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Moral-hazard-free insurance contract design under the rank-dependent utility theory

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Abstract This paper investigates a Pareto optimal insurance contract design problem within a behavioral finance framework. In this context, the insured evaluates contracts using the rank-dependent utility (RDU, for short) theory, while the insurer applies the expected value premium principle. The analysis incorporates the incentive compatibility constraint, ensuring that the contracts, called moral-hazard-free, are free from the moral hazard issues identified in Bernard et al. [4]. Initially, the problem is formulated as a non-concave maximization problem involving Choquet expectation. It is then transformed into a quantile optimization problem and addressed using the calculus of variations method. The optimal contracts are characterized by a double-obstacle ordinary differential equation for a semi-linear second-order elliptic operator with nonlocal boundary conditions, which seems new in the financial economics literature. We present a straightforward numerical scheme and a numerical example to compute the optimal contracts. Let θ and m_0 represent the relative safety loading and the mass of the potential loss at 0, respectively. We discover that every moral-hazard-free contract is optimal for infinitely many RDU-insured individuals if $0<\theta<\frac{m_0}{1-m_0}$. Conversely, certain contracts, such as the full coverage contract, are never optimal for any RDU-insured individual if $\theta>\frac{m_0}{1-m_0}$. Additionally, we derive all the Pareto optimal contracts when either the compensation or the retention violates the monotonicity constraint.

Keywords Pareto optimal/efficient insurance, Rank-dependent utility theory, Quantile optimization, Probability weighting/distortion function, Double-obstacle ordinary differential equation, Calculus of variations

2020 Mathematics Subject Classification 91G05, 91B30, 91B16

1. Introduction

Probability weighting function (also called probability distortion function) (see [31, 37]) plays a key role in many behavioral theories of choice under uncertainty, such as Kahneman and Tversky's [26, 38] cumulative prospect theory, Yaari's [46] dual model, Lopes' [28] SP/A model and Quiggin's [32] rank-dependent utility (RDU, for short) theory. These behavioral finance theories provide satisfactory explanations of the many paradoxes for which the classical expected

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utility (EU) theory fails to account (see, e.g. [1, 16, 17, 29]).

In recent years, much attention has been paid to the theoretical study of behavioral finance models under uncertainty (such as portfolio choice and optimal stopping models) involving probability weighting function; see, e.g., [21, 22, 24, 25, 34, 39, 40, 41, 44]. If the objective functional of a problem is law-invariant and satisfies some monotonicity condition (see Xu [41, Section 3] for a detailed discussion) and the market is complete, then the problem may be tackled by the following approach. Instead of looking for the optimal strategy directly for the original stochastic control or optimization problem, one first reduces the problem to a corresponding quantile optimization problem, in which the decision variable becomes a quantile function (or simply called a quantile, which is the inverse of a probability distribution function). With this change, solving a stochastic control or optimization problem is reduced to solving a static deterministic quantile optimization problem. In the second step, the deduced quantile optimization problem is tackled by deterministic optimization techniques, such as completing the square method (see [22]), convex analysis (see [20, 21, 39, 44]) and calculus of variations method (see [40]). The last step is to recover the optimal strategy for the original problem by appealing to some proper hedging theories, such as the backward stochastic differential equation theory for portfolio choice problems (see [21]) and the Skorokhod embedding theory for optimal stopping problems (see [44]). As every quantile function is non-decreasing, any quantile optimization problem must take this monotonicity (as a minimum) constraint into consideration. This becomes the main hurdle when tackling many quantile optimization problems.

Researchers generally tackle quantile optimization problems in isolation and often under fairly strong assumptions (see, e.g., [21, 25]) due to lack of a systematic method, until Xia and Zhou [40] provided a systematic approach via calculus of variations method. Xia and Zhou demonstrated the utility of their approach by solving a portfolio choice problem under the RDU theory. Shortly after, the author [42] introduced an alternative simple method, namely the change of variables and relaxation method, to solve a class of quantile optimization problems including the one considered in [40]. For this class of problems, the constraints on the quantiles are almost minimum: beyond the monotonicity constraint (which is necessary for quantile optimization problems, as previously noted), the only other constraint arising from the models is the so-called budget constraint, which, mathematically speaking, is a one-dimension linear constraint that can be easily dealt by Lagrangian method.

Probability weighting function is also widely used in the risk-sharing literature. In the context of insurance, the primary risk-sharing problem is to design an insurance contract between insurer and insured that achieves Pareto optimality/efficiency. Although there is plenty of work done on designing optimal compensations under the RDU theory ([4, 9, 12, 15, 18, 45]), most studies assume that the probability weighting function takes a special shape, such as convex, concave or inverse-S-shaped. If both the objective and constraints in an insurance problem are law-invariance, it is possible to tackle the problem by the aforementioned routine to solve investment problems, namely translating it into a quantile optimization problem, determining the optimal quantile, and translating back to discern the optimal contract.

However, there is a key difference in how quantile optimization problems are formulated for investment problems (called “first-type”) and that for optimal insurance contract problems (called “second-type”). When considering the latter, one should to take some special concerns from insurer and insured into account. To avoid certain moral hazard issue (see, Bernard et al. [4] and Xu et al. [45]), both compensation and retention functions are a prior non-decreasing for the optimal contracts. Both Huberman, Mayers and Smith Jr [23] and Picard [30] called the non-decreasing condition of compensation and retention the *incentive compatibility constraint* for optimal insurance contracts. Mathematically speaking, this leads to a new class of quantile optimization problems, in which the derivatives of decision quantiles are bounded. This is an infinite-dimension constraint. Bernard et al. [4] studied an insurance contract design model under the RDU theory, but ignored the incentive compatibility constraint. As a consequence, their result suffers from a moral hazard issue that provides incentives for the insured to falsely report actual losses. Xu et al. [45] examined the same model but took the incentive compatibility constraint into consideration to avoid this moral hazard issue; but due to technical difficulties, they can only partially solve the problem by imposing restrictive assumptions on the loss and probability weighting function. Ghossoub [18] revisited the same problem by imposing a state-verification cost that the insurer can incur in order to verify the loss severity, hence automatically ruling out any ex post moral hazard that could otherwise arise from possible misreporting of the loss by the insured. When taking the incentive compatibility condition into consideration, the risk-sharing market is comonotone. Such markets have been studied by many authors. For instance, Boonen et al. [5] consider fairly general preferences and study equilibrium in the market.

To the best of our knowledge, no systematic approach has been developed to solve the second-type quantile optimization problems. Although the calculus of variations method has been used in the insurance literature, most applications did not take the constraint of incentive compatibility into account. For example, Spence and Zeckhauser [36] used this method to solve an optimal insurance problem under the EU theory without considering the constraint. However, because the optimal contract turns out to be the classical deductible, it coincidentally satisfies the compatibility constraint. As mentioned above, if problems are considered in behavioral finance framework (such as the RDU theory), the optimal contracts can lead to moral hazard issue.

In this paper we investigate the Pareto optimal (PO) insurance contracts in a behavioral finance framework, in which the insured evaluates contracts by the RDU theory and the insurer is risk neutral and maximizes the expected profit with linear expected value premium principle. The incentive compatibility constraint is taken into account when modeling, so the contracts are free of the aforementioned moral hazard issue arisen in Bernard et al. [4]. The problem is formulated as a non-concave optimization problem which involves Choquet expectation (see (2.5)). To solve it, we first translate the problem into a quantile optimization problem of the second-type by change of variables in Section 3. Then apply calculus of variations method to get the optimality condition for the latter in Section 4. Finally we derive all the PO insurance contracts in Theorem 4.5 and study their properties in Section 5. In this process, an ordinary integro-differential equation (OIDE) (4.5) and an ordinary differential equation (ODE) (4.7) play the key role. We also derive all the PO insurance contracts if either the compensations or the retentions are not required to be non-decreasing in Section 6.

The main mathematical contribution of this paper is to express the solution by a double-obstacle ODE for a semi-linear second-order elliptic operator with nonlocal boundary conditions (Theorem 4.5). At first sight, the ODE (4.5) looks like a standard double-obstacle problem in the financial economics literature, however, the obstacles are put on the highest order gradient of the unknown function; by contrast, they are put on the lower order gradients in the literature. To the best of our knowledge, it seems it is the first time that this type of double-obstacle problems appears in the financial economics literature. We prove the existence and uniqueness of the solution to the ODE (4.7) as well as the OIDE (4.5) from the pure optimization point view. Furthermore, we also provide a numerical scheme to calculate all the PO contracts in Section 4.2.

The problem can be solved numerically. We discover that the optimal solution obtained in [40] and [42] can be expressed by a single-obstacle problem for a linear second-order elliptic operator. Last, if either the compensation or the retention losses the monotonicity constraint, we derive all the PO contracts as well. In the former case, all the PO contracts are given explicitly via a free parameter and the concave envelope of a known function (see Theorem 6.1). Moreover, two equivalent conditions for these contracts to be moral-hazard-free are provided in Remark 6.1 and Remark 6.2.

The main economic contributions of this paper are as follows. First, we reveals in Proposition 5.1 the classical assertion that all the PO contracts are of deductible type in the EU theory framework (see, e.g. [36]). Second, we give an equivalent condition under which the PO contracts are of deductible type in Theorem 5.2. Next, we find that the relative safety loading θ of the insurer and the mass m_0 of the potential loss at 0 play the key role in determining the type of the PO contracts. In fact, if the relative safety loading is small or the mass m_0 is big, namely $0<\theta<\frac{m_0}{1-m_0}$, then every moral-hazard-free contract will be accepted by infinitely many RDU insureds (see Theorem 5.4); by contrast, if the relative safety loading is big or the mass m_0 is small, namely $\theta>\frac{m_0}{1-m_0}$, some contracts (such as the full coverage contract) will be rejected by every RDU insured (see Corollary 5.3). Last, we find that some PO contracts may not be moral-hazard-free if the incentive compatibility constraint is ignored when modeling.

The rest of this paper is organized as follows. In Section 2, we introduce a PO insurance problem. In Section 3, the problem is turned into a concave quantile optimization problem via change of variables. Section 4 is devoted to solving the quantile optimization problem by calculus of variations method and providing a numerical scheme with an example to calculate the PO contracts. In Section 5, we discuss properties of the PO contracts. Section 6 studies two models where the compensations or the retentions are not required to be non-decreasing. Section 7 concludes the paper. The proofs of some results are provided in Appendix A.

Notations

Throughout the paper, we fix an atom-less probability space $(\Omega,\mathcal{F},\mathbb{P})$. For any random variable Y, we denote its probability distribution function by F_Y; and define its quantile function (or the left-continuous inverse function of F_Y) by

$$F_Y^{-1}(p):=\inf\big\{z\geqslant 0 \;\big|\; F_Y(z)\geqslant p\big\},\quad p\in(0,1).$$

Since quantile functions are non-decreasing, we can define

$$F_Y^{-1}(0):=\lim_{p\to 0+}F_Y^{-1}(p)\;\text{ and }\;F_Y^{-1}(1):=\lim_{p\to 1-}F_Y^{-1}(p).$$

It is not hard to verify that $F_Y^{-1}(0)=\operatorname{ess\,inf} Y$ and $F_Y^{-1}(1)=\operatorname{ess\,sup} Y$. In particular, $F_Y^{-1}(1)<\infty$ if and only if Y is (essentially) upper bounded.

All the quantile functions by definition are non-decreasing and left-continuous. However, they may not be continuous in general. In this paper, we will mainly deal with absolutely continuous quantiles due to the incentive compatibility constraint involved.

We denote by $C^{2-}([0,1])$ the set of functions $f:[0,1]\to\mathbb{R}$ which are differentiable with derivatives f' being absolutely continuous on $[0,1]$. Clearly $C^2([0,1])\subsetneqq C^{2-}([0,1])\subsetneqq C^1([0,1])$.

In our below argument, “almost surely” (a.s.) and “almost everywhere” (a.e.) are often suppressed for simplicity in some circumstances when no confusion occurs.

2. Problem formulation

In Pareto optimal (PO, for short; also called Pareto efficient) insurance problem, one seeks the best way for the insurer (an insurance company) and the insured ("She") to share a potential loss to achieve Pareto optimality or efficiency.

We fix a random variable $X \geqslant 0$ to denote potential loss that is covered by the insurance contract. Let $I(x)$ be the loss borne by the insurer when a real loss x occurs. It is called the *compensation* (or *indemnity*) function in the insurance literature. A compensation is called full coverage if $I(x) \equiv x$; called deductible (with deductible d) if $I(x) \equiv \max\{x-d, 0\}$. Let $\mathscr{C}$ denote the set of acceptable compensations that will be specified shortly.

Following the literature, we call a pair $(\mathscr{P}, I) \in \mathbb{R} \times \mathscr{C}$ an insurance contract, where $\mathscr{P} \in \mathbb{R}$ is a premium that the insured pays to the insurer at initial time and $I \in \mathscr{C}$ is an acceptable compensation. We use $\mathcal{U}_{\text{insured}}: \mathbb{R} \times \mathscr{C} \to \mathbb{R}$ to denote the insured's utility and $\mathcal{U}_{\text{insurer}}: \mathbb{R} \times \mathscr{C} \to \mathbb{R}$ the insure's utility. Economically speaking, the insured (resp. insurer) would prefer to choose a contract $(\mathscr{P}', I')$ rather than another one $(\mathscr{P}, I)$ if $\mathcal{U}_{\text{insured}}(\mathscr{P}', I') > \mathcal{U}_{\text{insured}}(\mathscr{P}, I)$ (resp. $\mathcal{U}_{\text{insurer}}(\mathscr{P}', I') > \mathcal{U}_{\text{insurer}}(\mathscr{P}, I)$). The insured should be happier if she pays less premium for the same compensation or receives a higher compensation without paying more premium, so we assume $\mathcal{U}_{\text{insured}}(\mathscr{P}, I)$ is decreasing in $\mathscr{P}$ and increasing in I. Similarly, we assume $\mathcal{U}_{\text{insurer}}(\mathscr{P}, I)$ is increasing in $\mathscr{P}$ and decreasing in I.

