Probability, Uncertainty and Quantitative Risk
Vol. 10, No. 1, 2025, 31–58

https://www.aimsciences.org/PUQR
DOI:10.3934/puqr.2025003

Existence, uniqueness and comparison theorem on unbounded solutions of general time-interval BSDEs with sub-quadratic generators

Chuang Gu, Yan Wang, Shengjun Fan*

School of Mathematics, China University of Mining and Technology, Xuzhou 221116, *China*

Email: guchuang1026@163.com, wangyan_ shuxue@163.com, shengjunfan@cumt.edu.cn

Abstract This study addresses the existence, uniqueness, and comparison theorem for unbounded solutions of one-dimensional backward stochastic differential equations (BSDEs) with sub-quadratic generators, considering both finite and infinite terminal times. Initially, we establish the existence of unbounded solutions for BSDEs where the generator g satisfies a time-varying one-sided linear growth condition in the first unknown variable y and a time-varying sub-quadratic growth condition in the second unknown variable z. Next, the uniqueness and comparison theorems for unbounded solutions are proven under a time-varying extended convexity assumption. These findings extend the results in [12] to the general time-interval BSDEs. Finally, we propose and verify several sufficient conditions for ensuring uniqueness, utilizing innovative approaches applied for the first time, even in the context of finite time-interval BSDEs.

Keywords Existence and uniqueness, Unbounded solutions, Backward stochastic differential equation, Comparison theorem, General time-interval, Sub-quadratic growth

2020 Mathematics Subject Classification 60H10, 60K35

1. Introduction

Throughout this study, let d denote a positive integer, and $(B_t)_{t\geqslant 0}$ represent a standard d-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The filtration $(\mathcal{F}_t)_{t\geqslant 0}$ represents the completed natural σ-algebra generated by $(B_t)_{t\geqslant 0}$, with $\mathcal{F}_T := \mathcal{F}$, where the terminal time T is either a positive real number or $T = +\infty$. Let $|x|$ denote the Euclidian norm for $x \in \mathbb{R}^d$ and $x \cdot y$ represent the standard scalar inner product for $x, y \in \mathbb{R}^d$. We consider the following one-dimensional backward stochastic differential equation (BSDE for short):

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \cdot \mathrm{d}B_s, \quad t \in [0, T], \tag{1.1}$$

where the terminal value ξ is an $\mathcal{F}_T$-measurable random variable, and the generator

Received 4 July 2024; Accepted 27 November 2024; Early access 24 January 2025
*Corresponding author

$g(\omega,t,y,z):\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^d\mapsto\mathbb{R}$ is an $(\mathcal{F}_t)$-progressively measurable stochastic process for each (y,z) and it is almost everywhere continuous in (y,z). The triple (ξ,T,g) is referred to as the parameters of BSDE (1.1). A solution to BSDE (1.1) is a pair of $(\mathcal{F}_t)$-progressively measurable processes $(Y_t,Z_t)_{t\in[0,T]}$ that take values in $\mathbb{R}\times\mathbb{R}^d$ and satisfy (1.1). For brevity, BSDE (1.1) is often denoted as BSDE (ξ,g). General nonlinear BSDEs were first studied in 1990 by Pardoux and Peng [21], who established the existence and uniqueness of L^2 solutions for multidimensional BSDEs with square-integrable parameters and uniformly Lipschitz continuous generators. Since then, BSDEs have garnered substantial attention from numerous researchers and have evolved into essential tools in various fields, including partial differential equations, mathematical finance, stochastic control, and nonlinear mathematical expectation. For more details, interested readers are referred to [5, 8, 9, 12, 17–19, 22–24].

In this study, we focus on BSDE (1.1) with generator g satisfying the following condition:

$$\mathrm{d}\mathbb{P}\times\mathrm{d}t\text{-}a.e.,\quad |g(\omega,t,y,z)|\leqslant f_t(\omega)+\beta(t)|y|+\gamma(t)|z|^{\alpha},\quad\forall\,(y,z)\in\mathbb{R}\times\mathbb{R}^d,\tag{1.2}$$

where $(f_t)_{t\in[0,T]}$ is a nonnegative $(\mathcal{F}_t)$-progressively measurable process, $\beta(\cdot)$ and $\gamma(\cdot)$ are two nonnegative deterministic functions depending solely on the time variable t. Additionally, let $\alpha\in(1,2)$. It is well established that L^2 solutions, $L^p\ (p>1)$ solutions, L^1 solutions, and bounded solutions for general time-interval BSDEs have been extensively studied. For instance, when the generator g satisfies condition (1.2) with $\alpha=1$, Chen and Wang [4] were the first to propose and analyze the existence and uniqueness of L^2 solutions for BSDEs over general time-interval. Subsequently, Fan [10], Fan and Jiang [14], Zong and Hu [26, 27] established the existence and uniqueness of $L^p\ (p>1)$ solutions for such BSDEs. Moreover, Fan [10, 11] addressed the existence and uniqueness of L^1 solutions under the assumption that the generator g satisfies condition (1.2) with $\alpha\in(0,1)$. For bounded solutions of general time-interval BSDEs with a quadratic generator g satisfying (1.2) with $\alpha=2$, see Fan [10], Luo and Fan [20], and Kobylanski [19] for finite time-interval cases. Notably, Bahlali et al. [1] established an existence and uniqueness result for L^2 solutions of finite time-interval quadratic BSDEs, where the generator g takes the form $f(y)|z|^2$. In this case, f must be globally integrable on $\mathbb{R}$, as f being a positive constant alone is insufficient to ensure the existence of a solution. The existence and uniqueness of the $L^p\ (p>1)$ solutions for such quadratic BSDEs was further investigated by Yang [25], under the condition that f is globally integrable on $\mathbb{R}$ and bounded on any compact subset of $\mathbb{R}$.

On the other hand, Briand and Hu [2] introduced a localization procedure to establish the existence of unbounded solutions for finite time-interval quadratic BSDEs with terminal conditions $(\xi,f.)$ satisfying only a certain exponential moment condition. This represents the weakest integrability requirement necessary to ensure the existence of unbounded solutions. The uniqueness of unbounded solutions was later proven by Briand and Hu [3] for finite time-interval quadratic BSDEs with convex generators and unbounded terminal conditions $(\xi,f.)$ satisfying all exponential moment conditions, utilizing the θ-difference technique. Further studies on unbounded solutions for finite time-interval quadratic BSDEs can be found in Delbaen et al. [6, 7] and Fan et al. [13]. Recently, using the localization procedure, Fan and Hu [12] explored the existence of unbounded solutions for a finite time-interval BSDEs with terminal conditions $(\xi,f.)$ satisfying all sub-exponential moment conditions. This was the assumption that the generator g satisfies (1.2) with $\alpha\in(1,2)$, and that both $\beta(\cdot)$ and $\gamma(\cdot)$ are nonnegative constants. They also applied the θ-difference technique to establish the uniqueness of unbounded solutions under an extended convexity condition for the sub-quadratic generator g, which is typically satisfied by a local Lipschitz perturbation of a proper convex function. Notably, all the studies discussed in this context pertain to finite time-interval BSDEs. This raises two questions: do these results hold for general time-interval BSDEs, and can the extended convexity condition be further weakened? The present study aims to address whether the results of [12] can be extended to general time-interval BSDEs.

This study focuses on the existence, uniqueness, and comparison theorem for unbounded solutions of general time-interval BSDEs with sub-quadratic generators g satisfying (1.2) with $\alpha \in (1,2)$ and terminal conditions $(\xi, f.)$ satisfying all sub-exponential moment conditions. Our results generalize the corresponding ones of Fan and Hu [12] to the infinite time-interval case. Firstly, to establish the existence of unbounded solutions for general time-interval BSDEs, we introduce two new time-varying assumptions, (EX1) and (EX2), on the generator g (see the beginning of Section 3 for details). These assumptions generalize the assumption (H1$''$) of Fan and Hu [12] to the infinite time-interval case. We would emphasis that the deterministic functions $\beta(\cdot)$ and $\gamma(\cdot)$ in (EX1) and (EX2) must satisfy $\beta(\cdot) \in L^1([0,T];\mathbb{R}_+)$ and $\gamma(\cdot) \in L^{\frac{2}{2-\alpha}}([0,T];\mathbb{R}_+)$, which appear to be the most appropriate integrability conditions for ensuring the existence of unbounded solutions. When $\alpha = 1$, $\frac{2}{2-\alpha}$ equals 2, this is just the case in Chen and Wang [4], Fan [10], and Zong and Hu [26, 27]. Our proof begins by establishing a uniform a priori estimate for the first process $Y.$ in any bounded solution $(Y., Z.)$ of BSDE (ξ, g). This is achieved by constructing a suitable test function $\psi(s,x)$ and applying Itô-Tanaka's formula to $\psi(s, \hat{Y}_s)$, where $\hat{Y}.$ is defined in (3.5). We then verify the existence of unbounded solutions for BSDE (ξ, g) using the localization procedure proposed by Briand and Hu [2] (see the first two steps of the proof of Theorem 3.2 in Section 3 for details). Secondly, to establish a comparison theorem for unbounded solutions of general time-interval BSDEs, we introduce a new time-varying extended convexity assumption (UN) (see the beginning of Section 4.1 for details). Due to the presence of the term $\gamma(t)\,[\ln(e+|z_2|)]^{\frac{\alpha^*}{2}}$, assumption (UN) not only extends assumption (H2') of Fan and Hu [12] to the infinite time-interval case but is also strictly weaker, even for finite time-interval cases. The main idea of the proof involves applying the θ-difference technique proposed by Briand and Hu [3]. However, the weaker assumption (UN) introduces a new challenge, which we overcome (see the first step of the proof of Theorem 4.2 in Section 4 for details). Finally, we propose several weaker sufficient conditions under which the time-varying extended convexity assumption (UN) holds. These conditions allow for a broader class of perturbations of proper convex functions than those considered in Fan and Hu [12] (see Propositions 5.4 and 5.5, as well as Examples 5.7 and 5.8 for details). Some of these conditions are explored here for the first time, even for finite time-interval BSDEs.

The remainder of this paper is organized as follows. Section 2 introduces the necessary notations and a key lemma that will be used in subsequent sections. Section 3 presents and proves an existence result for unbounded solutions of general time-interval BSDEs with sub-quadratic generators. The main result, stated in Theorem 3.2, extends Theorem 2.2 of Fan and Hu [12] to general time-interval BSDEs. In Section 4, we establish a comparison theorem for unbounded solutions of general time-interval BSDEs with sub-quadratic generators. This result, detailed in Theorem 4.2, serves as an extension of Theorem 3.2 in Fan and Hu [12]. Additionally, Theorem 4.3 provides an existence and uniqueness result for unbounded solutions of such BSDEs, generalizing Theorem 3.9 of Fan and Hu [12] to the infinite time-interval case. Section 5 introduces and proves three sufficient conditions for ensuring uniqueness (see Propositions 5.1, 5.4 and 5.5). Remarks and examples are included in Sections 3, 4, and 5 to illustrate the results. Notable examples include Remarks 3.1, 3.3 and 3.4 in Section 3, Remark 4.1 in Section 4, and Remarks 5.2, 5.3, 5.6, as well as Examples 5.7 and 5.8 in Section 5. Finally, three technical lemmas used to prove Propositions 5.4 and 5.5 are provided in the Appendix. These lemmas are noteworthy and hold independent interest.

2. Preliminaries

In this section, we introduce the notations and a key lemma that will be used later. Let $a \vee b$ denote the maximum of two real numbers a and b, $a^+ := a \vee 0$, and $a^- := (-a)^+$. For $\alpha \in (1,2)$, let α^* represent the conjugate of α, defined such that $1/\alpha + 1/\alpha^* = 1$ or

$$\alpha^* := \frac{\alpha}{\alpha - 1} > 2.$$

Denote $\mathbb{R}_+ := [0, +\infty)$, $\mathbb{R}_- := (-\infty, 0]$, $\mathbf{1}_A = 1$ if $x \in A$ and 0 otherwise, and $\operatorname{sgn}(x) := \mathbf{1}_{x>0} - \mathbf{1}_{x \leqslant 0}$. Recall that an $(\mathcal{F}_t)$-progressively measurable scalar process $(X_t)_{t\in[0,T]}$ is said to belong to class (D) if the family of random variables $\{X_\tau\}$, where τ ranges over all $(\mathcal{F}_t)$-stopping times valued in $[0,T]$, is uniformly integrable. For any real $p \geqslant 1$, we define the following spaces:

- $L^\infty(\Omega, \mathcal{F}_T, \mathbb{P})$ denotes the set of all $\mathcal{F}_T$-measurable bounded random variables.
- $\mathcal{S}^\infty(0,T;\mathbb{R})$ denotes the set of all $(\mathcal{F}_t)$-progressively measurable, continuous, and bounded processes.
- $\mathcal{M}^p(0,T;\mathbb{R}^d)$ denotes the set of all $(\mathcal{F}_t)$-progressively measurable $\mathbb{R}^d$-valued processes $(Z_t)_{t\in[0,T]}$ such that

$$\| Z \|_{\mathcal{M}^p} := \left(\mathbb{E}\left[\left(\int_0^T |Z_t|^2 \mathrm{d}t \right)^{\frac{p}{2}} \right] \right)^{\frac{1}{p}} < +\infty.$$

- $L^p([0,T];\mathbb{R}_+)$ denotes the set of all nonnegative deterministic functions $u(\cdot)$ such that

$$\int_0^T u^p(t)\mathrm{d}t < +\infty.$$

Let $\alpha \in (1,2)$, $\beta(\cdot) \in L^1([0,T];\mathbb{R}_+)$, $\gamma(\cdot) \in L^{\frac{2}{2-\alpha}}([0,T];\mathbb{R}_+)$, and $(f_t)_{t\in[0,T]}$ be a nonnegative $(\mathcal{F}_t)$-progressively measurable process with sub-exponential moments of any order, i.e.,

$$\mathbb{E}\left[\exp\left(p \left(\int_0^T f_s \mathrm{d}s \right)^{\frac{2}{\alpha^*}} \right) \right] < +\infty, \quad \forall\, p > 1.$$

The assumptions and definitions introduced above will be utilized throughout this study.

The following Lemma 2.1 can be viewed as a direct consequence of Lemma 3.4 and Theorems 3.2 and 3.3 in Fan [10]. This lemma will play a crucial role in proving our main results.

Lemma 2.1 *Assume that $\xi \in L^\infty(\Omega, \mathcal{F}_T, \mathbb{P})$, and the generator g satisfies the following two assumptions:*

(H1) $\mathrm{d}\mathbb{P} \times \mathrm{d}t$*-a.e.,* $g(\omega, t, \cdot, \cdot) : \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}$ *is continuous*;

(H2) There exists a function $u(t) \in L^1([0,T];\mathbb{R}_+)$ such that $\mathrm{d}\mathbb{P} \times \mathrm{d}t$-a.e., for each $(y,z) \in \mathbb{R} \times \mathbb{R}^d$,

$$|g(\omega, t, y, z)| \leqslant u(t).$$

Then, the BSDE (ξ, g) *admits a minimal bounded solution* $(Y_t, Z_t)_{t\in[0,T]}$ *in* $\mathcal{S}^\infty(0,T;\mathbb{R})\times\mathcal{M}^2(0,T;\mathbb{R}^d)$*.*

Furthermore, assume that $\xi, \xi' \in L^\infty(\Omega, \mathcal{F}_T, \mathbb{P})$*, and both generators* g *and* g' *satisfy (H1) and (H2). Let* $(Y_t, Z_t)_{t\in[0,T]}$ *and* $(Y'_t, Z'_t)_{t\in[0,T]}$ *be the minimal bounded solutions of the BSDEs* (ξ, g) *and* (ξ', g')*, respectively, in the space of* $\mathcal{S}^\infty(0,T;\mathbb{R})\times\mathcal{M}^2(0,T;\mathbb{R}^d)$*. If* $\mathbb{P}$-a.s.*,* $\xi \leqslant \xi'$*, and*

$$\mathrm{d}\mathbb{P}\times\mathrm{d}t\text{-}a.e., \quad g(t, Y'_t, Z'_t) \leqslant g'(t, Y'_t, Z'_t) \ \ (\text{resp. } g(t, Y_t, Z_t) \leqslant g'(t, Y_t, Z_t)),$$

then $\mathbb{P}$-a.s.*, for each* $t\in[0,T]$*,* $Y_t \leqslant Y'_t$*.*

3. Existence of the unbounded solution

Let us first present the following assumptions on the generator g, which can be compared to assumption (H1$''$) in Fan and Hu [12].

