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Non-homogeneous stochastic linear-quadratic optimal control problems with multidimensional state and regime switching

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Abstract In this paper, we explore non-homogeneous stochastic linear-quadratic (LQ) optimal control problems with multidimensional states and regime switching. We focus on the corresponding stochastic Riccati equation (SRE), which mirrors that of the homogeneous stochastic LQ optimal control problem, and the adjoint backward stochastic differential equation (BSDE), which arises from the non-homogeneous terms in the state equation and cost functional. We solve both the SRE and adjoint BSDE using the contraction mapping method, which helps represent the closed-loop optimal control and the optimal value of our problems. In particular, we extend some results of Hu et al. [7] to the multidimensional case.

Keywords Non-homogeneous stochastic LQ problem, Regime switching, Multidimensional state, BSDE, Unbounded coefficients

2020 Mathematics Subject Classification 93E20, 60H30

1. Introduction

The linear-quadratic (LQ) optimal control problem is a widely researched and classical topic in control theory. Wonham studied the stochastic LQ optimal control problem with deterministic coefficients in [24] and discussed the corresponding Riccati equation in detail in [25]. Bismut [1] first addressed the stochastic Riccati equation (SRE) in 1976; however, he was only able to solve some special cases. Peng [17] used Bellman's principle and Wonham's method to establish existence and uniqueness results for a more general SRE. After extensive research (see [8–11]), Tang [22] effectively resolved the most comprehensive SRE. We refer to [16, 20, 21] for further advancements in stochastic LQ optimal control problems and SREs.

In fields like engineering, management, and economics, different states often require different equations, leading to LQ problems with Poisson jumps. Wu and Wang [26] introduced a stochastic LQ problem whose state equation is driven by both Brownian motion and Poisson process, solving the related Riccati equation whose coefficients partially degenerate to zero. Hu and Øksendal [5] first studied the non-homogeneous stochastic LQ problem with Poisson jumps

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although they did not solve the Riccati equations. Yu [29] obtained the existence and uniqueness results for forward–backward stochastic equations with Poisson jumps under monotone conditions, applying these to backward stochastic LQ problems. Li et al. [12] discussed the stochastic LQ problem with Poisson processes under indefinite conditions.

Another form of state switching under varying equations is referred to as regime switching. Li and Zhou [13] studied a stochastic LQ problem with Markovian jumps in parameter values over a finite time horizon, while Li et al. [14] explored the infinite time horizon case. Liu et al. [15] examined near-optimal controls for regime switching LQ problems with indefinite control weight costs. Recent breakthroughs include Zhang et al. [30], who established both open-loop and closed-loop solvability for stochastic LQ optimal control problems in Markovian regime switching systems with deterministic coefficients. Hu et al. [6] investigated the stochastic LQ control problem with regime switching, random coefficients, and cone control constraints in a one-dimensional state equation. Wen et al. [23], Chen and Luo [2] used different methods to solve multidimensional stochastic LQ optimal control problems with regime switching and the corresponding matrix-valued Riccati equations. The LQ problem with regime switching has numerous applications in mathematical finance, especially in mean-variance portfolio selection and asset–liability management, see [7, 18, 27, 28, 31].

In this paper, we focus on solving non-homogeneous stochastic LQ optimal control problems with multidimensional states and regime switching. We begin by revisiting the matrix-valued SRE, also discussed in our previous work [2] for homogeneous problems. Compared with [2], we now establish both the existence and uniqueness of the solution for the Riccati equation. Next, we address the adjoint BSDE, which is determined by the SRE and non-homogeneous terms in the state equation and cost functional. Unlike the scalar-valued case in Hu et al. [7], our adjoint BSDEs are multidimensional with unbounded coefficients under each regime, rendering the transformation method used by [7] ineffective. To the best of our knowledge, the only related work dealing with this kind of multidimensional BSDEs with unbounded coefficients is Delbaen and Tang [3]; however, their results cannot be applied to our adjoint BSDEs. By using a contraction mapping method, we achieve solvability of the adjoint BSDEs, extending some results from [3]. Finally, we derive the optimal control and optimal value for the non-homogeneous stochastic LQ problem, represented by the solutions of the SRE and adjoint BSDE.

The rest of this paper is organized as follows: In Section 2, we formulate the multidimensional stochastic LQ optimal control problems with regime switching and present our three main results. Section 3 provides insights into the multidimensional BSDEs with unbounded coefficients. The proofs of these theorems are detailed sequentially in Section 4.

2. Formulation of the problem and main results

Let $T>0$ be fixed and $(\Omega, \mathcal{F}, \mathbb{P})$ be a fixed complete probability space on which are defined a standard one-dimensional Brownian motion $W=\{W(t); 0 \leqslant t \leqslant T\}$ and a continuous-time stationary Markov chain α_t valued in a finite state space $\mathcal{M}=\{1,2,\ldots,\ell\}$ with $\ell>1$. We assume that $W(t)$ and α_t are independent processes. The Markov chain has a generator $Q=(q_{ij})_{\ell\times\ell}$ with $q_{ij} \geqslant 0$ for $i \neq j$ and $\sum_{j=1}^{\ell} q_{ij}=0$ for every $i \in \mathcal{M}$. We define the filtrations $\mathcal{F}_t=\sigma\{W(s), \alpha_s: 0 \leqslant s \leqslant t\} \vee \mathcal{N}$ and $\mathcal{F}_t^W=\sigma\{W(s): 0 \leqslant s \leqslant t\} \vee \mathcal{N}$, where $\mathcal{N}$ is the totality of all the $\mathbb{P}$-null sets of $\mathcal{F}$. For a random variable η, $\|\eta\|_\infty$ denotes the L^∞-norm of η, i.e., $\|\eta\|_\infty := \operatorname{ess\,sup}_{\omega} |\eta(\omega)|$. Equalities and inequalities between random variables and processes are understood in the P-a.s. and $P \otimes \mathrm{d}t$-a.e. sense, respectively.

We use the following notation throughout the paper:

$\mathbb{R}^n:$ the n-dimensional Euclidean space with the Euclidean norm $|\cdot|$;
$\mathbb{R}^{m\times n}:$ the Euclidean space of all $(m\times n)$ real matrices;
$\mathbb{S}^n:$ the space of all symmetric $(n\times n)$ real matrices;
$I_n:$ the identity matrix of size n;
$M^\top:$ the transpose of a matrix M;
$tr(M):$ the trace of a matrix M;
$\langle\cdot,\cdot\rangle:$ the Frobenius inner product on $\mathbb{R}^{n\times m}$, which is defined by $\langle A,B\rangle = tr(A^\top B)$;
$|M|:$ the Frobenius norm of a matrix M, defined by $\big(tr(MM^\top)\big)^{\frac{1}{2}}$;

Furthermore, we introduce the following spaces of random processes: for the Euclidean spaces $\mathbb{H}=\mathbb{R}^n,\mathbb{R}^{n\times m},\mathbb{S}^n$, $p\geqslant 2$ and all $\{\mathcal{F}_t\}_{t\geqslant 0}$-stopping time $\tau\leqslant T$:

$$L^\infty_{\mathcal{F}}(\Omega;\mathbb{H})=\{\xi:\Omega\to\mathbb{H}\mid \xi \text{ is } \mathcal{F}_T\text{-measurable, and essentially bounded }\};$$

$$\mathcal{H}^p_{\mathcal{F}}(0,T;\mathbb{H})=\Bigg\{\phi:[0,T]\times\Omega\to\mathbb{H}\mid\phi(\cdot)\text{ is an }\{\mathcal{F}_t\}_{t\geqslant0}\text{-adapted process with }\mathbb{E}\left(\int_0^T|\phi(t)|\mathrm{d}t\right)^p<\infty\Bigg\};$$

$$L^p_{\mathcal{F}}(0,T;\mathbb{H})=\Bigg\{\phi:[0,T]\times\Omega\to\mathbb{H}\mid\phi(\cdot)\text{ is an }\{\mathcal{F}_t\}_{t\geqslant0}\text{-adapted process with }\mathbb{E}\left(\int_0^T|\phi(t)|^2\mathrm{d}t\right)^{\frac{p}{2}}<\infty\Bigg\};$$

$$L^p_{\mathcal{F}}(\Omega;C(0,T;\mathbb{H}))=\Bigg\{\phi:[0,T]\times\Omega\to\mathbb{H}\mid\phi(\cdot)\text{ is an }\{\mathcal{F}_t\}_{t\geqslant0}\text{-adapted process and continuous with }\mathbb{E}\left(\sup_{t\in[0,T]}|\phi(t)|^p\right)<\infty\Bigg\};$$

$$L^\infty_{\mathcal{F}}(0,T;\mathbb{H})=\Bigg\{\phi:[0,T]\times\Omega\to\mathbb{H}\mid\phi(\cdot)\text{ is an }\{\mathcal{F}_t\}_{t\geqslant0}\text{-adapted essentially bounded process}\Bigg\};$$

$$L^{1,\mathrm{bmo}}_{\mathcal{F}}(0,T;\mathbb{H})=\Bigg\{\phi:[0,T]\times\Omega\to\mathbb{H}\mid\phi(\cdot)\text{ is an }\mathcal{F}\text{-progressively measurable process with }\sup_{0\leqslant\tau\leqslant T}\left\|\mathbb{E}\left[\int_\tau^T|\phi(s)|\mathrm{d}s\mid\mathcal{F}_\tau\right]\right\|_\infty<\infty\Bigg\};$$

$$L^{2,\mathrm{bmo}}_{\mathcal{F}}(0,T;\mathbb{H})=\Bigg\{\phi:[0,T]\times\Omega\to\mathbb{H}\mid\phi(\cdot)\text{ is an }\mathcal{F}\text{-progressively measurable process with }\sup_{0\leqslant\tau\leqslant T}\left\|\mathbb{E}\left[\int_\tau^T|\phi(s)|^2\mathrm{d}s\mid\mathcal{F}_\tau\right]\right\|_\infty^{\frac{1}{2}}<\infty\Bigg\}.$$

$L^\infty_{\mathcal{F}^W}(\Omega;\mathbb{H})$, $\mathcal{H}^p_{\mathcal{F}^W}(0,T;\mathbb{H})$, $L^p_{\mathcal{F}^W}(0,T;\mathbb{H})$, $L^p_{\mathcal{F}^W}(\Omega;C(0,T;\mathbb{H}))$, $L^\infty_{\mathcal{F}^W}(0,T;\mathbb{H})$ and $L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(0,T;\mathbb{H})$ are defined in a same manner by replacing $\mathcal{F}$ by $\mathcal{F}^W$.

