(\frac{r}{\varepsilon},X^{\varepsilon,t,\zeta}_r\right)-\bar{b}(X^{\varepsilon,t,\zeta}_r)\right)\mathrm{d}r\right|^2\\
&\quad+16\mathbb{E}\int_t^T\|\sigma\left(\frac{r}{\varepsilon},X^{\varepsilon,t,\zeta}_r\right)-\bar{\sigma}(X^{\varepsilon,t,\zeta}_r)\|^2\mathrm{d}r\\
&=:C\int_t^T\mathbb{E}\sup_{s\in[t,r]}|X^{\varepsilon,t,\zeta}_s-\bar{X}^{t,\zeta}_s|^2\mathrm{d}r+J_1+J_2.
\end{aligned} \tag{14}$$

For J_1, we obtain by $(\mathbf{H}^1_{b,\sigma})$ and $(\mathbf{H}^2_{b,\sigma})$ that

$$\begin{aligned}J_1 \leqslant\, & 8\mathbb{E}\sup_{s\in[t,T]}\left|\int_t^s \left(b\left(\frac{r}{\varepsilon}, X_r^{\varepsilon,t,\zeta}\right) - \bar{b}(X_r^{\varepsilon,t,\zeta}) - b\left(\frac{r}{\varepsilon}, X_{r(\delta)+t}^{\varepsilon,t,\zeta}\right) + \bar{b}(X_{r(\delta)+t}^{\varepsilon,t,\zeta})\right)\mathrm{d}r\right|^2 \\ & + 8\mathbb{E}\sup_{s\in[t,T]}\left|\int_t^s \left(b\left(\frac{r}{\varepsilon}, X_{r(\delta)+t}^{\varepsilon,t,\zeta}\right) - \bar{b}(X_{r(\delta)+t}^{\varepsilon,t,\zeta})\right)\mathrm{d}r\right|^2 \\ \leqslant\, & C\int_t^T \mathbb{E}|X_r^{\varepsilon,t,\zeta} - X_{r(\delta)+t}^{\varepsilon,t,\zeta}|^2\mathrm{d}r + 16\mathbb{E}\sup_{s\in[t,T]}\left|\int_t^{[\frac{s-t}{\delta}]\delta+t} \left(b\left(\frac{r}{\varepsilon}, X_{r(\delta)+t}^{\varepsilon,t,\zeta}\right) - \bar{b}(X_{r(\delta)+t}^{\varepsilon,t,\zeta})\right)\mathrm{d}r\right|^2 \\ & + 16\mathbb{E}\sup_{s\in[t,T]}\left|\int_{[\frac{s-t}{\delta}]\delta+t}^s \left(b\left(\frac{r}{\varepsilon}, X_{r(\delta)+t}^{\varepsilon,t,\zeta}\right) - \bar{b}(X_{r(\delta)+t}^{\varepsilon,t,\zeta})\right)\mathrm{d}r\right|^2 \\ =:\, & J_{11} + J_{12} + J_{13},\end{aligned}$$

where δ is a fixed positive number depending on ε, $r(\delta) := [\frac{r-t}{\delta}]\delta$, and $[\frac{r-t}{\delta}]$ denotes the integer part of $\frac{r-t}{\delta}$.

For J_{11}, (12) implies that

$$J_{11} \leqslant CT(\delta + \delta^2).$$

For J_{12}, it follows from $(\mathbf{H}^2_{b,\sigma})$ that

$$\begin{aligned}J_{12} \leqslant & 16\left[\frac{T-t}{\delta}\right]\sum_{k=0}^{[\frac{T-t}{\delta}]-1}\mathbb{E}\left|\int_{k\delta+t}^{(k+1)\delta+t}\left(b\left(\frac{r}{\varepsilon}, X_{k\delta+t}^{\varepsilon,t,\zeta}\right) - \bar{b}(X_{k\delta+t}^{\varepsilon,t,\zeta})\right)\mathrm{d}r\right|^2 \\ \leqslant & 16(T-t)\delta\sum_{k=0}^{[\frac{T-t}{\delta}]-1}\mathbb{E}\left|\frac{\varepsilon}{\delta}\int_{\frac{k\delta+t}{\varepsilon}}^{\frac{(k+1)\delta+t}{\varepsilon}}\left(b(u, X_{k\delta+t}^{\varepsilon,t,\zeta}) - \bar{b}(X_{k\delta+t}^{\varepsilon,t,\zeta})\right)\mathrm{d}u\right|^2 \\ \leqslant & 16(T-t)^2\kappa_1\left(\frac{\delta}{\varepsilon}\right)\sup_{0\leqslant k\leqslant[\frac{T-t}{\delta}]-1}\mathbb{E}(1 + |X_{k\delta+t}^{\varepsilon,t,\zeta}|^2) \\ \leqslant & C\kappa_1\left(\frac{\delta}{\varepsilon}\right).\end{aligned}$$

For J_{13}, it holds by the linear growth of $b, \bar{b}$ and (11) that

$$\begin{aligned}J_{13} & \leqslant 16\delta\mathbb{E}\sup_{s\in[t,T]}\int_{[\frac{s-t}{\delta}]\delta+t}^s\left|b\left(\frac{r}{\varepsilon}, X_{r(\delta)+t}^{\varepsilon,t,\zeta}\right) - \bar{b}\left(X_{r(\delta)+t}^{\varepsilon,t,\zeta}\right)\right|^2\mathrm{d}r \\ & \leqslant C\delta\mathbb{E}\int_t^T\left(1 + |X_{r(\delta)+t}^{\varepsilon,t,\zeta}|^2\right)\mathrm{d}r \\ & \leqslant C\delta.\end{aligned}$$

By combining the above deductions, we obtain that

$$J_1 \leqslant C\left(\delta + \delta^2 + \kappa_1\left(\frac{\delta}{\varepsilon}\right)\right). \tag{15}$$

The same computation as that for J_1 then yields that

$$J_2 \leqslant C\left(\delta + \delta^2 + \kappa_2\left(\frac{\delta}{\varepsilon}\right)\right). \tag{16}$$

Finally, inserting (15) and (16) into (14), we obtain by the Gronwall inequality that

$$\mathbb{E}\sup_{s\in[t,T]}|X_s^{\varepsilon,t,\zeta} - \bar{X}_s^{t,\zeta}|^2 \leqslant C\left(\delta + \delta^2 + \kappa_1\left(\frac{\delta}{\varepsilon}\right) + \kappa_2\left(\frac{\delta}{\varepsilon}\right)\right).$$

By taking $\delta=\varepsilon^{\gamma}$ for $0<\gamma<1$, we obtain (7).

Next, by the Chebyshev inequality and (7), it holds that for any $\theta>0$,

$$\mathbb{P}\Big(\sup_{s\in[t,T]}|X_s^{\varepsilon,t,\zeta}-\bar{X}_s^{t,\zeta}|>\theta\Big)\leqslant\frac{\mathbb{E}\Big(\sup\limits_{s\in[t,T]}|X_s^{\varepsilon,t,\zeta}-\bar{X}_s^{t,\zeta}|^2\Big)}{\theta^2}\leqslant\frac{C}{\theta^2}\left(\delta+\delta^2+\kappa_1\left(\varepsilon^{\gamma-1}\right)+\kappa_2\left(\varepsilon^{\gamma-1}\right)\right),$$

which implies that

$$\sup_{s\in[t,T]}|X_s^{\varepsilon,t,\zeta}-\bar{X}_s^{t,\zeta}|\xrightarrow{\mathbb{P}}0,$$

as ε tends to 0. In addition, it follows from (11) and (13) that

$$\sup_{\varepsilon}\mathbb{E}\sup_{s\in[t,T]}|X_s^{\varepsilon,t,\zeta}-\bar{X}_s^{t,\zeta}|^{2p+2}\leqslant C(1+\mathbb{E}|\zeta|^{2p+2}).$$

Therefore, by the Vitali convergence theorem one can obtain that

$$\lim_{\varepsilon\to0}\mathbb{E}\left(\sup_{s\in[t,T]}|X_s^{\varepsilon,t,\zeta}-\bar{X}_s^{t,\zeta}|^{2p}\right)=0,$$

which completes the proof. □

5. Proofs of Theorem 3.5

In this section, we prove Theorem 3.5. First, we prepare some key lemmas.

Lemma 5.1 *Suppose that* $(\mathbf{H}_{b,\sigma}^1)$, $(\mathbf{H}_{\varphi})$, $(\mathbf{H}_g)$, *and* $(\mathbf{H}_f^1)$ *hold, and* $\mathbb{E}|\zeta|^{2p+2}<\infty$ *for any* $p\geqslant\frac{q_1-2}{2}\vee q_2\vee q_3$. *Then, there exists a constant* $C>0$ *such that*

$$\mathbb{E}\sup_{s\in[t,T]}|Y_s^{\varepsilon,t,\zeta}|^2+\int_t^T\mathbb{E}\|Z_r^{\varepsilon,t,\zeta}\|^2\mathrm{d}r+\mathbb{E}|K^{\varepsilon,t,\zeta}|_t^T\leqslant C(1+\mathbb{E}|\zeta|^{2q_2+2}+\mathbb{E}|\zeta|^{2q_3+2}).\tag{17}$$

Proof By the Itô formula, it holds that

$$\begin{aligned}&|Y_s^{\varepsilon,t,\zeta}|^2+\int_s^T\|Z_r^{\varepsilon,t,\zeta}\|^2\mathrm{d}r\\&=\left|g\left(X_T^{\varepsilon,t,\zeta}\right)\right|^2-2\int_s^T\langle Y_r^{\varepsilon,t,\zeta},\mathrm{d}K_r^{\varepsilon,t,\zeta}\rangle-2\int_s^T\langle Y_r^{\varepsilon,t,\zeta},Z_r^{\varepsilon,t,\zeta}\mathrm{d}W_r\rangle\\&\quad+2\int_s^T\left\langle Y_r^{\varepsilon,t,\zeta},f_1\left(\frac{r}{\varepsilon},X_r^{\varepsilon,t,\zeta},Y_r^{\varepsilon,t,\zeta}\right)+f_2\left(Z_r^{\varepsilon,t,\zeta}\right)\right\rangle\mathrm{d}r.\end{aligned}\tag{18}$$