(EX1) g exhibits a time-varying one-sided linear growth in y and a time-varying sub-quadratic growth in z, i.e., $\mathrm{d}\mathbb{P}\times\mathrm{d}t$-$a.e.$, for each $(y,z)\in\mathbb{R}\times\mathbb{R}^d$,

$$\mathrm{sgn}(y) g(\omega, t, y, z) \leqslant f_t(\omega) + \beta(t)|y| + \gamma(t)|z|^\alpha.$$

(EX2) $\mathrm{d}\mathbb{P}\times\mathrm{d}t$-$a.e.$, $g(\omega, t, \cdot, \cdot): \mathbb{R}\times\mathbb{R}^d \mapsto \mathbb{R}$ is continuous, and g exhibits a time-varying general growth in y and a quadratic growth in z. Specifically, there exists a constant $c>0$ such that $\mathrm{d}\mathbb{P}\times\mathrm{d}t$-$a.e.$, for each $(y,z)\in\mathbb{R}\times\mathbb{R}^d$,

$$|g(\omega, t, y, z)| \leqslant f_t(\omega) + \beta(t)\psi(|y|) + c|z|^2,$$

where $\psi(\cdot)$ is a nonnegative, increasing, and continuous function that satisfies $\psi(0)=0$.

Remark 3.1 *Recall that the time-varying functions* $\beta(\cdot)$ *and* $\gamma(\cdot)$ *must satisfy the following integrability conditions:*

$$\beta(\cdot) \in L^1([0,T];\mathbb{R}_+) \quad \text{and} \quad \gamma(\cdot) \in L^{\frac{2}{2-\alpha}}([0,T];\mathbb{R}_+). \tag{3.1}$$

It is evident that (3.1) is trivially satisfied when $T<+\infty$ *and both* $\beta(\cdot)$ *and* $\gamma(\cdot)$ *are constants. Moreover, regardless of whether* $T<+\infty$ *or* $T=+\infty$*, the functions* $\beta(\cdot)$ *and* $\gamma(\cdot)$ *satisfying (3.1) can be unbounded. Consequently, assumptions (EX1) and (EX2) extend the scope of assumption (H1$''$) in Fan and Hu [12], where* $\beta(\cdot)$ *and* $\gamma(\cdot)$ *are constants and* $T<+\infty$*.*

Theorem 3.2 presents an existence theorem for the unbounded solution of a general time-interval BSDE under assumptions (EX1) and (EX2), which is the main result of this section.

Theorem 3.2 *Assume that* ξ *is a terminal value and* g *is a generator satisfying assumptions (EX1) and (EX2). If* $|\xi| + \int_0^T f_t \mathrm{d}t$ *has sub-exponential moments of any order, i.e.,*

$$\mathbb{E}\left[\exp\left(p\left(|\xi| + \int_0^T f_s \mathrm{d}s\right)^{\frac{2}{\alpha^*}}\right)\right] < +\infty, \quad \forall\, p>1, \tag{3.2}$$

then BSDE (ξ, g) *admits a solution* $(Y_t, Z_t)_{t\in[0,T]}$ *such that* $\mathbb{P}$-a.s.*,*

$$\exp\left(|Y_t|^{\frac{2}{\alpha^*}}\right) + \mathbb{E}_t\left[\int_t^T |Z_s|^2 \mathrm{d}s\right] \leqslant K\mathbb{E}_t\left[\exp\left(K\left(|\xi| + \int_t^T f_s \mathrm{d}s\right)^{\frac{2}{\alpha^*}}\right)\right], \quad t\in[0,T], \tag{3.3}$$

where $K>0$ *is a constant that depends only on* $(\alpha, T, \beta(\cdot), \gamma(\cdot))$*. Furthermore, for each* $p>1$*, there exists a constant* $K_p>0$ *that depends only on* $(\alpha, T, p, \beta(\cdot), \gamma(\cdot))$ *such that* $\mathbb{P}$-a.s.*,*

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(p\left(\sup_{s\in[t,T]}|Y_s|\right)^{\frac{2}{\alpha^*}}\right)\right]+\mathbb{E}\left[\left(\int_t^T|Z_s|^2\mathrm{d}s\right)^{\frac{p}{2}}\right]\\
&\leqslant K_p\mathbb{E}\left[\exp\left(K_p\left(|\xi|+\int_t^T f_s\mathrm{d}s\right)^{\frac{2}{\alpha^*}}\right)\right],\quad t\in[0,T].
\end{aligned}\tag{3.4}$$

Proof The proof is structured into the following two steps.

Step 1 We prove that inequalities (3.3) and (3.4) hold for any bounded solution $(Y_t,Z_t)_{t\in[0,T]}$ of the BSDE (ξ,g), provided that $|\xi|\leqslant\bar c$ and $|f_t|\leqslant\bar c e^{-t}$ hold for each $t\in[0,T]$ and some constant $\bar c>0$. The proof is primarily inspired by Lemma 2.7 in Fan et al. [16]. Let $t\in[0,T]$ be fixed. Define the test function

$$\psi(s,x):=\exp\left(\mu(s)x^{\frac{2}{\alpha^*}}\right),\quad (s,x)\in[t,T]\times[x_0,+\infty),$$

where $x_0>(\alpha^*/2)^{\alpha^*/2}$ and $\mu(\cdot):[t,T]\to\mathbb{R}_+$ is an undetermined deterministic function satisfying $\mu(\cdot)\geqslant 1$ and $\mu'(\cdot)>0$. Note that the larger the α^*, the smaller the value of $\psi(s,x)$. A straightforward computation gives that for each $(s,x)\in[t,T]\times[x_0,+\infty)$,

$$\begin{aligned}
\psi_s(s,x)&=\psi(s,x)\mu'(s)x^{\frac{2}{\alpha^*}}>0,\\
\psi_x(s,x)&=\psi(s,x)\mu(s)\frac{2}{\alpha^*}x^{\frac{2}{\alpha^*}-1}>0,\\
\psi_{xx}(s,x)&=\psi(s,x)\mu(s)\frac{2}{\alpha^*}x^{\frac{2}{\alpha^*}-2}\left[\frac{2}{\alpha^*}\mu(s)x^{\frac{2}{\alpha^*}}-\left(1-\frac{2}{\alpha^*}\right)\right]\\
&\geqslant\psi(s,x)\mu(s)\frac{2}{\alpha^*}x^{\frac{2}{\alpha^*}-2}\left[\frac{2}{\alpha^*}x_0^{\frac{2}{\alpha^*}}-1\right]>0.
\end{aligned}$$

For such a case, $x\mapsto\psi(s,x)$ is strictly convex on $[x_0,+\infty)$. Note that $\lim_{\alpha^*\to+\infty}(\alpha^*/2)^{\alpha^*/2}=+\infty$. When α^* is getting larger, x_0 is getting larger. Define $A(s):=\int_0^s\beta(r)\mathrm{d}r$ and

$$\hat Y_s:=e^{A(s)}|Y_s|+k+\int_t^s e^{A(r)}f_r\mathrm{d}r,\quad s\in[t,T],\tag{3.5}$$

where k is an undetermined constant, depending only on α, that is greater than x_0. Let $L(\cdot)$ denote the local time of $Y(\cdot)$ at 0. Considering assumption (EX1) and applying Itô-Tanaka's formula to $\psi(s,\hat Y_s)$, we obtain

$$\begin{aligned}
\mathrm{d}\psi(s,\hat Y_s)=&\left[\psi_s(s,\hat Y_s)+\psi_x(s,\hat Y_s)e^{A(s)}\left(-\mathrm{sgn}(Y_s)g(s,Y_s,Z_s)+\beta(s)|Y_s|+f_s\right)\right.\\
&\left.+\frac12\psi_{xx}(s,\hat Y_s)e^{2A(s)}|Z_s|^2\right]\mathrm{d}s+\psi_x(s,\hat Y_s)\mathrm{d}L_s+\psi_x(s,\hat Y_s)e^{A(s)}\mathrm{sgn}(Y_s)Z_s\cdot\mathrm{d}B_s\\
\geqslant&\ \psi(s,\hat Y_s)\left[\mu'(s)\hat Y_s^{\frac{2}{\alpha^*}}-\frac{2}{\alpha^*}\mu(s)\hat Y_s^{\frac{2}{\alpha^*}-1}e^{A(s)}\gamma(s)|Z_s|^\alpha\right.\\
&\left.+\frac{1}{\alpha^*}\mu(s)\hat Y_s^{\frac{2}{\alpha^*}-2}e^{2A(s)}\left(\frac{2}{\alpha^*}\mu(s)\hat Y_s^{\frac{2}{\alpha^*}}-\left(1-\frac{2}{\alpha^*}\right)\right)|Z_s|^2\right]\mathrm{d}s\\
&+\psi_x(s,\hat Y_s)e^{A(s)}\mathrm{sgn}(Y_s)Z_s\cdot\mathrm{d}B_s,\quad s\in[t,T].
\end{aligned}$$

For each $r\in[t,T]$ and each integer $m\geqslant 1$, we define the following stopping time:

$$\sigma_m^r:=\inf\left\{s\in[r,T]:\int_r^s\left(\psi_x(\tau,\hat Y_\tau)\right)^2e^{2A(\tau)}|Z_\tau|^2\mathrm{d}\tau\geqslant m\right\}\wedge T$$

with the convention $\inf \emptyset = +\infty$. By integrating over $[r, \sigma_m^r]$, we obtain that for each $m \geqslant 1$ and $r \in [t, T]$,

$$\begin{aligned}
\psi(r, \hat{Y}_r) + X_r^m \leqslant \psi(\sigma_m^r, \hat{Y}_{\sigma_m^r}) &+ \int_r^{\sigma_m^r} \psi(s, \hat{Y}_s) \Bigg[-\mu'(s)\hat{Y}_s^{\frac{2}{\alpha^*}} + \frac{2}{\alpha^*}\mu(s)\hat{Y}_s^{\frac{2}{\alpha^*}-1} e^{2A(s)}\gamma(s)|Z_s|^\alpha \\
&- \frac{1}{\alpha^*}\mu(s)\hat{Y}_s^{\frac{2}{\alpha^*}-2} e^{2A(s)} \left(\frac{2}{\alpha^*}\mu(s)\hat{Y}_s^{\frac{2}{\alpha^*}} - \left(1 - \frac{2}{\alpha^*}\right)\right) |Z_s|^2 \Bigg] \mathrm{d}s \\
&\leqslant \psi(\sigma_m^r, \hat{Y}_{\sigma_m^r}) + \int_r^{\sigma_m^r} \psi(s, \hat{Y}_s) \Bigg[-\mu'(s)\hat{Y}_s^{\frac{2}{\alpha^*}} + \frac{1}{\alpha^*}\mu(s)\hat{Y}_s^{\frac{2}{\alpha^*}-2} e^{2A(s)} \\
&\cdot \left(2\gamma(s)\hat{Y}_s|Z_s|^\alpha - \frac{2}{\alpha^*}\mu(s)\hat{Y}_s^{\frac{2}{\alpha^*}}|Z_s|^2 + |Z_s|^2 \right) \Bigg] \mathrm{d}s,
\end{aligned}$$

where

$$X_r^m := \int_r^{\sigma_m^r} \psi_x(s, \hat{Y}_s) e^{A(s)} \mathrm{sgn}(Y_s) Z_s \cdot \mathrm{d}B_s.$$

Using Young's inequality, we have

$$\begin{aligned}
2\gamma(s)\hat{Y}_s|Z_s|^\alpha &= \left(\hat{Y}_s^{\frac{2}{\alpha^*}}|Z_s|^2 \right)^{\frac{\alpha}{2}} \left(2\gamma(s)\hat{Y}_s^{1-\frac{\alpha}{\alpha^*}} \right) \\
&\leqslant \frac{1}{\alpha^*}\hat{Y}_s^{\frac{2}{\alpha^*}}|Z_s|^2 + \widehat{K}\gamma^{\frac{2}{2-\alpha}}(s)\hat{Y}_s^2
\end{aligned}$$

with

$$\widehat{K} := (2-\alpha)\left(\frac{\alpha-1}{\alpha^2}\right)^{\frac{-\alpha}{2-\alpha}} > 0.$$

Noticing $\mu(s) \geqslant 1$ and $\hat{Y}_s \geqslant k$, we choose $k := \left((\alpha^*)^2 + \alpha^*\right)^{\frac{\alpha^*}{2}} > x_0$ such that $\hat{Y}_s^{\frac{2}{\alpha^*}} \geqslant 2\ln\hat{Y}_s$ and then

$$\psi(s, \hat{Y}_s)\hat{Y}_s^{\frac{2}{\alpha^*}-2} \geqslant \exp\left(\hat{Y}_s^{\frac{2}{\alpha^*}}\right)\hat{Y}_s^{\frac{2}{\alpha^*}-2} \geqslant 1.$$

According to the previous inequalities, we obtain that for each $m \geqslant 1$ and $r \in [t, T]$,

$$\begin{aligned}
&\psi(r, \hat{Y}_r) + X_r^m \\
&\leqslant \psi(\sigma_m^r, \hat{Y}_{\sigma_m^r}) + \int_r^{\sigma_m^r} \Bigg\{ \psi(s, \hat{Y}_s)\hat{Y}_s^{\frac{2}{\alpha^*}} \left[-\mu'(s) + \frac{\widehat{K}}{\alpha^*} e^{2A(s)}\gamma^{\frac{2}{2-\alpha}}(s)\mu(s) \right] \\
&\quad - \frac{\mu(s)}{\alpha^*}\psi(s, \hat{Y}_s)\hat{Y}_s^{\frac{2}{\alpha^*}-2} e^{2A(s)} \left(\frac{1}{\alpha^*}\hat{Y}_s^{\frac{2}{\alpha^*}} - 1 \right) |Z_s|^2 \Bigg\} \mathrm{d}s \\
&\leqslant \psi(\sigma_m^r, \hat{Y}_{\sigma_m^r}) + \int_r^{\sigma_m^r} \Bigg\{ \psi(s, \hat{Y}_s)\hat{Y}_s^{\frac{2}{\alpha^*}} \left[-\mu'(s) + \frac{\widehat{K}}{\alpha^*} e^{2A(s)}\gamma^{\frac{2}{2-\alpha}}(s)\mu(s) \right] \\
&\quad - \frac{1}{\alpha^*} \left(\frac{1}{\alpha^*}k^{\frac{2}{\alpha^*}} - 1 \right) |Z_s|^2 \Bigg\} \mathrm{d}s, \\
&\leqslant \psi(\sigma_m^r, \hat{Y}_{\sigma_m^r}) + \int_r^{\sigma_m^r} \Bigg\{ \psi(s, \hat{Y}_s)\hat{Y}_s^{\frac{2}{\alpha^*}} \left[-\mu'(s) + \frac{\widehat{K}}{\alpha^*} e^{2A(s)}\gamma^{\frac{2}{2-\alpha}}(s)\mu(s) \right] - |Z_s|^2 \Bigg\} \mathrm{d}s. \qquad (3.6)
\end{aligned}$$

