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Law of large numbers for m-dependent random vectors under sublinear expectations

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Abstract Sublinear expectation relaxes the linear property of classical expectation to subadditivity and positive homogeneity, which can be expressed as $\mathbb{E}(\cdot) = \sup_{\theta\in\Theta} E_\theta(\cdot)$ for a certain set of linear expectations $\{E_\theta : \theta \in \Theta\}$. Such a framework can capture the uncertainty and facilitate a robust method of measuring risk loss reasonably. This study established a law of large numbers for m-dependent random vectors within the framework of sublinear expectation. Consequently, the corresponding explicit rate of convergence were derived. The results of this study can be considered as an extension of the Peng's law of large numbers [22].

Keywords Law of large numbers, m-dependence, Sublinear expectations, Rate of convergence, Random vectors

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1. Introduction

In probability and statistical theory, the classical linear expectation can be determined exactly and is dependent on the certain probability measure, which is a rarity in our real world. In contrast, the uncertainty of probability itself is a common and challenging issue in several application scenarios such as uncertain statistics, measures of risk, and supper-hedging in finance [1, 4, 5, 7, 9]. To overcome this, [18, 20, 21] formulated a new framework, referred to as sublinear expectation. This framework relaxes the linear property of the classical linear expectation to subadditivity and positive homogeneity (Definition 2.1 (iii) and (iv)). A sublinear expectation $\mathbb{E}$ can be expressed as $\mathbb{E}(X) = \sup_{\theta\in\Theta} E_\theta(X)$ (Lemma 2.2 in the subsequent section), where the linear expectations $\{E_\theta : \theta \in \Theta\}$ can be considered as an uncertain model of probabilities $\{P_\theta : \theta \in \Theta\}$. Therefore, sublinear expectation can capture the uncertainty and provides a robust method of measuring risk loss.

Within the existing profound theoretical results of probability theory, the law of large numbers

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(LLN) and the central limit theorem (CLT) are two important theorems, which have been extensively used in several practical fields such as statistics, data science, financial pricing, and risk controls. Consequently, concerns regarding whether these two profound results hold parallelly under sublinear expectations are natural and reasonable. Thus, another study provided an affirmative answer to this question. This study, focused on LLN under sublinear expectations. [18] first formulated an LLN for random variables satisfying moment condition $\mathbb{E}(X_iX_{i+j}) = \mathbb{E}(-X_iX_{i+j}) = 0$ for $i, j = 1, 2, \ldots$. Subsequently, [21] proposed the notions of *identically distributed* and *independent* under sublinear expectations (Definitions 2.2 and 2.3 below), based on which he formulated a more general form of LLN for independent and identically distributed random vectors. [14] extended the LLN proposed by Peng to the nonidentically distributed case. [8] obtained a rate of convergence for Peng's LLN, and formulated a new LLN considering rate of convergence. Further, [24] established Stein's method for LLN under sublinear expectations, and obtained the rate of convergence of Peng's LLN for one-dimensional random variables. Furthermore, [13] derived the same result as that of [24] employing Chatterji's inequality.

The above theoretical results all necessitated independence of random variables or vectors, which is challenging to achieve in the real world. [12] proposed a notion of *m-dependence*, which is a weaker and reasonable replacement of independence. In the framework of classical linear expectations, [2, 6, 15, 23] extensively studied such dependence and developed the corresponding limit theorems. However, the parallel results in sublinear expectations have seldom been accomplished. [17] formulated a CLT for m-dependent random variables under sublinear expectations. Other studies attempted to improve this result. For instance, [10] derived a CLT for one-dimensional m-dependent random variables under the Lindeberg condition. [11] established a CLT for linear processes generated via m-dependent random variables. As evident, LLN cannot be derived from CLT directly under the framework of sublinear expectation. To the best of our knowledge, there are no studies focused on LLN for m-dependent random variables or vectors under sublinear expectations, particularly for nonasymptotic convergence bound. Thus, this study formulated an LLN for m-dependent random vectors under sublinear expectations and derived the corresponding explicit rate of convergence. The result can be regarded as an extension of Peng's LLN.

The remainder of this paper is organized as follows. Section 2 describes the framework of sublinear expectation and the corresponding notions. Section 3 presents the primary results on the law of large numbers. Furthermore, all Proofs are presented in Section 4.