We say a contract $(\mathscr{P}, I) \in \mathbb{R} \times \mathscr{C}$ is a Pareto optimal/efficient contract if there is no other contract $(\mathscr{P}', I') \in \mathbb{R} \times \mathscr{C}$ such that

$$\begin{aligned} \mathcal{U}_{\text{insured}}(\mathscr{P}', I') &\geqslant \mathcal{U}_{\text{insured}}(\mathscr{P}, I), \\ \mathcal{U}_{\text{insurer}}(\mathscr{P}', I') &\geqslant \mathcal{U}_{\text{insurer}}(\mathscr{P}, I), \end{aligned}$$

and

$$\mathcal{U}_{\text{insured}}(\mathscr{P}', I') + \mathcal{U}_{\text{insurer}}(\mathscr{P}', I') > \mathcal{U}_{\text{insured}}(\mathscr{P}, I) + \mathcal{U}_{\text{insurer}}(\mathscr{P}, I).$$

In other words, it is impossible to improve one of the insured's and insurer's utilities without reducing the other one's. All the PO contracts form a set, called Pareto frontier. This frontier forms the insurance company's menu for its clients. Please refer to [2] for a recent study on PO contracts.

Assume both $\mathcal{U}_{\text{insured}}(\mathscr{P}, I)$ and $\mathcal{U}_{\text{insurer}}(\mathscr{P}, I)$ are continuous functions in $\mathscr{P}$. It is not hard to show that a contract $(\mathscr{P}^*, I^*) \in \mathbb{R} \times \mathscr{C}$ is PO if and only if there exists $\gamma \in \mathbb{R}$ such that $(\mathscr{P}^*, I^*)$ is an optimal solution to the problem[1]

$$\begin{aligned} &\sup_{(\mathscr{P}, I) \in \mathbb{R} \times \mathscr{C}} \quad \mathcal{U}_{\text{insured}}(\mathscr{P}, I) \\ &\text{s.t.} \qquad \mathcal{U}_{\text{insurer}}(\mathscr{P}, I) \geqslant \gamma. \end{aligned} \tag{2.1}$$

The last inequality is often called the participation constraint for the insurer. If we runs through all the possible values of γ, we will get all the PO contracts, which form the Pareto frontier.

Different choices of the insured's and the insurer's utilities will lead to different PO Pareto frontiers. We now specify our choice in this paper for further analysis.

We start with the set of acceptable compensations $\mathscr{C}$. Let $R(x)$ be the loss borne by the insured when a real loss x occurs. It is called the *retention* function. Let $\mathscr{R}$ denote the set of the retention functions. Mathematically speaking, one always has

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[1] Please refer to [2, Theorem 3.1.] for a similar result.

$$I(0)=R(0)=0, \quad I(x)+R(x)=x, \quad x\geqslant 0. \tag{2.2}$$

Therefore, $\mathscr{R}=\{\star-I(\star): I\in\mathscr{C}\}$. On the other hand, economically speaking, both the insurer and the insured should bear a greater financial responsibility when a bigger loss occurs. For, if one party borne less responsibility, it could result in the moral hazard issue (detailed discussions can be found in [45]). Therefore, mathematically speaking, one also requires that

$$I(x)\geqslant I(y), \quad R(x)\geqslant R(y), \quad x\geqslant y\geqslant 0. \tag{2.3}$$

This is called the *incentive compatibility constraint* (see, e.g. [23, 30, 45]). In our model, we request all the compensations and retentions should satisfy the above constraints (2.2) and (2.3). We call them *moral-hazard-free* since they avoid the moral hazard arisen in Bernard et al. [4]. For instance, all deductible compensations (which satisfy $I(x)\equiv(x-d)^{+}$ for some $d>0$ called deductible; also called stop-loss compensations) and proportional coverage compensations (which satisfy $I(x)\equiv cx$ for some $0\leqslant c\leqslant 1$) are moral-hazard-free. Cleary we can combine the constraints (2.2) and (2.3) into the following one:

$$I(0)=0, \quad 0\leqslant I(x)-I(y)\leqslant x-y, \quad x\geqslant y\geqslant 0.$$

In another words, we have the classical result of Denneberg:

$$\begin{aligned}\mathscr{C}=\Big\{I:[0,\infty)\to[0,\infty)\;\Big|\;&I\text{ is absolutely}\\&\text{continuous with } I(0)=0 \text{ and } 0\leqslant I'\leqslant\ 1\text{ a.e.}\Big\}.\end{aligned}$$

Recalling that $\mathscr{R}=\{\star-I(\star): I\in\mathscr{C}\}$, the above implies

$$\begin{aligned}\mathscr{R}=\Big\{R:[0,\infty)\to[0,\infty)\;\Big|\;&R\text{ is absolutely}\\&\text{continuous with } R(0)=0 \text{ and } 0\leqslant R'\leqslant\ 1\text{ a.e.}\Big\},\end{aligned}$$

so $\mathscr{C}=\mathscr{R}$. Because the derivative of the compensation function I (as well as that of the retention function R) is bounded in $[0,1]$, on one hand, economically speaking, it avoids the potential moral hard issue; on the other hand, mathematically speaking, it also makes the related PO moral-hazard-free insurance contracts hard to find since it is an infinity-dimension constraint.

We now specify the insurer's utility $\mathcal{U}_{\text{insurer}}(\mathscr{P},I)$. Following the existing literature (see, e.g, [3, 4, 19, 33, 45]), we assume the insurer evaluates compensations by the expected value premium principle $(1+\theta)\mathbb{E}[I(X)]$ and aims at maximizing the utility

$$\mathcal{U}_{\text{insurer}}(\mathscr{P},I)=\mathscr{P}-(1+\theta)\mathbb{E}[I(X)].$$

The constant $\theta>0$ is called the *relative safety loading*. Its value will have a significant impact on the type of the PO contracts as we will show.

Now let us introduce the insured's utility $\mathcal{U}_{\text{insured}}(\mathscr{P},I)$. In this paper, we consider a RDU insured, whose uses a kind of nonlinear Choquet expectation to evaluate the utility of her wealth. Precisely, we assume

$$\mathcal{U}_{\text{insured}}(\mathscr{P},I)=\mathscr{E}\Big(u\big(\beta_{\text{insured}}-\mathscr{P}-X+I(X)\big)\Big).$$

Here the constant β_{insured} stands for the final wealth of the insured, so $\beta_{\text{insured}}-\mathscr{P}-X+I(X)$ represents the insured's net wealth after claim, the utility function $u:\mathbb{R}\to\mathbb{R}$ is assumed to be twice differentiable and to satisfy $u'>0$ and $u''<0$, the expectation $\mathscr{E}$ for a random variable Y is defined in this paper as

$$\mathscr{E}(Y) := \int_0^1 F_Y^{-1}(p)\mathrm{d}\left(1 - w(1-p)\right), \tag{2.4}$$

where F_Y^{-1} denotes the quantile function of Y, w is a probability weighting function in the set

$$\mathscr{W} := \Big\{ w : [0,1] \to [0,1] \;\Big|\; w \text{ is strictly increasing with } w(0) = 0 \text{ and } w(1) = 1 \Big\}.$$

Note that $\mathscr{E}$ is a kind of *nonlinear Choquet expectation* unless w is the identical function in which case $\mathscr{E}$ becomes the usual mathematical expectation $\mathbb{E}$. Because of this, many existing techniques are failed to apply to our problem below.

Under the above specific setting, the problem (2.1) becomes

$$\begin{aligned} &\sup_{(\mathscr{P},I)\in\mathbb{R}\times\mathscr{C}} \mathscr{E}\Big(u\big(\beta_{\text{insured}} - \mathscr{P} - X + I(X)\big)\Big) \\ &\text{s.t.} \quad \mathscr{P} - (1+\theta)\mathbb{E}[I(X)] \geqslant \gamma. \end{aligned}$$

Note that the target is decreasing in $\mathscr{P}$, so any optimal contract $(\mathscr{P}^*, I^*)$ shall make the constraint tight, namely

$$\mathscr{P}^* = \gamma + (1+\theta)\mathbb{E}[I^*(X)].$$

Therefore, it suffices to study the problem

$$\sup_{I\in\mathscr{C}} \mathscr{E}\Big(u\big(\beta_{\text{insured}} - \gamma - (1+\theta)\mathbb{E}[I(X)] - X + I(X)\big)\Big). \tag{2.5}$$

To study the above problem, we still need to specify the distribution of the potential loss X. We put the following technical assumption on X.

Assumption 2.1 *The quantile function F_X^{-1} of the potential loss X is upper bounded. Furthermore, it is absolutely continuous on $[0,1]$ and $(F_X^{-1})'(p) > 0$ for a.e. $p \in (m_0, 1)$, where $m_0 := F_X(0) < 1$.*

It is very important to remark that Assumption 2.1 allows X to have a positive mass m_0 at 0. This is the most important and common case in insurance practice. We will show that the value of m_0 has a significant impact on the type of the PO contracts. Under Assumption 2.1, X is essentially bounded, so our subsequent arguments only deal with bounded random variables, which simplifies our argument a lot. Of course, one can extend our analysis to cover unbounded risks via a more careful discussion. Clearly the probability distribution function F_X of X is continuous on $[0,1]$ and strictly increasing on $[m_0, 1]$. Moreover, $F_X^{-1}(F_X(x)) = x$ for all $\operatorname{ess\,inf} X \leqslant x \leqslant \operatorname{ess\,sup} X$, $F_X^{-1}(p) = 0$ for $p \leqslant m_0$ and $F_X^{-1}(p) > 0$ for $p > m_0$. These facts may be used in the subsequent analysis without claim.

For $\beta \in \mathbb{R}$, suppose $R_\beta^*(\star)$ is an optimal solution to the following problem,

$$\sup_{R\in\mathscr{R}} \mathscr{E}\Big(u\big(\beta + (1+\theta)\mathbb{E}[R(X)] - R(X)\big)\Big). \tag{2.6}$$

Then

$$I^*(\star) := \star - R^*_{\beta_{\text{insured}}-\gamma-(1+\theta)\mathbb{E}[X]}(\star)$$

is an optimal solution to the problem (2.5), and

$$\Big(\gamma + (1+\theta)\mathbb{E}\Big[X - R^*_{\beta_{\text{insured}}-\gamma-(1+\theta)\mathbb{E}[X]}(X)\Big], \star - R^*_{\beta_{\text{insured}}-\gamma-(1+\theta)\mathbb{E}[X]}(\star)\Big)$$

is a PO contract. In fact it is not hard to see that every PO contract is of the form

$$\big(\beta_{\text{insured}} - \beta - (1+\theta)\mathbb{E}\big[R^*_\beta(X)\big], \star - R^*_\beta(\star)\big)$$

for some $\beta \in \mathbb{R}$. If β goes through all values in $\mathbb{R}$, we then will get all the PO contracts as well as the PO frontier. We notice that even for the same optimal retention function $R^*_\beta(\star)$, different insurers with different β_{insured} will prefer to pay different premiums. The optimal premium linearly grows with respect to the insurer's final wealth.

Now our problem is reduced to solving (2.6). Without confusion, we also call its solutions (which are indeed optimal retentions) PO moral-hazard-free contracts. A PO contract is called deductible if the compensation in the contract is a deductible one.

Remark 2.1 *Xu, Zhou and Zhuang [45] solved an optimal compensation design problem under the following additional assumptions. First, the probability weighting function w is inverse- S -shaped, that is, it is strictly concave on $[0,a]$ and strictly convex on $[a,1]$ for some $0<a<1$. Second, the function $\frac{u''}{u'}$ is non-decreasing. Third, it holds that*

$$\frac{w''}{w'}(p) < \frac{u''}{u'}\big(\beta_{\text{insured}} - \gamma - (1+\theta)\mathbb{E}[X] - F_X^{-1}(p)\big)(F_X^{-1})'(p), \quad p \in [0,a].$$

These assumptions restrict the applications of their results. Their problem can be solved without these assumptions by our method. We encourage the interested readers to do it.

3. Quantile optimization problem

The probability weighting function w makes $\mathscr{E}$ a nonlinear Choquet expectation, so (2.6) is a challenging non-concave optimization problem. Since the objective function of the problem (2.6) is law-invariant[2], we can adopt the so-called quantile optimization method to tackle it; see [8–10, 20–22, 35, 39, 40, 42, 44] for the recent applications of this method.

Our first step to tackle the problem (2.6) is to make a change of variable to find an equivalent *concave* quantile optimization problem. This clearly will reduce the difficulty of solving it.