Now, we pick

$$\mu(s) := \mu(0)\exp\left(\frac{\widehat{K}}{\alpha^*} \int_0^s e^{2A(r)}\gamma^{\frac{2}{2-\alpha}}(r)\mathrm{d}r \right) \quad \text{and} \quad \mu(0) \geqslant 1, \quad s \in [t, T]. \qquad (3.7)$$

Then, given that $\mathbb{E}_r[X_r^m]=0$, by taking the conditional expectation with respect to $\mathcal{F}_r$ in (3.6), we obtain that for each $m \geqslant 1$ and $r \in [t,T]$,

$$\psi(r,\hat{Y}_r)+\mathbb{E}_r\left[\int_r^{\sigma_m^r}|Z_s|^2\mathrm{d}s\right] \leqslant \mathbb{E}_r\left[\psi(\sigma_m^r,\hat{Y}_{\sigma_m^r})\right]. \tag{3.8}$$

Furthermore, by applying Fatou's lemma and Lebesgue's dominated convergence theorem, along with the boundedness of $Y.$ and $f.$, and sending $m \to \infty$ in (3.8), we obtain

$$\begin{aligned}&\exp\left(\mu(r)\left(e^{A(r)}|Y_r|+k+\int_t^r e^{A(s)}f_s\mathrm{d}s\right)^{\frac{2}{\alpha^*}}\right)+\mathbb{E}_r\left[\int_r^T|Z_s|^2\mathrm{d}s\right]\\&\leqslant \mathbb{E}_r\left[\exp\left(\mu(T)\left(|\xi|e^{A(T)}+k+\int_t^T e^{A(s)}f_s\mathrm{d}s\right)^{\frac{2}{\alpha^*}}\right)\right],\quad r\in[t,T].\end{aligned} \tag{3.9}$$

Letting $r=t$ in the last inequality, we deduce that for each $t \in [0,T]$,

$$\begin{aligned}&\exp\left(\mu(t)\left(e^{A(t)}|Y_t|+k\right)^{\frac{2}{\alpha^*}}\right)+\mathbb{E}_t\left[\int_t^T|Z_s|^2\mathrm{d}s\right]\\&\leqslant \mathbb{E}_t\left[\exp\left(\mu(T)\left(|\xi|e^{A(T)}+k+\int_t^T e^{A(s)}f_s\mathrm{d}s\right)^{\frac{2}{\alpha^*}}\right)\right],\end{aligned} \tag{3.10}$$

which yields that

$$\exp\left(|Y_t|^{\frac{2}{\alpha^*}}\right)+\mathbb{E}_t\left[\int_t^T|Z_s|^2\mathrm{d}s\right]\leqslant K\mathbb{E}_t\left[\exp\left(K\left(|\xi|+\int_t^T f_s\mathrm{d}s\right)^{\frac{2}{\alpha^*}}\right)\right] \tag{3.11}$$

with

$$K:=\left(\exp\left(\mu(T)k^{\frac{2}{\alpha^*}}\right)\right)\vee\left(\mu(T)e^{A(T)}\right).$$

Thus, the desired inequality (3.3) holds for any bounded solution $(Y_t,Z_t)_{t\in[0,T]}$ of the BSDE (ξ,g) when $|\xi|\leqslant \bar{c}$ and $|f_t|\leqslant \bar{c}e^{-t}$ hold for each $t\in[0,T]$ and some constant $\bar{c}>0$.

Furthermore, using Doob's martingale inequality, inequality (3.9), and the property $(a+b)^\lambda \leqslant a^\lambda+b^\lambda$ for $a,b\geqslant 0$ and $\lambda\in(0,1)$, we deduce that for each $p>1$ and $t\in[0,T]$,

$$\begin{aligned}&\mathbb{E}\left[\exp\left(p\left(\sup_{r\in[t,T]}|Y_r|\right)^{\frac{2}{\alpha^*}}\right)\right]\leqslant \mathbb{E}\left[\sup_{r\in[t,T]}\exp\left(p\mu(r)\left(e^{A(r)}|Y_r|+k+\int_t^r e^{A(s)}f_s\mathrm{d}s\right)^{\frac{2}{\alpha^*}}\right)\right]\\&\leqslant \mathbb{E}\left[\sup_{r\in[t,T]}\left\{\mathbb{E}_r\left[\exp\left(\mu(T)\left(|\xi|e^{A(T)}+k+\int_t^T e^{A(s)}f_s\mathrm{d}s\right)^{\frac{2}{\alpha^*}}\right)\right]\right\}^p\right]\\&\leqslant \left(\frac{p}{p-1}\right)^p\mathbb{E}\left[\exp\left(p\mu(T)\left(|\xi|e^{A(T)}+k+\int_t^T e^{A(s)}f_s\mathrm{d}s\right)^{\frac{2}{\alpha^*}}\right)\right]\\&\leqslant \left(\frac{p}{p-1}\right)^p e^{p\mu(T)k^{\frac{2}{\alpha^*}}}\mathbb{E}\left[\exp\left(p\mu(T)e^{A(T)}\left(|\xi|+\int_t^T f_s\mathrm{d}s\right)^{\frac{2}{\alpha^*}}\right)\right]<+\infty.\end{aligned} \tag{3.12}$$

On the other hand, using inequalities (3.6) and (3.7), we can deduce that for each $m\geqslant 1$ and $p>1$,

$$\mathbb{E}\left[\left(\int_r^{\sigma_m^r}|Z_s|^2\mathrm{d}s\right)^{\frac{p}{2}}\right]\leqslant \mathbb{E}\left[\left(\psi(\sigma_m^r,\hat{Y}_{\sigma_m^r})+|X_r^m|\right)^{\frac{p}{2}}\right]$$
$$\leqslant 2^{p-1}\left\{\mathbb{E}\left[\left(\psi(\sigma_m^r,\hat{Y}_{\sigma_m^r})\right)^{\frac{p}{2}}\right]+\mathbb{E}\left[|X_r^m|^{\frac{p}{2}}\right]\right\},\quad r\in[t,T].$$

Considering the definition of X_r^m and the fact that $\hat{Y}_t\geqslant k\geqslant 1$, and applying Burkholder-Davis-Gundy's inequality and Hölder's inequality, we observe that

$$2^{p-1}\mathbb{E}\left[|X_r^m|^{\frac{p}{2}}\right]\leqslant 2^{p-1}(\mu(T))^{\frac{p}{2}}e^{\frac{p}{2}A(T)}\mathbb{E}\left[\left(\int_r^{\sigma_m^r}\left(\psi(s,\hat{Y}_s)\right)^2|Z_s|^2\mathrm{d}s\right)^{\frac{p}{4}}\right]$$
$$\leqslant 2^{p-1}(\mu(T))^{\frac{p}{2}}e^{\frac{p}{2}A(T)}\mathbb{E}\left[\sup_{s\in[r,\sigma_m^r]}\left(\psi(s,\hat{Y}_s)\right)^{\frac{p}{2}}\left(\int_r^{\sigma_m^r}|Z_s|^2\mathrm{d}s\right)^{\frac{p}{4}}\right]$$
$$\leqslant 2^{2p-3}(\mu(T))^{p}e^{pA(T)}\mathbb{E}\left[\sup_{s\in[r,\sigma_m^r]}\left(\psi(s,\hat{Y}_s)\right)^{p}\right]+\frac{1}{2}\mathbb{E}\left[\left(\int_r^{\sigma_m^r}|Z_s|^2\mathrm{d}s\right)^{\frac{p}{2}}\right].$$

Then,

$$\mathbb{E}\left[\left(\int_r^{\sigma_m^r}|Z_s|^2\mathrm{d}s\right)^{\frac{p}{2}}\right]\leqslant 2^{p}\mathbb{E}\left[\left(\psi(\sigma_m^r,\hat{Y}_{\sigma_m^r})\right)^{\frac{p}{2}}\right]+2^{2p-2}(\mu(T))^{p}e^{pA(T)}\mathbb{E}\left[\sup_{s\in[r,\sigma_m^r]}\left(\psi(s,\hat{Y}_s)\right)^{p}\right]$$
$$\leqslant (8\mu(T))^{p}e^{pA(T)}\mathbb{E}\left[\sup_{s\in[r,\sigma_m^r]}\left(\psi(s,\hat{Y}_s)\right)^{p}\right],\ r\in[t,T].$$

From the last inequality, by letting $m\to\infty$ and $r=t$, and applying Fatou's lemma, Lebesgue's dominated convergence theorem, and inequality (3.12), we obtain that for each $t\in[0,T]$,

$$\mathbb{E}\left[\left(\int_t^{T}|Z_s|^2\mathrm{d}s\right)^{\frac{p}{2}}\right]\leqslant (8\mu(T))^{p}e^{pA(T)}\mathbb{E}\left[\sup_{s\in[t,T]}\left(\psi(s,\hat{Y}_s)\right)^{p}\right]$$
$$=(8\mu(T))^{p}e^{pA(T)}\mathbb{E}\left[\sup_{r\in[t,T]}\exp\left(p\mu(r)\left(e^{A(r)}|Y_r|+k+\int_t^r e^{A(s)}f_s\mathrm{d}s\right)^{\frac{2}{\alpha^*}}\right)\right]$$
$$\leqslant (8\mu(T))^{p}e^{pA(T)}\left(\frac{p}{p-1}\right)^{p}e^{p\mu(T)k^{\frac{2}{\alpha^*}}}\mathbb{E}\left[\exp\left(p\mu(T)e^{A(T)}\left(|\xi|+\int_t^T f_s\mathrm{d}s\right)^{\frac{2}{\alpha^*}}\right)\right].\tag{3.13}$$

Combining inequalities (3.12) and (3.13), for each $p>1$, $t\in[0,T]$, we have

$$\mathbb{E}\left[\exp\left(p\left(\sup_{s\in[t,T]}|Y_s|\right)^{\frac{2}{\alpha^*}}\right)\right]+\mathbb{E}\left[\left(\int_t^{T}|Z_s|^2\mathrm{d}s\right)^{\frac{p}{2}}\right]$$
$$\leqslant K_p\mathbb{E}\left[\exp\left(K_p\left(|\xi|+\int_t^T f_s\mathrm{d}s\right)^{\frac{2}{\alpha^*}}\right)\right],\tag{3.14}$$

where

$$K_p:=\left(\left(\frac{p}{p-1}\right)^{p}\left((8\mu(T))^{p}e^{pA(T)}+1\right)e^{p\mu(T)k^{\frac{2}{\alpha^*}}}\right)\vee\left(p\mu(T)e^{A(T)}\right).$$

Thus, the desired inequality (3.4) holds for any bounded solution $(Y_t, Z_t)_{t\in[0,T]}$ of the BSDE (ξ, g), provided that $|\xi| \leqslant \bar{c}$ and $|f_t| \leqslant \bar{c}e^{-t}$ for each $t \in [0,T]$ and some constant $\bar{c} > 0$.

Step 2 Building on Step 1, we employ the localization procedure of Briand and Hu [2] to construct the desired solution. For each pair of positive integers n, $q \geqslant 1$ and $(\omega, t, y, z) \in \Omega \times [0,T] \times \mathbb{R}^{1+d}$, let

$$\xi^{n,q} := \xi^+ \wedge n - \xi^- \wedge q \quad \text{and} \quad g^{n,q}(\omega,t,y,z) := g^+(\omega,t,y,z) \wedge (ne^{-t}) - g^-(\omega,t,y,z) \wedge (qe^{-t}).$$

It is immediately apparent that $|\xi^{n,q}| \leqslant |\xi| \wedge (n \vee q)$ and $|g^{n,q}| \leqslant |g| \wedge [(n \vee q)e^{-t}]$. Moreover, it can also be verified that the generator $g^{n,q}$ satisfies assumptions (H1), (H2), (EX1), and (EX2) with $u(t) = (n \vee q)e^{-t}$ and $f_t \wedge [(n \vee q)e^{-t}]$ replacing f_t. Consequently, by Lemma 2.1, the following BSDE $(\xi^{n,q}, g^{n,q})$ admits a minimal bounded solution $(Y_t^{n,q}, Z_t^{n,q})_{t\in[0,T]}$ in the space of $\mathcal{S}^\infty(0,T;\mathbb{R}) \times \mathcal{M}^2(0,T;\mathbb{R}^d)$:

$$Y_t^{n,q} = \xi^{n,q} + \int_t^T g^{n,q}(s, Y_s^{n,q}, Z_s^{n,q})\mathrm{d}s - \int_t^T Z_s^{n,q} \cdot \mathrm{d}B_s, \quad t \in [0,T].$$

From the comparison result stated in Lemma 2.1, we know that $(Y_t^{n,q})_{t\in[0,T]}$ is non-decreasing in n and non-increasing in q. Furthermore, by virtue of the assertion (3.11) from Step 1, for each $t \in [0,T]$ and $n, q \geqslant 1$, we have

$$\begin{aligned}\exp\left(|Y_t^{n,q}|^{\frac{2}{\alpha^*}}\right) &\leqslant K\mathbb{E}_t\left[\exp\left(K\left(|\xi^{n,q}| + \int_t^T f_s \wedge [(n\vee q)e^{-s}]\mathrm{d}s\right)^{\frac{2}{\alpha^*}}\right)\right] \\ &\leqslant K\mathbb{E}_t\left[\exp\left(K\left(|\xi| + \int_0^T f_s\mathrm{d}s\right)^{\frac{2}{\alpha^*}}\right)\right],\end{aligned}$$

which implies

$$|Y_t^{n,q}| \leqslant \left(\ln K + \ln\left\{\mathbb{E}_t\left[\exp\left(K\left(|\xi| + \int_0^T f_s\mathrm{d}s\right)^{\frac{2}{\alpha^*}}\right)\right]\right\}\right)^{\frac{\alpha^*}{2}}.$$

Thus, in view of the last inequality and assumption (EX2), we can apply the localization procedure initially developed by Briand and Hu [2] to obtain an $(\mathcal{F}_t)$-progressively measurable $\mathbb{R}^d$-valued process $(Z_t)_{t\in[0,T]}$, such that $Z_\cdot^{n,q}$ converges to $Z_\cdot$ as n, q tend to infinity. Consequently, the pair $(Y_\cdot := \inf_q \sup_n Y_\cdot^{n,q}, Z_\cdot)$ is a solution to the BSDE (ξ, g). Moreover, the desired inequalities (3.3) and (3.4) for $(Y_\cdot, Z_\cdot)$ follow immediately from Step 1 and the definition of $(Y_\cdot, Z_\cdot)$. This completes the proof. □

Remark 3.3 *From the above proof, it is straightforward to verify that in the assumptions of Theorem 3.2, if $|\xi|$ is replaced by ξ^+ and (EX1) is replaced by the following (EX1′):*

(EX1′) $\mathrm{d}\mathbb{P}\times$ $\mathrm{d}t$*-a.e., for each $(y, z) \in \mathbb{R} \times \mathbb{R}^d$,*

$$\mathbf{1}_{\{y>0\}} g(\omega, t, y, z) \leqslant f_t(\omega) + \beta(t)|y| + \gamma(t)|z|^\alpha.$$