For convenience, we recall the following definition from [3]:

Definition 2.1 *Let $M(\cdot)\in L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(0,T;\mathbb{H})$ and $\varepsilon>0$. A finite sequence of stopping times $0=t_0\leqslant t_1\leqslant\cdots\leqslant t_h=T$ is said to $\varepsilon-$slice $M(\cdot)$ in $L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(0,T;\mathbb{H})$ if $\|M(\cdot)\|_{L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(t_i,t_{i+1};\mathbb{H})}\leqslant\varepsilon$, for $i=0,1,\cdots,h-1$. If such a sequence of stopping times exists, we say that $M(\cdot)$ is $\varepsilon-$sliceable in $L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(0,T;\mathbb{H})$.*

We now introduce the non-homogeneous multidimensional stochastic LQ optimal problem with regime switching. We consider the following n-dimensional controlled linear stochastic differential equation over the finite time interval $[0,T]$:

$$\begin{cases} \mathrm{d}X(t)=\left[A\left(t,\alpha_t\right)X(t)+B\left(t,\alpha_t\right)u(t)+b\left(t,\alpha_t\right)\right]\mathrm{d}t \\ \qquad\qquad +\left[C\left(t,\alpha_t\right)X(t)+D\left(t,\alpha_t\right)u(t)+\sigma\left(t,\alpha_t\right)\right]\mathrm{d}W(t), \quad t\in[0,T], \\ X(0)=x, \quad \alpha_0=i_0, \end{cases} \tag{2.1}$$

where $A(t,\omega,i),B(t,\omega,i),C(t,\omega,i),D(t,\omega,i)$ are all $\left\{\mathcal{F}_t^W\right\}_{t\geqslant 0}$-adapted processes of suitable sizes for $i\in\mathcal{M}$ and $x\in\mathbb{R}^n$ is an initial state, $i_0\in\mathcal{M}$ is an initial regime. The solution $X=\{X(t);0\leqslant t\leqslant T\}$ of (2.1), valued in $\mathbb{R}^n$, is called a state process; the process $u=\{u(t);0\leqslant t\leqslant T\}$ of (2.1), valued in $\mathbb{R}^m$, is called a control which influences the state X, and is taken from the space $\mathcal{U}[0,T]:=L^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$.

To measure the performance of control $u(\cdot)$, we introduce the following quadratic cost functional:

$$\begin{aligned} J\left(x,i_0,u(\cdot)\right):=\mathbb{E}\Bigg[\int_0^T \Big(&\left\langle Q(t,\alpha_t)\left(X(t)-q(t,\alpha_t)\right),X(t)-q(t,\alpha_t)\right\rangle+\left\langle R(t,\alpha_t)\left(u(t)-r(t,\alpha_t)\right),\right. \\ &\left. u(t)-r(t,\alpha_t)\right\rangle\Big)\mathrm{d}t+\left\langle G(\alpha_T)\left(X(T)-g(\alpha_T)\right),X(T)-g(\alpha_T)\right\rangle\Bigg]. \end{aligned} \tag{2.2}$$

For state equation (2.1) and cost functional (2.2), we introduce the following assumption:

($\mathscr{A}1$) For all $i\in\mathcal{M}$,

$$\begin{cases} A(t,\omega,i),\ C(t,\omega,i)\in L^\infty_{\mathcal{F}^W}(0,T;\mathbb{R}^{n\times n}), \\ B(t,\omega,i),\ D(t,\omega,i)\in L^\infty_{\mathcal{F}^W}(0,T;\mathbb{R}^{n\times m}), \\ Q(t,\omega,i)\in L^\infty_{\mathcal{F}^W}(0,T;\mathbb{S}^n), \\ R(t,\omega,i)\in L^\infty_{\mathcal{F}^W}(0,T;\mathbb{S}^m), \\ G(\omega,i)\in L^\infty_{\mathcal{F}^W}(\Omega;\mathbb{S}^n), \\ b(t,\omega,i),\ \sigma(t,\omega,i),\ q(t,\omega,i)\in L^\infty_{\mathcal{F}^W}(0,T;\mathbb{R}^n), \\ r(t,\omega,i)\in L^\infty_{\mathcal{F}^W}(0,T;\mathbb{R}^m), \\ g(\omega,i)\in L^\infty_{\mathcal{F}^W}(\Omega;\mathbb{R}^n). \end{cases}$$

Under condition ($\mathscr{A}1$), for any initial state x and any control $u(\cdot)\in\mathcal{U}[0,T]$, standard SDE theory shows that equation (2.1) has a unique solution $X(\cdot)\in L^2_{\mathcal{F}}(\Omega;C(0,T;\mathbb{R}^n))$. We refer to such $(X(\cdot),u(\cdot))$ as an admissible pair.

The following problem, called the stochastic LQ optimal control problem with regime switching, can then be formulated:

Problem (SLQ) For any initial pair $(x,i_0)\in\mathbb{R}^n\times\mathcal{M}$, find a control $u^*\in\mathcal{U}[0,T]$ such that

$$J\left(x,i_0,u^*\right)=\inf_{u\in\mathcal{U}[0,T]}J\left(x,i_0,u\right)\equiv V\left(x,i_0\right). \tag{2.3}$$

Any element $u^*\in\mathcal{U}[0,T]$ satisfying (2.3) is called an optimal control of Problem (SLQ) corresponding to the initial pair $(x,i_0)\in\mathbb{R}^n\times\mathcal{M}$. The corresponding state process $X^*(\cdot)\equiv X(\cdot;u^*)$ is referred to as an optimal state process. We also define $V\left(x,i_0\right)$ the value function of

Problem (SLQ). Our objective is to solve Problem (SLQ). In classical LQ problems, it is commonly observed that positive definite coefficients in (2.2) provide a sufficient condition for the solvability of the Riccati equation, and subsequently the LQ problem, as illustrated in [22]. Therefore, we also introduce the following assumption:

($\mathscr{A}$2) For all $i \in \mathcal{M}, t \in [0,T]$ and some $\lambda > 0$,

$$R(t,i) \geqslant \lambda I_m, \quad Q(t,i) \geqslant 0, \quad G(i) \geqslant 0.$$

Remark 2.2 *When $R(\cdot,i) > 0$, $Q(\cdot,i) - S(\cdot,i)^\top R(\cdot,i)^{-1} S(\cdot,i) \geqslant 0$, $i \in \mathcal{M}$, the SLQ problem with the state equation*

$$\begin{cases} \mathrm{d}X(t) = \left[A(t,\alpha_t) X(t) + B(t,\alpha_t) u(t) + b(t,\alpha_t)\right] \mathrm{d}t \\ \qquad\qquad + \left[C(t,\alpha_t) X(t) + D(t,\alpha_t) u(t) + \sigma(t,\alpha_t)\right] \mathrm{d}W(t), \quad t \in [0,T], \\ X(0) = x, \alpha_0 = i_0 \end{cases}$$

and the cost functional

$$\begin{aligned} J_1(x, i_0, u(\cdot)) = E\Big[& \langle G(\alpha_T)(X(T) - g(\alpha_T)), X(T) - g(\alpha_T) \rangle \\ & + \int_0^T \left\langle \begin{pmatrix} Q(t,\alpha_t) & S(t,\alpha_t)^\top \\ S(t,\alpha_t) & R(t,\alpha_t) \end{pmatrix} \begin{pmatrix} X(t) - q(t,\alpha_t) \\ u(t) - r(t,\alpha_t) \end{pmatrix}, \begin{pmatrix} X(t) - q(t,\alpha_t) \\ u(t) - r(t,\alpha_t) \end{pmatrix} \right\rangle \mathrm{d}t \Big] \end{aligned}$$

is equivalent to another one with

$$\begin{cases} \mathrm{d}X(t) = \left[\widetilde{A}(t,\alpha_t) X(t) + B(t,\alpha_t) \widetilde{u}(t) + b(t,\alpha_t)\right] \mathrm{d}t \\ \qquad\qquad + \left[\widetilde{C}(t,\alpha_t) X(t) + D(t,\alpha_t) \widetilde{u}(t) + \sigma(t,\alpha_t)\right] \mathrm{d}W(t), \quad t \in [0,T], \\ X(0) = x, \quad \alpha_0 = i_0 \end{cases}$$

and

$$\begin{aligned} J_2(x, i_0, \widetilde{u}(\cdot)) = E\Big[& \langle G(\alpha_T)(X(T) - g(\alpha_T)), X(T) - g(\alpha_T) \rangle \\ & + \int_0^T \left\langle \begin{pmatrix} \widetilde{Q}(t,\alpha_t) & 0 \\ 0 & R(t,\alpha_t) \end{pmatrix} \begin{pmatrix} X(t) - q(t,\alpha_t) \\ \widetilde{u}(t) - \widetilde{r}(t,\alpha_t) \end{pmatrix}, \begin{pmatrix} X(t) - q(t,\alpha_t) \\ \widetilde{u}(t) - \widetilde{r}(t,\alpha_t) \end{pmatrix} \right\rangle \mathrm{d}t \Big], \end{aligned}$$

where

$$\begin{aligned} &\widetilde{A}(t,\alpha_t) = A(t,\alpha_t) - B(t,\alpha_t) R(t,\alpha_t)^{-1} S(t,\alpha_t), \\ &\widetilde{C}(t,\alpha_t) = C(t,\alpha_t) - D(t,\alpha_t) R(t,\alpha_t)^{-1} S(t,\alpha_t), \\ &\widetilde{r}(t,\alpha_t) = r(t,\alpha_t) + R(t,\alpha_t)^{-1} S(t,\alpha_t) q(t,\alpha_t), \\ &\widetilde{Q}(t,\alpha_t) = Q(t,\alpha_t) - S(t,\alpha_t)^\top R(t,\alpha_t)^{-1} S(t,\alpha_t), \\ &\widetilde{u}(t) = u(t) + R(t,\alpha_t)^{-1} S(t,\alpha_t) X(t). \end{aligned}$$

Therefore, we only need to consider $S(\cdot,i) = 0, i \in \mathcal{M}$ in this paper.