2. Preliminaries

In this section, we briefly recall Peng's framework of sublinear expectation and introduce the notion of *m-dependent* under sublinear expectations. First, we define certain notations used in this paper. Let Ω be a given set and $\mathcal{H}$ be a linear space of real valued functions defined on Ω such that if random variables $X_1, \ldots, X_p \in \mathcal{H}$ then $\varphi(X_1, \ldots, X_p) \in \mathcal{H}$ for each $\varphi \in C_{b,lip}(\mathbb{R}^p)$. Here, where $C_{b,lip}(\mathbb{R}^p)$ denotes the linear space of bounded Lipschitz functions. Let $\nabla\varphi$ and Hessφ denote the gradient and Hessian matrix of φ, respectively. We refer to $X \in \mathcal{H}^p$, if $X = (X_1, \ldots, X_p)$ and $X_i \in \mathcal{H}$ for all $1 \leqslant i \leqslant p$. For a $q_1 \times q_2$ matrix A, let $\|A\|_2 = \lambda_{\max}^{1/2}(AA^\top)$ be the spectral norm of A, where $A^\top$ is the transpose of A. For a p-dimensional vector $x = (x_1, \ldots, x_p) \in \mathbb{R}^p$, let $|x|_2 = (\sum_{i=1}^p x_i^2)^{1/2}$ be the ℓ^2-norm of x. Further, let $a \cdot b$ be the inner product of two vector a and b. For two sequences of positive numbers $\{a_n\}$ and $\{b_n\}$, we obtain $a_n \lesssim b_n$ or $b_n \gtrsim a_n$ if there exists a constant $c > 0$ such that $\limsup_{n\to\infty} a_n/b_n \leqslant c$. If $a_n \lesssim b_n$ and $b_n \lesssim a_n$, we say $a_n \asymp b_n$.

Definition 2.1 (Sublinear expectation). *A sublinear expectation* $\mathbb{E}$ *on* $\mathcal{H}$ *is a functional* $\mathbb{E}:\mathcal{H}\to\mathbb{R}$ *that satisfies the following properties: for all* $X,Y\in\mathcal{H}$*,*

(i) *Monotonicity:* $X\geqslant Y$ *implies* $\mathbb{E}(X)\geqslant\mathbb{E}(Y)$.
(ii) *Constant preservation:* $\mathbb{E}(c)=c$*, for any* $c\in\mathbb{R}$.
(iii) *Sub-additivity:* $\mathbb{E}(X+Y)\leqslant\mathbb{E}(X)+\mathbb{E}(Y)$.
(iv) *Positive homogeneity:* $\mathbb{E}(\lambda X)=\lambda\mathbb{E}(X)$*, for any* $\lambda\geqslant 0$.

The triple $(\Omega,\mathcal{H},\mathbb{E})$ is referred to as the sublinear expectation space. If only (iii) and (iv) are satisfied, such an $\mathbb{E}$ is referred to as a *sublinear functional*. The following properties of sublinear expectation can be checked easily.

Lemma 2.1 *For any* $X,Y\in\mathcal{H}$*, it holds that*

(a) $\mathbb{E}(X)-\mathbb{E}(-Y)\leqslant\mathbb{E}(X+Y)\leqslant\mathbb{E}(X)+\mathbb{E}(Y)$.
(b) $\mathbb{E}(X+Y)=\mathbb{E}(X)+\mathbb{E}(Y)$ *if* $\mathbb{E}(Y)=-\mathbb{E}(-Y)$.
(c) $|\mathbb{E}(X)-\mathbb{E}(Y)|\leqslant\mathbb{E}(|X-Y|)$.

Based on Definition 2.1 (ii) and Lemma 2.1 (a), we also have $\mathbb{E}(X+c)=\mathbb{E}(X)+c$ and $-\mathbb{E}(-X)\leqslant\mathbb{E}(X)$ for any $X\in\mathcal{H}$ and constant $c\in\mathbb{R}$. Further, based on Lemma 2.1 (a), we know that the sublinear expectation $\mathbb{E}$ reduces to a classical linear expectation if $\mathbb{E}(-Y)=-\mathbb{E}(Y)$. The following lemma states that a sublinear expectation can be expressed as a supremum of linear expectations.