Then, for any solution $(Y_t, Z_t)_{t\in[0,T]}$ of the BSDE (ξ, g) such that $\sup_{t\in[0,T]} Y_t^+$ has sub-exponential moments of any order, there exists a constant $k \geqslant 1$ and a differentiable function $\mu(\cdot) : [0,T] \mapsto [1, +\infty)$ such that $\mathbb{P}$-a.s., *for each $t \in [0,T]$,*

$$\exp\left(\mu(t)\left(e^{A(t)}Y_t^+ + k\right)^{\frac{2}{\alpha^*}}\right) + \mathbb{E}_t\left[\int_t^T \mathbf{1}_{\{Y_s>0\}}|Z_s|^2\mathrm{d}s\right]$$
$$\leqslant \mathbb{E}_t\left[\exp\left(\mu(T)\left(\xi^+e^{A(T)} + k + \int_t^T e^{A(s)}f_s\mathrm{d}s\right)^{\frac{2}{\alpha^*}}\right)\right].$$

Consequently, there exists a constant $K>0$, depending only on $(\alpha, T, \beta(\cdot), \gamma(\cdot))$, such that $\mathbb{P}$-a.s., *for each $t\in[0,T]$,*

$$\exp\left(\left(Y_t^+\right)^{\frac{2}{\alpha^*}}\right) + \mathbb{E}_t\left[\int_t^T \mathbf{1}_{\{Y_s>0\}}|Z_s|^2\mathrm{d}s\right] \leqslant K\mathbb{E}_t\left[\exp\left(K\left(\xi^+ + \int_t^T f_s\mathrm{d}s\right)^{\frac{2}{\alpha^*}}\right)\right].$$

In the above proof, it suffices to use $Y_\cdot^+$, $\mathbf{1}_{Y_\cdot>0}Y_\cdot$, and $\frac{1}{2}L_\cdot$ in place of $|Y_\cdot|$, $\mathrm{sgn}(Y_\cdot)$, *and $L_\cdot$, respectively.*

Remark 3.4 *From* (3.5) *and* (3.7) *in the proof of Theorem 3.2, it is evident that the integrability conditions for the functions $\beta(\cdot)$ and $\gamma(\cdot)$ in* (3.1) *are likely the most appropriate to ensure the existence of an unbounded solution. When $\alpha=1$, $2/(2-\alpha)=2$, which corresponds to the cases studied in Chen and Wang [4], Fan [10] and Zong and Hu [26, 27]. Moreover, to apply existing results on the minimal bounded solution of a general time-interval BSDE, a time-varying truncation for the generator g becomes necessary, particularly for the case of $T=+\infty$. This is illustrated in the definition of $g^{n,p}$ in the second step of the proof of Theorem 3.2. Similar arguments can be found in Fan [12] and Fan et al. [15].*

We would like to note that similar arguments to those presented in Remarks 3.1 and 3.4 can be applied in the subsequent sections. For simplicity, we omit them.

4. Uniqueness of the unbounded solution

In this section, we establish a general comparison theorem and a general uniqueness and existence theorem for unbounded solutions of BSDEs, extending the corresponding results from Fan and Hu [12].

4.1 Comparison theorem

Next, we introduce the following time-varying extended convexity assumption for the generator g.

(UN) The generator g satisfies either of the following conditions:

(i) $\mathrm{d}\mathbb{P}\times\mathrm{d}t$-*a.e.*, for each $(y_i,z_i)\in\mathbb{R}^{1+d}$, $i=1,2$ and $\theta\in(0,1)$,

$$\begin{aligned}&\mathbf{1}_{\{y_1-\theta y_2>0\}}(g(\omega,t,y_1,z_1)-\theta g(\omega,t,y_2,z_2))\\ &\leqslant (1-\theta)(f_t(\omega)+\beta(t)|y_2|+\gamma(t)\left[\ln(e+|z_2|)\right]^{\frac{\alpha^*}{2}}+\beta(t)|\delta_\theta y|+\gamma(t)|\delta_\theta z|^\alpha);\end{aligned} \tag{4.1}$$

(ii) $\mathrm{d}\mathbb{P}\times\mathrm{d}t$-*a.e.*, for each $(y_i,z_i)\in\mathbb{R}^{1+d}$, $i=1,2$ and $\theta\in(0,1)$,

$$\begin{aligned}&-\mathbf{1}_{\{y_1-\theta y_2<0\}}(g(\omega,t,y_1,z_1)-\theta g(\omega,t,y_2,z_2))\\ &\leqslant (1-\theta)(f_t(\omega)+\beta(t)|y_2|+\gamma(t)\left[\ln(e+|z_2|)\right]^{\frac{\alpha^*}{2}}+\beta(t)|\delta_\theta y|+\gamma(t)|\delta_\theta z|^\alpha),\end{aligned} \tag{4.2}$$

where

$$\delta_\theta y := \frac{y_1-\theta y_2}{1-\theta},\quad \delta_\theta z := \frac{z_1-\theta z_2}{1-\theta}.$$

Assume further that $\gamma(\cdot):[0,T]\mapsto\mathbb{R}_+$ satisfies

$$0<\int_0^T\gamma(t)\mathrm{d}t<+\infty. \tag{4.3}$$

Remark 4.1 (i) *It is clear that even in the case where $T<+\infty$ and both $\beta(\cdot)$ and $\gamma(\cdot)$ are nonnegative constants, the presence of the term $\gamma(t)\left[\ln(e+|z_2|)\right]^{\alpha^*/2}$ makes assumption (UN) strictly weaker than assumption (H2′) in Fan and Hu [12]. Additionally, it is easy to verify that if $g(\omega,t,y,z)$ satisfies conditions (i) of (UN), then $\hat{g}(\omega,t,y,z):=-g(\omega,t,-y,-z)$ satisfies condition (ii) of (UN).*

(ii) *Using a similar argument as on page 29 of Fan et al. [16], we conclude that if g satisfies assumption (UN), then $\mathrm{d}\mathbb{P}\times\mathrm{d}t$-a.e., $g(\omega,t,\cdot,\cdot)$ is locally Lipschitz continuous on $\mathbb{R}\times\mathbb{R}^d$.*

(iii) *It is straightforward to verify that if the generator g_i satisfies inequality (4.1) (resp. (4.2)) for $i=1,2$, then for any $k_i>0$, $i=1,2$, the generators $k_1g_1+k_2g_2$, $g_1\vee g_2$, and $g_1\wedge g_2$ also satisfy inequality (4.1) (resp. (4.2)).*

(iv) *Setting $y_1=y_2=y$ and $z_1=z_2=z$ in* (4.1) *and* (4.2) *respectively yields that for some constant $k>0$ depending only on α,*

$$\mathbf{1}_{\{y>0\}}g(\omega,t,y,z)\leqslant f_t(\omega)+2\beta(t)|y|+k\gamma(t)|z|^\alpha$$

and

$$-\mathbf{1}_{\{y<0\}}g(\omega,t,y,z)\leqslant f_t(\omega)+2\beta(t)|y|+k\gamma(t)|z|^\alpha.$$

These inequalities imply that the generator g exhibits a time-varying one-sided linear growth in y and a time-varying sub-quadratic growth in z.

(v) *Setting $z_1=z_2=z$ in* (4.1) *or* (4.2) *and then letting $\theta\to1$ yields*

$$\mathbf{1}_{\{y_1-y_2>0\}}(g(\omega,t,y_1,z)-g(\omega,t,y_2,z))\leqslant\beta(t)|y_1-y_2|.$$

This implies that the generator g satisfies a time-varying monotonicity condition in y.

Theorem 4.2 presents a general comparison theorem for unbounded solutions of BSDEs under assumption (UN), serving as the main result of this subsection.

Theorem 4.2 *Assume that ξ and ξ' are two terminal values satisfying $\mathbb{P}$-a.s., $\xi\leqslant\xi'$, g and g' are two generators, and let $(Y_t,Z_t)_{t\in[0,T]}$ and $(Y'_t,Z'_t)_{t\in[0,T]}$ be the corresponding solutions to the BSDEs (ξ,g) and (ξ',g'), respectively, such that for each $p>1$,*

$$\mathbb{E}\left[\exp\left(p\left(\sup_{t\in[0,T]}(|Y_t|+|Y'_t|)\right)^{\frac{2}{\alpha^*}}\right)\right]<+\infty \tag{4.4}$$

and $Z_\cdot,Z'_\cdot\in\mathcal{M}^p(0,T;\mathbb{R}^d)$. If g (resp. g') *satisfies assumption (UN) and*

$$\mathrm{d}\mathbb{P}\times\mathrm{d}t\text{-a.e.},\quad g(t,Y'_t,Z'_t)\leqslant g'(t,Y'_t,Z'_t)\ \ (\text{resp. } g(t,Y_t,Z_t)\leqslant g'(t,Y_t,Z_t)), \tag{4.5}$$

then $\mathbb{P}$-a.s., for each $t\in[0,T]$, $Y_t\leqslant Y'_t$.

Proof The proof is organized into the following four steps.

Step 1 Define

$$\hat{f}_t:=f_t+\beta(t)|Y'_t|+\gamma(t)\left[\ln\left(e+|Z'_t|\right)\right]^{\frac{\alpha^*}{2}},\quad t\in[0,T].$$

First, we prove that $\int_0^T \hat{f}_t \mathrm{d}t$ has sub-exponential moments of any order, i.e.,

$$\mathbb{E}\left[\exp\left(p\left(\int_0^T \hat{f}_s \mathrm{d}s\right)^{\frac{2}{\alpha^*}}\right)\right] < +\infty, \quad \forall\, p > 1. \tag{4.6}$$

In fact, by virtue of $\beta(\cdot) \in L^1([0,T];\mathbb{R}_+)$ and inequality (4.4), we have for each $p > 1$,

$$\mathbb{E}\left[\exp\left(p\left(\int_0^T \beta(t)|Y_t'|\mathrm{d}t\right)^{\frac{2}{\alpha^*}}\right)\right] \leqslant \mathbb{E}\left[\exp\left(p\left(\sup_{t\in[0,T]}|Y_t'|\int_0^T \beta(t)\mathrm{d}t\right)^{\frac{2}{\alpha^*}}\right)\right]$$
$$= \mathbb{E}\left[\exp\left(p\left(\int_0^T \beta(t)\mathrm{d}t\right)^{\frac{2}{\alpha^*}}\left(\sup_{t\in[0,T]}|Y_t'|\right)^{\frac{2}{\alpha^*}}\right)\right] < +\infty. \tag{4.7}$$

On the other hand, by combining inequality (4.3), Hölder's inequality, and the basic inequality $(a+b)^q \leqslant 2^q(a^q+b^q)$ for $a,b>0$ and $q>0$, we deduce that for each $p>1$,

$$\mathbb{E}\left[\left(\int_0^T \gamma(t)|Z_t'|\mathrm{d}t\right)^{\delta_p}\right] \leqslant \mathbb{E}\left[\left(\int_0^T \gamma(t)(1+|Z_t'|^{\alpha})\mathrm{d}t\right)^{\delta_p}\right]$$
$$\leqslant 2^{\delta_p}\left(\int_0^T \gamma(t)\mathrm{d}t\right)^{\delta_p} + 2^{\delta_p}\mathbb{E}\left[\left(\int_0^T \gamma^{\frac{2}{2-\alpha}}(t)\mathrm{d}t\right)^{\frac{(2-\alpha)}{2}\delta_p}\left(\int_0^T |Z_t'|^2\mathrm{d}t\right)^{\frac{\alpha}{2}\delta_p}\right] < +\infty,$$

where

$$\delta_p := p\left(\int_0^T \gamma(t)\mathrm{d}t\right)^{\frac{2}{\alpha^*}}.$$

Note that $\mathrm{d}\mu_t := \frac{\gamma(t)\mathrm{d}t}{\int_0^T \gamma(t)\mathrm{d}t}$ defines a probability measure on $[0,T]$, and $[\ln(k_\alpha + x)]^{\frac{\alpha^*}{2}}$, with $k_\alpha := \exp\left(\frac{\alpha^*}{2}\right)$, is a concave function on $\mathbb{R}_+$. Applying Jensen's inequality to the concave function and using the previous inequality, we conclude that

$$\mathbb{E}\left[\exp\left(p\left(\int_0^T \gamma(t)\left(\ln(e+|Z_t'|)\right)^{\frac{\alpha^*}{2}}\mathrm{d}t\right)^{\frac{2}{\alpha^*}}\right)\right]$$
$$\leqslant \mathbb{E}\left[\exp\left(\delta_p\left(\int_0^T \left(\ln(k_\alpha+|Z_t'|)\right)^{\frac{\alpha^*}{2}}\mathrm{d}\mu_t\right)^{\frac{2}{\alpha^*}}\right)\right]$$
$$\leqslant \mathbb{E}\left[\exp\left(\delta_p \ln\left(k_\alpha + \int_0^T |Z_t'|\mathrm{d}\mu_t\right)\right)\right] = \mathbb{E}\left[\left(k_\alpha + \int_0^T |Z_t'|\mathrm{d}\mu_t\right)^{\delta_p}\right]$$
$$\leqslant 2^{\delta_p}\left(k_\alpha^{\delta_p} + \mathbb{E}\left[\left(\int_0^T |Z_t'|\mathrm{d}\mu_t\right)^{\delta_p}\right]\right)$$
$$= 2^{\delta_p}\left(k_\alpha^{\delta_p} + \frac{1}{\left(\int_0^T \gamma(t)\mathrm{d}t\right)^{\delta_p}}\mathbb{E}\left[\left(\int_0^T \gamma(t)|Z_t'|\mathrm{d}t\right)^{\delta_p}\right]\right) < +\infty. \tag{4.8}$$

In light of inequalities (4.7) and (4.8), along with integrability of process $f_\cdot$, the desired assertion (4.6) follows directly from Hölder's inequality.

Step 2 Assume that the generator g satisfies condition (i) of assumption (UN), and that $\mathrm{d}\mathbb{P}\times\mathrm{d}t$-$a.e.$, $g(t,Y'_t,Z'_t)\leqslant g'(t,Y'_t,Z'_t)$. To proceed, we utilize the θ-difference technique developed by Briand and Hu [3]. For each fixed $\theta\in(0,1)$, define

$$\delta_\theta U_\cdot := \frac{Y_\cdot-\theta Y'_\cdot}{1-\theta} \quad \text{and} \quad \delta_\theta V_\cdot := \frac{Z_\cdot-\theta Z'_\cdot}{1-\theta}.$$

Then, the pair $(\delta_\theta U_\cdot,\delta_\theta V_\cdot)$ satisfies the following BSDE:

$$\delta_\theta U_t=\delta_\theta U_T+\int_t^T \delta_\theta g(s,\delta_\theta U_s,\delta_\theta V_s)\mathrm{d}s-\int_t^T \delta_\theta V_s\cdot \mathrm{d}B_s,\quad t\in[0,T],$$

where $\mathrm{d}\mathbb{P}\times\mathrm{d}s$-$a.e.$, for each $(y,z)\in\mathbb{R}\times\mathbb{R}^d$,

$$\begin{aligned}\delta_\theta g(s,y,z):=&\frac{1}{1-\theta}\left[g(s,(1-\theta)y+\theta Y'_s,(1-\theta)z+\theta Z'_s)-\theta g(s,Y'_s,Z'_s)\right]\\&+\frac{\theta}{1-\theta}[g(s,Y'_s,Z'_s)-g'(s,Y'_s,Z'_s)].\end{aligned}\tag{4.9}$$

It follows from (UN) and (4.9) that $\mathrm{d}\mathbb{P}\times\mathrm{d}s$-$a.e.$, for each $(y,z)\in\mathbb{R}\times\mathbb{R}^d$,

$$\begin{aligned}\mathbf{1}_{\{y>0\}}\delta_\theta g(s,y,z)&\leqslant f_s+\beta(s)|Y'_s|+\gamma(s)\left[\ln\left(e+|Z'_s|\right)\right]^{\frac{\alpha^*}{2}}+\beta(s)|y|+\gamma(s)|z|^\alpha\\&=\hat{f}_s+\beta(s)|y|+\gamma(s)|z|^\alpha,\end{aligned}\tag{4.10}$$

which together with inequality (4.6) implies that the generator $\delta_\theta g$ satisfies assumption (EX1$'$) with $\hat{f}_\cdot$ in place of $f_\cdot$, and

$$\delta_\theta U_T^+=\frac{(\xi-\theta\xi')^+}{1-\theta}=\frac{[\xi-\theta\xi+\theta(\xi-\xi')]^+}{1-\theta}\leqslant \xi^+.$$

Then, according to Remark 3.3, there exists a constant $K>0$ depending only on $(\alpha,T,\beta(\cdot),\gamma(\cdot))$, such that for each $t\in[0,T]$,

$$\exp\left(\delta_\theta U_t^+\right)^{\frac{2}{\alpha^*}}\leqslant K\mathbb{E}_t\left[\exp\left(K\left(\xi^++\int_0^T f_s\mathrm{d}s\right)^{\frac{2}{\alpha^*}}\right)\right],$$

and then

$$(Y_t-\theta Y'_t)^+\leqslant(1-\theta)\left\{\ln\left[K\mathbb{E}_t\left[\exp\left(K\left(\xi^++\int_0^T \hat{f}_s\mathrm{d}s\right)^{\frac{2}{\alpha^*}}\right)\right]\right]\right\}^{\frac{\alpha^*}{2}}.$$

Consequently, the desired conclusion follows by letting $\theta\to1$ in the last inequality. Specifically, $\mathbb{P}$-a.s., for each $t\in[0,T]$, $Y_t\leqslant Y'_t$.