Recall that the nonlinear Choquet expectation $\mathscr{E}$ is defined by (2.4). As is well-known, by change of variable, we have

$$\mathscr{E}\big(u(Y)\big) = \int_0^1 u\left(F_Y^{-1}(p)\right) \mathrm{d}\left(1 - w(1-p)\right). \tag{3.1}$$

Next, because our probability space is atom-less, there exists a random variable ξ, which is uniformly distributed on $(0,1)$, such that $X = F_X^{-1}(\xi)$ almost surely (see, e.g. [41]). We introduce the real-valued quantile function

$$G(p) := R\big(F_X^{-1}(p)\big), \quad p \in [0,1]. \tag{3.2}$$

Then it is a non-decreasing function and satisfies

$$G(\xi) = R\big(F_X^{-1}(\xi)\big) = R(X),$$

so

$$\mathbb{E}[R(X)] = \mathbb{E}[G(\xi)] = \int_0^1 G(t)\mathrm{d}t.$$

Take

---

[2] Law-invariant means the value of its objective only relays on the distribution of X, but not relay on the sample path of X.

$$
\begin{aligned}
Y &= \beta + (1+\theta)\mathbb{E}[R(X)] - R(X) \\
&= \beta + (1+\theta)\int_0^1 G(t)\mathrm{d}t - G(\xi).
\end{aligned}
$$

Using the last expression, it is easy to verify that the quantile function of Y is given by

$$
F_Y^{-1}(p) = \beta + (1+\theta)\int_0^1 G(t)\mathrm{d}t - G(1-p), \quad \text{a.e. } p \in [0,1].
$$

Inserting it into (3.1), we get

$$
\begin{aligned}
&\mathscr{E}\Big(u\big(\beta + (1+\theta)\mathbb{E}[R(X)] - R(X)\big)\Big) \\
&= \int_0^1 u\Big(\beta + (1+\theta)\int_0^1 G(t)\mathrm{d}t - G(1-p)\Big)\mathrm{d}\,\big(1 - w(1-p)\big) \\
&= \int_0^1 u\Big(\beta + (1+\theta)\int_0^1 G(t)\mathrm{d}t - G(p)\Big)\mathrm{d}w(p).
\end{aligned}
$$

We have now written the objective function of the problem (2.6) into the quantile formulation.

It is yet to rewrite the compatibility constraint on $R \in \mathscr{R}$ in terms of the quantile function G. It is not hard to show $R \in \mathscr{R}$ if and only if $G \in \mathscr{G}$,[3] where

$$
\begin{aligned}
\mathscr{G} = \Big\{&G : [0,1] \to [0,\infty) \;\big|\; G \text{ is absolutely} \\
&\text{continuous with } G(0) = 0 \text{ and } 0 \leqslant G' \leqslant h \;\text{ a.e.}\Big\},
\end{aligned}
$$

and

$$
h(p) := (F_X^{-1})'(p) \geqslant 0, \quad \text{a.e. } p \in [0,1]. \tag{3.3}
$$

Thanks to Assumption 2.1, we have

$$
0 \leqslant \int_0^p h(t)\mathrm{d}t = F_X^{-1}(p) \leqslant F_X^{-1}(1) = \operatorname{ess\,sup} X < \infty, \quad p \in [0,1]. \tag{3.4}
$$

After the above change of variables, the study of the optimization the problem (2.6) under compatibility constraint is reduced to that of the following second-type quantile optimization problem

$$
\sup_{G \in \mathscr{G}} \int_0^1 u\Big(\beta + (1+\theta)\int_0^1 G(t)\mathrm{d}t - G(p)\Big)\mathrm{d}w(p). \tag{3.5}
$$

In this problem, the objective functional is concave with respect to the decision variable G and the constraint set $\mathscr{G}$ is convex, so it is a concave optimization problem, which, intuitively speaking, is expected to be easier to study than the non-concave optimization the problem (2.6).

Our first result is about the existence and uniqueness of the solution to the problem (3.5).

Lemma 3.1 *The problem* (3.5) *admits a unique optimal solution.*

Proof By virtue of the Arzelá-Ascoli theorem, the existence can be proved by a standard compact argument. The uniqueness is due to the strictly concavity of u and the positivity of w'. □

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[3] For more details we refer to [45].

Remark 3.1 *By the definition* (3.2) *and Assumption 2.1, we have* $G(p) = R\big(F_X^{-1}(p)\big) = R(0) = 0$ *for* $p \leqslant m_0$. *This can also be seen from the constraint* $\mathscr{G}$. *In fact,* $h(p) = (F_X^{-1})'(p) = 0$ *for* $p \leqslant m_0$ *, so* $G'(p) = 0$ *for a.e.* $p \leqslant m_0$. *Consequently* G *is a constant on* $[0, m_0]$ *, which must be zero as* $G(0) = 0$. *By virtue of this fact, we can see the objective in* (3.5) *can be written as*

$$u\Big(\beta + (1+\theta)\int_{m_0}^{1} G(t)\mathrm{d}t\Big)w(m_0) + \int_{m_0}^{1} u\Big(\beta + (1+\theta)\int_{m_0}^{1} G(t)\mathrm{d}t - G(p)\Big)\mathrm{d}w(p).$$

Therefore, the optimal solution to (3.5) *only depends the shape of* w *on* $[m_0, 1]$.

Following the change of variable argument in [42], we further simplify the problem (3.5) to remove w from the objective functional.

Let ν be the inverse function of w. Also let

$$Q(p) := G(\nu(p)), \quad p \in [0,1]. \tag{3.6}$$

Then

$$\begin{aligned}
&\int_0^1 u\Big(\beta + (1+\theta)\int_0^1 G(t)\mathrm{d}t - G(p)\Big)\mathrm{d}w(p)\\
&= \int_0^1 u\Big(\beta + (1+\theta)\int_0^1 G(\nu(t))\mathrm{d}\nu(t) - G(\nu(p))\Big)\mathrm{d}w(\nu(p))\\
&= \int_0^1 u\Big(\beta + (1+\theta)\int_0^1 Q(t)\mathrm{d}\nu(t) - Q(p)\Big)\mathrm{d}p.
\end{aligned}$$

It is clearly that G is non-decreasing if and only if so is Q. Moreover, since $\nu' > 0$, we see that $G'(p) \leqslant h(p)$ for a.e. $p \in [0,1]$ if and only if $Q'(p) = G'(\nu(p))\nu'(p) \leqslant h(\nu(p))\nu'(p)$ for a.e. $p \in [0,1]$. Therefore, $G \in \mathscr{G}$ if and only if $Q \in \mathscr{Q}$, where

$$\mathscr{Q} := \Big\{Q : [0,1] \to [0,\infty) \;\Big|\; Q \text{ is absolutely continuous with } Q(0) = 0 \text{ and } 0 \leqslant Q' \leqslant \hbar \text{ a.e.}\Big\},$$

and

$$\hbar(p) := h(\nu(p))\nu'(p) \geqslant 0, \quad \text{a.e. } p \in [0,1]. \tag{3.7}$$

Thanks to (3.4),

$$\int_0^p \hbar(t)\mathrm{d}t = \int_0^p h(\nu(t))\mathrm{d}\nu(t) = F_X^{-1}(\nu(p)) \leqslant F_X^{-1}(1) = \operatorname{ess\,sup} X, \quad p \in [0,1]. \tag{3.8}$$

By the above change of variables, solving the problem (3.5) is now reduced to solving the following concave optimization problem

$$\sup_{Q \in \mathscr{Q}} \int_0^1 u\left(\beta + (1+\theta)\int_0^1 Q(t)\mathrm{d}\nu(t) - Q(p)\right)\mathrm{d}p. \tag{3.9}$$

The above change of variables is invertible, so by Lemma 3.1 the problem (3.9) also admits a unique solution.

Lemma 3.2 *A quantile* $\overline{Q}$ *is the optimal solution to the problem* (3.9) *if and only if*

$$\overline{R}(x) \equiv \overline{Q}\left(w(F_X(x))\right)$$

is the optimal solution to the problem (2.6).

Proof By virtue of the change of variables (3.2) and (3.6), we have the relation

$$R(x) \equiv R(F_X^{-1}(F_X(x))) \equiv G(F_X(x)) \equiv Q(\nu^{-1}(F_X(x))) \equiv Q\big(w(F_X(x))\big). \tag{3.10}$$

This indicates the desired result. □

It follows from (3.8) that, for any $Q \in \mathscr{Q}$,

$$0 = Q(0) \leqslant Q(1) = \int_0^1 Q'(p)\mathrm{d}p \leqslant \int_0^1 \hbar(t)\mathrm{d}t = F_X^{-1}(1) = \operatorname{ess\,sup} X < \infty.$$

Hence the constraint set $\mathscr{Q}$ is a compact set under the supreme normal. So we will not emphasis issues such as integrability and boundedness in the subsequent argument.

4. Optimal solution

The problem (3.9) is a second-type quantile optimization problem, in which the derivatives of decision quantiles are subject to both a lower and an upper bound constraint. The existing change of variables and relaxation method introduced in [42] can only deal with the first-type problems, in which the derivatives of decision quantiles are subject to a solo lower bound constraint. It cannot be applied to solve the second-type problems. In this paper, we suggest a novel calculus variations method to solve the second-type problems including (3.9).

4.1 Optimality conditions and optimal solution

Because u is concave, the problem (3.9) is a concave optimization problem. So the calculus variations method is expected to provide not only a necessary but also a sufficient optimality condition. One would need to do extra analysis if a non-concave utility function (such as S-shaped utility in Kahneman and Tversky's [26, 38] cumulative prospect theory) would be considered in the model.

Lemma 4.1 (Optimality condition I) *Suppose* $\overline{Q} \in \mathscr{Q}$. *Then* $\overline{Q}$ *is the optimal solution to the problem* (3.9) *if and only if it satisfies*

$$\begin{aligned}&\int_0^1 u'\Big(\beta + (1+\theta)\int_0^1 \overline{Q}(t)\mathrm{d}\nu(t) - \overline{Q}(p)\Big)\\ &\times \bigg[\Big((1+\theta)\int_0^1 Q(t)\mathrm{d}\nu(t) - Q(p)\Big) - \Big((1+\theta)\int_0^1 \overline{Q}(t)\mathrm{d}\nu(t) - \overline{Q}(p)\Big)\bigg]\mathrm{d}p \leqslant 0, \quad \textit{for any } Q \in \mathscr{Q}.\end{aligned} \tag{4.1}$$

Its proof is provided in Appendix A.

By this result it suffices to find a $\overline{Q} \in \mathscr{Q}$ to satisfy the condition (4.1), so (4.1) can be regarded a *maximum principle* for our problem. It is very hard, if not impossible, to verify (4.1) since one has to compare $\overline{Q}$ with all the other quantiles in $\mathscr{Q}$, which is, intuitively speaking, of the same level of difficulty as the problem (3.9).

Our next step is to find an equivalent condition to (4.1) that can be easily verified. To this end, write

$$\Phi(p) = -\int_p^1 u'\Big(\beta + (1+\theta)\int_0^1 \overline{Q}(t)\mathrm{d}\nu(t) - \overline{Q}(s)\Big)\mathrm{d}s, \quad p \in [0,1].$$

Then the inequality in (4.1) is equivalent to

$$\int_0^1 \Phi'(p)\bigg[\Big((1+\theta)\int_0^1 Q(t)\mathrm{d}\nu(t) - Q(p)\Big) - \Big((1+\theta)\int_0^1 \overline{Q}(t)\mathrm{d}\nu(t) - \overline{Q}(p)\Big)\bigg]\mathrm{d}p \leqslant 0.$$

Applying integration by parts to the integral and using $\Phi(1)=0$ and $Q(0)=\overline{Q}(0)=0$, the left hand side in above becomes

$$-\Phi(0)(1+\theta)\int_0^1 \big(Q(t)-\overline{Q}(t)\big)\mathrm{d}\nu(t)+\int_0^1 \Phi(p)\big(Q'(p)-\overline{Q}'(p)\big)\mathrm{d}p.$$

Applying integration by parts to the first integral in above and by virtue of $Q(0)=\overline{Q}(0)=0$, the above is equal to

$$-\Phi(0)(1+\theta)\int_0^1 (1-\nu(t))\big(Q'(t)-\overline{Q}'(t)\big)\mathrm{d}t+\int_0^1 \Phi(p)\big(Q'(p)-\overline{Q}'(p)\big)\mathrm{d}p.$$

Therefore, the condition (4.1) is equivalent to the following one

$$\int_0^1 \big(\Phi(p)-\Phi(0)(1+\theta)(1-\nu(p))\big)\big(Q'(p)-\overline{Q}'(p)\big)\mathrm{d}p \leqslant 0 \text{ for any } Q\in\mathscr{Q}. \tag{4.2}$$

This condition is equivalent to saying that the linear mapping

$$Q\mapsto \int_0^1 \big(\Phi(p)-\Phi(0)(1+\theta)(1-\nu(p))\big)Q'(p)\mathrm{d}p, \quad Q\in\mathscr{Q}, \tag{4.3}$$

is maximized at $\overline{Q}$. Because $0\leqslant Q'\leqslant \hbar$ for $Q\in\mathscr{Q}$, we conclude that

$$\overline{Q}\in\mathscr{Q} \text{ and } \begin{cases}\overline{Q}'(p)=\hbar(p), & \text{if } \Phi(p)>\Phi(0)(1+\theta)(1-\nu(p));\\ \overline{Q}'(p)\in[0,\hbar(p)], & \text{if } \Phi(p)=\Phi(0)(1+\theta)(1-\nu(p));\\ \overline{Q}'(p)=0, & \text{if } \Phi(p)<\Phi(0)(1+\theta)(1-\nu(p)),\end{cases} \text{ for a.e. } p\in[0,1]. \tag{4.4}$$

It is easy to check that (4.4) also implies (4.2), so the condition (4.4) is equivalent to (4.1). We would like to point out that, in contrast to (4.1), the condition (4.4) is easy to verify since it only depends on $\overline{Q}$ itself.