Lemma 2.2 (Theorem 1.2.1 of [22]). *Let* $\mathbb{E}$ *be a functional defined on a linear space* $\mathcal{H}$ *that satisfies subadditivity and positive homogeneity. Then, there exists a family of linear functionals* $E_\theta:\mathcal{H}\to\mathbb{R}$*, indexed by* $\theta\in\Theta$*, such that*

$$\mathbb{E}(X)=\sup_{\theta\in\Theta}E_\theta(X)\ \text{ for } X\in\mathcal{H}\,.$$

Furthermore, if $\mathbb{E}$ *is a sublinear expectation, then the corresponding* E_θ *is a linear expectation.*

Lemma 2.2 establishes a bridge between sublinear and linear expectations. In this study, the theoretical analysis is primarily based on this statement. Under the framework of sublinear expectation, we adopted the following notions of *identical distribution* and *independence*, which were designed by [21, 22].

Definition 2.2 (Identical distribution). *Let* X_1 *and* X_2 *be two p-dimensional random vectors defined on the sublinear expectation spaces* $(\Omega,\mathcal{H},\mathbb{E}_1)$ *and* $(\Omega,\mathcal{H},\mathbb{E}_2)$*, respectively. They are referred to as identically distributed, denoted by* $X_1\overset{d}{=}X_2$*, if*

$$\mathbb{E}_1\{\varphi(X_1)\}=\mathbb{E}_2\{\varphi(X_2)\}\ \text{for all}\ \varphi\in C_{b,Lip}(\mathbb{R}^p)\,.$$

Remark 2.1 *By Proposition 1.3.2 of [22], we know that* $X_1\overset{d}{=}X_2$ *implies* $\mathbb{E}_1\{\psi(X_1)\}=\mathbb{E}_2\{\psi(X_2)\}$ *for all* $\psi\in C_{l,Lip}(\mathbb{R}^p)$*, where* $C_{l,Lip}(\mathbb{R}^p)$ *denotes the linear space of local Lipschitz function* $\varphi(\cdot)$ *satisfying* $|\varphi(x)-\varphi(y)|\leqslant C(1+|x|^a+|y|^a)|x-y|_2$*,* $\forall x,y\in\mathbb{R}^p$*, for some* $C>0$ *and nonnegative integer a depending on* $\varphi(\cdot)$.

Definition 2.3 (Independence). *In a sublinear expectation space* $(\Omega,\mathcal{H},\mathbb{E})$*, a random vector* $Y\in\mathcal{H}^{p_1}$ *is considered as independent of another random vector* $X\in\mathcal{H}^{p_2}$ *unde* $\mathbb{E}$ *if for each test function* $\varphi\in C_{b,Lip}(\mathbb{R}^{p_1+p_2})$ *we have*

$$\mathbb{E}\{\varphi(X,Y)\}=\mathbb{E}[\mathbb{E}\{\varphi(x,Y)\}_{x=X}]\,.$$

Remark 2.2 *If $\mathbb{E}$ is a linear expectation, the above definition simply indicates the classical independence. In contrast to linear expectations, "Y is independent of X" does not imply that "X is independent of Y" under sublinear expectations Further, details can be obtained from Example 1.3.15 of [22].*

Based on the above independence between two random vectors, we introduce the following definitions of independent random and m-dependent random vectors.

Definition 2.4 (Independent random vectors). *The sequence of random vectors $\{X_i\}_{i=1}^{\infty}$ is considered to be independent, if X_{j+1} is independent of $(X_1, \dots, X_j)$ for each $j \geqslant 1$.*

Definition 2.5 (m-dependent random vectors). *The sequence of random vectors $\{X_i\}_{i=1}^{\infty}$ is considered to be m-dependent if $(X_s, \dots, X_t)$ is independent of $(X_1, \dots, X_r)$ for any $t \geqslant s \geqslant r$ such that $s - r > m$.*

Remark 2.3 *Definition 2.4 has been extensively used in the study of limit theorems under sublinear expectations, including those by [14, 16, 19, 25]. In the case that $\mathbb{E}$ is a linear expectation, Definition 2.5 coincides with the classical notion of m-dependent introduced by [12]. Moreover, if $\mathbb{E}$ is a sublinear expectation, Definition 2.5 is equivalent to the m-dependence defined by [10, 11, 17] for either the one-dimensional or multi-dimensional case.*

3. Main results

In this section, first, we describe the two laws of large numbers for independent random vectors without the assumption of identical distribution, wherein the rates of convergence are derived. Consequently, we formulate a law of large numbers for m-dependent random vectors under sublinear expectations, which extends the Peng's law of large numbers [22] to the local dependent case (m-dependent) and derives the explicit rate of convergence. Finally, we present two corollaries to discuss our main results.