Taking the expectation on two sides, we know by Lemma 2.2 and (8), (9) that for any $v\in\partial\varphi(0)$,

$$\begin{aligned}\mathbb{E}|Y_s^{\varepsilon,t,\zeta}|^2+\int_s^T\mathbb{E}\|Z_r^{\varepsilon,t,\zeta}\|^2\mathrm{d}r&\leqslant C(1+\mathbb{E}|\zeta|^{2q_2+2})+2\int_s^T\mathbb{E}|Y_r^{\varepsilon,t,\zeta}||v|\mathrm{d}r+C\int_s^T\mathbb{E}|Y_r^{\varepsilon,t,\zeta}|^2\mathrm{d}r\\&\quad+CT+C\int_s^T\mathbb{E}|X_r^{\varepsilon,t,\zeta}|^{2q_3+2}\mathrm{d}r+\frac{1}{2}\int_s^T\mathbb{E}\|Z_r^{\varepsilon,t,\zeta}\|^2\mathrm{d}r,\end{aligned}$$

and

$$\mathbb{E}|Y_s^{\varepsilon,t,\zeta}|^2+\frac{1}{2}\int_s^T\mathbb{E}\|Z_r^{\varepsilon,t,\zeta}\|^2\mathrm{d}r\leqslant C(1+\mathbb{E}|\zeta|^{2q_2+2}+\mathbb{E}|\zeta|^{2q_3+2})+(|v|^2+C)T+C\int_s^T\mathbb{E}|Y_r^{\varepsilon,t,\zeta}|^2\mathrm{d}r.\tag{19}$$

The Gronwall inequality implies that

$$\sup_{s\in[t,T]}\mathbb{E}|Y_s^{\varepsilon,t,\zeta}|^2\leqslant C(1+\mathbb{E}|\zeta|^{2q_2+2}+\mathbb{E}|\zeta|^{2q_3+2}),$$

which, together with (19), yields that

$$\int_t^T\mathbb{E}\|Z_r^{\varepsilon,t,\zeta}\|^2\mathrm{d}r\leqslant C(1+\mathbb{E}|\zeta|^{2q_2+2}+\mathbb{E}|\zeta|^{2q_3+2}).$$

In the following, we investigate (18), and obtain by the BDG inequality that

$$\begin{aligned}\mathbb{E}\sup_{s\in[t,T]}|Y_s^{\varepsilon,t,\zeta}|^2&\leqslant C(1+\mathbb{E}|\zeta|^{2q_2+2})+2\int_t^T\mathbb{E}|Y_r^{\varepsilon,t,\zeta}||v|\mathrm{d}r+C\mathbb{E}\left(\int_t^T|Y_r^{\varepsilon,t,\zeta}|^2\|Z_r^{\varepsilon,t,\zeta}\|^2\mathrm{d}r\right)^{1/2}\\&\quad+CT+C\int_t^T\mathbb{E}|X_r^{\varepsilon,t,\zeta}|^{2q_3+2}\mathrm{d}r+C\int_t^T\mathbb{E}|Y_r^{\varepsilon,t,\zeta}|^2\mathrm{d}r+C\int_t^T\mathbb{E}\|Z_r^{\varepsilon,t,\zeta}\|^2\mathrm{d}r\\&\leqslant C(1+\mathbb{E}|\zeta|^{2q_2+2}+\mathbb{E}|\zeta|^{2q_3+2})+C\int_t^T\mathbb{E}\sup_{s\in[t,r]}|Y_s^{\varepsilon,t,\zeta}|^2\mathrm{d}r+\frac{1}{2}\mathbb{E}\sup_{r\in[t,T]}|Y_r^{\varepsilon,t,\zeta}|^2,\end{aligned}$$

which, together with the Gronwall inequality, yields that

$$\mathbb{E}\sup_{r\in[t,T]}|Y_r^{\varepsilon,t,\zeta}|^2\leqslant C(1+\mathbb{E}|\zeta|^{2q_2+2}+\mathbb{E}|\zeta|^{2q_3+2}).$$

Finally, according to Lemma 2.3 and (18), it holds that

$$\begin{aligned}2M_1|K^{\varepsilon,t,\zeta}|_t^T&\leqslant|g(X_T^{\varepsilon,t,\zeta})|^2+2M_2\int_t^T|Y_s^{\varepsilon,t,\zeta}|\mathrm{d}s+2M_3T-2\int_t^T\left\langle Y_s^{\varepsilon,t,\zeta},Z_s^{\varepsilon,t,\zeta}\mathrm{d}W_s\right\rangle\\&\quad+2\int_t^T\left\langle Y_s^{\varepsilon,t,\zeta},f_1\left(\frac{s}{\varepsilon},X_s^{\varepsilon,t,\zeta},Y_s^{\varepsilon,t,\zeta}\right)+f_2(Z_s^{\varepsilon,t,\zeta})\right\rangle\mathrm{d}s.\end{aligned}$$

Hence, by (8) and (9),

$$\mathbb{E}|K^{\varepsilon,t,\zeta}|_t^T\leqslant C(1+\mathbb{E}|\zeta|^{2q_2+2}+\mathbb{E}|\zeta|^{2q_3+2}).$$

Thus, the proof is complete. □

By the same deduction as that of (17), we then obtain the following result.

Lemma 5.2 *Suppose that* $(\mathbf{H}_{b,\sigma}^1)$, $(\mathbf{H}_{b,\sigma}^2)$, $(\mathbf{H}_\varphi)$, $(\mathbf{H}_g)$, $(\mathbf{H}_f^1)$, *and* $(\mathbf{H}_f^2)$ *hold, and* $\mathbb{E}|\zeta|^{2p+2}<\infty$ *for any* $p\geqslant\frac{q_1-2}{2}\vee q_2\vee q_3$. *There then exists a constant* $C>0$ *such that*

$$\mathbb{E}\sup_{s\in[t,T]}|\bar{Y}_s^{t,\zeta}|^2+\int_t^T\mathbb{E}\|\bar{Z}_r^{t,\zeta}\|^2\mathrm{d}r+\mathbb{E}|\bar{K}^{t,\zeta}|_t^T\leqslant C(1+\mathbb{E}|\zeta|^{2q_2+2}+\mathbb{E}|\zeta|^{2q_3+2}).\tag{20}$$

Lemma 5.3 *Under the assumptions of Theorem 3.5, it holds that*

$$\lim_{\varrho\to0}\sup_{s\in[t,T]}\sup_{s\leqslant r\leqslant s+\varrho}\mathbb{E}|Y_r^{\varepsilon,t,\zeta}-Y_{s+\varrho}^{\varepsilon,t,\zeta}|^2=0,\tag{21}$$

$$\lim_{\varrho\to0}\sup_{s\in[t,T]}\sup_{s\leqslant r\leqslant s+\varrho}\mathbb{E}|\bar{Y}_r^{t,\zeta}-\bar{Y}_{s+\varrho}^{t,\zeta}|^2=0.\tag{22}$$

Proof As the proofs of (21) and (22) are similar, we prove only (21).

First, we know that for $\varrho>0$ and $t\leqslant s\leqslant r\leqslant s+\varrho\leqslant T$,

$$
\begin{aligned}
Y_r^{\varepsilon,t,\zeta}=Y_{s+\varrho}^{\varepsilon,t,\zeta}-K_{s+\varrho}^{\varepsilon,t,\zeta}+K_r^{\varepsilon,t,\zeta}+\int_r^{s+\varrho}\left[f_1\left(\frac{u}{\varepsilon},X_u^{\varepsilon,t,\zeta},Y_u^{\varepsilon,t,\zeta}\right)+f_2(Z_u^{\varepsilon,t,\zeta})\right]\mathrm{d}u\\
-\int_r^{s+\varrho}Z_u^{\varepsilon,t,\zeta}\mathrm{d}W_u.
\end{aligned}
$$

By the Itô formula, it then holds that

$$
\begin{aligned}
&|Y_r^{\varepsilon,t,\zeta}-Y_{s+\varrho}^{\varepsilon,t,\zeta}|^2+\int_r^{s+\varrho}\|Z_u^{\varepsilon,t,\zeta}\|^2\mathrm{d}u\\
=&-2\int_r^{s+\varrho}\left\langle Y_u^{\varepsilon,t,\zeta}-Y_{s+\varrho}^{\varepsilon,t,\zeta},\mathrm{d}K_u^{\varepsilon,t,\zeta}\right\rangle+2\int_r^{s+\varrho}\left\langle Y_u^{\varepsilon,t,\zeta}-Y_{s+\varrho}^{\varepsilon,t,\zeta},f_1(\frac{u}{\varepsilon},X_u^{\varepsilon,t,\zeta},Y_u^{\varepsilon,t,\zeta})\right\rangle\mathrm{d}u\\
&+2\int_r^{s+\varrho}\left\langle Y_u^{\varepsilon,t,\zeta}-Y_{s+\varrho}^{\varepsilon,t,\zeta},f_2(Z_u^{\varepsilon,t,\zeta})\right\rangle\mathrm{d}u-2\int_r^{s+\varrho}\left\langle Y_u^{\varepsilon,t,\zeta}-Y_{s+\varrho}^{\varepsilon,t,\zeta},Z_u^{\varepsilon,t,\zeta}\mathrm{d}W_u\right\rangle. \qquad (23)
\end{aligned}
$$

Next, we compute $-2\int_r^{s+\varrho}\left\langle Y_u^{\varepsilon,t,\zeta}-Y_{s+\varrho}^{\varepsilon,t,\zeta},\mathrm{d}K_u^{\varepsilon,t,\zeta}\right\rangle$. Take any $a\in\mathrm{Int}(\mathcal{D}(\partial\varphi))$. There is a $\theta_0>0$ such that for any $R>0$ and $0<\theta<\theta_0$,

$$\left\{x\in B(a,R):d(x,(\overline{\mathcal{D}(\partial\varphi)})^c)\geqslant\theta\right\}\neq\emptyset,$$

where $B(a,R):=\{x\in\mathbb{R}^d:|x-a|\leqslant R\}$, $d(\cdot,\cdot)$ is the Euclidean distance in $\mathbb{R}^d$, and $(\overline{\mathcal{D}(\partial\varphi)})^c$ denotes the complement of $\overline{\mathcal{D}(\partial\varphi)}$. We set

$$g_R(\theta):=\sup\left\{|z|:z\in\partial\varphi(x)\text{ for all }x\in B(a,R)\text{ with }d\left(x,(\overline{\mathcal{D}(\partial\varphi)})^c\right)\geqslant\theta\right\},$$

and by the local boundedness of $\partial\varphi$ on $\mathrm{Int}(\mathcal{D}(\partial\varphi))$ (cf. [2]), it holds that

$$g_R(\theta)<+\infty.$$

We again set

$$h_R(\varrho):=\inf\left\{\theta\in(0,\theta_0):g_R(\theta)\leqslant\varrho^{-1/2}\right\},\quad\varrho>0,$$

and we obtain that

$$g_R\left(\varrho+h_R(\varrho)\right)\leqslant\varrho^{-1/2}\quad\text{and}\quad\lim_{\varrho\downarrow0}h_R(\varrho)=0.$$