Although (4.4) is easier to verify, it is still not easy to derive the optimal solution $\overline{Q}$ from it. We now express the condition (4.4) through an ordinary integro-differential equation (OIDE). Later, this OIDE will be further reduced to an ordinary differential equation (ODE) from which we can compute the optimal solution $\overline{Q}$. To this end, we first introduce an elementary technical lemma whose proof is given in Appendix A.

Lemma 4.2 *Let a, b, c and b be real quantities such that $b\leqslant c$. Then*

$$\min\{\max\{a-c,\ d\},\ a-b\}=0$$

if and only if

$$\begin{cases}a=c, & if\ d<0;\\ a\in[b,c], & if\ d=0;\\ a=b, & if\ d>0.\end{cases}$$

This together with the previous analysis leads to the following new optimality condition.

Lemma 4.3 (Optimality condition II) *Suppose $\overline{Q}:[0,1]\to\mathbb{R}$ is an absolutely continuous function. Then $\overline{Q}$ is the optimal solution to the problem* (3.9) *if and only if it satisfies the following OIDE:*

$$\begin{cases}\min\Big\{\max\big\{\overline{Q}'(p)-\hbar(p),\ \Phi(0)(1+\theta)(1-\nu(p))-\Phi(p)\big\},\ \overline{Q}'(p)\Big\}=0, \quad a.e.\ p\in[0,1],\\ \overline{Q}(0)=0,\end{cases} \tag{4.5}$$

where

$$\Phi(p) = -\int_p^1 u'\left(\beta + (1+\theta)\int_0^1 \overline{Q}(t)\mathrm{d}\nu(t) - \overline{Q}(s)\right)\mathrm{d}s. \tag{4.6}$$

Proof This is an immediate consequence of the optimality condition (4.4) and Lemma 4.2. □

As OIDEs are not easy to solve in general, our next goal is to reduce (4.5) to an easily solved ODE problem. The following technical result which is proved in Appendix A will be used frequently in this process.

Lemma 4.4 *Let a, b, c and b be real quantities. Then* $\min\{\max\{a,b\},\ c\} = 0$ *if and only if* $\min\{\max\{ak, b\ell\}, cm\} = 0$ *for any real quantities* $k, \ell, m \geqslant 0$.

Recall that $C^{2-}([0,1])$ is defined as the set of functions $f : [0,1] \to \mathbb{R}$ which are differentiable with derivatives f' being absolutely continuous on $[0,1]$ and $C^2([0,1]) \subsetneqq C^{2-}([0,1]) \subsetneqq C^1([0,1])$.

Definition 4.1 *A function $f \in C^{2-}([0,1])$ is called good, if it satisfies all the following conditions:*

(1) *f is convex;*
(2) $f' > 0$;
(3) *f is linear on the set $\{p \in [0,1] : f(0)(1+\theta)(1-m_0) - f(p) \geqslant 0\}$.*

Now we are ready to present an important ODE with boundary conditions. It will play a key role in solving the problem (3.9).

$$\begin{cases} \min\Big\{ \max\big\{\Phi''(p) + \hbar(p)u''\big((u')^{-1}\big(\Phi'(p)\big)\big),\ \Phi(0)(1+\theta)(1-\nu(p)) - \Phi(p)\big\}, \Phi''(p)\Big\} = 0, \\ \qquad \text{a.e. } p \in [0,1], \\ \Phi(1) = 0, \quad \beta = (1+\theta)\int_0^1 (u')^{-1}\big(\Phi'(t)\big)\mathrm{d}\nu(t) - (u')^{-1}\big(\Phi'(0)\big)\theta. \end{cases} \tag{4.7}$$

Our main result, which states the connection between the above ODE and the optimal solution to the problem (3.9), is given as follows.

Theorem 4.5 (Optimal solution) *We have the following assertions.*

(1) *If $\overline{Q} \in \mathscr{Q}$ is the optimal solution to the problem* (3.9), *then the function*

$$\Phi(p) := -\int_p^1 u'\left(\beta + (1+\theta)\int_0^1 \overline{Q}(t)\mathrm{d}\nu(t) - \overline{Q}(s)\right)\mathrm{d}s, \quad p \in [0,1], \tag{4.8}$$

is a solution to the ODE (4.7) *in* $C^{2-}([0,1])$. *Moreover, Φ is a good function.*

(2) *If Φ is a solution to the ODE* (4.7) *in* $C^{2-}([0,1])$, *then the functions*

$$\overline{Q}(p) := (u')^{-1}\big(\Phi'(0)\big) - (u')^{-1}\big(\Phi'(p)\big), \quad p \in [0,1],$$

and

$$\overline{R}(x) := (u')^{-1}\big(\Phi'(0)\big) - (u')^{-1}\big(\Phi'\big(w(F_X(x))\big)\big), \quad x \in [0,\infty),$$

are, respectively, the optimal solutions to the problems (3.9) *and* (2.6).

As a consequence, the ODE (4.7) *admits a unique solution in* $C^{2-}([0,1])$ *and the solution is a good function.*

Its proof is provided in Appendix A.

Because (4.5) is an OIDE while (4.7) is an ODE, intuitively speaking, the latter is easier to study than the former.

At first sight the ODE like (4.7) looks like a standard double-obstacle problem, however, the obstacles are put on the highest (second) order gradient of the unknown function; by contrast, they are usually put on the lower order gradient(s) in the financial economics literature (see, e.g., Dai and Yi [14], Dai, Xu and Zhou [13] with obstacles on the first order gradients). To the best of our knowledge, it seems it is the first time that such type of double-obstacle problems appears in the financial economics literature.

Thanks to Remark 3.1, for any $Q \in \mathscr{Q}$, we have $Q(p) = 0$ for $p \in [0, w(m_0)]$, so the function Φ defined in (4.8) satisfies

$$\Phi(p) = u'\left(\beta + (1+\theta)\int_0^1 \overline{Q}(t)\mathrm{d}\nu(t)\right)(p - m_0) + \Phi(m_0)$$

for $p \leqslant w(m_0)$. Using the fact that $\hbar(p) = 0$ for $p \leqslant w(m_0)$, this can also be deduced from (4.7) directly.

Example 4.1 *We may simplify the ODE* (4.7) *for the most widely used three utility functions.*

- *For the exponential utility $u(x) = -\alpha^{-1}e^{-\alpha x}$, $\alpha > 0$, the ODE* (4.7) *becomes*

$$\begin{cases} \min\Big\{\max\big\{\Phi''(p) - \alpha\hbar(p)\Phi'(p),\ \Phi(0)(1+\theta)(1-\nu(p)) - \Phi(p)\big\}, \Phi''(p)\Big\} = 0, \\ \qquad a.e.\ p \in [0,1], \\ \Phi(1) = 0, \quad \beta = \alpha^{-1}\Big((\log\Phi'(0))\theta - (1+\theta)\int_0^1 (\log\Phi'(t))\mathrm{d}\nu(t)\Big). \end{cases} \tag{4.9}$$

In this case the operator $\Phi''(p) - \alpha\hbar(p)\Phi'(p)$ grows linearly in Φ'.

- *For the power utility $u(x) = \alpha^{-1}x^{\alpha}$, $\alpha < 0$ or $0 < \alpha < 1$, the ODE* (4.7) *becomes*

$$\begin{cases} \min\Big\{\max\big\{\Phi''(p) - (1-\alpha)\hbar(p)(\Phi'(p))^{\frac{2-\alpha}{1-\alpha}},\ \Phi(0)(1+\theta)(1-\nu(p)) - \Phi(p)\big\}, \Phi''(p)\Big\} = 0, \\ \qquad a.e.\ p \in [0,1], \\ \Phi(1) = 0, \quad \beta = (1+\theta)\int_0^1 (\Phi'(t))^{\frac{1}{\alpha-1}}\mathrm{d}\nu(t) - (\Phi'(0))^{\frac{1}{\alpha-1}}\theta. \end{cases} \tag{4.10}$$

If $\alpha < 0$, then $1 < \frac{2-\alpha}{1-\alpha} < 2$, so the operator $\Phi''(p) - (1-\alpha)\hbar(p)(\Phi'(p))^{\frac{2-\alpha}{1-\alpha}}$ is super-linear but not quadratic growth in Φ'. If $0 < \alpha < 1$, then $\frac{2-\alpha}{1-\alpha} > 2$, so the operator is beyond quadratic growth in Φ'.

- *For the logarithmic utility $u(x) = \log x$, the ODE* (4.7) *becomes*

$$\begin{cases} \min\Big\{\max\big\{\Phi''(p) - \hbar(p)(\Phi'(p))^2,\ \Phi(0)(1+\theta)(1-\nu(p)) - \Phi(p)\big\}, \Phi''(p)\Big\} = 0, \\ \qquad a.e.\ p \in [0,1], \\ \Phi(1) = 0, \quad \beta = (1+\theta)\int_0^1 (\Phi'(t))^{-1}\mathrm{d}\nu(t) - (\Phi'(0))^{-1}\theta. \end{cases} \tag{4.11}$$

This can be regarded as the limit case $\alpha = 0$ in (4.10).

Although (4.7) is an ODE, it is not easy to study for the following reasons: First, $\Phi(0)$ appears in the operator, so the operator is nonlocal, which is in general harder than problems with local operator. Second, as mentioned earlier, the obstacles are put on the highest order gradient of the unknown function rather than lower order gradients; the corresponding penalty approximating problem is a fully nonlinear one rather than a semi-linear one. Third, the operator $\Phi'' + \hbar u''\big((u')^{-1}(\Phi')\big)$ may not be linear growth in Φ' (see, e.g,., (4.10) and (4.11)). Fourth, the last boundary condition is nonlocal and not the classical Dirichlet and/or Neumann boundary conditions. In fact, its solvability including the existence and uniqueness of the solution is an interesting issue from the pure ODE point of view, although we have resolved this issue from the pure optimization point of view.

The author introduced a change of variables and relaxation method in [42] to solve the first-type quantile optimization problem (e.g. the problem in [40]). The optimal solution is expressed via the concave envelope of some known function (we remark that a concave envelope can be expressed as the solution to a single-obstacle problem). The model does not involve the compatibility constraint, and it can be regarded as the special case $\hbar(p)=+\infty$ for $p\in(w(m_0),1]$ in our model. In this case,

$$\Phi''(p)+\hbar(p)u''\big((u')^{-1}\big(\Phi'(p)\big)\big)=-\infty,\quad p\in(w(m_0),1],$$

so the ODE in (4.7) becomes

$$\begin{cases}\Phi''(p)=0, & \text{a.e. } p\in[0,w(m_0)],\\ \min\Big\{\Phi(0)(1+\theta)(1-\nu(p))-\Phi(p),\ \Phi''(p)\Big\}=0, & \text{a.e. } p\in(w(m_0),1].\end{cases}$$

From this, one can reveal the optimal solution in [40] and [42]. Roughly speaking, the first-type quantile optimization problems only involve one-side constraint, while the second-type problems require two-side constraint, so the classical single-obstacle ODE becomes a double-obstacle ODE. As a result, no closed-form solution is available for the second-type problem in general.

We cannot find an existing numerical scheme to solve (4.7) directly due to the last nonlocal condition. Recalling that our original target is to find all the PO contracts, this inspires us to consider all $\beta\in\mathbb{R}$ rather than a particular one. This allows us forgetting the last nonlocal condition in (4.7). This observation leads to the next result, which will play a key role in designing a numerical scheme to calculate the PO frontier in the next section.

Corollary 4.6 *A function Φ in $C^{2-}([0,1])$ corresponds to a PO contract if and only if it satisfies*

$$\begin{cases}\min\Big\{\max\big\{\Phi''(p)+\hbar(p)u''\big((u')^{-1}\big(\Phi'(p)\big)\big),\ \sigma(1+\theta)(1-\nu(p))-\Phi(p)\big\},\Phi''(p)\Big\}=0,\\ \qquad a.e.\ p\in[0,1],\\ \Phi(0)=\sigma,\quad \Phi(1)=0,\end{cases}\tag{4.12}$$

for some $\sigma<0$; in which case

$$\overline{Q}(p):=(u')^{-1}\big(\Phi'(0)\big)-(u')^{-1}\big(\Phi'(p)\big),\quad p\in[0,1],$$

and

$$\overline{R}(x):=(u')^{-1}\left(\Phi'(0)\right)-(u')^{-1}\left(\Phi'\left(w(F_X(x))\right)\right),\quad x\in[0,\infty),$$

are, respectively, the optimal solutions to problems (3.9) *and* (2.6) *with*

$$\beta:=(1+\theta)\int_0^1(u')^{-1}\big(\Phi'(t)\big)\mathrm{d}\nu(t)-(u')^{-1}\big(\Phi'(0)\big)\theta.$$

Moreover, the solution Φ to (4.12) *is a good function.*

Proof This is an immediate consequence of Theorem 4.5. □

The ODE (4.12) is a problem with Dirichlet boundary conditions, but we failed to find an existing numerical scheme that can be applied directly to it. Therefore, we will provide an alternative way to solve it in the following section.