Proposition 3.1 presents a law of large numbers for independent random vectors, which can be regarded as an extension of Theorem 2.3 of [8] to the nonidentically distributed case. The proof of Proposition 3.1 is presented in Section 4.1.

Proposition 3.1 *Let $\{X_i\}_{i=1}^{\infty}$ be a p-dimensional sequence of independent random vectors under a sublinear expectation $\mathbb{E}$ such that $\mathbb{E} = \sup_{\theta\in\Theta} E_\theta$ for a family of linear expectation $\{E_\theta : \theta \in \Theta\}$. Let $\mathcal{M}_i = \{E_\theta(X_i) : \theta \in \Theta\}$ and $\mathcal{P}_i$ is the convex hull of closure of $\mathcal{M}_i$. We write $S_k = \sum_{i=1}^{k}(X_i - \mu_i)/n$ with $\mu_i = \arg\sup_{\mu\in\mathcal{P}_i}\{\mu \cdot \nabla\varphi(S_{i-1})\}$ and $S_0 = 0$. For $\varphi \in C_{b,lip}(\mathbb{R}^p)$. Thus, we we have*

$$|\mathbb{E}\{\varphi(S_n)\} - \varphi(0)| \leqslant \frac{\lambda^*}{2n^2}\sum_{i=1}^{n}\left[\sup_{\theta\in\Theta} E_\theta\{|X_i - E_\theta(X_i)|_2^2\} + \mathrm{diam}^2(\mathcal{P}_i)\right],$$

where $\lambda^* = \sup_{x\in\mathbb{R}^p} \|\mathrm{Hess}\varphi(x)\|_2$.

For any $A \subset \mathbb{R}^p$ and $x \in \mathbb{R}^p$, we define the distance $d_A(x) = \inf_{w\in A} |x - w|_2$ and the diameter $\mathrm{diam}(A) = \sup_{x,y\in A} |x - y|_2$. Applying Proposition 3.1, the next proposition describes a law of large numbers for independent and nonidentically distributed random vectors based on the distance $d_A(x)$. The proof is presented in Section 4.2. We write $\bar{X} = n^{-1}\sum_{i=1}^{n} X_i$.

Proposition 3.2 *Assume that the conditions of Proposition 3.1 hold. Further, we assume $\{E_\theta[X_i] : \theta \in \Theta\} = \{E_\theta[X_1] : \theta \in \Theta\}$ for all $2 \leqslant i \leqslant n$. If the convex hull of the closure of $\{E_\theta(X_1) : \theta \in \Theta\}$ is bounded convex polytope $\mathcal{P}$ with K vertices, we have*

$$\mathbb{E}\{d_{\mathcal{P}}^2(\bar{X})\} \leqslant \frac{K}{n^2}\sum_{i=1}^{n}\sup_{\theta\in\Theta} E_\theta\{|X_i - E_\theta(X_i)|_2^2\} + \frac{K\mathrm{diam}^2(\mathcal{P})}{n}.$$

Proposition 3.2 alleviates the identical distributed constriction in the Peng's law of large numbers, which facilitates the derivation of our main results. To present the main results, we introduce certain new notation. Let $\{X_i\}_{i=1}^\infty$ be a sequence of p-dimensional random vectors under a sublinear expectation $\mathbb{E}$ such that $\mathbb{E} = \sup_{\theta\in\Theta} E_\theta$ for a family of linear expectation $\{E_\theta : \theta \in \Theta\}$. We define

$$\bar{\sigma}_n^2 := \sup_{\theta\in\Theta}\max_{1\leqslant i\leqslant n} E_\theta\{|X_i - E_\theta(X_i)|_2^2\},$$

which can be considered as an extension of equation (2.1) in [8] to the multi-dimensional case without the identically distributed constraint.

Based on Proposition 3.2, applying *large-and-small-blocks* technique, Theorem 3.1 describes a law of large numbers for m-dependent random vectors under sublinear expectations. The proof is presented in Section 4.3. Further applications of *large-and-small-blocks* technique can be found in [3]. Theorem 3.1 is an extension of the Peng's law of large numbers to local dependent case (m-dependent) without the identically distributed constraint and provides the corresponding rate of convergence.