We take $\varrho_R>0$ such that $\varrho_R+h_R(\varrho_R)<\theta_0$. For $0<\varrho<\varrho_R\wedge1$, we let $Y_{s+\varrho}^{\varepsilon,\varrho,R}$ be the projection of $Y_{s+\varrho}^{\varepsilon,t,\zeta}$ onto $\left\{x\in B(a,R):d\left(x,(\overline{\mathcal{D}(\partial\varphi)})^c\right)\geqslant\varrho+h_R(\varrho)\right\}$. Thus, for $\Pi_{s+\varrho}^{\varepsilon,\varrho,R}\in\partial\varphi(Y_{s+\varrho}^{\varepsilon,\varrho,R})$, $\sup\limits_{v\in[t,T]}|Y_v^{\varepsilon,t,\zeta}-a|\leqslant R$, and $0<r-s<\varrho$, it holds that

$$
\begin{aligned}
&-2\int_r^{s+\varrho}\left\langle Y_u^{\varepsilon,t,\zeta}-Y_{s+\varrho}^{\varepsilon,t,\zeta},\mathrm{d}K_u^{\varepsilon,t,\zeta}\right\rangle\\
=&-2\int_r^{s+\varrho}\left\langle Y_u^{\varepsilon,t,\zeta}-Y_{s+\varrho}^{\varepsilon,\varrho,R},\mathrm{d}K_u^{\varepsilon,t,\zeta}\right\rangle-2\int_r^{s+\varrho}\left\langle Y_{s+\varrho}^{\varepsilon,\varrho,R}-Y_{s+\varrho}^{\varepsilon,t,\zeta},\mathrm{d}K_u^{\varepsilon,t,\zeta}\right\rangle\\
\leqslant&-2\int_r^{s+\varrho}\left\langle Y_u^{\varepsilon,t,\zeta}-Y_{s+\varrho}^{\varepsilon,\varrho,R},\Pi_{s+\varrho}^{\varepsilon,\varrho,R}\right\rangle\mathrm{d}u+2\left(\varrho+h_R(\varrho)\right)\left|K^{\varepsilon,t,\zeta}\right|_t^T\\
\leqslant&\,4\varrho^{1/2}(R+|a|)+2\left(\varrho+h_R(\varrho)\right)\left|K^{\varepsilon,t,\zeta}\right|_t^T,
\end{aligned}
$$

and furthermore, by (23),

$$\begin{aligned}
&|Y_r^{\varepsilon,t,\zeta}-Y_{s+\varrho}^{\varepsilon,t,\zeta}|^2 I_{\{\sup\limits_{v\in[t,T]}|Y_v^{\varepsilon,t,\zeta}-a|\leqslant R\}}+\int_r^{s+\varrho}\|Z_u^{\varepsilon,t,\zeta}\|^2\mathrm{d}u I_{\{\sup\limits_{v\in[t,T]}|Y_v^{\varepsilon,t,\zeta}-a|\leqslant R\}}\\
&\leqslant 4\varrho^{1/2}(R+|a|)+2\left(\varrho+h_R(\varrho)\right)\left|K^{\varepsilon,t,\zeta}\right|_t^T+C\int_r^{s+\varrho}|Y_u^{\varepsilon,t,\zeta}-Y_{s+\varrho}^{\varepsilon,t,\zeta}|^2 I_{\{\sup\limits_{v\in[t,T]}|Y_v^{\varepsilon,t,\zeta}-a|\leqslant R\}}\mathrm{d}u\\
&\quad+C\varrho+C\int_r^{s+\varrho}|X_u^{\varepsilon,t,\zeta}|^{2q_3+2}\mathrm{d}u+C\int_r^{s+\varrho}|Y_u^{\varepsilon,t,\zeta}|^2\mathrm{d}u+\frac{1}{2}\int_r^{s+\varrho}\|Z_u^{\varepsilon,t,\zeta}\|^2\mathrm{d}u I_{\{\sup\limits_{v\in[t,T]}|Y_v^{\varepsilon,t,\zeta}-a|\leqslant R\}}\\
&\quad+2\left|\int_r^{s+\varrho}\left\langle Y_u^{\varepsilon,t,\zeta},Z_u^{\varepsilon,t,\zeta}\mathrm{d}W_u\right\rangle\right|+(R+|a|)\left|\int_r^{s+\varrho}Z_u^{\varepsilon,t,\zeta}\mathrm{d}W_u\right|.
\end{aligned}$$

In the following, we take the expectation on both sides of the above inequality, and obtain by the BDG inequality and the Hölder inequality that

$$\begin{aligned}
&\sup_{s\leqslant r\leqslant s+\varrho}\mathbb{E}|Y_r^{\varepsilon,t,\zeta}-Y_{s+\varrho}^{\varepsilon,t,\zeta}|^2 I_{\{\sup\limits_{v\in[t,T]}|Y_v^{\varepsilon,t,\zeta}-a|\leqslant R\}}+\mathbb{E}\int_s^{s+\varrho}\|Z_u^{\varepsilon,t,\zeta}\|^2\mathrm{d}u I_{\{\sup\limits_{v\in[t,T]}|Y_v^{\varepsilon,t,\zeta}-a|\leqslant R\}}\\
&\leqslant 4\varrho^{1/2}(R+|a|)+2\left(\varrho+h_R(\varrho)\right)\mathbb{E}\left|K^{\varepsilon,t,\zeta}\right|_t^T+C\mathbb{E}\int_s^{s+\varrho}|Y_u^{\varepsilon,t,\zeta}-Y_{s+\varrho}^{\varepsilon,t,\zeta}|^2 I_{\{\sup\limits_{v\in[t,T]}|Y_v^{\varepsilon,t,\zeta}-a|\leqslant R\}}\mathrm{d}u\\
&\quad+C\varrho+C\int_s^{s+\varrho}\mathbb{E}|X_u^{\varepsilon,t,\zeta}|^{2q_3+2}\mathrm{d}u+C\int_s^{s+\varrho}\mathbb{E}|Y_u^{\varepsilon,t,\zeta}|^2\mathrm{d}u+\frac{1}{2}\mathbb{E}\int_s^{s+\varrho}\|Z_u^{\varepsilon,t,\zeta}\|^2\mathrm{d}u I_{\{\sup\limits_{v\in[t,T]}|Y_v^{\varepsilon,t,\zeta}-a|\leqslant R\}}\\
&\quad+C\left(\mathbb{E}\sup_{u\in[s,s+\varrho]}|Y_u^{\varepsilon,t,\zeta}|^2\right)^{1/2}\left(\int_s^{s+\varrho}\mathbb{E}\|Z_u^{\varepsilon,t,\zeta}\|^2\mathrm{d}u\right)^{1/2}+C(R+|a|)\left(\int_s^{s+\varrho}\mathbb{E}\|Z_u^{\varepsilon,t,\zeta}\|^2\mathrm{d}u\right)^{1/2}.
\end{aligned}$$

Thus, it follows from the Gronwall inequality that

$$\begin{aligned}
&\sup_{s\leqslant r\leqslant s+\varrho}\mathbb{E}|Y_r^{\varepsilon,t,\zeta}-Y_{s+\varrho}^{\varepsilon,t,\zeta}|^2 I_{\{\sup\limits_{v\in[t,T]}|Y_v^{\varepsilon,t,\zeta}-a|\leqslant R\}}\\
&\leqslant \left(4\varrho^{1/2}(R+|a|)+2\left(\varrho+h_R(\varrho)\right)\mathbb{E}\left|K^{\varepsilon,t,\zeta}\right|_t^T\right)+C\varrho+C(1+R+|a|)\left(\int_s^{s+\varrho}\mathbb{E}\|Z_u^{\varepsilon,t,\zeta}\|^2\mathrm{d}u\right)^{1/2}.
\end{aligned}$$

Based on this and the absolute continuity of the integration, we let $\varrho\to 0$ first and then $R\to\infty$ to conclude (21), which completes the proof. □

Proof of Theorem 3.5

Step 1 We estimate $\int_t^T\mathbb{E}\|Z_r^{\varepsilon,t,\zeta}-\bar{Z}_r^{t,\zeta}\|^2\mathrm{d}r$.

First, combining (2) with (10), we obtain that

$$\begin{aligned}
Y_s^{\varepsilon,t,\zeta}-\bar{Y}_s^{t,\zeta}=\;&g\left(X_T^{\varepsilon,t,\zeta}\right)-g\left(\bar{X}_T^{t,\zeta}\right)-\left(K_T^{\varepsilon,t,\zeta}-K_s^{\varepsilon,t,\zeta}\right)+\left(\bar{K}_T^{t,\zeta}-\bar{K}_s^{t,\zeta}\right)\\
&+\int_s^T\left(f_1\left(\frac{r}{\varepsilon},X_r^{\varepsilon,t,\zeta},Y_r^{\varepsilon,t,\zeta}\right)-\bar{f}_1\left(\bar{X}_r^{t,\zeta},\bar{Y}_r^{t,\zeta}\right)\right)\mathrm{d}r\\
&+\int_s^T\left(f_2\left(Z_r^{\varepsilon,t,\zeta}\right)-f_2\left(\bar{Z}_r^{t,\zeta}\right)\right)\mathrm{d}r-\int_s^T\left(Z_r^{\varepsilon,t,\zeta}-\bar{Z}_r^{t,\zeta}\right)\mathrm{d}W_r.
\end{aligned}$$