4.2 A numerical scheme to solve the problem

By Theorem 4.5, solving problems (3.9) and (2.6) is reduced to solving (4.7). But for a given β, it is very hard to directly solve (4.7), even numerically. Corollary 4.6 motives us to solve the problem for all the possible values of $\beta \in \mathbb{R}$. Hence we propose the following numerical scheme. It turns out that we can use this method to solve (4.7) and (2.6) for any fixed β as well.

(1) First, fix any value $\sigma < 0$. Then, for each $\varpi > 0$, consider the following problem[4]

$$\begin{cases} \min\Big\{\max\big\{\Phi''(p) + \hbar(p)u''\big((u')^{-1}\big(\Phi'(p)\big)\big),\ \sigma(1+\theta)(1-\nu(p)) - \Phi(p)\big\}, \Phi''(p)\Big\} = 0, \\ \qquad \text{a.e. } p \in [0,1], \\ \Phi(0) = \sigma, \quad \Phi'(0) = \varpi. \end{cases} \tag{4.13}$$

This is an initial value double-obstacle problem with a semi-linear *local* operator. It is different from (4.7) only on the boundary conditions. Since (4.13) is an initial value problem, we can solve it by existing numerical schemes such as the finite difference method. We denote its solution by $\Phi_{\sigma,\varpi}$.

(2) By comparison theorem for nonlinear ODE (see, e.g. Lieberman [27]), we can show that $\Phi_{\sigma,\varpi}$ is non-decreasing in ϖ and there exists a $\varpi(\sigma)$ (depending on σ) such that $\Phi_{\sigma,\varpi(\sigma)}(1) = 0$ (assuming $\hbar$ is a continuous function). Moreover, the map $\sigma \mapsto \varpi(\sigma)$ is non-increasing. By virtue of this, one can check that $\Phi_{\sigma,\varpi(\sigma)}$ solves (4.7) with β replaced by

$$\beta(\sigma) = (1+\theta)\int_0^1 (u')^{-1}\big(\Phi'_{\sigma,\varpi(\sigma)}(t)\big)\mathrm{d}\nu(t) - (u')^{-1}(\varpi(\sigma))\theta. \tag{4.14}$$

Since $\Phi_{\sigma,\varpi}$ is a convex function, we have

$$\Phi_{\sigma,\varpi}(p) \geqslant \Phi_{\sigma,\varpi}(0) + \Phi'_{\sigma,\varpi}(0)p, \quad p \in [0,1].$$

In particular, it implies

$$0 = \Phi_{\sigma,\varpi(\sigma)}(1) \geqslant \Phi_{\sigma,\varpi(\sigma)}(0) + \Phi'_{\sigma,\varpi(\sigma)}(0) = \sigma + \varpi(\sigma),$$

that is, $\varpi(\sigma) \leqslant -\sigma$. Therefore, it suffices to consider the case $0 < \varpi \leqslant -\sigma$ in the first step in order to find $\varpi(\sigma)$.

(3) The ODE (4.7) has at most one solution for each fixed $\beta \in \mathbb{R}$, so different values of $\sigma < 0$ must lead to different values of $\beta(\sigma)$. Hence, we will get all the PO contracts if we go through all $\sigma < 0$. Since $\Phi'_{\sigma,\varpi}$ is continuous with respective to σ and ϖ, $\beta(\sigma)$ is continuous in σ. The map $\sigma \mapsto \beta(\sigma)$ is injective and continuous, so $\beta(\sigma)$ is a monotone function of σ. On the other hand, the solution to (4.13) is evidently convex, so by (4.14) and the concavity of u, one has

$$\beta(\sigma) \leqslant (1+\theta)\int_0^1 (u')^{-1}\big(\Phi'_{\sigma,\varpi(\sigma)}(0)\big)\mathrm{d}\nu(t) - (u')^{-1}(\varpi(\sigma))\theta = (u')^{-1}(\varpi(\sigma)),$$

which leads to

---

[4] Clearly the solution Φ to (4.7) must be convex and satisfy $\Phi(1) = 0$ and $\Phi' > 0$ by Theorem 4.5. Therefore, it suffices to consider $\varpi > 0$ and $\sigma < 0$.

$$\lim_{\sigma\to-\infty}\beta(\sigma)=\lim_{\varpi(\sigma)\to\infty}\beta(\sigma)\leqslant\lim_{\varpi(\sigma)\to\infty}(u')^{-1}(\varpi(\sigma))<\infty.$$

Because there is no upper bound for β in the problem (2.6), we conclude that $\beta(\sigma)$ must be increasing in σ. Because $\beta(\sigma)$ is continuous and strictly increasing in σ, we can numerically solve (4.7) efficiently for any fixed $\beta\in\mathbb{R}$, say, by the bisection method.

4.3 A numerical example

In this section we provide a numerical example to illustrate the theoretical results obtained thus far.

We set the function ν as

$$\nu(p)=\begin{cases}\dfrac{25}{12}(p-2p^2), & p\in\left[0,\dfrac{1}{5}\right];\\ \dfrac{1}{444}\left(625p^3-375p^2+260p+69\right), & p\in\left(\dfrac{1}{5},\dfrac{2}{5}\right];\\ \dfrac{1}{444}\left(375p^2-40p+109\right), & p\in\left(\dfrac{2}{5},1\right].\end{cases}\tag{4.15}$$

This is a strictly increasing mapping $[0,1]$ to itself. Let the probability weighting function w be defined as the inverse of ν. Then both w and ν are $C^{2-}([0,1])$ functions with positive derivatives. The function ν and its inverse, the probability weighting function w are illustrated in Figure 1 and Figure 2, respectively.

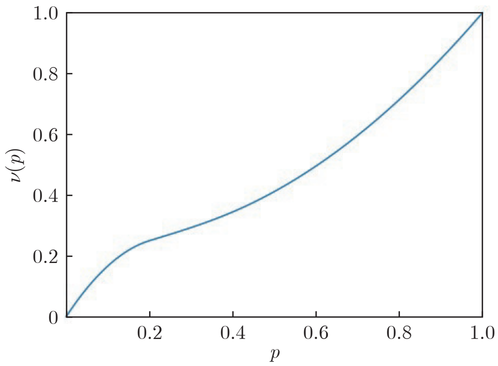

Figure 1 The function ν in solid blue color, defined in (4.15), is a continuous function and strictly increasing

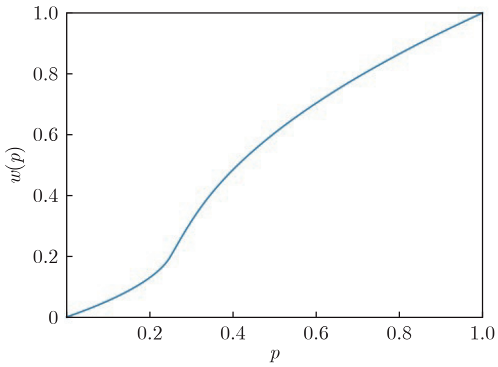

Figure 2 The probability weighting function w in solid blue color, defined as the inverse function of ν given in (4.15), is a continuous function and strictly increasing

We choose a power utility function $u(x) = 2\sqrt{x}$, set the relative safety loading as $\theta = 0.2$, and define the quantile of the potential loss X by

$$F_X^{-1}(\nu(p)) = \begin{cases} 0, & p \in \left[0, \frac{1}{5}\right]; \\ \frac{2}{3}(1875p^4 - 2500p^3 + 1200p^2 - 240p + 617) \\ \qquad - \left(\frac{74}{375p^2 - 150p + 52}\right)^2, & p \in \left(\frac{1}{5}, \frac{2}{5}\right]; \\ \frac{14}{3} - \left(\frac{666}{-2500p^3 + 4125p^2 - 750p + 268}\right)^2, & p \in \left(\frac{2}{5}, 1\right]. \end{cases} \tag{4.16}$$

One can check that F_X^{-1} is continuous differentiable and fulfills Assumption 2.1 with a mass $m_0 = \frac{1}{4}$ at 0. Moreover, the corresponding function $\hbar$ defined by (3.7) is a continuous function. The pictures of F_X^{-1} and $\hbar$ are shown in Figure 3 and Figure 4, respectively.

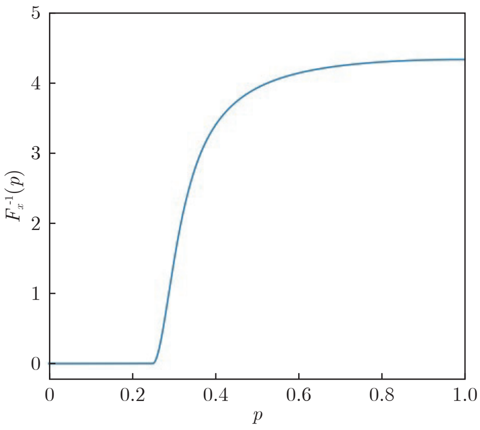

Figure 3 The quantile function F_X^{-1} in solid blue color, defined in (4.16), is continuous differentiable and fulfills Assumption 1 with a mass $m_0 = \frac{1}{4}$ at 0

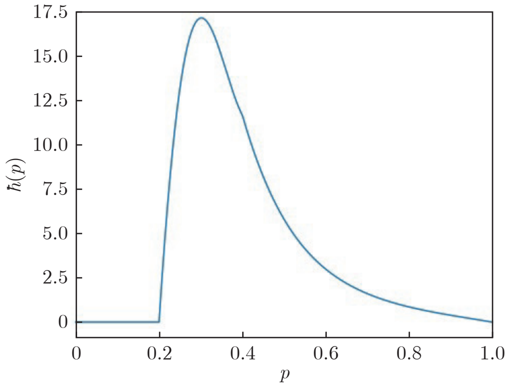

Figure 4 The function $\hbar$ in solid blue color, defined in (3.7), is a continuous function and being 0 for small p

Under the above setting, we can use the numerical scheme proposed in Section 4.2 to compute the function Φ and the optimal retention $\overline{R}$ for $\beta = -0.928$. Our scheme shows that $\sigma = -1$ and $\varpi = 0.5$ in this case. The pictures of Φ and $\overline{R}$ are presented in Figure 5 and Figure 6, respectively.

It can be seen from Figure 5 that $\Phi(p)$ is convex and different from $\sigma(1+\theta)(1-\nu(p))$ if p is near to 0 or 1. If p is small, then $\Phi(p)$ is linear since $\Phi(p)$ is a good function. If p is in the

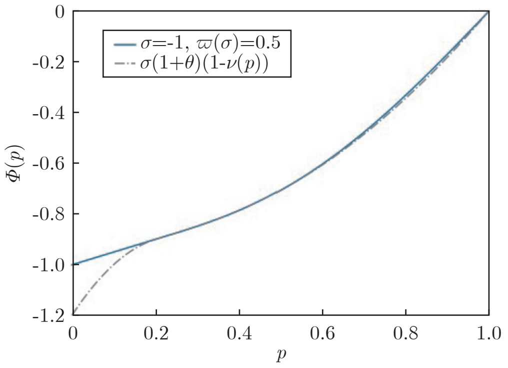

Figure 5 The function Φ in solid blue color coincides with the curve $\sigma(1+\theta)(1-\nu(p))$ in dotted gray color in the middle range. But they are diverse for small and large p

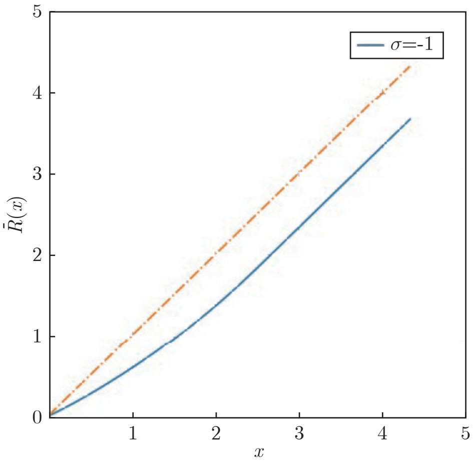

Figure 6 The optimal retention $\overline{R}$ in solid blue color satisfies $0<\overline{R}'<1$ for small p and $\overline{R}'=1$ for large p. The dotted orange line is the identical function

middle range, then $\Phi(p)$ coincides with $\sigma(1+\theta)(1-\nu(p))$, which indicates that the derivative constraint is not tight, namely $0<\overline{R}'<1$. This can be seen from Figure 6. If p is large, $\Phi(p)$ dominates $\sigma(1+\theta)(1-\nu(p))$, which corresponds to the upper tight case, namely $\overline{R}'=1$. This can also be seen from Figure 6.

5. On the PO contracts

In this section we discuss some properties of the PO contracts.

We first reproduce a classical result, that is, all the PO contracts are of deductible type if there is no probability distortion. We then provide a sufficient and necessary condition under which the optimal solution to the problem (2.6) corresponds to a deductible compensation with deductible d. Finally, we identify a sufficient condition to guarantee a moral-hazard-free retention is optimal to the problem (2.6) for infinite many RDU insureds.

5.1 PO contracts of deductible type

In this section we investigate under what conditions a deductible compensation is PO. Our first result reveals the classical assertion that all the PO contracts are of deductible type in the

EU theory framework (see, e.g. [36]). A proof based on our result is given in Appendix A.