2.1 SRE and adjoint BSDE

Inspired by classic methods for non-homogeneous LQ optimal control problems, we consider the following $n \times n$-dimensional SRE:

$$
\begin{cases}
\mathrm{d}P(t,i) = -\big[P(t,i)A(t,i) + A(t,i)^\top P(t,i) + C(t,i)^\top P(t,i)C(t,i) + \Lambda(t,i)C(t,i) \\
\qquad\qquad + C(t,i)^\top \Lambda(t,i) + Q(t,i) + \sum_{j=1}^{l} q_{ij}P(t,j) - \big(P(t,i)B(t,i) + C(t,i)^\top P(t,i)D(t,i) \\
\qquad\qquad + \Lambda(t,i)D(t,i)\big) \big(R(t,i) + D(t,i)^\top P(t,i)D(t,i)\big)^{-1} \big(B(t,i)^\top P(t,i) \\
\qquad\qquad + D(t,i)^\top P(t,i)C(t,i) + D(t,i)^\top \Lambda(t,i)\big)\big]\,\mathrm{d}t + \Lambda(t,i)\mathrm{d}W(t), \\
R(t,i) + D(t,i)^\top P(t,i)D(t,i) > 0, \quad t \in [0,T], \\
P(T,i) = G(i), \quad i \in \mathcal{M},
\end{cases}
\tag{2.4}
$$

and n-dimensional adjoint BSDE:

$$
\begin{cases}
\mathrm{d}K(t,i) = -\Big[\big(A(t,i) - B(t,i)\Gamma(t,i)\big)^\top K(t,i) + \big(C(t,i) - D(t,i)\Gamma(t,i)\big)^\top L(t,i) \\
\qquad\qquad + \Gamma(t,i)^\top \big(D(t,i)^\top P(t,i)\sigma(t,i) - R(t,i)r(t,i)\big) + Q(t,i)q(t,i) - P(t,i)b(t,i) \\
\qquad\qquad - \big(C(t,i)^\top P(t,i) + \Lambda(t,i)\big)\sigma(t,i) + \sum_{j=1}^{l} q_{ij}K(t,j)\Big]\,\mathrm{d}t + L(t,i)\mathrm{d}W(t), \quad t \in [0,T], \\
K(T,i) = G(i)g(i), \quad i \in \mathcal{M},
\end{cases}
\tag{2.5}
$$

where

$$
\Gamma(t,i) = \big(R(t,i)+D(t,i)^\top P(t,i)D(t,i)\big)^{-1}\big(B(t,i)^\top P(t,i)+D(t,i)^\top P(t,i)C(t,i)+D(t,i)^\top \Lambda(t,i)\big).
$$

Definition 2.3 *A vector process* $(P(\cdot,i),\Lambda(\cdot,i))_{i=1}^{\ell}$ *is called a solution of SRE* (2.4)*, if it satisfies* (2.4)*, and* $(P(\cdot,i),\Lambda(\cdot,i)) \in L^\infty_{\mathcal{F}^W}(\Omega; C(0,T;\mathbb{S}^n)) \times L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(0,T;\mathbb{S}^n)$ *for all* $i \in \mathcal{M}$.

Definition 2.4 *A vector process* $(K(\cdot,i),L(\cdot,i))_{i=1}^{\ell}$ *is called a solution of BSDE* (2.5)*, if it satisfies* (2.4)*, and* $(K(\cdot,i),L(\cdot,i)) \in L^\infty_{\mathcal{F}^W}(\Omega; C(0,T;\mathbb{R}^n)) \times L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(0,T;\mathbb{R}^n)$ *for all* $i \in \mathcal{M}$.

Before presenting our main results, we recall the following lemma from standard matrix analysis, which directly follows from [4, Theorem 7.4.1.1]. This lemma will be utilized throughout this paper.

Lemma 2.5 *We assume* $A, B \in \mathbb{S}^n$ *with* B *being positive semidefinite. Then, with* $\lambda_{max}(A)$ *denoting the largest eigenvalue of* A *, we have*

$$
\mathrm{tr}(AB) \leqslant \lambda_{\max}(A) \cdot \mathrm{tr}(B).
$$

Corollary 2.6 *Let* $A \in \mathbb{R}^{n\times m}, B \in \mathbb{R}^{m\times k}$. *We have*

$$
|AB| \leqslant |A| \cdot |B|.
$$

2.2 Main results

In our previous work [2], we established the existence of the solution for SRE (2.4) using Piccard iteration. We will now demonstrate that the uniqueness of the solution can also be achieve through the contraction mapping method.

Theorem 2.7 *Let* $(\mathscr{A}1)$ *and* $(\mathscr{A}2)$ *hold. There exists a sufficiently small constant* $L_\sigma > 0$ *, when*

$$
|D(t,i)R(t,i)^{-1}D(t,i)^\top| \leqslant L_\sigma, \quad i \in \mathcal{M}, \quad t \in [0,T], \tag{2.6}
$$

SRE (2.4) *has a unique solution* $(P(\cdot,i),\Lambda(\cdot,i))_{i=1}^{l}$ *such that* $(P(\cdot,i),\Lambda(\cdot,i))\in L_{\mathcal{F}^W}^{\infty}(0,T;\mathbb{S}^n)\times L_{\mathcal{F}^W}^{2,\mathrm{bmo}}(0,T;\mathbb{S}^n)$ *and* $P(\cdot,i)\geqslant 0$ for all $i\in\mathcal{M}$.

Based on the solvability of SRE (2.4), we propose the following theorem, showing the existence and uniqueness results for the BSDE solution (2.5), even with unbounded coefficients.

Theorem 2.8 *Under the conditions of Theorem 2.7, BSDE* (2.5) *has a unique solution* $(K(\cdot,i),L(\cdot,i))_{i=1}^{l}$ *such that* $(K(\cdot,i),L(\cdot,i))\in L_{\mathcal{F}^W}^{\infty}(0,T;\mathbb{R}^n)\times L_{\mathcal{F}^W}^{2,\mathrm{bmo}}(0,T;\mathbb{R}^n)$ *for all* $i\in\mathcal{M}$.

By leveraging the solvability of SRE (2.4) and BSDE (2.5), we can effectively address Problem (SLQ). The following theorem provides an optimal feedback control for Problem (SLQ) and the optimal value.

Theorem 2.9 *Under the conditions of Theorem 2.7, Problem (SLQ) admits an optimal control, expressed as a feedback function of time* t *, regime* i *, and state* X,

$$\begin{aligned}u^*(t,X,i)=&-\big(R(t,i)+D(t,i)^\top P(t,i)D(t,i)\big)^{-1}\\&\cdot\big[\big(B(t,i)^\top P(t,i)+D(t,i)^\top P(t,i)C(t,i)+D(t,i)^\top\Lambda(t,i)\big)X\\&+D(t,i)^\top P(t,i)\sigma(t,i)-R(t,i)r(t,i)-B(t,i)^\top K(t,i)-D(t,i)^\top L(t,i)\big].\end{aligned}\tag{2.7}$$

Moreover, the corresponding optimal value is

$$\begin{aligned}V(x,i_0)=&\langle P(0,i_0)x,x\rangle-2\langle K(0,i_0),x\rangle+\mathbb{E}\big[\langle G(\alpha_T)g(\alpha_T),g(\alpha_T)\rangle\big]\\&+\mathbb{E}\bigg[\int_0^T\big[\langle Q(t,\alpha_t)q(t,\alpha_t),q(t,\alpha_t)\rangle+\langle R(t,\alpha_t)r(t,\alpha_t),r(t,\alpha_t)\rangle+\langle P(t,\alpha_t)\sigma(t,\alpha_t),\sigma(t,\alpha_t)\rangle\\&-2\langle K(t,\alpha_t),b(t,\alpha_t)\rangle-2\langle L(t,\alpha_t),\sigma(t,\alpha_t)\rangle-\langle\big(R(t,\alpha_t)+D(t,\alpha_t)^\top P(t,\alpha_t)D(t,\alpha_t)\big)^{-1}\\&\cdot\big(D(t,\alpha_t)^\top P(t,\alpha_t)\sigma(t,\alpha_t)-R(t,\alpha_t)r(t,\alpha_t)-B(t,\alpha_t)^\top K(t,\alpha_t)-D(t,\alpha_t)^\top L(t,\alpha_t)\big),\\&D(t,\alpha_t)^\top P(t,\alpha_t)\sigma(t,\alpha_t)-R(t,\alpha_t)r(t,\alpha_t)-B(t,\alpha_t)^\top K(t,\alpha_t)-D(t,\alpha_t)^\top L(t,\alpha_t)\rangle\big)\mathrm{d}t\bigg],\end{aligned}$$

where $(P(\cdot,i),\Lambda(\cdot,i))_{i=1}^{l}$ *is the unique solution of* (2.4) *and* $(K(\cdot,i),L(\cdot,i))_{i=1}^{l}$ *is the unique solution of* (2.5).

3. Multidimensional BSDEs with unbounded coefficients

In this section, we discuss the following k-dimensional BSDE with unbounded coefficients, which takes a more general form of equation (2.5):

$$\begin{cases}\mathrm{d}Y(t)=-[\alpha(t)^\top Y(t)+\beta(t)^\top Z(t)+\gamma(t)^\top Z(t)+\eta(t)]\mathrm{d}t+Z(t)\mathrm{d}W(t),\quad t\in[0,T],\\Y(T)=\xi,\end{cases}\tag{3.1}$$

with $\alpha(\cdot)\in L_{\mathcal{F}^W}^{2,\mathrm{bmo}}(0,T;\mathbb{R}^{k\times k})$, $\beta(\cdot)\in L_{\mathcal{F}^W}^{\infty}(0,T;\mathbb{R}^{k\times k})$, $\eta(\cdot)\in L_{\mathcal{F}^W}^{1,\mathrm{bmo}}(0,T;\mathbb{R}^k)$ and $\gamma(\cdot)$ is δ-sliceable in $L_{\mathcal{F}^W}^{2,\mathrm{bmo}}(0,T;\mathbb{R}^{k\times k})$ for some $\delta\in(0,1)$.

Theorem 3.1 *The equation* (3.1) *admits a unique solution* $(Y(\cdot),Z(\cdot))$ *such that* $(Y(\cdot),Z(\cdot))\in L_{\mathcal{F}^W}^{\infty}(0,T;\mathbb{R}^k)\times L_{\mathcal{F}^W}^{2,\mathrm{bmo}}(0,T;\mathbb{R}^k)$.

Proof We denote $(y(\cdot),z(\cdot))\in L_{\mathcal{F}^W}^{\infty}(0,T;\mathbb{R}^k)\times L_{\mathcal{F}^W}^{2,\mathrm{bmo}}(0,T;\mathbb{R}^k)$ and consider the following BSDEs:

$$\begin{cases}\mathrm{d}Y(t)=-[\alpha(t)^\top y(t)+\beta(t)^\top z(t)+\gamma(t)^\top z(t)+\eta(t)]\mathrm{d}t+Z(t)\mathrm{d}W(t),\quad t\in[0,T],\\Y(T)=\xi.\end{cases}\tag{3.2}$$

From [19, Proposition 2.1], BSDE (3.2) has a unique solution $(y(\cdot),z(\cdot))\in L^p_{\mathcal{F}^W}(\Omega;C(0,T;\mathbb{R}^k))\times L^2_{\mathcal{F}^W}(0,T;\mathbb{R}^k)$.