Theorem 3.1 (LLN for m-dependent random vector). *Let $\{X_i\}_{i=1}^\infty$ be a p-dimensional sequence of m-dependent random vectors under a sublinear expectation $\mathbb{E}$ such that $\mathbb{E} = \sup_{\theta\in\Theta} E_\theta$. We assume that (i) if $m \geqslant 1$, given any $\theta \in \Theta$, $E_\theta(X_i) = E_\theta(X_1)$ for all $2 \leqslant i \leqslant n$; and (ii) if $m = 0$, $\{E_\theta[X_i] : \theta \in \Theta\} = \{E_\theta[X_1] : \theta \in \Theta\}$ for all $2 \leqslant i \leqslant n$. If the convex hull of the closure of $\{E_\theta(X_1) : \theta \in \Theta\}$ is bounded convex polytope $\mathcal{P}$ with K vertices, we obtain*

$$\mathbb{E}\{d_{\mathcal{P}}^2(\bar{X})\} \leqslant \frac{K^{1/2}\kappa_m}{n^{1/2}} + \frac{K\mathrm{diam}^2(\mathcal{P})}{n},$$

where $\kappa_m = 8\sqrt{2}m^{1/2}(\bar{\sigma}_n^2 \vee 1)^{3/4} I(m \geqslant 1) + (K/n)^{1/2}\bar{\sigma}_n^2 I(m = 0)$.

Remark 3.1 *If $m = 0$ and $X_{i+1} \overset{d}{=} X_1$ for each $i = 1, 2, \ldots$, Theorem 3.1 reduces to the result of the independent and identically distributed case,*

$$\mathbb{E}\{d_{\mathcal{P}}^2(\bar{X})\} \leqslant \frac{K\bar{\sigma}_n^2}{n} + \frac{K\mathrm{diam}^2(\mathcal{P})}{n},$$

where $\bar{\sigma}_n^2 = \sup_{\theta\in\Theta} E_\theta\{|X_1 - E_\theta(X_1)|_2^2\}$ for this case. It achieves the rate of convergence for independent and identically distributed case derived in Theorem 2.2 of [8]. Moreover, the formulated law of large numbers under sublinear expectation can be compatible with the nonidentically distributed and local dependent case for $m \geqslant 1$, where Theorem 3.1 yields

$$\mathbb{E}\{d_{\mathcal{P}}^2(\bar{X})\} \leqslant \frac{8\sqrt{2}(Km)^{1/2}(\bar{\sigma}_n^2 \vee 1)^{3/4}}{n^{1/2}} + \frac{K\mathrm{diam}^2(\mathcal{P})}{n}.$$

Notably, if $\max\{K, \bar{\sigma}_n^2, m\} = O(1)$, the result lacks the factor $n^{-1/2}$ compared to the independent case for the first term in the rate of convergence. This may be because, to deal with the local dependence, our large-and-small-blocks method rejected $L \asymp n^{1/2}$ small blocks of size m, which resulted in a total $O(n^{1/2})$ reduction of data points in the sample size.

Based on Theorem 3.1, we can also obtain the rate of convergence of $\mathbb{E}\{d_{\mathcal{P}}^2(\bar{X})\}$ when $\mathcal{P}$ is a regular convex set that can be approximated by polytope. The next corollary states a result when $\mathcal{P}$ is a circle. The proof is presented in Section 4.4.

Corollary 3.1 *Under the conditions of Theorem 3.1, if $\mathcal{P}$ is a circle with radius R, we have*

$$\mathbb{E}\{d_{\mathcal{P}}^2(\bar{X})\} \leqslant \left(\frac{7}{4}\right)^{1/5} \frac{2\sqrt{\pi}\kappa_m}{n^{2/5}} + \left(\frac{49}{16}\right)^{1/5} \frac{64\pi R^2}{n^{4/5}}\,,$$

with $\kappa_m = 8\sqrt{2}m^{1/2}(\bar{\sigma}_n^2 \vee 1)^{3/4} I(m \geqslant 1) + \{7/(4n^2)\}^{1/5}\sqrt{\pi}\bar{\sigma}_n^2 I(m=0)$.

Selecting $\mathcal{P} = [a,b]$ for some $a, b \in \mathbb{R}$, which is a polytope with $K = 2$ vertices, and applying Theorem 3.1, we obtain the following one-dimensional law of large numbers for m-dependent random variables under sublinear expectations.