Step 3 We now consider the case where the generator g' satisfies condition (i) of assumption (UN), and $\mathrm{d}\mathbb{P}\times\mathrm{d}t$-$a.e.$, $g(t,Y_t,Z_t)\leqslant g'(t,Y_t,Z_t)$. In this case, the generator $\delta_\theta g$ in (4.9) should be replaced by the following equality:

$$\begin{aligned}\delta_\theta g(s,y,z):=&\frac{1}{1-\theta}\left[g(s,Y_s,Z_s)-g'(s,Y_s,Z_s)\right]\\&+\frac{1}{1-\theta}[g'(s,(1-\theta)y+\theta Y'_s,(1-\theta)z+\theta Z'_s)-\theta g'(s,Y'_s,Z'_s)].\end{aligned}$$

It is straightforward to verify that $\delta_\theta g$ satisfies inequality (4.10). Using an identical argument as in Step 2, we arrive at the desired conclusion.

Step 4 Suppose that the generator g satisfies condition (ii) of assumption (UN), and that $\mathrm{d}\mathbb{P}\times\mathrm{d}t$-*a.e.*, $g(t,Y'_t,Z'_t)\leqslant g'(t,Y'_t,Z'_t)$. For each $t\in[0,T]$, let us define

$$\hat{\xi}:=-\xi,\quad \hat{Y}_t:=-Y_t,\quad \hat{Z}_t:=-Z_t,\quad \hat{g}(t,y,z):=-g(t,-y,-z)$$

and

$$\hat{\xi}':=-\xi',\quad \hat{Y}'_t:=-Y'_t,\quad \hat{Z}'_t:=-Z'_t,\quad \hat{g}'(t,y',z'):=-g'(t,-y',-z').$$

Then, $(\hat{Y}_t,\hat{Z}_t)_{t\in[0,T]}$ and $(\hat{Y}'_t,\hat{Z}'_t)_{t\in[0,T]}$ are solutions to the BSDEs $(\hat{\xi},\hat{g})$ and $(\hat{\xi}',\hat{g}')$, respectively. It is straightforward to verify that $\mathbb{P}$-a.s., $\hat{\xi}'\leqslant\hat{\xi}$, $\hat{g}$ satisfies condition (i) of assumption (UN), and $\mathrm{d}\,\mathbb{P}\times\,\mathrm{d}\,t$-*a.e.*,

$$\hat{g}'(t,\hat{Y}'_t,\hat{Z}'_t)=-g'(t,Y'_t,Z'_t)\leqslant -g(t,Y'_t,Z'_t)=\hat{g}(t,\hat{Y}'_t,\hat{Z}'_t).$$

Thus, by applying the results from Step 3, we conclude that $\mathbb{P}$-a.s., for each $t\in[0,T]$,

$$-Y'_t=\hat{Y}'_t\leqslant\hat{Y}_t=-Y_t,$$

which implies that $Y_t\leqslant Y'_t$. In the same way, we can prove the case where the generator g' satisfies condition (ii) of assumption (UN), and $\mathrm{d}\mathbb{P}\times\mathrm{d}t$-*a.e.*, $g(t,Y_t,Z_t)\leqslant g'(t,Y_t,Z_t)$. This completes the proof. □

4.2 Existence and uniqueness

Based on Theorems 3.2 and 4.2, the following general existence and uniqueness theorem for unbounded solutions of BSDEs follows immediately, extending Theorem 3.9 in Fan and Hu [12].

Theorem 4.3 *Assume that ξ is a terminal value and g is a generator satisfying assumptions (EX1), (EX2), and (UN). If $|\xi|+\int_0^T f_t\mathrm{d}t$ has sub-exponential moments of any order, i.e.,*

$$\mathbb{E}\left[\exp\left(p\left(|\xi|+\int_0^T f_s\mathrm{d}s\right)^{\frac{2}{\alpha^*}}\right)\right]<+\infty,\quad \forall\, p>1,$$

then the BSDE (ξ,g) admits a unique solution $(Y_t,Z_t)_{t\in[0,T]}$ such that $\sup_{t\in[0,T]}|Y_t|$ has sub-exponential moments of any order, and $Z_\cdot\in\mathcal{M}^p(0,T;\mathbb{R}^d)$ for all $p>1$.

5. Sufficient conditions ensuring the uniqueness

In this section, we propose and prove several sufficient conditions that ensure the assumption (UN) holds. Some of these conditions are explored here for the first time, even for the finite time-interval case. For narrative convenience, we first examine a special type of assumption (UN), denoted as (UN$'$).

(UN') g satisfies either of the following conditions:

(i) $\mathrm{d}\mathbb{P}\times\mathrm{d}t$-*a.e.*, for each $(y_i,z_i)\in\mathbb{R}^{1+d}$, $i=1,2$ and $\theta\in(0,1)$,

$$\begin{aligned}&\mathbf{1}_{\{y_1-\theta y_2>0\}}(g(\omega,t,y_1,z_1)-\theta g(\omega,t,y_2,z_2))\\ \leqslant\ &(1-\theta)(f_t(\omega)+\beta(t)|y_2|+\beta(t)|\delta_\theta y|+\gamma(t)|\delta_\theta z|^{\alpha});\end{aligned}\tag{5.1}$$

(ii) $\mathrm{d}\mathbb{P}\times\mathrm{d}t$-*a.e.*, for each $(y_i,z_i)\in\mathbb{R}^{1+d}$, $i=1,2$ and $\theta\in(0,1)$,

$$- \mathbf{1}_{\{y_1-\theta y_2<0\}}(g(\omega,t,y_1,z_1)-\theta g(\omega,t,y_2,z_2))$$
$$\leqslant (1-\theta)(f_t(\omega)+\beta(t)|y_2|+\beta(t)|\delta_\theta y|+\gamma(t)|\delta_\theta z|^\alpha), \tag{5.2}$$

where $\delta_\theta y$ and $\delta_\theta z$ defined in assumption (UN).

The Propositions 5.1 and 5.4 provide sufficient conditions ensuring that assumption (UN$'$) holds. These propositions represent the first two main results of this section.

Proposition 5.1 *Assume that the generator g satisfies the assumption (EX1). Then, assumption (UN$'$) holds for g if it satisfies either of the following conditions:*

(i) $\mathrm{d}\mathbb{P}\times\mathrm{d}t$-*a.e.*, $g(\omega,t,\cdot,\cdot)$ *is convex or concave;*

(ii) $g(t,y,z)=\gamma(t)l(y)q(z)$, *where both* $l:\mathbb{R}\mapsto\mathbb{R}$ *and* $q:\mathbb{R}^d\mapsto\mathbb{R}$ *are bounded and Lipschitz continuous functions,* $q(z)$ *has a bounded support, and* $\gamma(t)\in L^1([0,T];\mathbb{R}_+)\cap L^{\frac{2}{2-\alpha}}([0,T];\mathbb{R}_+)$.

The proof of Proposition 5.1 is analogous to the proofs of conditions (i) and (iii) in Proposition 3.5 of Fan and Hu [12], and is therefore omitted here.

Remark 5.2 *Let* $g(\omega,t,y,z):\equiv g_1(y)+g_2(y)$ *and* $\bar{g}(\omega,t,y,z):\equiv g_3(z)+g_4(z)$ *be defined as on page 38 of Fan and Hu [12]. It is straightforward to verify that* $\beta(t)g(\omega,t,y,z)$ *and* $\gamma(t)\bar{g}(\omega,t,y,z)$ *satisfy assumption (UN$'$). Additionally, it is clear that if* $g(\omega,t,y,z)$ *satisfies condition (i) of (UN$'$), then* $\hat{g}(\omega,t,y,z):=-g(\omega,t,-y,-z)$ *satisfies condition (ii) of (UN$'$).*

Next, let $a\geqslant 0$, $d=1$, $u(\cdot),k_1(\cdot),k_2(\cdot)\in L^1([0,T];\mathbb{R}_+)$, $v(\cdot),c_1(\cdot),c_2(\cdot),c_3(\cdot)\in L^1([0,T];\mathbb{R}_+)\cap L^{\frac{2}{2-\alpha}}([0,T];\mathbb{R}_+)$. In order to explore other sufficient conditions ensuring that assumption (UN$'$) holds, we introduce the following assumptions on the generator g.

(A1) g exhibits time-varying linear growth in y and time-varying sub-quadratic growth in z, i.e., $\mathrm{d}\mathbb{P}\times\mathrm{d}t$-*a.e.*, for each $(y,z)\in\mathbb{R}\times\mathbb{R}$, we have

$$|g(\omega,t,y,z)|\leqslant f_t(\omega)+u(t)|y|+v(t)|z|^\alpha.$$

(A2) $\mathrm{d}\mathbb{P}\times\mathrm{d}t$-*a.e.*, for each $z\in\mathbb{R}$, we have

(i) $g(\omega,t,\cdot,z)$ satisfies a time-varying monotonicity condition on $\mathbb{R}_-$, i.e.,

$$\forall\ (y_1,y_2)\in\mathbb{R}_-\times\mathbb{R}_-,\ \ \mathrm{sgn}(y_1-y_2)(g(\omega,t,y_1,z)-g(\omega,t,y_2,z))\leqslant k_1(t)|y_1-y_2|.$$

(ii) $g(\omega,t,\cdot,z)$ satisfies a time-varying Lipschitz continuity condition on $\mathbb{R}_+$, i.e.,

$$\forall\ (y_1,y_2)\in\mathbb{R}_+\times\mathbb{R}_+,\quad |g(\omega,t,y_1,z)-g(\omega,t,y_2,z)|\leqslant k_2(t)|y_1-y_2|.$$

(A3) $\mathrm{d}\mathbb{P}\times\mathrm{d}t$-*a.e.*, for each $y\in\mathbb{R}$, we have

(i) $g(\omega,t,y,\cdot)$ is a time-varying Lipschitz continuous function on $[-a,a]$, i.e.,

$$\forall\ (z_1,z_2)\in[-a,a]\times[-a,a],\quad |g(\omega,t,y,z_1)-g(\omega,t,y,z_2)|\leqslant c_1(t)|z_1-z_2|.$$

(ii) $g(\omega,t,y,\cdot)$ is convex on $(-\infty,-a]$ and $[a,+\infty)$, respectively.

(A4) $\mathrm{d}\mathbb{P}\times\mathrm{d}t$-*a.e.*, for each $y\in\mathbb{R}$, we have

$$\forall\ z\in[a,+\infty)\,,\quad g(\omega,t,y,z)-g(\omega,t,y,a)\geqslant -c_2(t)(z-a)$$

and

$$\forall\ z\in(-\infty,-a]\,,\quad g(\omega,t,y,z)-g(\omega,t,y,-a)\geqslant c_3(t)(z+a).$$

Remark 5.3 (i) *If $g(\omega,t,\cdot,z)$ satisfies the time-varying Lipschitz continuity condition on $\mathbb{R}$, i.e.,* $\mathrm{d}\mathbb{P}\times\mathrm{d}t$*-a.e., for each $z\in\mathbb{R}$, we have*

$$\forall\ (y_1,y_2)\in\mathbb{R}\times\mathbb{R},\quad |g(\omega,t,y_1,z)-g(\omega,t,y_2,z)|\leqslant\beta(t)|y_1-y_2|,$$

then $g(\omega,t,\cdot,z)$ satisfies assumption (A2). Consequently, assumption (A2) generalizes the time-varying Lipschitz continuity condition of the generator g with respect to y.

(ii) *If $g(\omega,t,y,\cdot)$ is convex on $\mathbb{R}$, then $g(\omega,t,y,\cdot)$ satisfies assumption (A3) with $a=0$. Consequently, assumption (A3) generalizes the convexity condition of the generator g with respect to z.*

(iii) *If $g(\omega,t,y,\cdot)$ satisfies condition (ii) of (A3) and has a right derivative at point a with a lower bound $-c_2(t)$ and a left derivative at the point $-a$ with an upper bound $c_3(t)$, then $g(\omega,t,y,\cdot)$ satisfies assumption (A4).*

Proposition 5.4 *Let $d=1$. If the generator g satisfies assumptions (H1), (A1), (A2), (A3), and (A4), then it also satisfies condition (i) of assumption (UN′).*

Proof Given arbitrary $(y_i,z_i)\in\mathbb{R}\times\mathbb{R}$, $i=1,2$ and $\theta\in(0,1)$, and let $\delta_\theta y$ and $\delta_\theta z$ be defined in assumption (UN′). It follows from assumptions (H1) and (A2), along with Lemma 6.1, that

$$\begin{aligned}&\mathbf{1}_{\{y_1>\theta y_2\}}[g(\omega,t,y_1,z_2)-\theta g(\omega,t,y_2,z_2)]\\&\leqslant(1-\theta)\left[(k_1(t)+k_2(t))|\delta_\theta y|+(k_1(t)+k_2(t))|y_2|+\mathbf{1}_{\{y_1>\theta y_2\}}g(\omega,t,y_2,z_2)\right].\end{aligned}\tag{5.3}$$