Proposition 5.1 *If there is no probability distortion, i.e.,* $w(p)\equiv p$ *, then all the PO contracts are of deductible type.*

Remark 5.1 *By Remark 3.1, the optimal solution solution to* (2.6) *does not depend on the shape of* w *on* $[0,m_0)$ *, so Proposition 5.1 still holds if the condition is replaced by* $w(p)=p$ *for* $p\in[m_0,1]$.

If the probability weighting function is nontrivial (i.e. $w(p)\not\equiv p$), the following result answers the question completely by providing an equivalent condition.

Theorem 5.2 *The optimal solution to the problem* (2.6) *corresponds to a deductible compensation with deductible* d *if and only if*

$$\begin{cases}(1+\theta)(1-\nu(p))\displaystyle\int_0^1 u'(g(s))\mathrm{d}s\geqslant\int_p^1 u'(g(s))\mathrm{d}s, & p\in[0,w(F_X(d))],\\ (1+\theta)(1-\nu(p))\displaystyle\int_0^1 u'(g(s))\mathrm{d}s\leqslant\int_p^1 u'(g(s))\mathrm{d}s, & p\in[w(F_X(d)),1],\end{cases}\tag{5.1}$$

where

$$g(s)=\beta+(1+\theta)\mathbb{E}[\min\{X,d\}]-\min\left\{F_X^{-1}(\nu(s)),\ d\right\}.$$

Moreover, the deductible d *satisfies* $F_X(d)\geqslant\frac{\theta}{1+\theta}$.

This result is proved in Appendix A. For $p\in[w(F_X(d)),1]$, we have a neat expression:

$$\begin{aligned}\int_p^1 u'(g(s))\mathrm{d}s&=\int_p^1 u'\Big(\beta+(1+\theta)\mathbb{E}[\min\{X,d\}]-\min\left\{F_X^{-1}(\nu(s)),\ d\right\}\Big)\mathrm{d}s\\&=\int_p^1 u'\big(\beta+(1+\theta)\mathbb{E}[\min\{X,d\}]-d\big)\mathrm{d}s\\&=u'\big(\beta+(1+\theta)\mathbb{E}[\min\{X-d,0\}]\big)(1-p).\end{aligned}$$

Corollary 5.3 *If* $\theta>\frac{m_0}{1-m_0}$ *, then the full coverage compensation is not optimal for any RDU insured.*

Proof By Theorem 5.2, we should have $F_X(d)\geqslant\frac{\theta}{1+\theta}>m_0$, so $d>0$. □

Economically speaking, if the insurance company asks too much, no RDU insured will buy the full coverage contract. Especially, if X has no mass at 0 (namely $m_0=0$), the condition in Corollary 5.3 is satisfied, therefore the full coverage compensation is not optimal for any RDU insured. On the other hand, by the proof of Proposition 5.1 and Remark 5.1, the full coverage compensation can be PO at least if $m_0>0$ and $w(p)=p$ for $p\in[m_0,1]$. Therefore, the relative safety loading and the mass of X at 0 play the critical role in determining the type of the PO contracts. Indeed, if the relative safety loading is small or X has a large mass at 0, namely $0<\theta<\frac{m_0}{1-m_0}$, the full coverage compensation can also be PO even if the probability weighting function is nontrivial. This will be shown in the following section.

Clearly one can use our idea to find equivalent conditions under which the optimal compensations will take a special form such as the proportional coverage compensations. For instance, Xu, Zhou and Zhuang [45] studied an optimal compensation problem, which is slight different from ours, but also under the RDU preference. Under certain assumptions (see Remark 2.1), they found that optimal compensation is of three-fold, that is

$$I(x) = \max\big\{\min\{x,\ d_1\},\ x - d_2\big\}.$$

Their assumptions are sufficient but not necessary. By contrast, one can find an equivalent condition for the above compensation being optimal to their problem by our approach. The interested readers may complete the details.

If the condition (5.1) is not satisfied, a PO contract may not be of deductible type. We are interested in which situation a moral-hazard-free contract is PO, that is, it is optimal for at least one RDU insured; in other words, economically speaking, when such a contract is acceptable in the insurance market. This will be addressed in the following section.

5.2 When is a moral-hazard-free contract PO?

In this section, we consider the following reverse problem. For a given potential loss X, a moral-hazard-free retention $R \in \mathscr{R}$ and a relative safety loading $\theta > 0$, is it possible to find at least one RDU insured such that the contract is optimal for her? Every RDU insured is characterized by her utility function u, probability weighting function w and wealth level β. So the question can be mathematically stated as follows: for any given reasonable (F_X, R, θ), can we find a triple (u, w, β) such that R is optimal to the problem (2.6)?

If the relative safety loading is big or X has a small mass at 0, namely $\theta > \frac{m_0}{1-m_0}$, the answer is negative as it is impossible to find a RDU insured such that the full coverage contract is optimal for her by Corollary 5.3. By contrast, if the relative safety loading is small or X has a large mass at 0, namely $0 < \theta < \frac{m_0}{1-m_0}$, the following result (whose proof is given in Appendix A) gives an affirmative answer to the question.

Theorem 5.4 *Suppose $R \in \mathscr{R}$ is a moral-hazard-free retention, X satisfies Assumption 2.1 and $0 < \theta < \frac{m_0}{1-m_0}$. Then R is optimal to the problem* (2.6) *for infinite many RDU insureds.*

Thanks to Corollary 5.3 and Theorem 5.4, our question is completely answered except for the marginal case $\theta = \frac{m_0}{1-m_0}$. The latter calls for a delicate treatment, which is left to the smart readers.

Given any moral-hazard-free retention, from the proof of Theorem 5.4, we can see that any choice of u that satisfies (A.3) will lead to an insured (u, w, β) who accepts the retention. Since (A.3) can be satisfied by essentially different utilities u (that is, they are not unique up to linear transformations), the retention is accepted by different types of insureds.

6. PO contracts with non-monotone compensations or retentions

In the previous sections, we consider the incentive compatibility constraint (2.3) in the formulation of the problem (2.5). One nature question is: what will the PO contracts look like if the incentive compatibility constraint is (fully or partially) ignored in the problem formulation at beginning? We give answers to this question when one of the retention and compensation is not monotone. Since the analysis is similar to the case with incentive compatibility constraint, we just point out the main differences in the arguments and leave the details to the interest readers.

6.1 PO contracts with non-monotone compensations

If we only require the retentions to be monotone but put no constraint on the compensations, that is,

$$\mathscr{R}=\Big\{R:[0,\infty)\to[0,\infty)\;\Big|\;R \text{ is absolutely continuous with } R(0)=0 \text{ and } R'\geqslant 0 \text{ a.e.}\Big\}. \tag{6.1}$$

Then the corresponding set of compensations becomes

$$\mathscr{C}=\Big\{I:[0,\infty)\to[0,\infty)\;\Big|\;I \text{ is absolutely continuous with } I(0)=0 \text{ and } I'\leqslant 1 \text{ a.e.}\Big\}.$$

In this case the compensations in $\mathscr{C}$ may not be monotone.

Theorem 6.1 *If the set of retentions $\mathscr{R}$ in the problem* (2.6) *is replaced by* (6.1), *then a retention $\overline{R}\in\mathscr{R}$ is optimal to the problem* (2.6) *if and only if it can be written as*

$$\overline{R}(x)=(u')^{-1}\big(\kappa\psi'(0)\big)-(u')^{-1}\big(\kappa\psi'(w(F_X(x)))\big), \tag{6.2}$$

where κ is the unique positive constant such that

$$\beta=(1+\theta)\int_0^1(u')^{-1}\left(\kappa\psi'(t)\right)\mathrm{d}\nu(t)-(u')^{-1}\left(\kappa\psi'(0)\right)\theta,$$

and ψ is the convex envelope of $(1+\theta)(\nu(p)-1)$ on $[w(m_0),1]$ with boundary values $\psi(1)=0$ and $\psi(w(m_0))=(1+\theta)(m_0-1)$, and $\psi(p)=(p-w(m_0))\psi'(w(m_0))+(1+\theta)(m_0-1)$ for $p\in[0,w(m_0))$.

Its proof is provided in Appendix A.

As we only have one-side constraint for R' in this model, the related ODE becomes a single-obstacle problem. Furthermore, since the operator in the ODE is linear, one can give an explicit solution.

Remark 6.1 *As the lower bound $\overline{R}'\geqslant 0$ is satisfied by the constraint* (6.1), *the optimal retention $\overline{R}$ given by* (6.2) *is moral-hazard-free if and only if $\overline{R}'\leqslant 1$ a.e., namely*

$$\frac{\kappa\psi''\big(w(F_X(x))\big)w'(F_X(x))F_X'(x)}{-u''\big((u')^{-1}\big(\kappa\psi'\big(w(F_X(x))\big)\big)\big)}\leqslant 1,\quad a.e.\ x\in[0,\operatorname{ess\,sup}X],$$

or equivalently,

$$\kappa\psi''(p)+\hbar(p)u''\big((u')^{-1}\big(\kappa\psi'(p)\big)\big)\leqslant 0,\quad a.e.\ p\in[w(m_0),1].$$

6.2 PO contracts with non-monotone retentions

If we only require the compensations to be monotone but put no constraint on the retentions, that is,

$$\mathscr{C}=\Big\{I:[0,\infty)\to[0,\infty)\;\Big|\;I \text{ is absolutely continuous with } I(0)=0 \text{ and } I'\geqslant 0 \text{ a.e.}\Big\},$$

then the corresponding set of retentions becomes

$$\mathscr{R}=\Big\{R:[0,\infty)\to[0,\infty)\;\Big|\;R \text{ is absolutely continuous with } R(0)=0 \text{ and } R'\leqslant 1 \text{ a.e.}\Big\}. \tag{6.3}$$

In this case the retentions in $\mathscr{R}$ may not be non-decreasing.

Theorem 6.2 *If the set of retentions $\mathscr{R}$ in the problem* (2.6) *is replaced by* (6.3), *then a retention $\overline{R}\in\mathscr{R}$ is optimal to the problem* (2.6) *if and only if it can be written as*

$$\overline{R}(x)=(u')^{-1}\big(\Phi'(0)\big)-(u')^{-1}\big(\Phi'\big(w(F_X(x))\big)\big) \tag{6.4}$$

for some $\Phi \in C^{2-}([0,1])$ *such that*

$$\begin{cases} \Phi''(p) = 0, \quad a.e.\ p \in [0, w(m_0)], \\ \max\left\{\Phi''(p) + \hbar(p)u''\left((u')^{-1}(\Phi'(p))\right), \Phi(0)(1+\theta)(1-\nu(p)) - \Phi(p)\right\} = 0, \\ \qquad a.e.\ p \in (w(m_0), 1], \\ \Phi(1) = 0. \end{cases} \tag{6.5}$$

Its proof is provided in Appendix A.

Same as the previous model, the ODE (6.5) is also a single-obstacle problem, but its operator is nonlinear in general so we cannot provide an explicit solution. However, if u is an exponential utility, the operator becomes linear (see (4.9)), so that one can show that Φ, after change of variable, is the convex envelope of some known function.

We have the following useful result whose proof is given in Appendix A.

Corollary 6.3 *If the probability weighting function* w *is concave (such as no distortion case* $w(p) \equiv p$ *), then the optimal retention* $\overline{R}$ *given by* (6.4) *is moral-hazard-free.*

Remark 6.2 *As the upper bound* $\overline{R}' \leqslant 1$ *is satisfied by the constraint* (6.3)*, the optimal retention* $\overline{R}$ *given by* (6.4) *is moral-hazard-free if and only if* $\overline{R}' \geqslant 0$. *The latter is equivalent to the solution* Φ *to* (6.5) *is convex.*

7. Concluding remarks

In our model, the preference $\mathcal{U}_{\text{insurer}}(\mathscr{P}, I)$ is a linear functional of I and therefore displays a constant marginal cost. More general convex functionals are considered in the literature; for instance,

$$\mathcal{U}_{\text{insurer}}(\mathscr{P}, I) = \mathscr{P} - \mathbb{E}[c(I(X))],$$

where $c : [0,\infty) \to [0,\infty)$ is a convex and increasing function with $c(0) = 0$, displaying increasing marginal cost. The shape of c plays a crucial role in determining the shape of Pareto optima. For instance, Carlier and Dana [6] showed the impact of the shape of the cost function on the existence of a deductible level at a Pareto optimum in a very general setting with *risk-averse agents.* In our model, the preferences admit a less abstract representation than in Carlier and Dana [6, 7] and the insurer is risk-neutral, but the RDU insured might not be risk-averse since the distortion function might not be convex (e.g., Chew et al. [11]). It would be very interesting to see how the interplay between the non-convexity of the distortion and the convexity of the cost function affects the shape of optimal indemnities (in particular, whether or not they have a deductible provision). We leave these challenging problems to the future work.

In this paper, we present a novel approach to computing all the PO moral-hazard-free insurance contracts under the RDU theory. Similar to [40, 42], the approach also works for problems under some other behavioral finance theories with law-invariant measure. For instance, one could use our method to consider the loss and gain parts in the cumulative prospect theory model, separately, and then combine them to get the optimal solution. Our method also allows us to find all the PO moral-hazard-free insurance contracts with general lower and/or upper bounds on the derivatives of the retentions and/or the compensations.

In our model, we considered the expected value premium principle for the insurer. It is possible to generalize our method to cope with models with other premium principles such as the variance premium principle and Wang's premium principle for the insurer. On the other hand, this paper only considered smooth probability weighting functions. It is possible to consider general discontinuous probability weighting functions in the model. We leave the above problems for our future research.