The Itô formula for $|Y_s^{\varepsilon,t,\zeta}-\bar{Y}_s^{t,\zeta}|^2$ and $(\mathbf{H}_g)$ $(\mathbf{H}_f^1)$ imply that

$$
\begin{aligned}
&|Y_s^{\varepsilon,t,\zeta}-\bar{Y}_s^{t,\zeta}|^2+\int_s^T \|Z_r^{\varepsilon,t,\zeta}-\bar{Z}_r^{t,\zeta}\|^2\mathrm{d}r\\
&=|g\left(X_T^{\varepsilon,t,\zeta}\right)-g\left(\bar{X}_T^{t,\zeta}\right)|^2-2\int_s^T \left\langle Y_r^{\varepsilon,t,\zeta}-\bar{Y}_r^{t,\zeta},\mathrm{d}\left(K_r^{\varepsilon,t,\zeta}-\bar{K}_r^{t,\zeta}\right)\right\rangle\\
&\quad-2\int_s^T \left\langle Y_r^{\varepsilon,t,\zeta}-\bar{Y}_r^{t,\zeta},\left(Z_r^{\varepsilon,t,\zeta}-\bar{Z}_r^{t,\zeta}\right)\mathrm{d}W_r\right\rangle\\
&\quad+2\int_s^T \left\langle Y_r^{\varepsilon,t,\zeta}-\bar{Y}_r^{t,\zeta},f_1\left(\frac{r}{\varepsilon},X_r^{\varepsilon,t,\zeta},Y_r^{\varepsilon,t,\zeta}\right)-\bar{f}_1\left(\bar{X}_r^{t,\zeta},\bar{Y}_r^{t,\zeta}\right)\right\rangle\mathrm{d}r\\
&\quad+2\int_s^T \left\langle Y_r^{\varepsilon,t,\zeta}-\bar{Y}_r^{t,\zeta},f_2\left(Z_r^{\varepsilon,t,\zeta}\right)-f_2\left(\bar{Z}_r^{t,\zeta}\right)\right\rangle\mathrm{d}r\\
&\overset{\text{Lemma 2.2}}{\leqslant} L_3^2\left(1+|X_T^{\varepsilon,t,\zeta}|^{q_2}+|\bar{X}_T^{t,\zeta}|^{q_2}\right)^2|X_T^{\varepsilon,t,\zeta}-\bar{X}_T^{t,\zeta}|^2\\
&\quad-2\int_s^T \left\langle Y_r^{\varepsilon,t,\zeta}-\bar{Y}_r^{t,\zeta},\left(Z_r^{\varepsilon,t,\zeta}-\bar{Z}_r^{t,\zeta}\right)\mathrm{d}W_r\right\rangle\\
&\quad+2\int_s^T \left\langle Y_r^{\varepsilon,t,\zeta}-\bar{Y}_r^{t,\zeta},f_1\left(\frac{r}{\varepsilon},X_r^{\varepsilon,t,\zeta},Y_r^{\varepsilon,t,\zeta}\right)-\bar{f}_1\left(X_r^{\varepsilon,t,\zeta},Y_r^{\varepsilon,t,\zeta}\right)\right\rangle\mathrm{d}r\\
&\quad+2\int_s^T \left\langle Y_r^{\varepsilon,t,\zeta}-\bar{Y}_r^{t,\zeta},\bar{f}_1\left(X_r^{\varepsilon,t,\zeta},Y_r^{\varepsilon,t,\zeta}\right)-\bar{f}_1\left(\bar{X}_r^{t,\zeta},\bar{Y}_r^{t,\zeta}\right)\right\rangle\mathrm{d}r\\
&\quad+2\int_s^T \left\langle Y_r^{\varepsilon,t,\zeta}-\bar{Y}_r^{t,\zeta},f_2\left(Z_r^{\varepsilon,t,\zeta}\right)-f_2\left(\bar{Z}_r^{t,\zeta}\right)\right\rangle\mathrm{d}r\\
&\leqslant L_3^2\left(1+|X_T^{\varepsilon,t,\zeta}|^{q_2}+|\bar{X}_T^{t,\zeta}|^{q_2}\right)^2|X_T^{\varepsilon,t,\zeta}-\bar{X}_T^{t,\zeta}|^2\\
&\quad-2\int_s^T \left\langle Y_r^{\varepsilon,t,\zeta}-\bar{Y}_r^{t,\zeta},\left(Z_r^{\varepsilon,t,\zeta}-\bar{Z}_r^{t,\zeta}\right)\mathrm{d}W_r\right\rangle\\
&\quad+2\int_s^T \left\langle Y_r^{\varepsilon,t,\zeta}-\bar{Y}_r^{t,\zeta},f_1\left(\frac{r}{\varepsilon},X_r^{\varepsilon,t,\zeta},Y_r^{\varepsilon,t,\zeta}\right)-\bar{f}_1\left(X_r^{\varepsilon,t,\zeta},Y_r^{\varepsilon,t,\zeta}\right)\right\rangle\mathrm{d}r\\
&\quad+L_4^2\int_s^T\left(1+|X_r^{\varepsilon,t,\zeta}|^{q_3}+|\bar{X}_r^{t,\zeta}|^{q_3}\right)^2|X_r^{\varepsilon,t,\zeta}-\bar{X}_r^{t,\zeta}|^2\mathrm{d}r\\
&\quad+\left(1+2L_4+L_5^2/\eta\right)\int_s^T|Y_r^{\varepsilon,t,\zeta}-\bar{Y}_r^{t,\zeta}|^2\mathrm{d}r+\eta\int_s^T\|Z_r^{\varepsilon,t,\zeta}-\bar{Z}_r^{t,\zeta}\|^2\mathrm{d}r, \qquad (24)
\end{aligned}
$$

where $0<\eta<1$ is a constant. By taking the expectation on two sides, we obtain that

$$
\begin{aligned}
&\mathbb{E}|Y_s^{\varepsilon,t,\zeta}-\bar{Y}_s^{t,\zeta}|^2+\int_s^T\mathbb{E}\|Z_r^{\varepsilon,t,\zeta}-\bar{Z}_r^{t,\zeta}\|^2\mathrm{d}r\\
&\leqslant C\left(1+\mathbb{E}|\zeta|^{4q_2}+\mathbb{E}|\zeta|^{4q_3}\right)^{1/2}\left(\mathbb{E}\sup_{s\in[t,T]}|X_s^{\varepsilon,t,\zeta}-\bar{X}_s^{t,\zeta}|^4\right)^{1/2}\\
&\quad+\left(1+2L_4+L_5^2/\eta\right)\int_s^T\mathbb{E}|Y_r^{\varepsilon,t,\zeta}-\bar{Y}_r^{t,\zeta}|^2\mathrm{d}s+\eta\int_s^T\mathbb{E}\|Z_r^{\varepsilon,t,\zeta}-\bar{Z}_r^{t,\zeta}\|^2\mathrm{d}r\\
&\quad+2\mathbb{E}\int_s^T\left\langle Y_r^{\varepsilon,t,\zeta}-\bar{Y}_r^{t,\zeta},f_1\left(\frac{r}{\varepsilon},X_r^{\varepsilon,t,\zeta},Y_r^{\varepsilon,t,\zeta}\right)-\bar{f}_1\left(X_r^{\varepsilon,t,\zeta},Y_r^{\varepsilon,t,\zeta}\right)\right\rangle\mathrm{d}r.
\end{aligned}
$$

Please note that

$$
\begin{aligned}
&2\sup_{s\in[t,T]}\mathbb{E}\left|\int_s^T \left\langle Y_r^{\varepsilon,t,\zeta}-\bar{Y}_r^{t,\zeta}, f_1\left(\frac{r}{\varepsilon}, X_r^{\varepsilon,t,\zeta}, Y_r^{\varepsilon,t,\zeta}\right)-\bar{f}_1\left(X_r^{\varepsilon,t,\zeta}, Y_r^{\varepsilon,t,\zeta}\right)\right\rangle \mathrm{d}r\right| \\
&\leqslant C\left(\sup_{s\in[t,T]}\sup_{s\leqslant r\leqslant s+\delta}\mathbb{E}|Y_r^{\varepsilon,t,\zeta}-Y_{s+\delta}^{\varepsilon,t,\zeta}|^2+\sup_{s\in[t,T]}\sup_{s\leqslant r\leqslant s+\delta}\mathbb{E}|\bar{Y}_r^{t,\zeta}-\bar{Y}_{s+\delta}^{t,\zeta}|^2\right)^{1/2} \\
&\quad +C\sup_{s\in[t,T]}\sup_{s\leqslant r\leqslant s+\delta}\mathbb{E}|Y_r^{\varepsilon,t,\zeta}-Y_{s+\delta}^{\varepsilon,t,\zeta}|^2+4CT\kappa_3^{1/2}\left(\frac{\delta}{\varepsilon}\right)+C\left(\delta+\delta^2\right) \\
&\quad +2\int_t^T\sup_{s\in[t,r]}\mathbb{E}|Y_s^{\varepsilon,t,\zeta}-\bar{Y}_s^{t,\zeta}|^2\mathrm{d}r, \qquad (25)
\end{aligned}
$$

where δ is the same as that in the proof for Theorem 3.2. Thus, it holds that

$$
\begin{aligned}
&\sup_{s\in[t,T]}\mathbb{E}|Y_s^{\varepsilon,t,\zeta}-\bar{Y}_s^{t,\zeta}|^2+(1-\eta)\int_t^T\mathbb{E}\|Z_r^{\varepsilon,t,\zeta}-\bar{Z}_r^{t,\zeta}\|^2\mathrm{d}r \\
&\leqslant \Gamma(\varepsilon)+(3+2L_4+L_5^2/\eta)\int_t^T\sup_{s\in[t,r]}\mathbb{E}|Y_s^{\varepsilon,t,\zeta}-\bar{Y}_s^{t,\zeta}|^2\mathrm{d}r, \qquad (26)
\end{aligned}
$$

where

$$
\begin{aligned}
\Gamma(\varepsilon):=\;&C(1+\mathbb{E}|\zeta|^{4q_2}+\mathbb{E}|\zeta|^{4q_3})^{1/2}(\mathbb{E}\sup_{s\in[t,T]}|X_s^{\varepsilon,t,\zeta}-\bar{X}_s^{t,\zeta}|^4)^{1/2} \\
&+C\sup_{s\in[t,T]}\sup_{s\leqslant r\leqslant s+\delta}\mathbb{E}|Y_r^{\varepsilon,t,\zeta}-Y_{s+\delta}^{\varepsilon,t,\zeta}|^2+4CT\kappa_3^{1/2}(\frac{\delta}{\varepsilon})+C(\delta+\delta^2) \\
&+C\left(\sup_{s\in[t,T]}\sup_{s\leqslant r\leqslant s+\delta}\mathbb{E}|Y_r^{\varepsilon,t,\zeta}-Y_{s+\delta}^{\varepsilon,t,\zeta}|^2+\sup_{s\in[t,T]}\sup_{s\leqslant r\leqslant s+\delta}\mathbb{E}|\bar{Y}_r^{t,\zeta}-\bar{Y}_{s+\delta}^{t,\zeta}|^2\right)^{1/2}.
\end{aligned}
$$

The Gronwall inequality implies that

$$
\sup_{s\in[t,T]}\mathbb{E}|Y_s^{\varepsilon,t,\zeta}-\bar{Y}_s^{t,\zeta}|^2\leqslant\Gamma(\varepsilon)e^{(4+6L_4^2)T}.
$$

Inserting the above inequality into (26), we obtain that

$$
\int_t^T\mathbb{E}\|Z_r^{\varepsilon,t,\zeta}-\bar{Z}_r^{t,\zeta}\|^2\mathrm{d}r\leqslant C\Gamma(\varepsilon). \qquad (27)
$$

Step 2 We prove $\lim_{\varepsilon\to0}\mathbb{E}\sup_{s\in[t,T]}|Y_s^{\varepsilon,t,\zeta}-\bar{Y}_s^{t,\zeta}|^2=0$.