First, we will prove that for some constants $\varepsilon,a,d,m>0$, $\Gamma(y(\cdot),z(\cdot)):=(Y(\cdot),Z(\cdot))$ is a contraction map on the closed convex set $\mathscr{B}_\varepsilon$ defined by:

$$\mathscr{B}_\varepsilon:=\Bigg\{(y(\cdot),z(\cdot))\in L^\infty_{\mathcal{F}^W}(T-\varepsilon,T;\mathbb{R}^k)\times L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(T-\varepsilon,T;\mathbb{R}^k):\frac{a}{d}\|y(\cdot)\|^2_{L^\infty_{\mathcal{F}^W}(T-\varepsilon,T;\mathbb{R}^k)}$$
$$+\|z(\cdot)\|^2_{L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(T-\varepsilon,T;\mathbb{R}^k)}\leqslant\frac{1}{1-d}\left[\|\xi\|^2_{L^\infty_{\mathcal{F}^W}(\Omega;\mathbb{R}^k)}+m\|\eta(\cdot)\|_{L^{1,\mathrm{bmo}}_{\mathcal{F}^W}(T-\varepsilon,T;\mathbb{R}^k)}\right]\Bigg\},$$

where $(y(\cdot),z(\cdot))$ is defined on $\Omega\times[T-\varepsilon,T]$ and ε,a,d will be determined later (see (3.4)). Applying Itô's formula to $|Y(t)|^2$ on $[T-\varepsilon,T]$ and taking conditional expectation, we have

$$\begin{aligned}
&|Y(t)|^2+\mathbb{E}_t\left[\int_t^T|Z(s)|^2\mathrm{d}s\right]\\
&=\mathbb{E}_t\left[|\xi|^2\right]+2\mathbb{E}_t\left[\int_t^T\langle\alpha(s)^\top y(s)+\beta(s)^\top z(s)+\gamma(s)^\top z(s)+\eta(s),Y(s)\rangle\mathrm{d}s\right]\\
&\leqslant\mathbb{E}_t\left[|\xi|^2\right]+\frac{a}{\varepsilon}\mathbb{E}_t\left[\int_t^T|y(s)|^2\mathrm{d}s\right]+\frac{\varepsilon}{a}\mathbb{E}_t\left[\int_t^T|\alpha(s)Y(s)|^2\mathrm{d}s\right]+b\mathbb{E}_t\left[\int_t^T|z(s)|^2\mathrm{d}s\right]\\
&\quad+\frac{1}{b}\mathbb{E}_t\left[\int_t^T|\beta(s)Y(s)|^2\mathrm{d}s\right]+c\mathbb{E}_t\left[\int_t^T|z(s)|^2\mathrm{d}s\right]+\frac{1}{c}\mathbb{E}_t\left[\int_t^T|\gamma(s)Y(s)|^2\mathrm{d}s\right]\\
&\quad+m\mathbb{E}_t\left[\int_t^T|\eta(s)|\mathrm{d}s\right]+\frac{1}{m}\mathbb{E}_t\left[\int_t^T|\eta(s)||Y(s)|^2\mathrm{d}s\right]\\
&\leqslant\|\xi\|^2_{L^\infty_{\mathcal{F}^W}(\Omega;\mathbb{R}^k)}+m\|\eta(\cdot)\|_{L^{1,\mathrm{bmo}}_{\mathcal{F}^W}(T-\varepsilon,T;\mathbb{R}^k)}+a\|y(\cdot)\|^2_{L^\infty_{\mathcal{F}^W}(T-\varepsilon,T;\mathbb{R}^k)}\\
&\quad+(b+c)\|z(\cdot)\|^2_{L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(T-\varepsilon,T;\mathbb{R}^k)}+\Bigg(\frac{\varepsilon}{a}\|\alpha(\cdot)\|^2_{L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(0,T;\mathbb{R}^{k\times k})}+\frac{\varepsilon}{b}\|\beta(\cdot)\|^2_{L^\infty_{\mathcal{F}^W}(0,T;\mathbb{R}^{k\times k})}\\
&\quad+\frac{1}{c}\|\gamma(\cdot)\|^2_{L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(T-\varepsilon,T;\mathbb{R}^{k\times k})}+\frac{1}{m}\|\eta(\cdot)\|_{L^{1,\mathrm{bmo}}_{\mathcal{F}^W}(T-\varepsilon,T;\mathbb{R}^k)}\Bigg)\|Y(\cdot)\|^2_{L^\infty_{\mathcal{F}^W}(T-\varepsilon,T;\mathbb{R}^k)}
\end{aligned}$$

for some $\varepsilon,a,b,c>0$, which implies

$$\begin{aligned}
&\|Y(\cdot)\|^2_{L^\infty_{\mathcal{F}^W}(T-\varepsilon,T;\mathbb{R}^k)}+\|Z(\cdot)\|^2_{L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(T-\varepsilon,T;\mathbb{R}^k)}\\
&\leqslant\|\xi\|^2_{L^\infty_{\mathcal{F}^W}(\Omega;\mathbb{R}^k)}+m\|\eta(\cdot)\|_{L^{1,\mathrm{bmo}}_{\mathcal{F}^W}(T-\varepsilon,T;\mathbb{R}^k)}+a\|y(\cdot)\|^2_{L^\infty_{\mathcal{F}^W}(T-\varepsilon,T;\mathbb{R}^k)}\\
&\quad+(b+c)\|z(\cdot)\|^2_{L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(T-\varepsilon,T;\mathbb{R}^k)}+\Bigg(\frac{\varepsilon}{a}\|\alpha(\cdot)\|^2_{L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(0,T;\mathbb{R}^{k\times k})}+\frac{\varepsilon}{b}\|\beta(\cdot)\|^2_{L^\infty_{\mathcal{F}^W}(0,T;\mathbb{R}^{k\times k})}\\
&\quad+\frac{1}{c}\|\gamma(\cdot)\|^2_{L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(T-\varepsilon,T;\mathbb{R}^{k\times k})}+\frac{1}{m}\|\eta(\cdot)\|_{L^{1,\mathrm{bmo}}_{\mathcal{F}^W}(T-\varepsilon,T;\mathbb{R}^k)}\Bigg)\|Y(\cdot)\|^2_{L^\infty_{\mathcal{F}^W}(T-\varepsilon,T;\mathbb{R}^k)}.
\end{aligned}\tag{3.3}$$

Since $\gamma(\cdot)$ is δ-sliceable in $L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(0,T;\mathbb{R}^{n\times n})$, Then, there are suitable $\varepsilon,a,b,c>0$ such that

$$\begin{gathered}
\varepsilon<\max_i|t_{i+1}-t_i|,\\
b+c<1,\\
\frac{\varepsilon}{a}\|\alpha(\cdot)\|^2_{L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(0,T;\mathbb{R}^{k\times k})}+\frac{\varepsilon}{b}\|\beta(\cdot)\|^2_{L^\infty_{\mathcal{F}^W}(0,T;\mathbb{R}^{k\times k})}\\
+\frac{1}{c}\|\gamma(\cdot)\|^2_{L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(T-\varepsilon,T;\mathbb{R}^{k\times k})}+\frac{1}{m}\|\eta(\cdot)\|_{L^{1,\mathrm{bmo}}_{\mathcal{F}^W}(0,T;\mathbb{R}^k)}+a<1.
\end{gathered}\tag{3.4}$$

Therefore, we denote

$$d=\max\left\{a\Big/\left(1-\frac{\varepsilon}{a}\|\alpha(\cdot)\|^2_{L^{2,\mathrm{bmo}}_{\mathcal{F}W}(0,T;\mathbb{R}^{k\times k})}-\frac{\varepsilon}{b}\|\beta(\cdot)\|^2_{L^{\infty}_{\mathcal{F}W}(0,T;\mathbb{R}^{k\times k})}-\frac{1}{c}\|\gamma(\cdot)\|^2_{L^{2,\mathrm{bmo}}_{\mathcal{F}W}(T-\varepsilon,T;\mathbb{R}^{k\times k})}\right.\right.$$
$$\left.\left.-\frac{1}{m}\|\eta(\cdot)\|_{L^{1,\mathrm{bmo}}_{\mathcal{F}W}(0,T;\mathbb{R}^k)}\right),b+c\right\}$$

and obtain

$$\begin{aligned}&\frac{a}{d}\|Y(\cdot)\|^2_{L^{\infty}_{\mathcal{F}W}(T-\varepsilon,T;\mathbb{R}^k)}+\|Z(\cdot)\|^2_{L^{2,\mathrm{bmo}}_{\mathcal{F}W}(T-\varepsilon,T;\mathbb{R}^k)}\\ \leqslant\ & d\left[\frac{a}{d}\|y(\cdot)\|^2_{L^{\infty}_{\mathcal{F}W}(T-\varepsilon,T;\mathbb{R}^k)}+\|z(\cdot)\|^2_{L^{2,\mathrm{bmo}}_{\mathcal{F}W}(T-\varepsilon,T;\mathbb{R}^k)}\right]+\|\xi\|^2_{L^{\infty}_{\mathcal{F}W}(\Omega;\mathbb{R}^k)}+\|\eta(\cdot)\|_{L^{1,\mathrm{bmo}}_{\mathcal{F}W}(T-\varepsilon,T;\mathbb{R}^k)}\\ \leqslant\ & \frac{1}{1-d}\left[\|\xi\|^2_{L^{\infty}_{\mathcal{F}W}(\Omega;\mathbb{R}^k)}+\|\eta(\cdot)\|_{L^{1,\mathrm{bmo}}_{\mathcal{F}W}(T-\varepsilon,T;\mathbb{R}^k)}\right],\end{aligned}$$

yielding $(y(\cdot),z(\cdot))\in\mathscr{B}_\varepsilon$. For $(y_1(\cdot),z_1(\cdot)),(y_2(\cdot),z_2(\cdot))\in\mathscr{B}_\varepsilon$, we denote

$$(Y_1(\cdot),Z_1(\cdot))=\Gamma(y_1(\cdot),z_1(\cdot)),\ (Y_2(\cdot),Z_2(\cdot))=\Gamma(y_2(\cdot),z_2(\cdot))$$

and

$$\begin{aligned}&\Delta Y(\cdot)=Y_1(\cdot)-Y_2(\cdot),\ \Delta Z(\cdot)=Z_1(\cdot)-Z_2(\cdot),\\ &\Delta y(\cdot)=y_1(\cdot)-y_2(\cdot),\ \Delta z(\cdot)=z_1(\cdot)-z_2(\cdot).\end{aligned}$$