Corollary 3.2 *Under the conditions of Theorem 3.1, if $\mathcal{P} = [\underline{\mu}, \overline{\mu}]$ with $\overline{\mu} = \mathbb{E}(X_1)$ and $\underline{\mu} = -\mathbb{E}(-X_1)$, we have*

$$\mathbb{E}\{d_{\mathcal{P}}^2(\bar{X})\} \leqslant \frac{\sqrt{2}\kappa_m}{n^{1/2}} + \frac{2\mathrm{diam}^2(\mathcal{P})}{n}\,, \tag{1}$$

where $\kappa_m = 8\sqrt{2}m^{1/2}(\bar{\sigma}_n^2 \vee 1)^{3/4} I(m \geqslant 1) + (2/n)^{1/2}\bar{\sigma}_n^2 I(m=0)$.

Remark 3.2 *Notice that $d_{\mathcal{P}}^2(\bar{X}) = (\bar{X} - \bar{\mu})_+^2 + (\bar{X} - \underline{\mu})_-^2$, where $(\cdot)_+ = \max(\cdot, 0)$ and $(\cdot)_- = \min(\cdot, 0)$. Then, we can rewrite (1) as*

$$\mathbb{E}\{(\bar{X} - \bar{\mu})_+^2 + (\bar{X} - \underline{\mu})_-^2\} \leqslant \frac{\sqrt{2}\kappa_m}{n^{1/2}} + \frac{2\mathrm{diam}^2(\mathcal{P})}{n}\,. \tag{2}$$

When $m = 0$ and $\bar{\mu} = \underline{\mu} = \mu$ for certain $\mu \in \mathbb{R}$, by (2), we have

$$\mathbb{E}(|\bar{X} - \mu|^2) \leqslant \frac{2\bar{\sigma}_n^2}{n}\,.$$

Consequently, $\lim_{n\to\infty}\mathbb{E}(|\bar{X} - \mu|^2) = 0$ if $\bar{\sigma}_n^2 = o(n)$, which implies the Peng's law of large numbers for one-dimensional case [18]. When $m \geqslant 1$, the Corollary 3.2 extends the Peng's law of large number to the local dependent case with explicit rate of convergence.

4. Proofs

4.1 Proof of Proposition 3.1

In this section, we will follow the same proof strategy employed in [8] to present Proposition 3.1. Let $Y_i = (X_i - \mu_i)/n$. Then, $S_k = \sum_{i=1}^k Y_i$. We write $S_{[k]} = \{S_1, \ldots, S_k\}$. For any random vectors X and Y, we denote $\mathbb{E}_X\{\varphi(X,Y)\} = \mathbb{E}\{\varphi(X,y)\}|_{y=Y}$. Based on Taylor's theorem, it holds that

$$\mathbb{E}_{Y_k}\{\varphi(S_k) - \varphi(S_{k-1})\} = \mathbb{E}_{Y_k}\left\{Y_k \cdot \nabla\varphi(S_{k-1}) + \int_0^1 (1-t)Y_k^\top \mathrm{Hess}\varphi(S_{k-1} + tY_k)Y_k\,\mathrm{d}t\right\}.$$

Notice that

$$\begin{aligned}&\left|\mathbb{E}_{Y_k}\{\varphi(S_k) - \varphi(S_{k-1})\} - \mathbb{E}_{Y_k}\{Y_k \cdot \nabla\varphi(S_{k-1})\}\right| \\ &\leqslant \mathbb{E}_{Y_k}\left\{\left|\int_0^1 (1-t)Y_k^\top \mathrm{Hess}\varphi(S_{k-1} + tY_k)Y_k\,\mathrm{d}t\right|\right\} \leqslant \frac{1}{2}\lambda^*\mathbb{E}(|Y_k|_2^2)\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}(|Y_k|_2^2) &= \frac{1}{n^2}\sup_{\theta\in\Theta} E_\theta(|X_k-\mu_k|_2^2)\\ &= \frac{1}{n^2}\sup_{\theta\in\Theta}[E_\theta\{|X_k-E_\theta(X_k)|_2^2\}+|E_\theta(X_k)-\mu_k|_2^2]\\ &\leqslant \frac{1}{n^2}\left[\sup_{\theta\in\Theta}E_\theta\{|X_k-E_\theta(X_k)|_2^2\}+\mathrm{diam}^2(\mathcal{P}_k)\right].\end{aligned}$$