For (24), it holds by the BDG inequality that

$$
\begin{aligned}
&\mathbb{E}\sup_{s\in[t,T]}|Y_s^{\varepsilon,t,\zeta}-\bar{Y}_s^{t,\zeta}|^2+\int_t^T\mathbb{E}\|Z_r^{\varepsilon,t,\zeta}-\bar{Z}_r^{t,\zeta}\|^2\mathrm{d}r \\
&\leqslant L_3^2\mathbb{E}\left(1+|X_T^{\varepsilon,t,\zeta}|^{q_2}+|\bar{X}_T^{t,\zeta}|^{q_2}\right)^2|X_T^{\varepsilon,t,\zeta}-\bar{X}_T^{t,\zeta}|^2 \\
&\quad +C\mathbb{E}\left(\int_t^T|Y_r^{\varepsilon,t,\zeta}-\bar{Y}_r^{t,\zeta}|^2\|Z_r^{\varepsilon,t,\zeta}-\bar{Z}_r^{t,\zeta}\|^2\mathrm{d}r\right)^{1/2} \\
&\quad +2\mathbb{E}\sup_{s\in[t,T]}\left|\int_s^T\left\langle Y_r^{\varepsilon,t,\zeta}-\bar{Y}_r^{t,\zeta}, f_1\left(\frac{r}{\varepsilon}, X_r^{\varepsilon,t,\zeta}, Y_r^{\varepsilon,t,\zeta}\right)-\bar{f}_1\left(X_r^{\varepsilon,t,\zeta}, Y_r^{\varepsilon,t,\zeta}\right)\right\rangle\mathrm{d}r\right| \\
&\quad +6L_4^2\mathbb{E}\int_t^T\left(1+|X_r^{\varepsilon,t,\zeta}|^{q_3}+|\bar{X}_r^{t,\zeta}|^{q_3}\right)^2|X_r^{\varepsilon,t,\zeta}-\bar{X}_r^{t,\zeta}|^2\mathrm{d}r
\end{aligned}
$$

$$
\begin{aligned}
&+\left(2+6L_4^2\right)\mathbb{E}\int_t^T |Y_r^{\varepsilon,t,\zeta}-\bar{Y}_r^{t,\zeta}|^2\mathrm{d}r+L_5\mathbb{E}\int_t^T \|Z_r^{\varepsilon,t,\zeta}-\bar{Z}_r^{t,\zeta}\|^2\mathrm{d}r\\
\leqslant{}& C\left(1+\mathbb{E}|\zeta|^{4q_2}+\mathbb{E}|\zeta|^{4q_3}\right)^{1/2}\left(\mathbb{E}\sup_{s\in[t,T]}|X_s^{\varepsilon,t,\zeta}-\bar{X}_s^{t,\zeta}|^4\right)^{1/2}+\frac{1}{2}\mathbb{E}\sup_{s\in[t,T]}|Y_s^{\varepsilon,t,\zeta}-\bar{Y}_s^{t,\zeta}|^2\\
&+\left(2+6L_4^2\right)\int_t^T \mathbb{E}\sup_{s\in[t,r]}|Y_s^{\varepsilon,t,\zeta}-\bar{Y}_s^{t,\zeta}|^2\mathrm{d}r+C\int_t^T \mathbb{E}\|Z_r^{\varepsilon,t,\zeta}-\bar{Z}_r^{t,\zeta}\|^2\mathrm{d}r\\
&+2\mathbb{E}\sup_{s\in[t,T]}\left|\int_s^T\left\langle Y_r^{\varepsilon,t,\zeta}-\bar{Y}_r^{t,\zeta},f_1\left(\frac{r}{\varepsilon},X_r^{\varepsilon,t,\zeta},Y_r^{\varepsilon,t,\zeta}\right)-\bar{f}_1\left(X_r^{\varepsilon,t,\zeta},Y_r^{\varepsilon,t,\zeta}\right)\right\rangle\mathrm{d}r\right|.
\end{aligned}
$$

In addition, by deduction similar to that for (25), we obtain that

$$
\begin{aligned}
&2\mathbb{E}\sup_{s\in[t,T]}\left|\int_s^T\left\langle Y_r^{\varepsilon,t,\zeta}-\bar{Y}_r^{t,\zeta},f_1(\frac{r}{\varepsilon},X_r^{\varepsilon,t,\zeta},Y_r^{\varepsilon,t,\zeta})-\bar{f}_1(X_r^{\varepsilon,t,\zeta},Y_r^{\varepsilon,t,\zeta})\right\rangle\mathrm{d}r\right|\\
\leqslant{}& C\left(\sup_{s\in[t,T]}\sup_{s\leqslant r\leqslant s+\delta}\mathbb{E}|Y_r^{\varepsilon,t,\zeta}-Y_{s+\delta}^{\varepsilon,t,\zeta}|^2+\sup_{s\in[t,T]}\sup_{s\leqslant r\leqslant s+\delta}\mathbb{E}|\bar{Y}_r^{t,\zeta}-\bar{Y}_{s+\delta}^{t,\zeta}|^2\right)^{1/2}\\
&+C\sup_{s\in[t,T]}\sup_{s\leqslant r\leqslant s+\delta}\mathbb{E}|Y_r^{\varepsilon,t,\zeta}-Y_{s+\delta}^{\varepsilon,t,\zeta}|^2+4CT\kappa_3^{1/2}(\frac{\delta}{\varepsilon})+C(\delta+\delta^2)\\
&+2\int_t^T\mathbb{E}\sup_{s\in[t,r]}|Y_s^{\varepsilon,t,\zeta}-\bar{Y}_s^{t,\zeta}|^2\mathrm{d}r,
\end{aligned}
$$

which, together with (27) and the Gronwall inequality, yields that

$$
\mathbb{E}\sup_{s\in[t,T]}|Y_s^{\varepsilon,t,\zeta}-\bar{Y}_s^{t,\zeta}|^2\leqslant C\Gamma(\varepsilon).
$$

Finally, we study the limit of $\mathbb{E}\sup_{s\in[t,T]}|Y_s^{\varepsilon,t,\zeta}-\bar{Y}_s^{t,\zeta}|^2$ as $\varepsilon\to0$. On the one hand, it holds by (6) that

$$
\lim_{\varepsilon\to0}\mathbb{E}\sup_{s\in[t,T]}|X_s^{\varepsilon,t,\zeta}-\bar{X}_s^{t,\zeta}|^4=0.
$$

On the other hand, we take $\delta=\varepsilon^\gamma$ for $0<\gamma<1$, and by (21) and (22) we obtain that

$$
\begin{gathered}
\lim_{\varepsilon\to0}\sup_{s\in[t,T]}\sup_{s\leqslant r\leqslant s+\delta}\mathbb{E}|Y_r^{\varepsilon,t,\zeta}-Y_{s+\delta}^{\varepsilon,t,\zeta}|^2=0,\\
\lim_{\varepsilon\to0}\sup_{s\in[t,T]}\sup_{s\leqslant r\leqslant s+\delta}\mathbb{E}|\bar{Y}_r^{t,\zeta}-\bar{Y}_{s+\delta}^{t,\zeta}|^2=0,\\
\lim_{\varepsilon\to0}\kappa_3^{1/2}\left(\frac{\delta}{\varepsilon}\right)=0.
\end{gathered}
$$

Combining the above deductions, we obtain that

$$
\lim_{\varepsilon\to0}\mathbb{E}\sup_{s\in[t,T]}|Y_s^{\varepsilon,t,\zeta}-\bar{Y}_s^{t,\zeta}|^2=0.
$$

Step 3 We prove (25).