Similar to (3.3), we can also obtain

$$\begin{aligned}&\|\Delta Y(\cdot)\|^2_{L^{\infty}_{\mathcal{F}W}(T-\varepsilon,T;\mathbb{R}^k)}+\|\Delta Z(\cdot)\|^2_{L^{2,\mathrm{bmo}}_{\mathcal{F}W}(T-\varepsilon,T;\mathbb{R}^k)}\\ \leqslant\ & a\|\Delta y(\cdot)\|^2_{L^{\infty}_{\mathcal{F}W}(T-\varepsilon,T;\mathbb{R}^k)}+(b+c)\|\Delta z(\cdot)\|^2_{L^{2,\mathrm{bmo}}_{\mathcal{F}W}(T-\varepsilon,T;\mathbb{R}^k)}\\ &+\left(\frac{\varepsilon}{a}\|\alpha(\cdot)\|^2_{L^{2,\mathrm{bmo}}_{\mathcal{F}W}(0,T;\mathbb{R}^{k\times k})}+\frac{\varepsilon}{b}\|\beta(\cdot)\|^2_{L^{\infty}_{\mathcal{F}W}(0,T;\mathbb{R}^{k\times k})}\right.\\ &\left.+\frac{1}{c}\|\gamma(\cdot)\|^2_{L^{2,\mathrm{bmo}}_{\mathcal{F}W}(T-\varepsilon,T;\mathbb{R}^{k\times k})}\right)\|\Delta Y(\cdot)\|^2_{L^{\infty}_{\mathcal{F}W}(T-\varepsilon,T;\mathbb{R}^k)},\end{aligned}$$

and thus

$$\begin{aligned}&\frac{a}{d}\|Y(\cdot)\|^2_{L^{\infty}_{\mathcal{F}W}(T-\varepsilon,T;\mathbb{R}^k)}+\|Z(\cdot)\|^2_{L^{2,\mathrm{bmo}}_{\mathcal{F}W}(T-\varepsilon,T;\mathbb{R}^k)}\\ \leqslant\ & d\left[\frac{a}{d}\|y(\cdot)\|^2_{L^{\infty}_{\mathcal{F}W}(T-\varepsilon,T;\mathbb{R}^k)}+\|z(\cdot)\|^2_{L^{2,\mathrm{bmo}}_{\mathcal{F}W}(T-\varepsilon,T;\mathbb{R}^k)}\right].\end{aligned}$$

We have now proven that BSDE (3.1) has a unique solution $(Y(\cdot),Z(\cdot))$ on $[T-\varepsilon,T]$ such that $(Y(\cdot),Z(\cdot))\in L^{\infty}_{\mathcal{F}W}(T-\varepsilon,T;\mathbb{R}^k)\times L^{2,\mathrm{bmo}}_{\mathcal{F}W}(T-\varepsilon,T;\mathbb{R}^k)$. By applying the same method, we can prove that BSDE (3.1) has a unique solution $(Y(\cdot),Z(\cdot))$ on $[(T-2\varepsilon)\vee t_{h-1},T-\varepsilon]$ such that $(Y(\cdot),Z(\cdot))\in L^{\infty}_{\mathcal{F}W}((T-2\varepsilon)\vee t_{h-1},T-\varepsilon;\mathbb{R}^k)\times L^{2,\mathrm{bmo}}_{\mathcal{F}W}((T-2\varepsilon)\vee t_{h-1},T-\varepsilon;\mathbb{R}^k)$. In conclusion, we can prove the solvability of BSDE (3.1) on [0,T]. □

As a natural extension, we consider the following BSDE:

$$\begin{cases}\mathrm{d}Y(t)=-f(t,Y(t),Z(t))\mathrm{d}t+Z(t)\mathrm{d}W(t), & t\in[0,T],\\ Y(T)=\xi,\end{cases}\tag{3.5}$$

where for any $(y,z)\in\mathbb{R}^k\times\mathbb{R}^k\to\mathbb{R}^k$, $f(\cdot,y,z)$ is $\mathcal{F}^W$-adapted and has the form

$$f(t,y,z)=g(t,y,z)+h(t,z)$$

and

$$\begin{aligned}
&|g(t,y,z)|\leqslant\tilde\alpha(t)|y|+\tilde\beta(t)|z|+\tilde\eta(t),\\
&|g(t,y,z)-g(t,\bar y,\bar z)|\leqslant\tilde\alpha(t)|y-\bar y|+\tilde\beta(t)|z-\bar z|,\\
&|h(t,z)|\leqslant\tilde\gamma(t)|z|,\\
&|h(t,z)-h(t,\bar z)|\leqslant\tilde\gamma(t)|z-\bar z|,
\end{aligned}$$

for $y,\bar y,z,\bar z\in\mathbb{R}^k$, and non-negative adapted processes $\tilde\alpha(\cdot),\tilde\beta(\cdot),\tilde\gamma(\cdot),\tilde\eta(\cdot)$ satisfy

- $\tilde\beta(\cdot)\in L^\infty_{\mathcal{F}^W}(0,T;\mathbb{R}^{k\times k})$,
- $\tilde\gamma(\cdot)$ is δ-sliceable in $L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(0,T;\mathbb{R}^{k\times k})$ for some $\delta\in(0,1)$,
- $|\tilde\alpha|\leqslant|\phi|^p,|\tilde\eta|\leqslant|\phi|^2$ for some $p\in(0,2)$ and $\phi\in L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(0,T;\mathbb{R}^{k\times k})$.

Similar to the proof of Theorem 3.1, we can obtain the following result:

Corollary 3.2 *The equation* (3.5) *admits a unique solution* $(Y(\cdot),Z(\cdot))$ *such that* $(Y(\cdot),Z(\cdot))\in L^\infty_{\mathcal{F}^W}(0,T;\mathbb{R}^k)\times L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(0,T;\mathbb{R}^k)$.

4. Proofs of our main results

4.1 Solvability of SRE (2.4)

In this subsection, we focus on the solvability of SRE (2.4). The cornerstone of our approach is constructing a contraction map, inspired by [2]:

Proof of Theorem 2.7 First of all, for $i\in\mathcal{M},\ t\in[0,T]$, we denote

$$\begin{aligned}
\Pi(t,i,P,\Lambda)&:=P(t,i)A(t,i)+A(t,i)^\top P(t,i)+C(t,i)^\top P(t,i)C(t,i)+\Lambda(t,i)C(t,i)+C(t,i)^\top\Lambda(t,i),\\
H(t,i,P,\Lambda)&:=-\left(P(t,i)B(t,i)+C(t,i)^\top P(t,i)D(t,i)+\Lambda(t,i)D(t,i)\right)\left(R(t,i)+D(t,i)^\top P(t,i)\right.\\
&\qquad\left.\cdot D(t,i)\right)^{-1}\left(B(t,i)^\top P(t,i)+D(t,i)^\top P(t,i)C(t,i)+D(t,i)^\top\Lambda(t,i)\right).
\end{aligned}$$

We consider $(p(\cdot,i),\lambda(\cdot,i))_{i=1}^l\in L^\infty_{\mathcal{F}^W}(0,T;\mathbb{S}^n)\times L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(0,T;\mathbb{S}^{n\times d}),\ p(\cdot,i)\geqslant0$ and the following equation:

$$\begin{cases}
\mathrm{d}P(t,i)=-[\Pi(t,i,P,\Lambda)+Q(t,i)+H(t,i,P,\Lambda)+q_{ii}P(t,i)+\sum_{j\neq i}q_{ij}p(t,j)]\mathrm{d}t+\Lambda(t,i)\mathrm{d}W(t),\\
R(t,i)+D(t,i)^\top P(t,i)D(t,i)>0,\quad t\in[0,T],\quad i\in\mathcal{M},\\
P(T,i)=G(i).
\end{cases}\tag{4.1}$$

From [2], BSDE (4.1) has a unique solution $(P(\cdot,i),\Lambda(\cdot,i))_{i=1}^l\in L^\infty_{\mathcal{F}^W}(0,T;\mathbb{S}^n)\times L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(0,T;\mathbb{S}^{n\times d})$ and $P(\cdot,i)\geqslant0$. We will now prove $\Theta\big((p(\cdot,i),\lambda(\cdot,i))_{i=1}^l\big):=(P(\cdot,i),\Lambda(\cdot,i))_{i=1}^l$ is a contraction map on the closed convex set $\mathscr{C}$ by

$$\begin{aligned}
\mathscr{C}:=\Bigg\{&(p(\cdot,i),\lambda(\cdot,i))_{i=1}^l\in L^\infty_{\mathcal{F}^W}(0,T;\mathbb{S}^n)\times L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(0,T;\mathbb{S}^n):p(\cdot,i)\geqslant0,\ \|e^{\frac12\rho\cdot}p(\cdot,i)\|^2_{L^\infty_{\mathcal{F}^W}(0,T;\mathbb{S}^n)}\\
&+\frac12\|e^{\frac12\cdot}\lambda(\cdot,i)\|^2_{L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(0,T;\mathbb{S}^n)}\leqslant2\left[\|e^{\frac12\rho T}G(i)\|^2_{L^\infty_{\mathcal{F}^W}(\Omega;\mathbb{S}^n)}+\|e^{\frac12\rho\cdot}Q(\cdot,i)\|^2_{L^\infty_{\mathcal{F}^W}(0,T;\mathbb{R}^n)}\right]\Bigg\}
\end{aligned}$$

where ρ is a positive constant determined later. Applying Itô's formula to $e^{\rho t}|P(t,i)|^2$ and taking conditional expectation, we have