Furthermore, define

$$\phi(\omega,t,y_1,|z|):=f_t(\omega)+u(t)|y_1|+v(t)|z|^\alpha,$$

where $\phi(\omega,t,y_1,\cdot):\mathbb{R}_+\mapsto\mathbb{R}_+$ is a non-decreasing continuous function. In light of assumptions (H1), (A1) (A3), and (A4), along with Lemma 6.2 and (iii) of Remark 5.3, we obtain that

$$\begin{aligned}&g(\omega,t,y_1,z_1)-\theta g(\omega,t,y_1,z_2)\\&\leqslant(1-\theta)\left[\phi(\omega,t,|y_1|,|\delta_\theta z|+2a)+2c(t)|\delta_\theta z|+11c(t)a+22\phi(\omega,t,|y_1|,a)\right]\\&=(1-\theta)\left[23f_t(\omega)+23u(t)|y_1|+v(t)||\delta_\theta z|+2a|^\alpha+22v(t)a^\alpha+2c(t)|\delta_\theta z|+11c(t)a\right]\\&\leqslant(1-\theta)\left[23f_t(\omega)+23u(t)|y_1|+2^\alpha v(t)(|\delta_\theta z|^\alpha+(2a)^\alpha)+22v(t)a^\alpha+2c(t)|\delta_\theta z|+11c(t)a\right]\\&\leqslant(1-\theta)\left[23f_t(\omega)+23u(t)|y_1-\theta y_2|+23u(t)\theta|y_2|+4v(t)|\delta_\theta z|^\alpha\right.\\&\qquad\left.+38v(t)a^\alpha+2c(t)|\delta_\theta z|+11c(t)a\right]\\&\leqslant(1-\theta)\left[\bar f_t(\omega)+23u(t)|y_2|+23u(t)|\delta_\theta y|+(4v(t)+2c(t))|\delta_\theta z|^\alpha\right],\end{aligned}$$

where $c(t):=c_1(t)\vee c_2(t)\vee c_3(t)$, and the process $\bar f_t(\omega):=23f_t(\omega)+38v(t)a^\alpha+2c(t)+11c(t)a$ has sub-exponential moments of any order. Combining inequality (5.3) and the last inequality, we obtain

$$\begin{aligned}&\mathbf{1}_{\{y_1>\theta y_2\}}[g(\omega,t,y_1,z_1)-\theta g(\omega,t,y_2,z_2)]\\&=\mathbf{1}_{\{y_1>\theta y_2\}}[g(\omega,t,y_1,z_1)-\theta g(\omega,t,y_1,z_2)]+\mathbf{1}_{\{y_1>\theta y_2\}}\theta[g(\omega,t,y_1,z_2)-\theta g(\omega,t,y_2,z_2)]\\&\quad-\mathbf{1}_{\{y_1>\theta y_2\}}\theta(1-\theta)g(\omega,t,y_2,z_2)\\&\leqslant\mathbf{1}_{\{y_1>\theta y_2\}}[g(\omega,t,y_1,z_1)-\theta g(\omega,t,y_1,z_2)]+(1-\theta)\theta\left[(k_1(t)+k_2(t))|\delta_\theta y|+(k_1(t)+k_2(t))|y_2|\right]\\&\leqslant(1-\theta)\left[\bar f_t(\omega)+(23u(t)+k_1(t)+k_2(t))|y_2|+(23u(t)+k_1(t)+k_2(t))|\delta_\theta y|\right.\\&\qquad\left.+(v(t)+2c(t))|\delta_\theta z|^\alpha\right].\end{aligned}$$

Then, the desired inequality (5.1) holds with $f_t(\omega):=\bar f_t(\omega)$, $\beta(t):=23u(t)+k_1(t)+k_2(t)$ and $\gamma(t):=v(t)+2c(t)$. This completes the proof. □

Next, let $d=1$, $\bar{u}(\cdot)\in L^1([0,T];\mathbb{R}_+)$, and $\bar{c}(\cdot),\bar{v}(\cdot)\in L^{\frac{2}{2-\alpha}}([0,T];\mathbb{R}_+)$ satisfying $0<\int_0^T\bar{c}(t)\mathrm{d}t<+\infty$ and $0<\int_0^T\bar{v}(t)\mathrm{d}t<+\infty$. To explore sufficient conditions ensuring that assumption (UN) holds, we introduce the following assumptions on the generator g.

(A5) g exhibits time-varying linear growth in y and time-varying logarithmic growth in z, i.e., $\mathrm{d}\mathbb{P}\times\mathrm{d}t$-$a.e.$, for each $(y,z)\in\mathbb{R}\times\mathbb{R}$, we have

$$|g(\omega,t,y,z)|\leqslant f_t(\omega)+\bar{u}(t)|y|+\bar{v}(t)\left[\ln(e+|z|)\right]^{\frac{\alpha^*}{2}}.$$

(A6) There exists a constant $a\geqslant 0$ such that $\mathrm{d}\mathbb{P}\times\mathrm{d}t$-$a.e.$, for each $y\in\mathbb{R}$, we have

(i) $g(\omega,t,y,\cdot)$ is a time-varying Lipschitz continuous function on $\mathbb{R}$, i.e.,

$$\forall\,(z_1,z_2)\in\mathbb{R}\times\mathbb{R},\quad |g(\omega,t,y,z_1)-g(\omega,t,y,z_2)|\leqslant\bar{c}(t)|z_1-z_2|.$$

(ii) $g(\omega,t,y,\cdot)$ decreases on $(-\infty,-a]$ and increases on $[a,+\infty)$.

Proposition 5.5 is the third main result of this section.

Proposition 5.5 *Let $d=1$. If the generator g satisfies assumptions (H1), (A2), (A5), and (A6), then it also satisfies condition (i) of assumption (UN).*

Proof Given arbitrary $(y_i,z_i)\in\mathbb{R}\times\mathbb{R}$, $i=1,2$, and $\theta\in(0,1)$, and let $\delta_\theta y$ and $\delta_\theta z$ be defined in assumption (UN). Define

$$\phi(\omega,t,y_1,|z|):=f_t(\omega)+\bar{u}(t)|y_1|+\bar{v}(t)\left[\ln(e+|z|)\right]^{\frac{\alpha^*}{2}},$$

where $\phi(\omega,t,y_1,\cdot):\mathbb{R}_+\mapsto\mathbb{R}_+$ is a non-decreasing continuous function. In light of assumptions (H1), (A5), and (A6), along with Lemma 6.3, we obtain

$$\begin{aligned}
&g(\omega,t,y_1,z_1)-\theta g(\omega,t,y_1,z_2)\\
\leqslant{}&(1-\theta)\left[4\bar{c}(t)|\delta_\theta z|+4\bar{c}(t)a+11\phi(\omega,t,|y_1|,a)+\phi(\omega,t,|y_1|,|z_2|)\right]\\
={}&(1-\theta)\Big[12f_t(\omega)+12\bar{u}(t)|y_1|+4\bar{c}(t)|\delta_\theta z|+4\bar{c}(t)a+\bar{v}(t)\left[\ln(e+|z_2|)\right]^{\frac{\alpha^*}{2}}\\
&+11\bar{v}(t)\left[\ln(e+a)\right]^{\frac{\alpha^*}{2}}\Big]\\
\leqslant{}&(1-\theta)\Big[12f_t(\omega)+12\bar{u}(t)|y_1-\theta y_2|+12\bar{u}(t)\theta|y_2|+4\bar{c}(t)|\delta_\theta z|+4\bar{c}(t)a\\
&+\bar{v}(t)\left[\ln(e+|z_2|)\right]^{\frac{\alpha^*}{2}}+11\bar{v}(t)\left[\ln(e+a)\right]^{\frac{\alpha^*}{2}}\Big]\\
\leqslant{}&(1-\theta)\left[\hat{f}_t(\omega)+12\bar{u}(t)|y_2|+12\bar{u}(t)|\delta_\theta y|+\bar{v}(t)\left[\ln(e+|z_2|)\right]^{\frac{\alpha^*}{2}}+4\bar{c}(t)|\delta_\theta z|^\alpha\right],
\end{aligned}$$

where the process $\hat{f}_t(\omega):=12f_t(\omega)+11\bar{v}(t)\left[\ln(e+a)\right]^{\frac{\alpha^*}{2}}+4\bar{c}(t)a+4\bar{c}(t)$ has sub-exponential moments of any order. Furthermore, in light of assumptions (H1) and (A2), it follows from Lemma 6.1 that inequality (5.3) holds. Combining inequality (5.3) and the last inequality, we obtain

$$\begin{aligned}
&\mathbf{1}_{\{y_1>\theta y_2\}}[g(\omega,t,y_1,z_1)-\theta g(\omega,t,y_2,z_2)]\\
={}&\mathbf{1}_{\{y_1>\theta y_2\}}[g(\omega,t,y_1,z_1)-\theta g(\omega,t,y_1,z_2)]+\mathbf{1}_{\{y_1>\theta y_2\}}\theta[g(\omega,t,y_1,z_2)-\theta g(\omega,t,y_2,z_2)]\\
&-\mathbf{1}_{\{y_1>\theta y_2\}}\theta(1-\theta)g(\omega,t,y_2,z_2)\\
\leqslant{}&\mathbf{1}_{\{y_1>\theta y_2\}}[g(\omega,t,y_1,z_1)-\theta g(\omega,t,y_1,z_2)]+(1-\theta)\theta\left[(k_1(t)+k_2(t))|\delta_\theta y|+(k_1(t)+k_2(t))|y_2|\right]\\
\leqslant{}&(1-\theta)\Bigg[\hat{f}_t(\omega)+(k_1(t)+k_2(t)+12\bar{u}(t))|y_2|+\bar{v}(t)\left[\ln(e+|z_2|)\right]^{\frac{\alpha^*}{2}}\\
&+(k_1(t)+k_2(t)+12\bar{u}(t))|\delta_\theta y|+4\bar{c}(t)|\delta_\theta z|^\alpha\Bigg].
\end{aligned}$$

Thus, the desired inequality (4.1) holds with $f_t(\omega) := \hat{f}_t(\omega)$, $\beta(t) := k_1(t) + k_2(t) + 12\bar{u}(t)$ and $\gamma(t) := \bar{v}(t) + 4\bar{c}(t)$. The proof is then complete. $\square$

Remark 5.6 *Let $d \geqslant 2$. It is straightforward to verify that for each $i = 1, 2, \ldots, d$, if $g^i(\cdot) : \mathbb{R} \to \mathbb{R}$ satisfies assumption (UN) (resp. (UN′)) with $\mathbb{R}^2$ instead of $\mathbb{R}^{1+d}$, then $g(\omega, t, y, z) := \sum_{i=1}^{d} g^i(z^i)$ satisfies assumption (UN) (resp. (UN′)), where $z = (z^1, z^2, \ldots, z^d)^{\mathrm{T}} \in \mathbb{R}^d$.*

The following examples demonstrate that Theorem 4.3 is not covered by any existing results.

Example 5.7 *Let $\beta(\cdot) \in L^1([0,T]; \mathbb{R}_+)$, $\gamma(\cdot) \in L^1([0,T]; \mathbb{R}_+) \cap L^{\frac{2}{2-\alpha}}([0,T]; \mathbb{R}_+)$. For $(\omega, t, y, z) \in \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d$ with $z = (z^1, z^2, \ldots, z^d)^{\mathrm{T}}$, define*

$$g(\omega, t, y, z) := |B_t(\omega)| + \beta(t)\left(\sqrt[3]{y}\mathbf{1}_{y\leqslant 0} + \sin y\mathbf{1}_{y>0}\right) + \gamma(t)\sum_{i=1}^{d} q(z^i) + \gamma(t)|z|^{\alpha},$$

where

$$q(x) := \begin{cases} -x, & x \leqslant -2; \\ -\frac{3}{2}x - 1, & -2 < x < 2; \\ -2x, & x \geqslant 2. \end{cases}$$

In light of Proposition 5.4, (iii) of Remark 4.1, and Remark 5.6, it is straightforward to verify that the generator g satisfies assumptions (EX1), (EX2), and (UN′), but it does not satisfy either the time-varying Lipschitz continuity condition or the convexity/concavity condition with respect to (y, z). In particular, we note that g is not time-varying locally Lipschitz continuous in y.

Example 5.8 *Let $\beta(\cdot) \in L^1([0,T]; \mathbb{R}_+)$, $\gamma(\cdot) \in L^{\frac{2}{2-\alpha}}([0,T]; \mathbb{R}_+)$ satisfying inequality (4.3). For $(\omega, t, y, z) \in \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d$ with $z = (z^1, z^2, \ldots, z^d)^{\mathrm{T}}$, define*

$$g(\omega, t, y, z) := |B_t(\omega)| + \beta(t)\sqrt{|y|}\mathbf{1}_{y\leqslant 0} + \gamma(t)\sum_{i=1}^{d}\left[\ln(e + |z^i|)\right]^{\frac{\alpha^*}{2}} + 2\gamma(t)|z|^{\alpha}.$$

According to Proposition 5.5, (iii) of Remark 4.1, and Remark 5.6, it is straightforward to verify that the generator g satisfies assumptions (EX1), (EX2), and (UN), but it does not satisfy either the time-varying Lipschitz continuity condition or the convexity/concavity condition with respect to (y, z). In particular, g is not time-varying locally Lipschitz continuous in y.

6. Appendix

Lemma 6.1 *Let $f(x) : \mathbb{R} \mapsto \mathbb{R}$ be a continuous function. Assume that there exist two constants k_1, $k_2 > 0$ such that*

(i) *$f(x)$ satisfies a monotonicity condition on $\mathbb{R}_-$, i.e.,*

$$\forall\ (x_1, x_2) \in \mathbb{R}_- \times \mathbb{R}_-, \quad \mathrm{sgn}(x_1 - x_2)(f(x_1) - f(x_2)) \leqslant k_1|x_1 - x_2|.$$

(ii) *$f(x)$ is Lipschitz continuous on $\mathbb{R}_+$, i.e.,*

$$\forall\ (x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+, \quad |f(x_1) - f(x_2)| \leqslant k_2|x_1 - x_2|.$$

Then, for each $(x_1, x_2) \in \mathbb{R} \times \mathbb{R}$ and $\theta \in (0,1)$, we have

$$\mathbf{1}_{\{x_1>\theta x_2\}}\frac{f(x_1) - \theta f(x_2)}{1-\theta} \leqslant (k_1 + k_2)\left|\frac{x_1 - \theta x_2}{1-\theta}\right| + (k_1 + k_2)|x_2| + \mathbf{1}_{\{x_1>\theta x_2\}}f(x_2). \tag{6.1}$$

Proof Given arbitrary $(x_1, x_2) \in \mathbb{R} \times \mathbb{R}$ and $\theta \in (0,1)$, note that inequality (6.1) is clearly true when $x_1 \leqslant \theta x_2$. We only need to consider the following three cases: $x_1 > \theta x_2 \geqslant 0$, $0 \geqslant x_1 > \theta x_2 > x_2$ and $x_1 > 0 > \theta x_2$.

(a) The case of $x_1 > \theta x_2 \geqslant 0$: In light of (ii), we have

$$\begin{aligned} f(x_1) - \theta f(x_2) &\leqslant |f(x_1) - f(x_2)| + (1-\theta) f(x_2) \leqslant k_2 |x_1 - x_2| + (1-\theta) f(x_2) \\ &\leqslant (1-\theta)\left[k_2 \left|\frac{x_1 - \theta x_2}{1-\theta}\right| + k_2 |x_2| + f(x_2)\right]. \end{aligned}$$

(b) The case of $0 \geqslant x_1 > \theta x_2 > x_2$: Using (i), we obtain

$$\begin{aligned} f(x_1) - \theta f(x_2) &= f(x_1) - f(x_2) + (1-\theta) f(x_2) \leqslant k_1 |x_1 - x_2| + (1-\theta) f(x_2) \\ &\leqslant (1-\theta)\left[k_1 \left|\frac{x_1 - \theta x_2}{1-\theta}\right| + k_1 |x_2| + f(x_2)\right]. \end{aligned}$$

(c) The case of $x_1 > 0 > \theta x_2$: It follows from (b) that

$$f(0) - \theta f(x_2) \leqslant (1-\theta)\left[k_1 \left|\frac{-\theta x_2}{1-\theta}\right| + k_1 |x_2| + f(x_2)\right].$$

Then, by combining (ii) and the last inequality, we have

$$\begin{aligned} f(x_1) - \theta f(x_2) &= f(x_1) - f(0) + f(0) - \theta f(x_2) \\ &\leqslant k_2 |x_1| + (1-\theta)\left[k_1 \left|\frac{-\theta x_2}{1-\theta}\right| + k_1 |x_2| + f(x_2)\right] \\ &= (1-\theta)\left[(k_1 + k_2)\left|\frac{x_1 - \theta x_2}{1-\theta}\right| + k_1 |x_2| + f(x_2)\right]. \end{aligned}$$

In conclusion, inequality (6.1) holds. □

Lemma 6.2 *Let $f(x) : \mathbb{R} \mapsto \mathbb{R}$ be a continuous function such that $|f(x)| \leqslant \phi(|x|)$ for each $x \in \mathbb{R}$, where $\phi(x) : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a nondecreasing continuous function. Assume that there exist two constants $a \geqslant 0$ and $k > 0$ such that*

(i) *$f(x)$ is Lipschitz continuous on $[-a, a]$, i.e.,*

$$\forall\ (x_1, x_2) \in [-a, a] \times [-a, a], \ \ |f(x_1) - f(x_2)| \leqslant k |x_1 - x_2|;$$

(ii) *$f(x)$ is convex on $(-\infty, -a]$ and $[a, +\infty)$, respectively;*
(iii) *Both $f'_-(-a)$ and $f'_+(a)$ are finite value.*

Then, for each $(x_1, x_2) \in \mathbb{R} \times \mathbb{R}$ and $\theta \in (0,1)$, we have

$$\frac{f(x_1) - \theta f(x_2)}{1-\theta} \leqslant \phi\left(\left|\frac{x_1 - \theta x_2}{1-\theta}\right| + 2a\right) + 2k_0 \left|\frac{x_1 - \theta x_2}{1-\theta}\right| + 11 k_0 a + 22\phi(a), \tag{6.2}$$

where

$$k_0 := |f'_-(-a)| \vee |f'_+(a)| \vee k.$$

Proof Without loss of generality, we assume that $a > 0$. The proof is organized into the following four steps.