We set

$$
I:=2\sup_{s\in[t,T]}\mathbb{E}\left|\int_s^T\left\langle Y_u^{\varepsilon,t,\zeta}-\bar{Y}_u^{t,\zeta},f_1\left(\frac{u}{\varepsilon},X_u^{\varepsilon,t,\zeta},Y_u^{\varepsilon,t,\zeta}\right)-\bar{f}_1(X_u^{\varepsilon,t,\zeta},Y_u^{\varepsilon,t,\zeta})\right\rangle\mathrm{d}u\right|,
$$

and

$$
\begin{aligned}
I \leqslant\ & 2\sup_{s\in[t,T]}\mathbb{E}\left|\int_s^T\left\langle Y_u^{\varepsilon,t,\zeta}-\bar{Y}_u^{t,\zeta}-Y_{u(\delta)}^{\varepsilon,t,\zeta}+\bar{Y}_{u(\delta)}^{t,\zeta}, f_1\left(\frac{u}{\varepsilon},X_u^{\varepsilon,t,\zeta},Y_u^{\varepsilon,t,\zeta}\right)-\bar{f}_1\left(X_u^{\varepsilon,t,\zeta},Y_u^{\varepsilon,t,\zeta}\right)\right\rangle \mathrm{d}u\right|\\
&+2\sup_{s\in[t,T]}\mathbb{E}\left|\int_s^T\left\langle Y_{u(\delta)}^{\varepsilon,t,\zeta}-\bar{Y}_{u(\delta)}^{t,\zeta}, f_1\left(\frac{u}{\varepsilon},X_u^{\varepsilon,t,\zeta},Y_u^{\varepsilon,t,\zeta}\right)-f_1\left(\frac{u}{\varepsilon},X_{u(\delta)}^{\varepsilon,t,\zeta},Y_{u(\delta)}^{\varepsilon,t,\zeta}\right)\right\rangle \mathrm{d}u\right|\\
&+2\sup_{s\in[t,T]}\mathbb{E}\left|\int_s^T\left\langle Y_{u(\delta)}^{\varepsilon,t,\zeta}-\bar{Y}_{u(\delta)}^{t,\zeta}, \bar{f}_1\left(X_{u(\delta)}^{\varepsilon,t,\zeta},Y_{u(\delta)}^{\varepsilon,t,\zeta}\right)-\bar{f}_1\left(X_u^{\varepsilon,t,\zeta},Y_u^{\varepsilon,t,\zeta}\right)\right\rangle \mathrm{d}u\right|\\
&+2\sup_{s\in[t,T]}\mathbb{E}\left|\int_s^T\left\langle Y_{u(\delta)}^{\varepsilon,t,\zeta}-\bar{Y}_{u(\delta)}^{t,\zeta}, f_1\left(\frac{u}{\varepsilon},X_{u(\delta)}^{\varepsilon,t,\zeta},Y_{u(\delta)}^{\varepsilon,t,\zeta}\right)-\bar{f}_1\left(X_{u(\delta)}^{\varepsilon,t,\zeta},Y_{u(\delta)}^{\varepsilon,t,\zeta}\right)\right\rangle \mathrm{d}u\right|\\
=:\ & I_1+I_2+I_3+I_4.
\end{aligned}
\tag{28}
$$

For I_1, by the Hölder inequality and the growth of $f_1,\bar{f}_1$, it holds that

$$
\begin{aligned}
I_1 &\leqslant 2\left(\mathbb{E}\int_t^T|Y_u^{\varepsilon,t,\zeta}-\bar{Y}_u^{t,\zeta}-Y_{u(\delta)}^{\varepsilon,t,\zeta}+\bar{Y}_{u(\delta)}^{t,\zeta}|^2\mathrm{d}u\right)^{1/2}\\
&\quad\times\left(\mathbb{E}\int_t^T\left|f_1\left(\frac{u}{\varepsilon},X_u^{\varepsilon,t,\zeta},Y_u^{\varepsilon,t,\zeta}\right)-\bar{f}_1(X_u^{\varepsilon,t,\zeta},Y_u^{\varepsilon,t,\zeta})\right|^2\mathrm{d}u\right)^{1/2}\\
&\leqslant C\left(\int_t^T(\mathbb{E}|Y_u^{\varepsilon,t,\zeta}-Y_{u(\delta)}^{\varepsilon,t,\zeta}|^2+\mathbb{E}|\bar{Y}_u^{t,\zeta}-\bar{Y}_{u(\delta)}^{t,\zeta}|^2)\mathrm{d}u\right)^{1/2}\\
&\quad\times\left(\int_t^T(1+\mathbb{E}|X_u^{\varepsilon,t,\zeta}|^{2q_3+2}+\mathbb{E}|Y_u^{\varepsilon,t,\zeta}|^2)\mathrm{d}u\right)^{1/2}\\
&\leqslant C\left(\sup_{s\in[t,T]}\sup_{s\leqslant u\leqslant s+\delta}\mathbb{E}|Y_u^{\varepsilon,t,\zeta}-Y_{s+\delta}^{\varepsilon,t,\zeta}|^2+\sup_{s\in[t,T]}\sup_{s\leqslant u\leqslant s+\delta}\mathbb{E}|\bar{Y}_u^{t,\zeta}-\bar{Y}_{s+\delta}^{t,\zeta}|^2\right)^{1/2}.
\end{aligned}
\tag{29}
$$

For I_2, the Hölder inequality and $(\mathbf{H}_f^1)$ imply that

$$
\begin{aligned}
I_2 &\leqslant \int_t^T\mathbb{E}|Y_{u(\delta)}^{\varepsilon,t,\zeta}-\bar{Y}_{u(\delta)}^{t,\zeta}|^2\mathrm{d}u+\int_t^T\mathbb{E}\left|f_1\left(\frac{u}{\varepsilon},X_u^{\varepsilon,t,\zeta},Y_u^{\varepsilon,t,\zeta}\right)-f_1\left(\frac{u}{\varepsilon},X_{u(\delta)}^{\varepsilon,t,\zeta},Y_{u(\delta)}^{\varepsilon,t,\zeta}\right)\right|^2\mathrm{d}u\\
&\leqslant \int_t^T\sup_{s\in[t,u]}\mathbb{E}|Y_s^{\varepsilon,t,\zeta}-\bar{Y}_s^{t,\zeta}|^2\mathrm{d}u+2L_4^2\int_t^T\mathbb{E}|Y_u^{\varepsilon,t,\zeta}-Y_{u(\delta)}^{\varepsilon,t,\zeta}|^2\mathrm{d}u\\
&\quad+2L_4^2\int_t^T\mathbb{E}(1+|X_u^{\varepsilon,t,\zeta}|^{q_3}+|X_{u(\delta)}^{\varepsilon,t,\zeta}|^{q_3})^2|X_u^{\varepsilon,t,\zeta}-X_{u(\delta)}^{\varepsilon,t,\zeta}|^2\mathrm{d}u\\
&\leqslant \int_t^T\sup_{s\in[t,u]}\mathbb{E}|Y_s^{\varepsilon,t,\zeta}-\bar{Y}_s^{t,\zeta}|^2\mathrm{d}u+C\left(\sup_{s\in[t,T]}\sup_{s\leqslant u\leqslant s+\delta}\mathbb{E}|Y_u^{\varepsilon,t,\zeta}-Y_{s+\delta}^{\varepsilon,t,\zeta}|^2\right)\\
&\quad+2L_4^2TC(1+\mathbb{E}|\zeta|^{4q_3})^{1/2}(\delta+\delta^2).
\end{aligned}
\tag{30}
$$

For I_3, by the same deduction as that for I_2, we obtain that

$$
\begin{aligned}
I_3 &\leqslant \int_t^T\sup_{s\in[t,u]}\mathbb{E}|Y_s^{\varepsilon,t,\zeta}-\bar{Y}_s^{t,\zeta}|^2\mathrm{d}u+C\left(\sup_{s\in[t,T]}\sup_{s\leqslant u\leqslant s+\delta}\mathbb{E}|Y_u^{\varepsilon,t,\zeta}-Y_{s+\delta}^{\varepsilon,t,\zeta}|^2\right)\\
&\quad+2L_4^2TC(1+\mathbb{E}|\zeta|^{4q_3})^{1/2}(\delta+\delta^2).
\end{aligned}
\tag{31}
$$

For I_4, we define $X_u^{\varepsilon,t,\zeta}=\zeta, Y_u^{\varepsilon,t,\zeta}=Y_t^{\varepsilon,t,\zeta}$, and $\bar{Y}_u^{t,\zeta}=\bar{Y}_t^{t,\zeta}$ for $u\in[0,t]$, and obtain that

$$
\begin{aligned}
I_4 &\leqslant 2\mathbb{E}\left|\int_0^T \left\langle Y^{\varepsilon,t,\zeta}_{u(\delta)} - \bar{Y}^{t,\zeta}_{u(\delta)}, f_1\left(\frac{u}{\varepsilon}, X^{\varepsilon,t,\zeta}_{u(\delta)}, Y^{\varepsilon,t,\zeta}_{u(\delta)}\right) - \bar{f}_1\left(X^{\varepsilon,t,\zeta}_{u(\delta)}, Y^{\varepsilon,t,\zeta}_{u(\delta)}\right)\right\rangle \mathrm{d}u\right| \\
&\quad + 2\sup_{s\in[0,T]} \mathbb{E}\left|\int_0^s \left\langle Y^{\varepsilon,t,\zeta}_{u(\delta)} - \bar{Y}^{t,\zeta}_{u(\delta)}, f_1\left(\frac{u}{\varepsilon}, X^{\varepsilon,t,\zeta}_{u(\delta)}, Y^{\varepsilon,t,\zeta}_{u(\delta)}\right) - \bar{f}_1\left(X^{\varepsilon,t,\zeta}_{u(\delta)}, Y^{\varepsilon,t,\zeta}_{u(\delta)}\right)\right\rangle \mathrm{d}u\right| \\
&\leqslant 4\sup_{s\in[0,T]} \mathbb{E}\left|\int_0^s \left\langle Y^{\varepsilon,t,\zeta}_{u(\delta)} - \bar{Y}^{t,\zeta}_{u(\delta)}, f_1\left(\frac{u}{\varepsilon}, X^{\varepsilon,t,\zeta}_{u(\delta)}, Y^{\varepsilon,t,\zeta}_{u(\delta)}\right) - \bar{f}_1\left(X^{\varepsilon,t,\zeta}_{u(\delta)}, Y^{\varepsilon,t,\zeta}_{u(\delta)}\right)\right\rangle \mathrm{d}u\right| \\
&\leqslant 4\sup_{s\in[0,T]} \mathbb{E}\left|\int_0^{[\frac{s}{\delta}]\delta} \left\langle Y^{\varepsilon,t,\zeta}_{u(\delta)} - \bar{Y}^{t,\zeta}_{u(\delta)}, f_1\left(\frac{u}{\varepsilon}, X^{\varepsilon,t,\zeta}_{u(\delta)}, Y^{\varepsilon,t,\zeta}_{u(\delta)}\right) - \bar{f}_1\left(X^{\varepsilon,t,\zeta}_{u(\delta)}, Y^{\varepsilon,t,\zeta}_{u(\delta)}\right)\right\rangle \mathrm{d}u\right| \\
&\quad + 4\sup_{s\in[0,T]} \mathbb{E}\left|\int_{[\frac{s}{\delta}]\delta}^s \left\langle Y^{\varepsilon,t,\zeta}_{u(\delta)} - \bar{Y}^{t,\zeta}_{u(\delta)}, f_1\left(\frac{u}{\varepsilon}, X^{\varepsilon,t,\zeta}_{u(\delta)}, Y^{\varepsilon,t,\zeta}_{u(\delta)}\right) - \bar{f}_1\left(X^{\varepsilon,t,\zeta}_{u(\delta)}, Y^{\varepsilon,t,\zeta}_{u(\delta)}\right)\right\rangle \mathrm{d}u\right| \\
&=: I_{41} + I_{42},
\end{aligned}
$$