$$
\begin{aligned}
& e^{\rho t}|P(t,i)|^2+\mathbb{E}_t\left[\int_t^T e^{\rho s}|\Lambda(s,i)|^2\mathrm{d}s\right] \\
={}& \mathbb{E}_t\left[e^{\rho T}|P(T,i)|^2\right]+2\mathbb{E}_t\left[\int_t^T\langle\Pi(s,i,P,\Lambda)+Q(s,i)+H(s,i,P,\Lambda)+q_{ii}P(s,i)\right. \\
& \left.+\sum_{j\neq i}q_{ij}p(s,j),P(s,i)\rangle\mathrm{d}s\right]-\rho\mathbb{E}_t\left[\int_t^T e^{\rho s}|P(s,i)|^2\mathrm{d}s\right] \\
\leqslant{}& \mathbb{E}_t\left[e^{\rho T}|P(T,i)|^2\right]+2L_A\mathbb{E}_t\left[\int_t^T e^{\rho s}|P(s,i)|^2\mathrm{d}s\right]+2L_C\mathbb{E}_t\left[\int_t^T e^{\rho s}|P(s,i)|^2\mathrm{d}s\right] \\
& +\frac{1}{2}\mathbb{E}_t\left[\int_t^T e^{\rho s}|\Lambda(s,i)|^2\mathrm{d}s\right]+\mathbb{E}_t\left[\int_t^T e^{\rho s}|P(s,i)|^2\mathrm{d}s\right]+\mathbb{E}_t\left[\int_t^T e^{\rho s}|Q(s,i)|^2\mathrm{d}s\right] \\
& +2(l-1)^2q^2T\mathbb{E}_t\left[\int_t^T e^{\rho s}|P(s,i)|^2\mathrm{d}s\right]+\frac{1}{2(l-1)T}\sum_{j\neq i}\mathbb{E}_t\left[\int_t^T e^{\rho s}|p(s,j)|^2\mathrm{d}s\right] \\
& -\rho\mathbb{E}_t\left[\int_t^T e^{\rho s}|P(s,i)|^2\mathrm{d}s\right] \\
\leqslant{}& \|e^{\frac{1}{2}\rho T}G(i)\|^2_{L^\infty_{\mathcal{F}^W}(\Omega;\mathbb{S}^n)}+\|e^{\frac{1}{2}\rho\cdot}Q(\cdot,i)\|^2_{L^\infty_{\mathcal{F}^W}(0,T;\mathbb{R}^n)}+\frac{1}{2(l-1)}\sum_{j\neq i}\|e^{\frac{1}{2}\rho\cdot}p(\cdot,j)\|^2_{L^\infty_{\mathcal{F}^W}(0,T;\mathbb{S}^n)} \\
& +\Big(2L_A+2L_C+2(l-1)^2q^2T+1-\rho\Big)\mathbb{E}_t\left[\int_t^T e^{\rho s}|P(s,i)|^2\mathrm{d}s\right]+\frac{1}{2}\mathbb{E}_t\left[\int_t^T e^{\rho s}|\Lambda(s,i)|^2\mathrm{d}s\right]
\end{aligned}
$$

where $q=\max_{j\neq i}q_{ij}$ and L_A,L_C are positive constants depending on $A(\cdot,i),C(\cdot,i)$. Now we choose $\rho=2L_A+2L_C+2(l-1)^2q^2T+1$ and obtain $(P(\cdot,i),\Lambda(\cdot,i))_{i=1}^l\in\mathscr{C}$ from $(p(\cdot,i),\lambda(\cdot,i))_{i=1}^l\in\mathscr{C}$. For $(p_1(\cdot,i),\lambda_1(\cdot,i))_{i=1}^l,(p_2(\cdot,i),\lambda_2(\cdot,i))_{i=1}^l\in\mathscr{C}$, we denote

$$
(P_1(\cdot,i),\Lambda_1(\cdot,i))_{i=1}^l=\Theta\big((p_1(\cdot,i),\lambda_1(\cdot,i))_{i=1}^l\big),\quad(P_2(\cdot,i),\Lambda_2(\cdot,i))_{i=1}^l=\Theta\big((p_2(\cdot,i),\lambda_2(\cdot,i))_{i=1}^l\big)
$$

and

$$
\begin{aligned}
&\Delta P(\cdot,i)=P_1(\cdot,i)-P_2(\cdot,i),\ \Delta\Lambda(\cdot,i)=\Lambda_1(\cdot,i)-\Lambda_2(\cdot,i),\\
&\Delta p(\cdot,i)=p_1(\cdot,i)-p_2(\cdot,i),\ \Delta\lambda(\cdot,i)=\lambda_1(\cdot,i)-\lambda_2(\cdot,i),
\end{aligned}
$$

and obtain the following BSDE:

$$
\begin{cases}
\mathrm{d}\Delta P(t,i)=-\Big[\Pi(t,i,\Delta P,\Delta\Lambda)+H(t,i,P_1,\Lambda_1)-H(s,i,P_2,\Lambda_2)\\
\qquad\qquad+q_{ii}\Delta P(t,i)+\sum_{j\neq i}q_{ij}\Delta p(t,j)\Big]\mathrm{d}t+\Delta\Lambda(t,i)\mathrm{d}W(t),\\
\Delta P(T,i)=0.
\end{cases}
\tag{4.2}
$$

Then we focus on

$$
\begin{aligned}
&H(t,i,P_1,\Lambda_1)-H(t,i,P_2,\Lambda_2)\\
=&\Big[-\big(\Delta P(t,i)B(t,i)+C(t,i)^\top\Delta P(t,i)D(t,i)\big)\big(R(t,i)+D(t,i)^\top P_1(t,i)D(t,i)\big)^{-1}\big(B(t,i)^\top P_1(t,i)\\
&+D^\top(t,i)P_1(t,i)C(t,i)\big)-\big(P_2(t,i)B(t,i)+C(t,i)^\top P_2(t,i)D(t,i)\big)\big(R(t,i)+D(t,i)^\top P_2(t,i)D(t,i)\big)^{-1}\\
&\cdot\big(B(t,i)^\top\Delta P(t,i)+D^\top(t,i)\Delta P(t,i)C(t,i)\big)+\big(P_2(t,i)B(t,i)+C(t,i)^\top P_2(t,i)D(t,i)\big)\\
&\cdot\big(R(t,i)+D(t,i)^\top P_2(t,i)D(t,i)\big)^{-1}D(t,i)^\top\Delta P(t,i)D(t,i)\big(R(t,i)+D(t,i)^\top P_1(t,i)D(t,i)\big)^{-1}\\
&\cdot\big(B(t,i)^\top P_1(t,i)+D^\top(t,i)P_1(t,i)C(t,i)\big)\Big]\\
&+\Big[-\big(\Delta P(t,i)B(t,i)+C(t,i)^\top\Delta P(t,i)D(t,i)\big)\big(R(t,i)+D(t,i)^\top P_1(t,i)D(t,i)\big)^{-1}D(t,i)^\top\Lambda_1(t,i)\\
&-\Lambda_2(t,i)D(t,i)\big(R(t,i)+D(t,i)^\top P_2(t,i)D(t,i)\big)^{-1}\big(B(t,i)^\top\Delta P(t,i)+D^\top(t,i)\Delta P(t,i)C(t,i)\big)\\
&+\Lambda_2(t,i)D(t,i)\big(R(t,i)+D(t,i)^\top P_2(t,i)D(t,i)\big)^{-1}D(t,i)^\top\Delta P(t,i)D(t,i)\big(R(t,i)\\
&+D(t,i)^\top P_1(t,i)D(t,i)\big)^{-1}\big(B(t,i)^\top P_1(t,i)+D^\top(t,i)P_1(t,i)C(t,i)\big)+\big(P_2(t,i)B(t,i)\\
&+C(t,i)^\top P_2(t,i)D(t,i)\big)\big(R(t,i)+D(t,i)^\top P_2(t,i)D(t,i)\big)^{-1}D(t,i)^\top\Delta P(t,i)D(t,i)\\
&\cdot\big(R(t,i)+D(t,i)^\top P_1(t,i)D(t,i)\big)^{-1}D(t,i)^\top\Lambda_1(t,i)+\Lambda_2(t,i)D(t,i)\big(R(t,i)+D(t,i)^\top P_2(t,i)D(t,i)\big)^{-1}\\
&\cdot D(t,i)^\top\Delta P(t,i)D(t,i)\big(R(t,i)+D(t,i)^\top P_1(t,i)D(t,i)\big)^{-1}D(t,i)^\top\Lambda_1(t,i)\Big]\\
&+\Big[-\big(P_2(t,i)B(t,i)+C(t,i)^\top P_2(t,i)D(t,i)\big)\big(R(t,i)+D(t,i)^\top P_2(t,i)D(t,i)\big)^{-1}D(t,i)^\top\Delta\Lambda(t,i)\\
&-\Delta\Lambda(t,i)D(t,i)\big(R(t,i)+D(t,i)^\top P_1(t,i)D(t,i)\big)^{-1}\big(B(t,i)^\top P_1(t,i)+D^\top(t,i)P_1(t,i)C(t,i)\big)\Big]\\
&+\Big[-\Lambda_2(t,i)D(t,i)\big(R(t,i)+D(t,i)^\top P_2(t,i)D(t,i)\big)^{-1}D(t,i)^\top\Delta\Lambda(t,i)\\
&-\Delta\Lambda(t,i)D(t,i)\big(R(t,i)+D(t,i)^\top P_1(t,i)D(t,i)\big)^{-1}D^\top(t,i)\Lambda_1(t,i)\Big]\\
=:&\,H_1(t,i)+H_2(t,i)+H_3(t,i)+H_4(t,i).
\end{aligned}
$$

Thanks to $(\mathscr{A}1)$ and $(\mathscr{A}2)$ and $(P_1(\cdot,i),\Lambda_1(\cdot,i))_{i=1}^l,(P_2(\cdot,i),\Lambda_2(\cdot,i))_{i=1}^l\in L^\infty_{\mathcal{F}^W}(0,T;\mathbb{S}^n)\times L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(0,T;\mathbb{S}^{n\times d})$, $P_1(\cdot,i),P_2(\cdot,i)\geqslant 0$, there exists non-negative adapted processes $\bar\alpha(\cdot),\bar\gamma(\cdot)\in L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(0,T;\mathbb{R})$ and $\bar\beta(\cdot)\in L^\infty_{\mathcal{F}^W}(0,T;\mathbb{R})$ such that for $i\in\mathcal{M},t\in[0,T]$,

$$|\Delta P(t,i)A(t,i)+A(t,i)^\top\Delta P(t,i)+C(t,i)^\top\Delta P(t,i)C(t,i)+H_1(t,i)+H_2(t,i)|\leqslant\bar\alpha(t)|\Delta P(t,i)|,$$

$$|\Delta\Lambda(t,i)C(t,i)+C(t,i)^\top\Delta\Lambda(t,i)+H_3(t,i)|\leqslant\bar\beta(t)|\Delta\Lambda(t,i)|,$$

$$|H_4(t,i)|\leqslant\bar\gamma(t)|\Delta\Lambda(t,i)|.$$

Here, we can choose $\bar\gamma(t)=2\max\big(|D(t,i)R(t,i)^{-1}D(t,i)^\top\Lambda_1(t,i)|,|D(t,i)R(t,i)^{-1}D(t,i)^\top\Lambda_2(t,i)|\big)$.