Step 1 We construct a continuous function $g(x)$ such that

- $g(x) = f(x)$, $\forall\ x \in (-\infty, -a] \cup [a, +\infty)$;
- $g(x)$ is convex on $(-\infty, x_0]$ and $[x_0, +\infty)$ for some $x_0 \in (-a, a)$, respectively.

In fact, we define $g(x)$ (Figure 1) as follows:

$$g(x)=\begin{cases} f(x), & x\in(-\infty,-a];\\ k_0(x+a)+f(-a), & x\in(-a,x_0];\\ -k_0(x-a)+f(a), & x\in[x_0,a);\\ f(x), & x\in[a,+\infty), \end{cases}$$

where, in light of (i),

$$x_0:=\frac{f(-a)-f(a)}{2k_0}\in(-a,a).$$

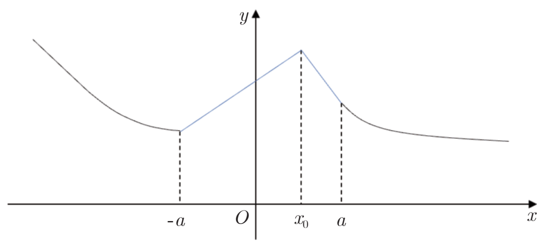

Figure 1 A graph of $g(x)$

By the definition of $g(x)$, we get

$$\forall\, x\in\mathbb{R},\quad |g(x)|\leqslant|f(x)|+k_0a+2\sup_{[-a,a]}|f(x)|\leqslant\phi(|x|)+k_0a+2\phi(a). \tag{6.3}$$

Step 2 Define $\bar g(x):=g(x+x_0)-g(x_0)$(Figure 2). Given arbitrary $(x_1,x_2)\in\mathbb{R}\times\mathbb{R}$ and $\theta\in(0,1)$, we prove that $\bar g(x)$ satisfies the following inequality

$$\frac{\bar g(x_1)-\theta\bar g(x_2)}{1-\theta}\leqslant\phi\left(\left|\frac{x_1-\theta x_2}{1-\theta}\right|+a\right)+k_0\left|\frac{x_1-\theta x_2}{1-\theta}\right|+2k_0a+5\phi(a). \tag{6.4}$$

First, $\bar g(0)=0$ and we have

- $\bar g(x)$ is Lipschitz continuous on $[-a-x_0,a-x_0]$, i.e.,

$$\forall\,(x_1,x_2)\in[-a-x_0,a-x_0]\times[-a-x_0,a-x_0],\quad |\bar g(x_1)-\bar g|(x_2)|\leqslant k_0|x_1-x_2|;$$

- $\bar g(x)$ is convex on $(-\infty,0]$ and $[0,+\infty)$, respectively.

On the other hand, from the definition of $\bar g$ and inequality (6.3), we know that

$$\forall\, x\in\mathbb{R},\quad |\bar g(x)|\leqslant|g(x+x_0)|+|g(x_0)|\leqslant\phi(|x|+a)+2k_0a+5\phi(a). \tag{6.5}$$

Next, we consider the following four cases: $x_1,x_2\geqslant0$, $x_1,x_2\leqslant0$, $x_2\leqslant0\leqslant x_1$ and $x_1\leqslant 0\leqslant x_2$.

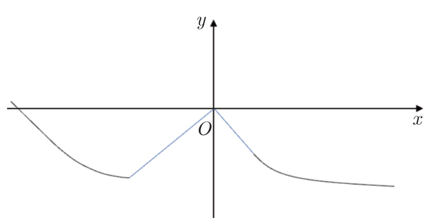

Figure 2 A graph of $\bar g(x)$

(a) The case of $x_1, x_2 \geqslant 0$: Define $\bar{g}_1(x)$ (Figure 3) as follows:

$$\bar{g}_1(x) := \begin{cases} \bar{g}(x), & x \in [0, +\infty); \\ -k_0 x, & x \in (-\infty, 0). \end{cases}$$

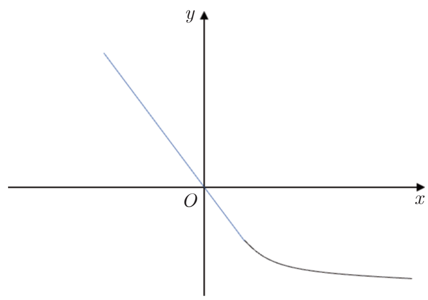

Figure 3 A graph of $\bar{g}_1(x)$

It is clear that $\bar{g}_1(x)$ is convex on $\mathbb{R}$ and satisfies $\forall\, x \in \mathbb{R}$, $\bar{g}_1(x) \leqslant |\bar{g}(x)| + k_0|x|$. Then, we have

$$\begin{aligned} \frac{\bar{g}(x_1) - \theta\bar{g}(x_2)}{1-\theta} = \frac{\bar{g}_1(x_1) - \theta\bar{g}_1(x_2)}{1-\theta} &= \frac{\bar{g}_1\left(\theta x_2 + (1-\theta)\dfrac{x_1 - \theta x_2}{1-\theta}\right) - \theta\bar{g}_1(x_2)}{1-\theta} \\ &\leqslant \bar{g}_1\left(\frac{x_1 - \theta x_2}{1-\theta}\right) \leqslant \left|\bar{g}\left(\frac{x_1 - \theta x_2}{1-\theta}\right)\right| + k_0\left|\frac{x_1 - \theta x_2}{1-\theta}\right|. \end{aligned}$$

(b) The case of $x_1, x_2 \leqslant 0$: Define $\bar{g}_2(x)$ (Figure 4) as follows:

$$\bar{g}_2(x) := \begin{cases} \bar{g}(x), & x \in (-\infty, 0]; \\ k_0 x, & x \in (0, +\infty). \end{cases}$$

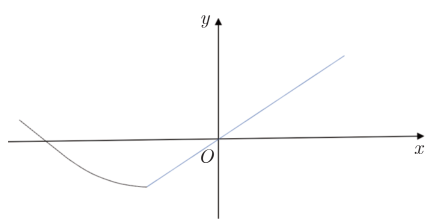

Figure 4 A graph of $\bar{g}_2(x)$

It is clear that $\bar{g}_2(x)$ is convex on $\mathbb{R}$ and satisfies

$$\forall\, x \in \mathbb{R}, \quad \bar{g}_2(x) \leqslant |\bar{g}(x)| + k_0|x|.$$

Then, we have

$$\begin{aligned} \frac{\bar{g}(x_1) - \theta\bar{g}(x_2)}{1-\theta} = \frac{\bar{g}_2(x_1) - \theta\bar{g}_2(x_2)}{1-\theta} &= \frac{\bar{g}_2\left(\theta x_2 + (1-\theta)\dfrac{x_1 - \theta x_2}{1-\theta}\right) - \theta\bar{g}_2(x_2)}{1-\theta} \\ &\leqslant \bar{g}_2\left(\frac{x_1 - \theta x_2}{1-\theta}\right) \leqslant \left|\bar{g}\left(\frac{x_1 - \theta x_2}{1-\theta}\right)\right| + k_0\left|\frac{x_1 - \theta x_2}{1-\theta}\right|. \end{aligned}$$

(c) The case of $x_2 \leqslant 0 \leqslant x_1$: It follows from (a) that

$$\frac{\bar{g}(x_1)-\theta\bar{g}(0)}{1-\theta}\leqslant \bar{g}_1\left(\frac{x_1}{1-\theta}\right)=\bar{g}\left(\frac{x_1}{1-\theta}\right). \tag{6.6}$$

In view of (b), we get

$$\frac{\bar{g}(0)-\theta\bar{g}(x_2)}{1-\theta}\leqslant \bar{g}_2\left(\frac{-\theta x_2}{1-\theta}\right)=k_0\left|\frac{-\theta x_2}{1-\theta}\right|. \tag{6.7}$$

Then, by combining inequalities (6.6) and (6.7), along with $\bar{g}(0)=0$, we obtain

$$\frac{\bar{g}(x_1)-\theta\bar{g}(x_2)}{1-\theta}=\frac{\bar{g}(x_1)-\theta\bar{g}(0)+\bar{g}(0)-\theta\bar{g}(x_2)}{1-\theta}\leqslant \bar{g}\left(\frac{x_1}{1-\theta}\right)+k_0\left|\frac{x_1-\theta x_2}{1-\theta}\right|.$$

(d) The case of $x_1\leqslant 0\leqslant x_2$: It follows from (b) that

$$\frac{\bar{g}(x_1)-\theta\bar{g}(0)}{1-\theta}\leqslant \bar{g}_2\left(\frac{x_1}{1-\theta}\right)=\bar{g}\left(\frac{x_1}{1-\theta}\right). \tag{6.8}$$

By virtue of (a), we have

$$\frac{\bar{g}(0)-\theta\bar{g}(x_2)}{1-\theta}\leqslant \bar{g}_1\left(\frac{-\theta x_2}{1-\theta}\right)=k_0\left|\frac{-\theta x_2}{1-\theta}\right|. \tag{6.9}$$

Then, by combining inequalities (6.8) and (6.9), along with $\bar{g}(0)=0$, we obtain

$$\frac{\bar{g}(x_1)-\theta\bar{g}(x_2)}{1-\theta}=\frac{\bar{g}(x_1)-\theta\bar{g}(0)+\bar{g}(0)-\theta\bar{g}(x_2)}{1-\theta}\leqslant \bar{g}\left(\frac{x_1}{1-\theta}\right)+k_0\left|\frac{x_1-\theta x_2}{1-\theta}\right|.$$

In conclusion, in light of inequality (6.5), inequality (6.4) holds.

Step 3 Given arbitrary $(x_1,x_2)\in\mathbb{R}\times\mathbb{R}$ and $\theta\in(0,1)$, we prove that $g(x)$ satisfies the following inequality

$$\frac{g(x_1)-\theta g(x_2)}{1-\theta}\leqslant \phi\left(\left|\frac{x_1-\theta x_2}{1-\theta}\right|+2a\right)+k_0\left|\frac{x_1-\theta x_2}{1-\theta}\right|+4k_0a+7\phi(a). \tag{6.10}$$

In fact, it follows from the definition of $g(x)$ that

$$|g(x_0)|\leqslant k_0a+2\sup_{[-a,a]}|f(x)|\leqslant k_0a+2\phi(a).$$

By combining inequality (6.4) with the last inequality, we obtain

$$\begin{aligned}
\frac{g(x_1)-\theta g(x_2)}{1-\theta}&=\frac{(g(x_1-x_0+x_0)-g(x_0))-\theta(g(x_2-x_0+x_0)-g(x_0))}{1-\theta}+g(x_0)\\
&\leqslant \frac{\bar{g}(x_1-x_0)-\theta\bar{g}(x_2-x_0)}{1-\theta}+k_0a+2\phi(a)\\
&\leqslant \phi\left(\left|\frac{(x_1-x_0)-\theta(x_2-x_0)}{1-\theta}\right|+a\right)+k_0\left|\frac{(x_1-x_0)-\theta(x_2-x_0)}{1-\theta}\right|+3k_0a+7\phi(a)\\
&\leqslant \phi\left(\left|\frac{x_1-\theta x_2}{1-\theta}-x_0\right|+a\right)+k_0\left|\frac{x_1-\theta x_2}{1-\theta}\right|+k_0|x_0|+3k_0a+7\phi(a)\\
&\leqslant \phi\left(\left|\frac{x_1-\theta x_2}{1-\theta}\right|+2a\right)+k_0\left|\frac{x_1-\theta x_2}{1-\theta}\right|+4k_0a+7\phi(a).
\end{aligned}$$

Hence, the inequality (6.10) holds.

Step 4 We prove that $f(x)$ satisfies (6.2). Define $h(x):=f(x)-g(x)$. Then, $h(x)$ satisfies (i) with Lipschitz constant k_0, and $h(x) \equiv 0$ when $|x| \geqslant a$. Note that $h(x)$ has a bound $M := k_0 a + 3 \sup_{[-a,a]} |f(x)| \leqslant k_0 a + 3\phi(a)$, for each $(x_1, x_2) \in \mathbb{R} \times \mathbb{R}$ and $\theta \in (0,1)$, we have

$$\begin{aligned} |h(\theta x_2) - h(x_2)| &= |h(\theta x_2) - h(x_2)| \mathbf{1}_{\{\theta \in (0,1/2]\}} + |h(\theta x_2) - h(x_2)| \mathbf{1}_{\{\theta \in (1/2,1)\}} \\ &\leqslant (1-\theta) \frac{2M}{1-\theta} \mathbf{1}_{\{\theta \in (0,1/2]\}} + (1-\theta) k_0 |x_2| \mathbf{1}_{\{|x_2| \leqslant 2a\}} \mathbf{1}_{\{\theta \in (1/2,1)\}} \\ &\leqslant (1-\theta)(4M + 2k_0 a). \end{aligned}$$

Then,

$$\begin{aligned} \frac{h(x_1) - \theta h(x_2)}{1-\theta} &\leqslant \frac{|h(x_1) - h(\theta x_2)|}{1-\theta} + \frac{|h(\theta x_2) - h(x_2)|}{1-\theta} + |h(x_2)| \\ &\leqslant k_0 \left| \frac{x_1 - \theta x_2}{1-\theta} \right| + 5M + 2k_0 a \leqslant k_0 \left| \frac{x_1 - \theta x_2}{1-\theta} \right| + 7k_0 a + 15\phi(a). \end{aligned} \tag{6.11}$$

Finally, in light of $f(x) = g(x) + h(x)$, and inequalities (6.10) and (6.11), we deduce that

$$\begin{aligned} \frac{f(x_1) - \theta f(x_2)}{1-\theta} &= \frac{g(x_1) - \theta g(x_2)}{1-\theta} + \frac{h(x_1) - \theta h(x_2)}{1-\theta} \\ &\leqslant \phi\left(\left| \frac{x_1 - \theta x_2}{1-\theta} \right| + 2a \right) + 2k_0 \left| \frac{x_1 - \theta x_2}{1-\theta} \right| + 11k_0 a + 22\phi(a). \end{aligned}$$

Therefore, the inequality (6.2) holds. □

Lemma 6.3 *Let $f(x) : \mathbb{R} \mapsto \mathbb{R}$ be a continuous function such that $|f(x)| \leqslant \phi(|x|)$ for each $x \in \mathbb{R}$, where $\phi(x) : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a nondecreasing continuous function. Assume that there exist two constants $a \geqslant 0$ and $k > 0$ such that*

(i) *$f(x)$ is Lipschitz continuous on $\mathbb{R}$, i.e.,*

$$\forall\, (x_1, x_2) \in \mathbb{R} \times \mathbb{R}, \quad |f(x_1) - f(x_2)| \leqslant k|x_1 - x_2|;$$

(ii) *$f(x)$ decreases on $(-\infty, -a]$ and increases on $[a, +\infty)$.*

Then, for each $(x_1, x_2) \in \mathbb{R} \times \mathbb{R}$ and $\theta \in (0,1)$, we have

$$\frac{f(x_1) - \theta f(x_2)}{1-\theta} \leqslant 4k \left| \frac{x_1 - \theta x_2}{1-\theta} \right| + 4ka + 11\phi(a) + \phi(|x_2|). \tag{6.12}$$

Proof Without lost of generality, we assume that $a > 0$ and $f(a) < f(-a)$. The proof is organized into the following three steps.