Appendix

A Proofs

Proof of Lemma 4.1 Suppose $\overline{Q}$ is the optimal solution to the problem (3.9). For any $Q \in \mathscr{Q}$ and constant $\varepsilon \in (0,1)$, define

$$Q_\varepsilon(p) = \overline{Q}(p) + \varepsilon(Q(p) - \overline{Q}(p)), \quad p \in [0,1].$$

Then it is easy to see $Q_\varepsilon \in \mathscr{Q}$. By virtue of that $\overline{Q}$ is the optimal solution to the problem (3.9) and applying Fatou's lemma, we get

$$\begin{aligned}
0 \geqslant & \liminf_{\varepsilon \to 0+} \frac{1}{\varepsilon}\left[\int_0^1 u\left(\beta + (1+\theta)\int_0^1 Q_\varepsilon(t)\mathrm{d}\nu(t) - Q_\varepsilon(p)\right)\mathrm{d}p \right.\\
& \left. - \int_0^1 u\left(\beta + (1+\theta)\int_0^1 \overline{Q}(t)\mathrm{d}\nu(t) - \overline{Q}(p)\right)\mathrm{d}p\right]\\
\geqslant & \int_0^1 u'\left(\beta + (1+\theta)\int_0^1 \overline{Q}(t)\mathrm{d}\nu(t) - \overline{Q}(p)\right)\\
& \times\left[\left((1+\theta)\int_0^1 Q(t)\mathrm{d}\nu(t) - Q(p)\right) - \left((1+\theta)\int_0^1 \overline{Q}(t)\mathrm{d}\nu(t) - \overline{Q}(p)\right)\right]\mathrm{d}p,
\end{aligned}$$

so (4.1) holds.

On the other hand, suppose $\overline{Q} \in \mathscr{Q}$ satisfies (4.1). Because u is concave, we have the elementary inequality $u(y) - u(x) \leqslant u'(x)(y-x)$ for any $x,\ y \in \mathbb{R}$. So

$$\begin{aligned}
& u\left(\beta + (1+\theta)\int_0^1 Q(t)\mathrm{d}\nu(t) - Q(p)\right) - u\left(\beta + (1+\theta)\int_0^1 \overline{Q}(t)\mathrm{d}\nu(t) - \overline{Q}(p)\right)\\
\leqslant & u'\left(\beta + (1+\theta)\int_0^1 \overline{Q}(t)\mathrm{d}\nu(t) - \overline{Q}(p)\right)\\
& \times\left[\left((1+\theta)\int_0^1 Q(t)\mathrm{d}\nu(t) - Q(p)\right) - \left((1+\theta)\int_0^1 \overline{Q}(t)\mathrm{d}\nu(t) - \overline{Q}(p)\right)\right]
\end{aligned}$$

for any $Q \in \mathscr{Q}$. Integrating both sides, it follows

$$\begin{aligned}
& \int_0^1 \left[u\left(\beta + (1+\theta)\int_0^1 Q(t)\mathrm{d}\nu(t) - Q(p)\right) - u\left(\beta + (1+\theta)\int_0^1 \overline{Q}(t)\mathrm{d}\nu(t) - \overline{Q}(p)\right)\right]\mathrm{d}p\\
\leqslant & \int_0^1 u'\left(\beta + (1+\theta)\int_0^1 \overline{Q}(t)\mathrm{d}\nu(t) - \overline{Q}(p)\right)\\
& \times\left[\left((1+\theta)\int_0^1 Q(t)\mathrm{d}\nu(t) - Q(p)\right) - \left((1+\theta)\int_0^1 \overline{Q}(t)\mathrm{d}\nu(t) - \overline{Q}(p)\right)\right]\mathrm{d}p.
\end{aligned}$$

The right hand side is non-positive by (4.1), so is the left hand side. Therefore, $\overline{Q}$ is an optimal solution to the problem (3.9). □

Proof of Lemma 4.2 The "if " part follows from the following facts.

- If $d < 0$, then $a = c$. Thanks to $c \geqslant b$,

$$\min\{\max\{a-c,\ d\},\ a-b\} = \min\{\max\{0,\ d\},\ c-b\} = \min\{0,\ c-b\} = 0.$$

- If $d = 0$, then $a \in [b, c]$, so

$$\min\{\max\{a-c,\ d\},\ a-b\} = \min\{\max\{a-c,\ 0\},\ a-b\} = \min\{0,\ a-b\} = 0.$$

- If $d > 0$, then $a = b$ and $\max\{a-c,\ d\} \geqslant d > 0$, so

$$\min\{\max\{a-c,\ d\},\ a-b\} = \min\{\max\{a-c,\ d\},\ 0\} = 0.$$

We next show the "only if " part. Notice it always holds that $\min\{\max\{x,\ y\},\ x\} = x$ for any real numbers x and y. So, thanks to $c \geqslant b$, we have

$$0 = \min\{\max\{a-c,\ d\},\ a-b\} \geqslant \min\{\max\{a-c,\ d\},\ a-c\} = a-c,$$

and

$$0 = \min\{\max\{a-c,\ d\},\ a-b\} \leqslant \min\{\max\{a-b,\ d\},\ a-b\} = a-b.$$

Therefore, $a \in [b, c]$. If $d < 0$ and $a \neq c$, then $a < c$, so

$$0 = \min\{\max\{a-c,\ d\},\ a-b\} \leqslant \max\{a-c,\ d\} < 0,$$

a contradiction. Hence $a = c$ if $d < 0$. Similarly we can prove $a = b$ if $d > 0$. □

Proof of Lemma 4.4 Suppose $\min\{\max\{a, b\},\ c\} = 0$ and $k, \ell, m \geqslant 0$. Then clearly $c \geqslant 0$.

- If $c > 0$, then $\max\{a, b\} = 0$, so $a \leqslant 0$, $b \leqslant 0$ and $ab = 0$. Hence $ak \leqslant 0$, $b\ell \leqslant 0$ and $akb\ell = 0$, which imply that $\max\{ak, b\ell\} = 0$. Thus, $\min\{\max\{ak, b\ell\}, cm\} = \min\{0, cm\} = 0$ as $cm \geqslant 0$.

- If $c = 0$, then $\max\{a, b\} \geqslant 0$, so $a \geqslant 0$ or $b \geqslant 0$. Hence $ak \geqslant 0$ or $b\ell \geqslant 0$, it follows that $\max\{ak, b\ell\} \geqslant 0$. Thus, $\min\{\max\{ak, b\ell\}, cm\} = \min\{\max\{ak, b\ell\}, 0\} = 0$.

The reverse assertion follows trivially by setting $k = \ell = m = 1$. □

Proof of Theorem 4.5

(1) Since $\overline{Q}$ is absolutely continuous, we have $\Phi \in C^{2-}([0,1])$. Moreover,

$$\overline{Q}(p) = \beta + (1+\theta)\int_0^1 \overline{Q}(t)\mathrm{d}\nu(t) - (u')^{-1}\big(\Phi'(p)\big), \tag{A.1}$$

and thus

$$\overline{Q}'(p) = \frac{\Phi''(p)}{-u''\big((u')^{-1}\big(\Phi'(p)\big)\big)}, \quad \text{a.e. } p \in [0,1].$$

Thanks to Lemma 4.3, $\overline{Q}$ satisfies (4.5), so

$$\min\left\{\max\left\{\frac{\Phi''(p)}{-u''\big((u')^{-1}\big(\Phi'(p)\big)\big)} - \hbar(p),\ \Phi(0)(1+\theta)(1-\nu(p)) - \Phi(p)\right\},\right.$$
$$\left.\frac{\Phi''(p)}{-u''\big((u')^{-1}\big(\Phi'(p)\big)\big)}\right\} = 0, \quad \text{a.e. } p \in [0,1].$$

By virtue of Lemma 4.4, the above equation is equivalent to the ODE in (4.7). We now show the last equation in (4.7). Multiplying $\nu'(p)$ on both sides in (A.1) and then integrating on $[0,1]$, we get

$$\int_0^1 \overline{Q}(p)\mathrm{d}\nu(p) = \beta + (1+\theta)\int_0^1 \overline{Q}(t)\mathrm{d}\nu(t) - \int_0^1 (u')^{-1}\big(\Phi'(p)\big)\mathrm{d}\nu(p),$$

so

$$\beta = \int_0^1 (u')^{-1}\big(\Phi'(p)\big)\mathrm{d}\nu(p) - \theta\int_0^1 \overline{Q}(t)\mathrm{d}\nu(t).$$

If $p=0$, (A.1) becomes

$$0 = \beta + (1+\theta)\int_0^1 \overline{Q}(t)\mathrm{d}\nu(t) - (u')^{-1}\big(\Phi'(0)\big).$$

Canceling the terms $\int_0^1 \overline{Q}(t)\mathrm{d}\nu(t)$ from the above two equations, we obtain the last equation in (4.7). It is left to show that Φ is a good function. Because $\overline{Q}$ is non-decreasing, by definition Φ is convex and $\Phi' > 0$. Suppose $q_0 = w(m_0)$. Since $\overline{Q} = 0$ on $[0,q_0]$, by definition Φ' is a constant on $[0,q_0]$. For $p \in (q_0,1)$ such that $\Phi(0)(1+\theta)(1-m_0) - \Phi(p) > 0$, we have $\Phi(0)(1+\theta)(1-\nu(p)) \geqslant \Phi(0)(1+\theta)(1-m_0) > \Phi(p)$, which together with (4.7) implies $\Phi''(p) = 0$ a.e. Since Φ' is continuous, we conclude Φ' is a constant on the set $\{p \in [0,1] : \Phi(0)(1+\theta)(1-m_0) - \Phi(p) \geqslant 0\}$. Therefore, Φ is a good function.

(2) Now suppose $\Phi \in C^{2-}([0,1])$ is a solution to (4.7). We set

$$\overline{Q}(p) = (u')^{-1}\big(\Phi'(0)\big) - (u')^{-1}\big(\Phi'(p)\big).$$

Then $\overline{Q}$ is absolutely continuous, $\overline{Q}(0) = 0$ and

$$\overline{Q}'(p) = \frac{\Phi''(p)}{-u''\big((u')^{-1}\big(\Phi'(p)\big)\big)}, \quad \text{a.e. } p \in [0,1].$$

By virtue of Lemma 4.3, we can rewrite the ODE in (4.7) as

$$\min\left\{\max\left\{\frac{\Phi''(p)}{-u''\big((u')^{-1}\big(\Phi'(p)\big)\big)} - \hbar(p),\ \Phi(0)(1+\theta)(1-\nu(p)) - \Phi(p)\right\},\right.$$
$$\left.\frac{\Phi''(p)}{-u''\big((u')^{-1}\big(\Phi'(p)\big)\big)}\right\} = 0, \quad \text{a.e. } p \in [0,1],$$

that is,

$$\min\left\{\max\left\{\overline{Q}'(p) - \hbar(p),\ \Phi(0)(1+\theta)(1-\nu(p)) - \Phi(p)\right\},\ \overline{Q}'(p)\right\} = 0, \quad \text{a.e. } p \in [0,1].$$

In other words, $\overline{Q}$ satisfies the ODE in (4.5). We now show (4.6). The last equation in (4.7) implies

$$(u')^{-1}\big(\Phi'(0)\big) = \beta + (1+\theta)\int_0^1 \Big[(u')^{-1}\big(\Phi'(0)\big) - (u')^{-1}\big(\Phi'(t)\big)\Big]\mathrm{d}\nu(t),$$

whose right hand side by definition is equal to

$$\beta + (1+\theta)\int_0^1 \overline{Q}(t)\mathrm{d}\nu(t).$$

Hence

$$\begin{aligned}\overline{Q}(p) &= (u')^{-1}\big(\Phi'(0)\big) - (u')^{-1}\big(\Phi'(p)\big)\\ &= \beta + (1+\theta)\int_0^1 \overline{Q}(t)\mathrm{d}\nu(t) - (u')^{-1}\big(\Phi'(p)\big),\end{aligned}$$

which implies

$$\Phi'(p) = u'\left(\beta + (1+\theta)\int_0^1 \overline{Q}(t)\mathrm{d}\nu(t) - \overline{Q}(p)\right).$$

Thanks to the boundary condition $\Phi(1) = 0$ in (4.7), the above equation implies (4.6). By Lemma 4.3, we see $\overline{Q}$ is the optimal solution to the problem (3.9). Consequently, by Lemma 3.2, $\overline{R}$ is the optimal solution to the problem (2.6).