Applying Itô's formula to $|\Delta P(t,i)|^2$ on $[T-\epsilon,T]$ and taking conditional expectation, we derive the following:

$$
\begin{aligned}
&|\Delta P(t,i)|^2+\mathbb{E}_t\left[\int_t^T|\Delta\Lambda(s,i)|^2\mathrm{d}s\right]\\
=&\,2\mathbb{E}_t\left[\int_t^T\langle\Pi(s,i,\Delta P,\Delta\Lambda)+H(s,i,P_1,\Lambda_1)-H(s,i,P_2,\Lambda_2)\right.
\end{aligned}
$$

$$
\begin{aligned}
&\quad +q_{ii}\Delta P(s,i)+\sum_{j\neq i} q_{ij}\Delta p(s,j), \Delta P(s,i)\rangle \mathrm{d}s\Bigg] \\
&\leqslant \epsilon^{\frac{1}{2}}\mathbb{E}_t\left[\int_t^T \bar{\alpha}(s)^2|\Delta P(s,i)|^2\mathrm{d}s\right]+\epsilon^{-\frac{1}{2}}\mathbb{E}_t\left[\int_t^T |\Delta P(s,i)|^2\mathrm{d}s\right]+\bar{a}\sum_{j\neq i}\mathbb{E}_t\left[\int_t^T |\Delta p(s,j)|^2\mathrm{d}s\right] \\
&\quad +\frac{(l-1)q^2}{\bar{a}}\mathbb{E}_t\left[\int_t^T |\Delta P(s,i)|^2\mathrm{d}s\right]+\bar{b}\mathbb{E}_t\left[\int_t^T |\Delta\Lambda(s,i)|^2\mathrm{d}s\right]+\frac{1}{\bar{b}}\mathbb{E}_t\left[\int_t^T \bar{\beta}(s)^2|\Delta P(s,i)|^2\mathrm{d}s\right] \\
&\quad +\bar{c}\mathbb{E}_t\left[\int_t^T |\Delta\Lambda(s,i)|^2\mathrm{d}s\right]+\frac{1}{\bar{c}}\mathbb{E}_t\left[\int_t^T \bar{\gamma}(s)^2|\Delta P(s,i)|^2\mathrm{d}s\right].
\end{aligned}
$$

for some $\epsilon,\bar{a},\bar{b},\bar{c}>0$, which implies

$$
\begin{aligned}
&\|\Delta P(\cdot,i)\|^2_{L^\infty_{\mathcal{F}^W}(T-\epsilon,T;\mathbb{S}^n)}+\|\Delta\Lambda(\cdot,i)\|^2_{L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(T-\epsilon,T;\mathbb{S}^n)} \\
&\leqslant \bar{a}\sum_{j\neq i}\|\Delta p(\cdot,j)\|^2_{L^\infty_{\mathcal{F}^W}(T-\epsilon,T;\mathbb{S}^n)}+(\bar{b}+\bar{c})\|\Delta\Lambda(\cdot,i)\|^2_{L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(T-\epsilon,T;\mathbb{S}^n)} \\
&\quad +\Bigg(\epsilon^{\frac{1}{2}}\|\bar{\alpha}(\cdot)\|^2_{L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(0,T;\mathbb{R}^{n\times n})}+\epsilon^{\frac{1}{2}}+\frac{(l-1)q^2\epsilon}{\bar{a}}+\frac{\epsilon}{\bar{b}}\left\|\bar{\beta}(\cdot)\right\|^2_{L^\infty_{\mathcal{F}^W}(0,T;\mathbb{R}^{n\times n})} \\
&\quad +\frac{1}{\bar{c}}\|\bar{\gamma}(\cdot)\|^2_{L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(0,T;\mathbb{R}^{n\times n})}\Bigg)\|\Delta P(\cdot,i)\|^2_{L^\infty_{\mathcal{F}^W}(T-\epsilon,T;\mathbb{S}^n)}.
\end{aligned}
$$

From (2.6), when $L_\sigma>0$ is sufficiently small, we have

$$
\|\bar{\gamma}(\cdot)\|_{L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(0,T;\mathbb{R}^{n\times n})}<1,
$$

then there exists suitable $\epsilon,\bar{a},\bar{b},\bar{c}>0$ such that

$$
\begin{cases}
1-\epsilon^{\frac{1}{2}}\|\bar{\alpha}(\cdot)\|^2_{L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(0,T;\mathbb{R}^{n\times n})}-\epsilon^{\frac{1}{2}}-\frac{(l-1)q^2\epsilon}{\bar{a}}-\frac{\epsilon}{\bar{b}}\left\|\bar{\beta}(\cdot)\right\|^2_{L^\infty_{\mathcal{F}^W}(0,T;\mathbb{R}^{n\times n})}-\frac{1}{\bar{c}}\|\bar{\gamma}(\cdot)\|^2_{L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(0,T;\mathbb{R}^{n\times n})}>0, \\
1-\bar{b}-\bar{c}>0, \\
\dfrac{\bar{a}(l-1)}{1-\epsilon^{\frac{1}{2}}\|\bar{\alpha}(\cdot)\|^2_{L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(0,T;\mathbb{R}^{n\times n})}-\epsilon^{\frac{1}{2}}-\frac{(l-1)q^2\epsilon}{\bar{a}}-\frac{\epsilon}{\bar{b}}\left\|\bar{\beta}(\cdot)\right\|^2_{L^\infty_{\mathcal{F}^W}(0,T;\mathbb{R}^{n\times n})}-\frac{1}{\bar{c}}\|\bar{\gamma}(\cdot)\|^2_{L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(0,T;\mathbb{R}^{n\times n})}}<1.
\end{cases}
\tag{4.3}
$$

Let

$$
\bar{d}=\frac{\bar{a}(l-1)}{1-\epsilon^{\frac{1}{2}}\|\bar{\alpha}(\cdot)\|^2_{L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(0,T;\mathbb{R}^{n\times n})}-\epsilon^{\frac{1}{2}}-\frac{(l-1)q^2\epsilon}{\bar{a}}-\frac{\epsilon}{\bar{b}}\left\|\bar{\beta}(\cdot)\right\|^2_{L^\infty_{\mathcal{F}^W}(0,T;\mathbb{R}^{n\times n})}-\frac{1}{\bar{c}}\|\bar{\gamma}(\cdot)\|^2_{L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(0,T;\mathbb{R}^{n\times n})}}
$$

and we can obtain

$$
\begin{aligned}
&\frac{\bar{a}(l-1)}{\bar{d}}\sum_{i=1}^{l}\|\Delta P(\cdot,i)\|^2_{L^\infty_{\mathcal{F}^W}(T-\epsilon,T;\mathbb{S}^n)}+(1-\bar{b}-\bar{c})\sum_{i=1}^{l}\|\Delta\Lambda(\cdot,i)\|^2_{L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(T-\epsilon,T;\mathbb{S}^n)} \\
&\leqslant \bar{a}(l-1)\sum_{i=1}^{l}\|\Delta p(\cdot,i)\|^2_{L^\infty_{\mathcal{F}^W}(T-\epsilon,T;\mathbb{S}^n)},
\end{aligned}
$$

which means SRE (2.4) has a unique solution $(P(\cdot,i),\Lambda(\cdot,i))_{i=1}^{l}$ on $[T-\epsilon,T]$ such that $(P(\cdot,i),\Lambda(\cdot,i))\in L^{\infty}_{\mathcal{F}^W}(T-\epsilon,T;\mathbb{S}^n)\times L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(T-\epsilon,T;\mathbb{S}^n)$. With this method we can solve SRE (2.4) on $[T-2\epsilon,T-\epsilon],[T-3\epsilon,T-2\epsilon],\dots,$ and finally $[0,T]$. □

4.2 Solvability of adjoint BSDE (2.5)

Based on SRE (2.4), we examine the solvability of BSDE (2.5), which is a special case of BSDE (3.1).

Proof of Theorem 2.8 For each $i\in\mathcal{M}$ and any $t\in[0,T]$, we set

$$\begin{aligned}
&\alpha(t,i)=A(t,i)-B(t,i)\Gamma(t,i),\\
&\beta(t,i)=C(t,i)-D(t,i)\left(R(t,i)+D(t,i)^\top P(t,i)D(t,i)\right)^{-1}\left(B(t,i)^\top P(t,i)+D(t,i)^\top P(t,i)C(t,i)\right),\\
&\gamma(t,i)=-D(t,i)\left(R(t,i)+D(t,i)^\top P(t,i)D(t,i)\right)^{-1}D(t,i)^\top\Lambda(t,i),\\
&\eta(t,i)=\Gamma(t,i)^\top\left(D(t,i)^\top P(t,i)\sigma(t,i)-R(t,i)r(t,i)\right)+Q(t,i)q(t,i)-P(t,i)b(t,i),\\
&\qquad\quad -C(t,i)^\top P(t,i)\sigma(t,i)+\Lambda(t,i)\sigma(t,i),\\
&\xi(i)=G(i)g(i).
\end{aligned}$$

Then, we can rewrite BSDE (2.5) as

$$\begin{cases}\mathrm{d}K(t,i)=-\left[\alpha(t,i)^\top K(t,i)+\beta(t,i)^\top L(t,i)+\gamma(t,i)^\top L(t,i)+\eta(t,i)+\sum_{j=1}^{l}q_{ij}K(t,j)\right]\mathrm{d}t\\ \qquad\qquad\quad +L(t,i)\mathrm{d}W(t),\quad t\in[0,T],\\ K(T,i)=\xi(i),\quad i\in\mathcal{M}.\end{cases}\tag{4.4}$$