where $u(\delta) := [\frac{u}{\delta}]\delta$. Then, $(\mathbf{H}^1_f)$ yields that

$$
\begin{aligned}
I_{41} &\leqslant 4\sup_{s\in[0,T]} \mathbb{E}\sum_{k=0}^{[\frac{s}{\delta}]-1}\left|\int_{k\delta}^{(k+1)\delta} \left\langle Y^{\varepsilon,t,\zeta}_{k\delta} - \bar{Y}^{t,\zeta}_{k\delta}, f_1\left(\frac{u}{\varepsilon}, X^{\varepsilon,t,\zeta}_{k\delta}, Y^{\varepsilon,t,\zeta}_{k\delta}\right) - \bar{f}_1\left(X^{\varepsilon,t,\zeta}_{k\delta}, Y^{\varepsilon,t,\zeta}_{k\delta}\right)\right\rangle \mathrm{d}u\right| \\
&\leqslant 4\sum_{k=0}^{[\frac{T}{\delta}]-1} \mathbb{E}\left|\int_{k\delta}^{(k+1)\delta} \left\langle Y^{\varepsilon,t,\zeta}_{k\delta} - \bar{Y}^{t,\zeta}_{k\delta}, f_1\left(\frac{u}{\varepsilon}, X^{\varepsilon,t,\zeta}_{k\delta}, Y^{\varepsilon,t,\zeta}_{k\delta}\right) - \bar{f}_1\left(X^{\varepsilon,t,\zeta}_{k\delta}, Y^{\varepsilon,t,\zeta}_{k\delta}\right)\right\rangle \mathrm{d}u\right| \\
&\leqslant 4\left[\frac{T}{\delta}\right] \sup_{0\leqslant k\leqslant [\frac{T}{\delta}]-1} \left(\mathbb{E}|Y^{\varepsilon,t,\zeta}_{k\delta} - \bar{Y}^{t,\zeta}_{k\delta}|^2\right)^{1/2} \\
&\quad \times \left(\mathbb{E}\left|\int_{k\delta}^{(k+1)\delta} \left(f_1\left(\frac{u}{\varepsilon}, X^{\varepsilon,t,\zeta}_{k\delta}, Y^{\varepsilon,t,\zeta}_{k\delta}\right) - \bar{f}_1\left(X^{\varepsilon,t,\zeta}_{k\delta}, Y^{\varepsilon,t,\zeta}_{k\delta}\right)\right)\mathrm{d}u\right|^2\right)^{1/2} \\
&\leqslant 4C\left[\frac{T}{\delta}\right]\delta \sup_{0\leqslant k\leqslant [\frac{T}{\delta}]-1} \left(\mathbb{E}\left|\frac{\varepsilon}{\delta}\int_{k\delta/\varepsilon}^{(k+1)\delta/\varepsilon} \left(f_1\left(u, X^{\varepsilon,t,\zeta}_{k\delta}, Y^{\varepsilon,t,\zeta}_{k\delta}\right) - \bar{f}_1\left(X^{\varepsilon,t,\zeta}_{k\delta}, Y^{\varepsilon,t,\zeta}_{k\delta}\right)\right)\mathrm{d}u\right|^2\right)^{1/2} \\
&\leqslant 4CT\kappa_3^{1/2}\left(\frac{\delta}{\varepsilon}\right).
\end{aligned}
\tag{32}
$$

By the Hölder inequality, it holds that

$$
\begin{aligned}
I_{42} &\leqslant 4\sup_{s\in[0,T]}\left(\int_{[\frac{s}{\delta}]\delta}^s \mathbb{E}\left|f_1\left(\frac{u}{\varepsilon}, X^{\varepsilon,t,\zeta}_{u(\delta)}, Y^{\varepsilon,t,\zeta}_{u(\delta)}\right) - \bar{f}_1(X^{\varepsilon,t,\zeta}_{u(\delta)}, Y^{\varepsilon,t,\zeta}_{u(\delta)})\right|^2 \mathrm{d}u\right)^{1/2} \\
&\quad \times \left(\int_{[\frac{s}{\delta}]\delta}^s \mathbb{E}|Y^{\varepsilon,t,\zeta}_{u(\delta)} - \bar{Y}^{t,\zeta}_{u(\delta)}|^2 \mathrm{d}u\right)^{1/2} \\
&\leqslant 4C\sup_{s\in[0,T]}\left(\int_{[\frac{s}{\delta}]\delta}^s (1+\mathbb{E}|X^{\varepsilon,t,\zeta}_{u(\delta)}|^{2q_3+2} + \mathbb{E}|Y^{\varepsilon,t,\zeta}_{u(\delta)})|^2)\mathrm{d}u\right)^{1/2} \\
&\quad \times \left(\int_{[\frac{s}{\delta}]\delta}^s (\mathbb{E}|Y^{\varepsilon,t,\zeta}_{u(\delta)}|^2 + \mathbb{E}|\bar{Y}^{t,\zeta}_{u(\delta)}|^2)\mathrm{d}u\right)^{1/2} \\
&\leqslant C\delta.
\end{aligned}
\tag{33}
$$

By combining (29)–(33) with (28), we conclude (25). Thus, the proof is complete. □

6. Applications

In this section, we apply Theorem 3.5 to nonlinear parabolic PDEs.

6.1 Application to parabolic variational inequalities

In this subsection, we require that $d=1, \zeta=x\in\mathbb{R}^m$, and we study the averaging for parabolic variational inequalities.

Let u^ε be the viscosity solution for the following parabolic variational inequality (see [26, Definition 4.2] for the definition of the related viscosity solutions),

$$\begin{cases} \frac{\partial u^\varepsilon(t,x)}{\partial t}+\mathscr{L}^\varepsilon u^\varepsilon(t,x)+f_1(\frac{t}{\varepsilon},x,u^\varepsilon(t,x))+f_2(\nabla u^\varepsilon(t,x)\sigma(\frac{t}{\varepsilon},x))\in\partial\varphi(u^\varepsilon(t,x)), \quad t\in[0,T], \\ u^\varepsilon(T,x)=g(x), \quad u^\varepsilon(t,x)\in\overline{\mathrm{Dom}(\varphi)}, \quad x\in\mathbb{R}^m, \end{cases} \tag{34}$$

where

$$\mathscr{L}^\varepsilon := \frac{1}{2}\sum_{i,j=1}^m(\sigma\sigma^*)_{ij}\left(\frac{t}{\varepsilon},x\right)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^m b_i\left(\frac{t}{\varepsilon},x\right)\frac{\partial}{\partial x_i},$$

and let $\bar{u}$ be the viscosity solution for the following parabolic variational inequality,

$$\begin{cases} \frac{\partial \bar{u}(t,x)}{\partial t}+\bar{\mathscr{L}}\bar{u}(t,x)+\bar{f}_1(x,\bar{u}(t,x))+f_2(\nabla\bar{u}(t,x)\bar{\sigma}(x))\in\partial\varphi(\bar{u}(t,x)), \quad t\in[0,T], \\ \bar{u}(T,x)=g(x), \quad \bar{u}(t,x)\in\overline{\mathrm{Dom}(\varphi)}, \quad x\in\mathbb{R}^m, \end{cases} \tag{35}$$

where

$$\bar{\mathscr{L}} := \frac{1}{2}\sum_{i,j=1}^m(\bar{\sigma}\bar{\sigma}^*)_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^m \bar{b}_i(x)\frac{\partial}{\partial x_i}.$$

We assume:

($\mathbf{H}_f^3$) For each $R>0$, there exists a continuous function $\alpha_R:\mathbb{R}_+\to\mathbb{R}_+, \alpha_R(0)=0$, such that

$$|f_1(s,x_1,y)-f_1(s,x_2,y)|\leqslant\alpha_R(|x_1-x_2|), \quad s\in[0,T], \quad |x_1|,|x_2|\leqslant R.$$

Theorem 6.1 *We assume that* $(\mathbf{H}_{b,\sigma}^1)$, $(\mathbf{H}_{b,\sigma}^2)$, $(\mathbf{H}_\varphi)$, $(\mathbf{H}_g)$, $(\mathbf{H}_f^1)$, $(\mathbf{H}_f^2)$ *and* $(\mathbf{H}_f^3)$ *hold. Then, for any* $(t,x)\in[0,T]\times\mathbb{R}^m$,

$$\lim_{\varepsilon\to 0}u^\varepsilon(t,x)=\bar{u}(t,x).$$

Proof First, by [26, Theorem 4.1 and 4.2], we know that $u^\varepsilon(t,x)=Y_t^{\varepsilon,t,x}, \bar{u}(t,x)=\bar{Y}_t^{t,x}$ are unique viscosity solutions for Eq.(34) and Eq.(35), respectively. Therefore, the result follows from Theorem 3.5. □

Remark 6.2 *If $f_1(s,x,y)$ is independent of s and $f_2(z)=0$, Theorem 6.1 is the same as [13, Theorem 4.1]. Therefore, our result is more general.*

6.2 Application to viscosity solutions for systems of parabolic variational inequalities

In this subsection, we require that $\zeta=x\in\mathbb{R}^m, f_2(z)=0$, and study the averaging for systems of parabolic variational inequalities.