Step 1 We construct a continuous function $g(x)$ such that

(1) $g(x) = f(x)$, $\forall\, x \in (-\infty, -a] \cup [a, +\infty)$;
(2) $g(x)$ decreases on $(-\infty, 0]$ and increases on $[0, +\infty)$.

In fact, we define $g(x)$ (Figure 5) as follows:

$$g(x) = \begin{cases} f(x), & x \in (-\infty, -a]; \\ -k_0(x+a) + f(-a), & x \in (-a, 0]; \\ f(a), & x \in [0, a); \\ f(x), & x \in [a, +\infty), \end{cases}$$

where given (i),

$$k_0 := \left| \frac{f(a) - f(-a)}{a} \right| \leqslant 2k.$$

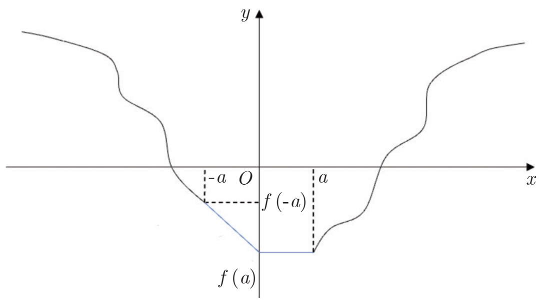

Figure 5 A graph of $g(x)$

We note that $g(x)$ satisfies (i) with Lipschitz constant $2k$, i.e.,

$$\forall\ (x_1, x_2) \in \mathbb{R} \times \mathbb{R}, \quad |g(x_1) - g(x_2)| \leqslant 2k|x_1 - x_2|. \tag{6.13}$$

By the definition of $g(x)$, we get

$$\forall\ x \in \mathbb{R}, \quad g(x) \leqslant f(x) + \sup_{[-a,a]} |f(x)| \leqslant \phi(|x|) + \phi(a). \tag{6.14}$$

Step 2 We prove that $g(x)$ satisfies the following inequality:

$$\frac{g(x_1) - \theta g(x_2)}{1 - \theta} \leqslant 2k \left| \frac{x_1 - \theta x_2}{1 - \theta} \right| + \phi(|x_2|) + \phi(a). \tag{6.15}$$

Given arbitrary $(x_1, x_2) \in \mathbb{R} \times \mathbb{R}$ and $\theta \in (0, 1)$, we need to consider the following six cases: $x_1 \geqslant x_2 \geqslant 0$, $x_1 \leqslant x_2 \leqslant 0$, $x_2 \geqslant x_1 \geqslant 0$, $x_2 \leqslant x_1 \leqslant 0$, $x_1 \leqslant 0 \leqslant x_2$ and $x_2 \leqslant 0 \leqslant x_1$.

(a) The case of $x_1 \geqslant x_2 \geqslant 0$: By inequality (6.13) and (6.14), we have

$$\begin{aligned} \frac{g(x_1) - \theta g(x_2)}{1 - \theta} &= \frac{g(x_1) - g(x_2)}{1 - \theta} + g(x_2) \leqslant 2k \left| \frac{x_1 - x_2}{1 - \theta} \right| + g(x_2) \\ &= 2k \frac{x_1 - \theta x_2 + \theta x_2 - x_2}{1 - \theta} + g(x_2) = 2k \frac{x_1 - \theta x_2}{1 - \theta} - 2kx_2 + g(x_2) \\ &\leqslant 2k \left| \frac{x_1 - \theta x_2}{1 - \theta} \right| + g(x_2) \leqslant 2k \left| \frac{x_1 - \theta x_2}{1 - \theta} \right| + \phi(|x_2|) + \phi(a). \end{aligned}$$

(b) The case of $x_1 \leqslant x_2 \leqslant 0$: On account of inequality (6.13) and (6.14), we have

$$\begin{aligned} \frac{g(x_1) - \theta g(x_2)}{1 - \theta} &= \frac{g(x_1) - g(x_2)}{1 - \theta} + g(x_2) \leqslant 2k \left| \frac{x_1 - x_2}{1 - \theta} \right| + g(x_2) \\ &= 2k \frac{-x_1 + \theta x_2 - \theta x_2 + x_2}{1 - \theta} + g(x_2) = 2k \frac{-x_1 + \theta x_2}{1 - \theta} + 2kx_2 + g(x_2) \\ &\leqslant 2k \left| \frac{x_1 - \theta x_2}{1 - \theta} \right| + g(x_2) \leqslant 2k \left| \frac{x_1 - \theta x_2}{1 - \theta} \right| + \phi(|x_2|) + \phi(a). \end{aligned}$$

(c) The case of $x_2 \geqslant x_1 \geqslant 0$: In light of (2), we have

$$\frac{g(x_1) - \theta g(x_2)}{1 - \theta} = \frac{g(x_1) - g(x_2)}{1 - \theta} + g(x_2) \leqslant g(x_2) \leqslant \phi(|x_2|) + \phi(a).$$

(d) The case of $x_2 \leqslant x_1 \leqslant 0$: Using (2), we have

$$\frac{g(x_1)-\theta g(x_2)}{1-\theta} = \frac{g(x_1)-g(x_2)}{1-\theta} + g(x_2) \leqslant g(x_2) \leqslant \phi(|x_2|) + \phi(a).$$

(e) The case of $x_1 \leqslant 0 \leqslant x_2$: It follows from (c) that

$$\frac{g(0)-\theta g(x_2)}{1-\theta} \leqslant \phi(|x_2|) + \phi(a).$$

Then, by combining inequality (6.13) and the last inequality, we obtain

$$\begin{aligned}\frac{g(x_1)-\theta g(x_2)}{1-\theta} &= \frac{g(x_1)-g(0)}{1-\theta} + \frac{g(0)-\theta g(x_2)}{1-\theta}\\ &\leqslant 2k\left|\frac{x_1}{1-\theta}\right| + \phi(|x_2|) + \phi(a) \leqslant 2k\left|\frac{x_1-\theta x_2}{1-\theta}\right| + \phi(|x_2|) + \phi(a).\end{aligned}$$

(f) The case of $x_2 \leqslant 0 \leqslant x_1$: According to (d), it follows that

$$\frac{g(0)-\theta g(x_2)}{1-\theta} \leqslant \phi(|x_2|) + \phi(a).$$

Then, using inequality (6.13) and the above inequality, we obtain

$$\begin{aligned}\frac{g(x_1)-\theta g(x_2)}{1-\theta} &= \frac{g(x_1)-g(0)}{1-\theta} + \frac{g(0)-\theta g(x_2)}{1-\theta}\\ &\leqslant 2k\left|\frac{x_1}{1-\theta}\right| + \phi(|x_2|) + \phi(a) \leqslant 2k\left|\frac{x_1-\theta x_2}{1-\theta}\right| + \phi(|x_2|) + \phi(a).\end{aligned}$$

In conclusion, inequality (6.15) holds.

Step 3 We prove that $f(x)$ satisfies inequality (6.12). Define $h(x) := f(x) - g(x)$. Then, $h(x)$ is a Lipschitz continuous function with Lipschitz constant $2k$, and $h(x) \equiv 0$ when $|x| \geqslant a$. Note that $h(x)$ has a bound $M := 2\sup_{[-a,a]} |f(x)| \leqslant 2\phi(a)$. For each $(x_1, x_2) \in \mathbb{R}\times\mathbb{R}$ and $\theta \in (0,1)$, we have

$$\begin{aligned}|h(\theta x_2) - h(x_2)| &= |h(\theta x_2) - h(x_2)|\mathbf{1}_{\{\theta\in(0,1/2]\}} + |h(\theta x_2) - h(x_2)|\mathbf{1}_{\{\theta\in(1/2,1)\}}\\ &\leqslant (1-\theta)\frac{2M}{1-\theta}\mathbf{1}_{\{\theta\in(0,1/2]\}} + 2(1-\theta)k|x_2|\mathbf{1}_{\{|x_2|\leqslant 2a\}}\mathbf{1}_{\{\theta\in(1/2,1)\}}\\ &\leqslant (1-\theta)(4M+4ka).\end{aligned}$$

Then,

$$\begin{aligned}\frac{h(x_1)-\theta h(x_2)}{1-\theta} &\leqslant \frac{|h(x_1)-h(\theta x_2)|}{1-\theta} + \frac{|h(\theta x_2)-h(x_2)|}{1-\theta} + |h(x_2)|\\ &\leqslant 2k\left|\frac{x_1-\theta x_2}{1-\theta}\right| + 5M + 4ka \leqslant 2k\left|\frac{x_1-\theta x_2}{1-\theta}\right| + 4ka + 10\phi(a).\end{aligned}\tag{6.16}$$

Finally, in light of $f(x) = g(x) + h(x)$, inequalities (6.15) and (6.16), we deduce that

$$\begin{aligned}\frac{f(x_1)-\theta f(x_2)}{1-\theta} &= \frac{g(x_1)-\theta g(x_2)}{1-\theta} + \frac{h(x_1)-\theta h(x_2)}{1-\theta}\\ &\leqslant 4k\left|\frac{x_1-\theta x_2}{1-\theta}\right| + 4ka + 11\phi(a) + \phi(|x_2|).\end{aligned}$$

Therefore, inequality (6.12) holds. □

Acknowledgements

This work is supported by the National Natural Science Foundation of China (Grant No. 12171471), the Natural Science Foundation of Jiangsu Province (Grant No. BK20231057), and the Graduate Innovation Program of China University of Mining and Technology (Grant No. 2023WLJCRCZL143). The authors would like to thank the anonymous referees for their careful reading and valuable suggestions.

References

[1] Bahlali, K., Eddahbi, M. and Ouknine, Y., Quadratic BSDE with L^2-terminal data: Krylov's estimate, Itô–Krylov's formula and existence results, The Annals of Probability, 2017, 45: 2377–2397.

[2] Briand, P. and Hu, Y., BSDE with quadratic growth and unbounded terminal value, Probability Theory and Related Fields, 2006, 136(4): 604–618.

[3] Briand, P. and Hu, Y., Quadratic BSDEs with convex generators and unbounded terminal conditions, Probability Theory and Related Fields, 2008, 141(3–4): 543–567.

[4] Chen, Z. and Wang, B., Infinite time interval BSDEs and the convergence of g-martingales, Journal of the Australian Mathematical Society, 2000, 69(2): 187–211.

[5] Delbaen, F., Risk measures for non-integrable random variables, Mathematical Finance, 2009, 19(2): 329–333.

[6] Delbaen, F., Hu, Y. and Richou, A., On the uniqueness of solutions to quadratic BSDEs with convex generators and unbounded terminal conditions, Annales de l'Institut Henri Poincaré-Probabilités et Statistiques, 2011, 47(2): 559–574.

[7] Delbaen, F., Hu, Y. and Richou, A., On the uniqueness of solutions to quadratic BSDEs with convex generators and unbounded terminal conditions: The critical case, Discrete and Continuous Dynamical Systems, 2015, 35(11): 5273–5283.

[8] Delbaen, F. and Tang, S., Harmonic analysis of stochastic equations and backward stochastic differential equations, Probability Theory and Related Fields, 2010, 146(1–2): 291–336.

[9] El Karoui, N., Peng, S. and Quenez, M. C., Backward stochastic differential equations in finance, Mathematical Finance, 1997, 7(1): 1–71.

[10] Fan, S., Bounded solutions, L^p $(p > 1)$ solutions and L^1 solutions for one dimensional BSDEs under general assumptions, Stochastic Processes and their Applications, 2016, 126(5): 1511–1552.

[11] Fan, S., Existence,uniqueness and stability of L^1 solutions for multidimensional BSDEs with generators of one-sided Osgood type, Journal of Theoretical Probability, 2018, 31(3): 1860–1899.

[12] Fan, S. and Hu, Y., Well-posedness of scalar BSDEs with sub-quadratic generators and related PDEs, Stochastic Processes and their Applications, 2021, 131: 21–50.

[13] Fan, S., Hu, Y. and Tang, S., On the uniqueness of solutions to quadratic BSDEs with non-convex generators and unbounded terminal conditions, Comptes Rendus de l'Académie des Sciences-Series I-Mathematics, 2020, 358(2): 227–235.

[14] Fan, S. and Jiang, L., L^p $(p > 1)$ solutions for one-dimensional BSDEs with linear-growth generators, Journal of Applied Mathematics and Computing, 2012, 38(1–2): 295–304.

[15] Fan, S., Jiang, L. and Tian, D., One-dimensional BSDEs with finite and infinite time horizons, Stochastic Processes and their Applications, 2011, 131(3): 427–440.

[16] Fan, S., Wang, T. and Yong, J., Multi-dimensional super-linear backward stochastic volterra integral equations, arXiv: 2211.04078, 2022.

[17] Hu, Y., Imkeller, P. and Müller, M., Utility maximization in incomplete markets, The Annals of Applied Probability, 2005, 15(3): 1691–1712.

[18] Jiang, L., Convexity, translation invariance and subadditivity for g-expectations and related risk measures, The Annals of Applied Probability, 2008, 18(1): 245–258.

[19] Kobylanski, M., Backward stochastic differential equations and partial differential equations with quadratic growth, The Annals of Probability, 2000, 28(2): 256–279.

[20] Luo, H. and Fan, S., Bounded solutions for general time interval BSDEs with quadratic growth coefficients and stochastic conditions, Stochastics and Dynamics, 2018, 18(5): 1850034.

[21] Pardoux, E. and Peng, S., Adapted solutions of a backward stochastic differential equation, Systems & Control Letters, 1990, 14(1): 55–61.

[22] Peng, S., A generalized stochastic maximum principle for optimal control problems, SIAM Journal on Control and Optimization, 1990, 28(4): 966–979.

[23] Peng, S., Probabilistic interpretation for systems of quasilinear parabolic partial differential equations, Stochastics and Stochastic Reports, 1991, 37(1–2): 61–74.

[24] Peng, S., Backward stochastic differential equations and applications to optimal control, Applied Mathematics and Optimization, 1993, 27(2): 125–144.

[25] Yang, H., L^p solutions of quadratic BSDEs, arXiv: 1506.08146v2, 2017.

[26] Zong, Z. and Hu, F., Infinite time interval backward stochastic differential equations with continuous coefficients, SpringerPlus, 2016, 5(1): 1733.

[27] Zong, Z. and Hu, F., L^p solutions of infinite time interval backward doubly stochastic differential equations under monotonicity and general increasing conditions, Journal of Mathematical Analysis and Applications, 2018, 458(2): 1486–1511.