The above argument also shows that any solution Φ in $C^{2-}([0,1])$ to (4.7) should satisfy (4.6). Since the optimal solution $\overline{Q}$ to the problem (3.9) is unique, we conclude that (4.7) has exactly one solution in $C^{2-}([0,1])$. By the first assertion, the solution is a good function. ☐

Proof of Proposition 5.1 Suppose $w(p) \equiv p$. Then (4.12) becomes

$$\begin{cases}\min\Big\{\max\big\{\Phi''(p) + h(p)u''\big((u')^{-1}\big(\Phi'(p)\big)\big),\ \sigma(1+\theta)(1-p) - \Phi(p)\big\}, \Phi''(p)\Big\} = 0,\\ \qquad \text{a.e. } p \in [0,1],\\ \Phi(0) = \sigma < 0, \quad \Phi(1) = 0.\end{cases} \tag{A.2}$$

Since Φ is continuous and convex, $f(p) = \sigma(1+\theta)(1-p) - \Phi(p)$ is a continuous concave function on $[0,1]$ with boundary values $f(0) = \sigma\theta < 0$ and $f(1) = 0$. Let

$$p_0 = \sup\{p \in [0,1] \colon f(p) < 0\}.$$

Then $0 < p_0 \leqslant 1$, $f < 0$ on $[0, p_0)$, $f(p_0) = 0$ and $f \geqslant 0$ on $[p_0, 1]$. By (A.2), we obtain $\Phi''(p) + h(p)u''\big((u')^{-1}\big(\Phi'(p)\big)\big) = 0$ for a.e. $p \in [0, p_0)$. As f is concave, there are two possible cases on $[p_0, 1]$:

(1) Either $f = 0$ on $[p_0, 1]$, which trivially implies $\Phi''(p) = (\sigma(1+\theta)(1-p) - f(p))'' = 0$ for $p \in (p_0, 1)$.

(2) Or $f > 0$ on $(p_0, 1)$, which by (A.2) also implies that $\Phi''(p) = 0$ for a.e. $p \in (p_0, 1)$.

Therefore, in both cases, we have

$$\overline{Q}'(p) = \frac{\Phi''(p)}{-u''\big((u')^{-1}\big(\Phi'(p)\big)\big)} = \begin{cases} h(p), & \text{a.e. } p \in [0, p_0);\\ 0, & \text{a.e. } p \in [p_0, 1].\end{cases}$$

By virtue of $\overline{Q}(0) = 0$ and (3.3), we obtain

$$\overline{Q}(p) = \int_0^{\min\{p,\ p_0\}} h(t)\mathrm{d}t = \min\big\{F_X^{-1}(p),\ F_X^{-1}(p_0)\big\}.$$

Thanks to (3.10), we see the corresponding compensation

$$\overline{I}(x) \equiv x - \overline{R}(x) \equiv x - \overline{Q}\big(F_X(x)\big) \equiv x - \min\{x,\ d\} = \max\{x - d,\ 0\}$$

is of deductible type, where $d = F_X^{-1}(p_0)$. In particular the compensation becomes the full coverage compensation, i.e., $d = 0$, if $p_0 \leqslant m_0$. ☐

Proof of Theorem 5.2 By (3.10), $R(x)=\min\{x,\ d\}$ is the optimal solution to the problem (2.6) if and only if

$$\overline{Q}(p):=\min\left\{F_X^{-1}(\nu(p)),\ d\right\}$$

is the optimal solution to the problem (3.9), which by Theorem 4.5 is also equivalent to that

$$\overline{\Phi}(p):=-\int_p^1 u'\left(\beta+(1+\theta)\int_0^1\overline{Q}(t)\mathrm{d}\nu(t)-\overline{Q}(s)\right)\mathrm{d}s$$

is a solution to (4.7) in $C^{2-}([0,1])$. Thanks to (3.8), we have

$$\frac{\overline{\Phi}''(p)}{-u''\big((u')^{-1}\big(\overline{\Phi}'(p)\big)\big)}=\overline{Q}'(p)=\begin{cases}\hbar(p), & \text{a.e. } p\in[0,w(F_X(d))];\\ 0, & p\in(w(F_X(d)),1].\end{cases}$$

By virtue of this, one can check $\overline{\Phi}$ is a solution to (4.7) holds true if and only if

$$\begin{cases}\overline{\Phi}(0)(1+\theta)(1-\nu(p))-\overline{\Phi}(p)\leqslant 0, & p\in[0,w(F_X(d))];\\ \overline{\Phi}(0)(1+\theta)(1-\nu(p))-\overline{\Phi}(p)\geqslant 0, & p\in[w(F_X(d)),1].\end{cases}$$

But this is equivalent to (5.1) because

$$\begin{aligned}\overline{\Phi}(p)&=-\int_p^1 u'\left(\beta+(1+\theta)\int_0^1\min\left\{F_X^{-1}(\nu(t)),\ d\right\}\mathrm{d}\nu(t)-\min\left\{F_X^{-1}(\nu(s)),\ d\right\}\right)\mathrm{d}s\\ &=-\int_p^1 u'\left(\beta+(1+\theta)\int_0^1\min\left\{F_X^{-1}(t),\ d\right\}\mathrm{d}t-\min\left\{F_X^{-1}(\nu(s)),\ d\right\}\right)\mathrm{d}s\\ &=-\int_p^1 u'\left(\beta+(1+\theta)\mathbb{E}[\min\{X,d\}]-\min\left\{F_X^{-1}(\nu(s)),\ d\right\}\right)\mathrm{d}s\\ &=-\int_p^1 u'(g(s))\mathrm{d}s.\end{aligned}$$

If $p=w(F_X(d))$, the condition (5.1) implies

$$(1+\theta)(1-F_X(d))=\frac{\int_{w(F_X(d))}^1 u'(g(s))\mathrm{d}s}{\int_0^1 u'(g(s))\mathrm{d}s}.$$

The right hand side is clearly no more than 1, so $F_X(d)\geqslant\frac{\theta}{1+\theta}$. ☐

Proof of Theorem 5.4 By Corollary 4.6, we only need to show that, for each good function Φ, there exist infinite pairs (u,w) such that Φ is a solution to (4.12), where $\sigma=\Phi(0)$, $\hbar(p)=\big(F_X^{-1}(\nu(p))\big)'$ and ν is the inverse function of w.

Define

$$q_0=\inf\{p\in[0,1]:\Phi(0)(1+\theta)(1-m_0)-\Phi(p)<0\}.$$

Since $\Phi(0)<\Phi(1)=0$ and $0<(1+\theta)(1-m_0)<1$, by continuity, we have $0<q_0<1$ and $\Phi(0)(1+\theta)(1-m_0)=\Phi(q_0)$. Set any feasible ν on $[0,q_0]$ such that $\nu(q_0)=m_0$. By the definition of good function, we see Φ is linear on $[0,q_0]$. Since $\nu(p)\leqslant\nu(q_0)=m_0$ for $p\in[0,q_0)$, $F_X^{-1}(\nu(p))=0$. Thus $\hbar(p)=\big(F_X^{-1}(\nu(p))\big)'=0$ and $\Phi''(p)=0$ for $p\in[0,q_0)$. By virtue of this,

$$\begin{aligned}&\min\Big\{\max\left\{\Phi''(p)+\hbar(p)u''\big((u')^{-1}\big(\Phi'(p)\big)\big),\ \Phi(0)(1+\theta)(1-\nu(p))-\Phi(p)\right\},\ \Phi''(p)\Big\}\\ &=\min\Big\{\max\left\{0,\ \Phi(0)(1+\theta)(1-\nu(p))-\Phi(p)\right\},\ 0\Big\}=0,\quad\text{a.e. } p\in[0,q_0].\end{aligned}$$

So the ODE in (4.12) is satisfied for a.e. $p \in [0, q_0]$.

We now consider the problem on $(q_0, 1]$. Let

$$\nu(p) = 1 - \frac{\Phi(p)}{\Phi(0)(1+\theta)}, \quad p \in (q_0, 1].$$

Since $\Phi' > 0$ and $\Phi(0) < 0$, ν is an absolutely continuous and strictly increasing function. Moreover,

$$\nu(1) = 1, \quad \nu(q_0+) = 1 - \frac{\Phi(q_0+)}{\Phi(0)(1+\theta)} = m_0 = \nu(q_0),$$

so $\nu \in \mathscr{W}$. Because $\Phi'(p)$ is continuous and non-decreasing and $F_X^{-1}(\nu(p))$ is strictly increasing on $[q_0, 1]$, there exists a utility function u which is in $C^2([0,\infty))$ with $u' > 0$ and $u'' < 0$ such that

$$\text{the function } p \mapsto (u')^{-1}(\Phi'(p)) + F_X^{-1}(\nu(p)) \text{ is strictly increasing on } [q_0, 1]. \tag{A.3}$$

It then follows

$$\frac{\Phi''(p)}{-u''\big((u')^{-1}\big(\Phi'(p)\big)\big)} - \big(F_X^{-1}(\nu(p))\big)' = \big(-(u')^{-1}(\Phi'(p)) - F_X^{-1}(\nu(p))\big)' \leqslant 0, \quad \text{a.e. } p \in (q_0, 1].$$

Notice $\Phi(0)(1+\theta)(1-\nu(p)) = \Phi(p)$ for $p \in (q_0, 1]$, so

$$\max\left\{\frac{\Phi''(p)}{-u''\big((u')^{-1}\big(\Phi'(p)\big)\big)} - \big(F_X^{-1}(\nu(p))\big)',\ \Phi(0)(1+\theta)(1-\nu(p)) - \Phi(p)\right\} = 0, \quad \text{a.e. } p \in (q_0, 1].$$

Because $\Phi'' \geqslant 0$, the above equation implies the ODE in (4.12) is satisfied on $(q_0, 1]$. Since there are infinite many choices of u that satisfies (A.3), the claim is proved. □

Proof of Theorem 6.1 The argument remains unchanged until (4.3). For any $Q \in \mathscr{Q}$, since there is no upper bound limitation for R' in (6.1), so is $Q'(p)$ for $p \in (w(m_0), 1]$. Recall $Q = 0$ on $[0, w(m_0)]$, so the optimality condition (4.3) implies $\Phi(p) \leqslant \Phi(0)(1+\theta)(1-\nu(p))$ and

$$\begin{cases} \overline{Q}'(p) \in [0,\infty), & \text{if } \Phi(p) = \Phi(0)(1+\theta)(1-\nu(p)), \\ \overline{Q}'(p) = 0, & \text{if } \Phi(p) < \Phi(0)(1+\theta)(1-\nu(p)), \end{cases} \quad \text{for a.e. } p \in (w(m_0), 1],$$

which can be equivalently expressed as

$$\min\Big\{\Phi(0)(1+\theta)(1-\nu(p)) - \Phi(p),\ \overline{Q}'(p)\Big\} = 0, \quad \text{a.e. } p \in (w(m_0), 1].$$

This leads to the following analog of (4.7),

$$\begin{cases} \Phi''(p) = 0, \quad \text{a.e. } p \in [0, w(m_0)], \\ \min\Big\{\Phi(0)(1+\theta)(1-\nu(p)) - \Phi(p), \Phi''(p)\Big\} = 0, \quad \text{a.e. } p \in (w(m_0), 1], \\ \Phi(1) = 0, \quad \beta = (1+\theta)\int_0^1 (u')^{-1}\big(\Phi'(t)\big)\mathrm{d}\nu(t) - (u')^{-1}\big(\Phi'(0)\big)\theta. \end{cases}$$

By virtue of Lemma 4.4 and $\Phi(0) < 0$, the ODE in above can be rewritten as

$$\begin{cases} \left(\dfrac{\Phi(p)}{-\Phi(0)}\right)'' = 0, \quad \text{a.e. } p \in [0, w(m_0)], \\ \min\left\{(1+\theta)(\nu(p)-1) - \dfrac{\Phi(p)}{-\Phi(0)},\ \left(\dfrac{\Phi(p)}{-\Phi(0)}\right)''\right\} = 0, \quad \text{a.e. } p \in (w(m_0), 1], \\ \dfrac{\Phi(1)}{-\Phi(0)} = 0. \end{cases}$$

It is easy to check $\psi \in C^{2-}([0,1])$ and ψ also satisfies the above ODE in place of $\frac{\Phi(p)}{-\Phi(0)}$. So $\frac{\Phi(p)}{-\Phi(0)}$ is identical to ψ on $[0,1]$. In another words, $\Phi = \kappa\psi$ with $\kappa = -\Phi(0) > 0$. □

Proof of Theorem 6.2 In this case, the requirement $R' \leqslant 1$ in (6.3) leads to $Q'(p) \leqslant \hbar(p)$ for $p \in (w(m_0), 1]$. Consequently, the optimality condition (4.3) implies $\Phi(p) \geqslant \Phi(0)(1+\theta)(1-\nu(p))$ and

$$\overline{Q}'(p) = \begin{cases} \hbar(p), & \text{if } \Phi(p) > \Phi(0)(1+\theta)(1-\nu(p)), \\ (-\infty, \hbar(p)], & \text{if } \Phi(p) = \Phi(0)(1+\theta)(1-\nu(p)), \end{cases} \quad \text{for a.e. } p \in (w(m_0), 1],$$

which is clearly equivalent to

$$\max\left\{\overline{Q}'(p) - \hbar(p), \Phi(0)(1+\theta)(1-\nu(p)) - \Phi(p)\right\} = 0, \quad \text{a.e. } p \in (w(m_0), 1].$$

This leads to the ODE (6.5) for Φ. □

Proof of Corollary 6.3 By Remark 6.2, it suffices to show the solution Φ to (6.5) is convex. Suppose on the contrary, the region $\{p \in (w(m_0), 1) : \Phi''(p) < 0\}$ was not empty. In this region, we would have $\Phi''(p) + \hbar(p)u''\big((u')^{-1}\big(\Phi'(p)\big)\big) < 0$; and consequently, $\Phi(p) = \Phi(0)(1+\theta)(1-\nu(p))$ by (6.5). This would imply $-\Phi(0)(1+\theta)\nu''(p) = \Phi''(p) < 0$. As $\Phi(0) < 0$, we conclude $\nu''(p) < 0$ which contradicts the concavity of w. □

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