Let v^ε be the viscosity solution for the following system of parabolic variational inequalities (see [20, Definition 4 and Remark 5] for the definition of the related viscosity solutions),

$$\begin{cases} \frac{\partial v^\varepsilon(t,x)}{\partial t}+\mathscr{L}^\varepsilon v^\varepsilon(t,x)+f_1(\frac{t}{\varepsilon},x,v^\varepsilon(t,x))\in\partial\varphi(v^\varepsilon(t,x)), \quad t\in[0,T], \\ v^\varepsilon(T,x)=g(x), \quad v^\varepsilon(t,x)\in\overline{\mathrm{Dom}}(\varphi), \quad x\in\mathbb{R}^m, \end{cases} \tag{36}$$

where

$$(\mathscr{L}^\varepsilon v_k^\varepsilon)(t,x) := \frac{1}{2}\sum_{i,j=1}^m(\sigma\sigma^*)_{ij}\left(\frac{t}{\varepsilon},x\right)\frac{\partial^2 v_k^\varepsilon(t,x)}{\partial x_i\partial x_j}+\sum_{i=1}^m b_i\left(\frac{t}{\varepsilon},x\right)\frac{\partial v_k^\varepsilon(t,x)}{\partial x_i}, \quad k=1,\cdots,d,$$

and let $\bar{v}$ be the viscosity solution for the following system of parabolic variational inequalities,

$$\begin{cases}\frac{\partial\bar{v}(t,x)}{\partial t}+\bar{\mathscr{L}}\bar{v}(t,x)+\bar{f}_1(x,\bar{v}(t,x))\in\partial\varphi(\bar{v}(t,x)), & t\in[0,T],\\ \bar{v}(T,x)=g(x),\quad \bar{v}(t,x)\in\overline{\mathrm{Dom}(\varphi)}, & x\in\mathbb{R}^m,\end{cases}\tag{37}$$

where

$$\bar{\mathscr{L}}\bar{v}_k(t,x):=\frac{1}{2}\sum_{i,j=1}^{m}(\bar{\sigma}\bar{\sigma}^*)_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^{m}\bar{b}_i(x)\frac{\partial}{\partial x_i},\quad k=1,\cdots,d.$$

We assume:

$(\mathbf{H}'_{\varphi})$ For all $\rho\in Dom(\varphi)$, there exists a neighborhood V of ρ such that $(\partial\varphi)_0$ is bounded on $\mathcal{D}(\partial\varphi)\cap V$, where for $\varrho\in\mathcal{D}(\partial\varphi)$, $(\partial\varphi)_0(\varrho)\in\mathbb{R}^d$ is unique such that $|(\partial\varphi)_0(\varrho)|=\inf|\partial\varphi(\varrho)|$. Moreover, if $\rho\in Dom(\varphi)$ and $y\in\mathbb{R}^d$ such that $\rho+y\in Dom(\varphi)$, there exists a neighborhood V of ρ such that

$$\forall\vartheta\in V\cap\mathcal{D}(\partial\varphi),\quad \exists t>0:\vartheta+ty\in\mathcal{D}(\partial\varphi).$$

Theorem 6.3 *Assume that* $(\mathbf{H}^1_{b,\sigma})$, $(\mathbf{H}^2_{b,\sigma})$, $(\mathbf{H}_g)$, $(\mathbf{H}^1_f)$, $(\mathbf{H}^2_f)$, *and* $(\mathbf{H}'_{\varphi})$ *hold. Then, for any* $(t,x)\in[0,T]\times\mathbb{R}^m$,

$$\lim_{\varepsilon\to0}v^{\varepsilon}(t,x)=\bar{v}(t,x).$$

Proof First, [20, Theorem 6 and 7] imply that $v^{\varepsilon}(t,x)=Y^{\varepsilon,t,x}_t,\bar{v}(t,x)=\bar{Y}^{t,x}_t$ are unique viscosity solutions for Eq.(36) and Eq.(37), respectively. Therefore, the result follows from Theorem 3.5. □

6.3 Application to the viscosity solutions for quasilinear parabolic PDEs

In this subsection, we require that $\varphi=0,d=1,\zeta=x\in\mathbb{R}^m$, and study the averaging for quasilinear parabolic PDEs.

Let w^{ε} be the viscosity solution for the following PDE,

$$\begin{cases}\frac{\partial w^{\varepsilon}(t,x)}{\partial t}+\mathscr{L}^{\varepsilon}w^{\varepsilon}(t,x)+f_1(\frac{t}{\varepsilon},x,w^{\varepsilon}(t,x))+f_2(\nabla w^{\varepsilon}(t,x)\sigma(\frac{t}{\varepsilon},x))=0, & t\in[0,T],\\ w^{\varepsilon}(T,x)=g(x),\quad x\in\mathbb{R}^m,\end{cases}\tag{38}$$

and let $\bar{w}$ be the viscosity solution for the following PDE,

$$\begin{cases}\frac{\partial\bar{w}(t,x)}{\partial t}+\bar{\mathscr{L}}\bar{w}(t,x)+\bar{f}_1(x,\bar{w}(t,x))+f_2(\nabla\bar{w}(t,x)\bar{\sigma}(x))=0, & t\in[0,T],\\ \bar{w}(T,x)=g(x),\quad x\in\mathbb{R}^m.\end{cases}\tag{39}$$

By [25, Theorem 4.3], we conclude that the relationship between w^{ε} and $\bar{w}$ is as follows.

Theorem 6.4 *Assume that* $(\mathbf{H}^1_{b,\sigma})$, $(\mathbf{H}^2_{b,\sigma})$, $(\mathbf{H}_g)$, $(\mathbf{H}^1_f)$, *and* $(\mathbf{H}^2_f)$ *hold. Then, for any* $(t,x)\in[0,T]\times\mathbb{R}^m$,

$$\lim_{\varepsilon\to0}w^{\varepsilon}(t,x)=\bar{w}(t,x).$$

7. Example

In this section, we present an example to illustrate our result.

Example 7.1 *Assume that* $m=l=d=1,\zeta=x\in\mathbb{R}$, *and consider the following forward-backward multivalued stochastic system:* $0\leqslant t\leqslant s\leqslant T$,

$$\begin{cases}\mathrm{d}X^{\varepsilon,t,x}_s=\frac{\frac{s}{\varepsilon}}{1+\frac{s}{\varepsilon}}\cos(X^{\varepsilon,t,x}_s)\mathrm{d}s+(1-e^{-\frac{s}{2\varepsilon}})\sin(X^{\varepsilon,t,x}_s)\mathrm{d}W_s,\\ X^{\varepsilon,t,x}_t=x,\\ \mathrm{d}Y^{\varepsilon,t,x}_s\in\partial\varphi(Y^{\varepsilon,t,x}_s)\mathrm{d}s+Z^{\varepsilon,t,x}_s\mathrm{d}W_s,\\ Y^{\varepsilon,t,x}_T=g(X^{\varepsilon,t,x}_T)\in\overline{\mathcal{D}(\partial\varphi)},\end{cases}\tag{40}$$

where φ, g *satisfy* $(\mathbf{H}_\varphi)$ *and* $(\mathbf{H}_g)$. *It is easy to see that* $b(s,x) = \frac{s}{1+s}\cos(x), \sigma(s,x) = (1-e^{-s/2})\sin(x), f_1(s,x,y) = f_2(z) = 0$ *satisfies* $(\mathbf{H}^1_{b,\sigma})$ *and* $(\mathbf{H}^1_f)$. *Thus, the system (40) has a unique solution* $(X^{\varepsilon,t,x}, Y^{\varepsilon,t,x}, K^{\varepsilon,t,x}, Z^{\varepsilon,t,x})$.

Taking $\bar{b}(x) = \cos(x)$, $\bar{\sigma}(x) = \sin(x)$, *we justify that*

$$\left|\frac{1}{\hat{T}}\int_0^{\hat{T}} b(s,x)\mathrm{d}s - \bar{b}(x)\right|^2 = \left|\frac{1}{\hat{T}}\int_0^{\hat{T}} \frac{s}{1+s}\cos(x)\mathrm{d}s - \cos(x)\right|^2$$

$$\leqslant \frac{1}{\hat{T}}\int_0^{\hat{T}} \frac{1}{(1+s)^2}\mathrm{d}s|x|^2 \leqslant \frac{1}{\hat{T}}(1+|x|^2),$$

and

$$\frac{1}{\hat{T}}\int_0^{\hat{T}} |\sigma(s,x) - \bar{\sigma}(x)|^2\,\mathrm{d}s = \frac{1}{\hat{T}}\int_0^{\hat{T}} \left|(1-e^{-s/2})\sin(x) - \sin(x)\right|^2\mathrm{d}s$$

$$\leqslant \frac{1}{\hat{T}}\int_0^{\hat{T}} e^{-s}\mathrm{d}s|x|^2 \leqslant \frac{1}{\hat{T}}(1+|x|^2).$$

That is, $(\mathbf{H}^2_{b,\sigma})$ *holds. Therefore, by Theorem 3.5, we obtain that*

$$\lim_{\varepsilon\to 0}\mathbb{E}\sup_{s\in[t,T]}|Y^{\varepsilon,t,x}_s - \bar{Y}^{t,x}_s|^2 = 0,$$

where $(\bar{X}^{t,x}, \bar{Y}^{t,x}, \bar{K}^{t,x}, \bar{Z}^{t,x})$ *is the unique solution for the following system:*

$$\begin{cases} \mathrm{d}\bar{X}^{t,x}_s = \cos(\bar{X}^{t,x}_s)\mathrm{d}s + \sin(\bar{X}^{t,x}_s)\mathrm{d}W_s, \\ \bar{X}^{t,x}_t = x, \\ \mathrm{d}\bar{Y}^{t,x}_s \in \partial\varphi(\bar{Y}^{t,x}_s)\mathrm{d}s + \bar{Z}^{t,x}_s\mathrm{d}W_s, \\ \bar{Y}^{t,x}_T = g(\bar{X}^{t,x}_T) \in \overline{\mathcal{D}(\partial\varphi)}. \end{cases}$$

Next, we investigate the following PDEs:

$$\begin{cases} \frac{\partial u^\varepsilon(t,x)}{\partial t} + \frac{1}{2}(1-e^{-\frac{t}{2\varepsilon}})^2\sin^2(x)\frac{\partial^2 u^\varepsilon(t,x)}{\partial x\partial x} + \frac{\frac{t}{\varepsilon}}{1+\frac{t}{\varepsilon}}\cos(x)\frac{\partial u^\varepsilon(t,x)}{\partial x} \in \partial\varphi(u^\varepsilon(t,x)), \quad t\in[0,T], \\ u^\varepsilon(T,x) = g(x), \quad x\in\mathbb{R}, \end{cases} \tag{41}$$

and

$$\begin{cases} \frac{\partial \bar{u}(t,x)}{\partial t} + \frac{1}{2}\sin^2(x)\frac{\partial^2 \bar{u}(t,x)}{\partial x\partial x} + \cos(x)\frac{\partial \bar{u}(t,x)}{\partial x} \in \partial\varphi(\bar{u}(t,x)), \quad t\in[0,T], \\ \bar{u}(T,x) = g(x), \quad x\in\mathbb{R}. \end{cases} \tag{42}$$

By Theorem 6.1, it holds that the viscosity solution $u^\varepsilon(t,x)$ *for Eq.(41) converges to the viscosity solution* $\bar{u}(t,x)$ *of Eq.(42).*

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