We define

$$\underline{\alpha}(t)=\begin{bmatrix}\alpha(t,1)&&&0\\&\alpha(t,2)&&\\&&\ddots&\\0&&&\alpha(t,l)\end{bmatrix}+\begin{bmatrix}q_{11}I_n&q_{12}I_n&\cdots&q_{1l}I_n\\q_{21}I_n&q_{22}I_n&\cdots&q_{2l}I_n\\\vdots&\vdots&\ddots&\vdots\\q_{l1}I_n&q_{l2}I_n&\cdots&q_{ll}I_n\end{bmatrix}^\top,$$

$$\underline{\beta}(t)=\begin{bmatrix}\beta(t,1)&&&0\\&\beta(t,2)&&\\&&\ddots&\\0&&&\beta(t,l)\end{bmatrix},\quad \underline{\gamma}(t)=\begin{bmatrix}\gamma(t,1)&&&0\\&\gamma(t,2)&&\\&&\ddots&\\0&&&\gamma(t,l)\end{bmatrix},$$

$$K(t)=\begin{bmatrix}K(t,1)\\K(t,2)\\\vdots\\K(t,l)\end{bmatrix},\quad L(t)=\begin{bmatrix}L(t,1)\\L(t,2)\\\vdots\\L(t,l)\end{bmatrix},\quad \underline{\eta}(t)=\begin{bmatrix}\eta(t,1)\\\eta(t,2)\\\vdots\\\eta(t,l)\end{bmatrix},\quad \underline{\xi}=\begin{bmatrix}\xi(1)\\\xi(2)\\\vdots\\\xi(l)\end{bmatrix},$$

and again rewrite BSDE (4.4) as

$$\begin{cases}\mathrm{d}K(t)=-\left[\underline{\alpha}(t)^\top K(t)+\underline{\beta}(t)^\top L(t)+\underline{\gamma}(t)^\top L(t)+\underline{\eta}(t)\right]\mathrm{d}t+L(t)\mathrm{d}W(t),\quad t\in[0,T],\\ K(T)=\underline{\xi}.\end{cases}\tag{4.5}$$

From (2.6), when $L_\sigma>0$ is sufficiently small, we have

$$\|\underline{\gamma}(\cdot)\|_{L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(0,T;\mathbb{R}^{nl\times nl})}<1.$$

From Theorem 3.1, BSDE (4.5) has a unique solution $(K(\cdot), L(\cdot))$ such that $(K(\cdot), L(\cdot)) \in L^{\infty}_{\mathcal{F}^W}(0,T;\mathbb{R}^{nl}) \times L^{2,\mathrm{bmo}}_{\mathcal{F}^W}(0,T;\mathbb{R}^{nl})$, which concludes the proof. □

4.3 Optimal control and optimal value

Thanks to the solvability of SRE (2.4) and BSDE (2.5), we can now represent the optimal control and optimal value using their unique solutions, as demonstrated in the following proof.

Proof of Theorem 2.9 From the proof of Lemma 4.2 in [7], it is evident that u^* is an admissible control. To avoid ambiguity, we will abbreviate $\psi(t,\alpha_t)$ to ψ, $\psi = A, B, C, D, b, \sigma, Q, R, q, p, P, \Lambda, K, L, X, u$. By applying Itô's formula to $t \mapsto \langle P(t,\alpha_t)X(t), X(t)\rangle$ and $t \mapsto -2\langle K(t,\alpha_t), X(t)\rangle$, and taking the expectation, it holds that:

$$
\begin{aligned}
&\mathbb{E}\left[\langle G(\alpha_T)X(T), X(T)\rangle\right] - \langle P(0,i_0)x, x\rangle \\
=\;& 2\mathbb{E}\left[\int_0^T \langle PBu + Pb + \Lambda Du + \Lambda\sigma + C^\top PDu + C^\top P\sigma, X\rangle \mathrm{d}t\right] \\
&+ \mathbb{E}\left[\int_0^T \langle (R + D^\top PD)^{-1}(B^\top P + D^\top PC + D^\top \Lambda)X, (B^\top P + D^\top PC + D^\top\Lambda)X\rangle \mathrm{d}t\right] \\
&+ \mathbb{E}\left[\int_0^T \langle P(Du+\sigma), Du+\sigma\rangle \mathrm{d}t\right] - \mathbb{E}\left[\int_0^T \langle QX, X\rangle \mathrm{d}t\right]
\end{aligned}
\tag{4.6}
$$

and

$$
\begin{aligned}
&-2\mathbb{E}\left[\langle G(\alpha_T)g(\alpha_T), X(T)\rangle\right] + 2\langle K(0,i_0), x\rangle \\
=\;& -2\mathbb{E}\left[\int_0^T \langle (R + D^\top PD)^{-1}(B^\top P + D^\top PC + D^\top\Lambda)X, B^\top K + D^\top L - D^\top P\sigma + Rr\rangle \mathrm{d}t\right] \\
&+ 2\mathbb{E}\left[\int_0^T \langle Qq - Pb - C^\top P\sigma - \Lambda\sigma, X\rangle \mathrm{d}t\right] - 2\mathbb{E}\left[\int_0^T \langle Bu + b, K\rangle \mathrm{d}t\right] \\
&- 2\mathbb{E}\left[\int_0^T \langle Du + \sigma, L\rangle \mathrm{d}t\right].
\end{aligned}
\tag{4.7}
$$

Now we denote

$$
\begin{aligned}
v(t,i) =\;& -\left(R(t,i) + D(t,i)P(t,i)D(t,i)^\top\right)^{-1} \\
&\cdot \left[\left(B(t,i)^\top P(t,i) + D(t,i)^\top P(t,i)C(t,i) + D(t,i)^\top\Lambda(t,i)\right)X \right.\\
&\left.+\, D(t,i)^\top P(t,i)\sigma(t,i) - R(t,i)r(t,i) - B(t,i)^\top K(t,i) - D(t,i)^\top L(t,i)\right]
\end{aligned}
$$

and add (4.6) and (4.7), then we can obtain

$$
\begin{aligned}
&\mathbb{E}\left[\langle G(\alpha_T)X(T), X(T)\rangle - 2\langle G(\alpha_T)g(\alpha_T), X(T)\rangle\right] - \langle P(0,i_0)x, x\rangle + 2\langle K(0,i_0), x\rangle \\
=\;& 2\mathbb{E}\left[\int_0^T \langle PBu + Pb + \Lambda Du + \Lambda\sigma + C^\top PDu + C^\top P\sigma + Qq - Pb - C^\top P\sigma - \Lambda\sigma, X\rangle \mathrm{d}t\right] \\
&+ \mathbb{E}\left[\int_0^T \langle (R + D^\top PD)^{-1}(B^\top P + D^\top PC + D^\top\Lambda)X, (B^\top P + D^\top PC + D^\top\Lambda)X\rangle \mathrm{d}t\right]
\end{aligned}
$$

$$
\begin{aligned}
&-2\mathbb{E}\left[\int_0^T \langle (R+D^\top PD)^{-1}(B^\top P+D^\top PC+D^\top\Lambda)X, B^\top K+D^\top L-D^\top P\sigma+Rr\rangle \mathrm{d}t\right]\\
&+\mathbb{E}\left[\int_0^T \langle P(Du+\sigma)-2L, Du+\sigma\rangle \mathrm{d}t\right]-\mathbb{E}\left[\int_0^T \langle QX,X\rangle \mathrm{d}t\right]-2\mathbb{E}\left[\int_0^T \langle Bu+b,K\rangle \mathrm{d}t\right]\\
=&\,\mathbb{E}\left[\int_0^T \langle (R+D^\top PD)(u-v),u-v\rangle \mathrm{d}t\right]-\mathbb{E}\left[\int_0^T \langle Ru,u-2r\rangle \mathrm{d}t\right]-\mathbb{E}\left[\int_0^T \langle QX,X-2q\rangle \mathrm{d}t\right]\\
&-\mathbb{E}\left[\int_0^T \langle (R+D^\top PD)^{-1}(D^\top P\sigma-Rr-B^\top K-D^\top L), D^\top P\sigma-Rr-B^\top K-D^\top L\rangle \mathrm{d}t\right]\\
&+\mathbb{E}\left[\int_0^T \langle P\sigma,\sigma\rangle \mathrm{d}t\right]-2\mathbb{E}\left[\int_0^T \langle b,K\rangle \mathrm{d}t\right]-2\mathbb{E}\left[\int_0^T \langle \sigma,L\rangle \mathrm{d}t\right],
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\mathbb{E}\left[\langle G(\alpha_T)(X(T)-g(\alpha_T)),X(T)-g(\alpha_T)\rangle\right]+\mathbb{E}\left[\int_0^T \big(\langle Q(X-q),X-q\rangle+\langle R(u-r),u-r\rangle\big)\mathrm{d}t\right]\\
=&\,\langle P(0,i_0)x,x\rangle-2\langle K(0,i_0),x\rangle+\mathbb{E}[\langle G(\alpha_T)g(\alpha_T),g(\alpha_T)\rangle]\\
&+\mathbb{E}\left[\int_0^T \langle (R+D^\top PD)^{-1}(u-v),u-v\rangle \mathrm{d}t\right]+\mathbb{E}\left[\int_0^T \langle Qq,q\rangle \mathrm{d}t\right]+\mathbb{E}\left[\int_0^T \langle Rr,r\rangle \mathrm{d}t\right]\\
&-\mathbb{E}\left[\int_0^T \langle (R+D^\top PD)^{-1}(D^\top P\sigma-Rr-B^\top K-D^\top L), D^\top P\sigma-Rr-B^\top K-D^\top L\rangle \mathrm{d}t\right]\\
&+\mathbb{E}\left[\int_0^T \langle P\sigma,\sigma\rangle \mathrm{d}t\right]-2\mathbb{E}\left[\int_0^T \langle b,K\rangle \mathrm{d}t\right]-2\mathbb{E}\left[\int_0^T \langle \sigma,L\rangle \mathrm{d}t\right].
\end{aligned}
$$

Since for any $t\in[0,T],\ i\in\mathcal{M}$, we have

$$
R(t,i)+D(t,i)^\top P(t,i)D(t,i)>0,
$$

then it holds that

$$
\begin{aligned}
&\mathbb{E}\left[\langle G(\alpha_T)(X(T)-g(\alpha_T)),X(T)-g(\alpha_T)\rangle\right]+E\left[\int_0^T \big(\langle Q(X-q),X-q\rangle+\langle R(u-p),u-p\rangle\big)\mathrm{d}t\right]\\
\geqslant&\,\langle P(0,i_0)x,x\rangle-2\langle K(0,i_0),x\rangle+\mathbb{E}\left[\langle G(\alpha_T)g(\alpha_T),g(\alpha_T)\rangle\right]\\
&-\mathbb{E}\left[\int_0^T \langle (R+D^\top PD)^{-1}(D^\top P\sigma-Rr-B^\top K-D^\top L), D^\top P\sigma-Rr-B^\top K-D^\top L\rangle \mathrm{d}t\right]\\
&+\mathbb{E}\left[\int_0^T \big(\langle Qq,q\rangle+\langle Rr,r\rangle+\langle P\sigma,\sigma\rangle-2\langle b,K\rangle-2\langle \sigma,L\rangle\big)\mathrm{d}t\right]
\end{aligned}
$$

and the equality holds when $u(t,i)=v(t,i)$. Now we obtain the optimal control and optimal value.

□

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