Then, it holds that

$$\begin{aligned}&|\mathbb{E}_{Y_k}\{\varphi(S_k)-\varphi(S_{k-1})\}-\mathbb{E}_{Y_k}\{Y_k\cdot\nabla\varphi(S_{k-1})\}|\\ &\leqslant \frac{\lambda^*}{2n^2}\left[\sup_{\theta\in\Theta}E_\theta\{|X_k-E_\theta(X_k)|_2^2\}+\mathrm{diam}^2(\mathcal{P}_k)\right].\end{aligned} \tag{3}$$

We next show $\mathbb{E}_{Y_k}\{Y_k\cdot\nabla\varphi(S_{k-1})\}=0$. Notice that

$$\begin{aligned}\mathbb{E}_{Y_k}\{X_k\cdot\nabla\varphi(S_{k-1})\} &= \mathbb{E}\{X_k\cdot\nabla\varphi(x)\}|_{x=S_{k-1}} = \sup_{\theta\in\Theta}E_\theta\{X_k\cdot\nabla\varphi(x)\}|_{x=S_{k-1}}\\ &= \sup_{\theta\in\Theta}E_\theta(X_k)\cdot\nabla\varphi(S_{k-1}) = \sup_{\mu\in\mathcal{M}_k}\mu\cdot\nabla\varphi(S_{k-1}).\end{aligned} \tag{4}$$

For $0\leqslant\lambda_1,\lambda_2\leqslant 1$ such that $\lambda_1+\lambda_2=1$ and $\mu_1,\mu_2\in\overline{\mathcal{M}}_k$, which is the closure of $\mathcal{M}_k$, we have

$$\begin{aligned}(\lambda_1\mu_1+\lambda_2\mu_2)\cdot\nabla\varphi(S_{k-1}) &= \lambda_1\mu_1\cdot\nabla\varphi(S_{k-1})+\lambda_2\mu_2\cdot\nabla\varphi(S_{k-1})\\ &\leqslant \sup_{\mu\in\overline{\mathcal{M}}_k}\mu\cdot\nabla\varphi(S_{k-1}) = \sup_{\mu\in\mathcal{M}_k}\mu\cdot\nabla\varphi(S_{k-1}).\end{aligned}$$

As $\mu_1,\mu_2\in\mathcal{P}_k$ and $\mathcal{P}_k$ is convex, we know $\lambda_1\mu_1+\lambda_2\mu_2\in\mathcal{P}_k$. Then, we have

$$\sup_{\mu\in\mathcal{P}_k}\mu\cdot\nabla\varphi(S_{k-1})\leqslant\sup_{\mu\in\mathcal{M}_k}\mu\cdot\nabla\varphi(S_{k-1})\leqslant\sup_{\mu\in\mathcal{P}_k}\mu\cdot\nabla\varphi(S_{k-1}).$$

Combined with (4), we obtain

$$\mathbb{E}_{Y_k}\{X_k\cdot\nabla\varphi(S_{k-1})\}=\sup_{\mu\in\mathcal{P}_k}\mu\cdot\nabla\varphi(S_{k-1}).$$

As $\mu_k=\arg\sup_{\mu\in\mathcal{P}_k}\{\mu\cdot\nabla\varphi(S_{k-1})\}$, we have

$$\begin{aligned}\mathbb{E}_{Y_k}\{Y_k\cdot\nabla\varphi(S_{k-1})\} &= \frac{1}{n}\mathbb{E}_{Y_k}\{(X_k-\mu_k)\cdot\nabla\varphi(S_{k-1})\}\\ &= \frac{1}{n}\mathbb{E}_{Y_k}\{X_k\cdot\nabla\varphi(S_{k-1})-\mu_k\cdot\nabla\varphi(S_{k-1})\}\\ &= \frac{1}{n}[\mathbb{E}_{Y_k}\{X_k\cdot\nabla\varphi(S_{k-1})\}-\mu_k\cdot\nabla\varphi(S_{k-1})]\\ &= \frac{1}{n}\left[\sup_{\mu\in\mathcal{P}_k}\mu\cdot\nabla\varphi(S_{k-1})-\mu_k\cdot\nabla\varphi(S_{k-1})\right]=0.\end{aligned}$$

Thus, (3) yields,

$$|\mathbb{E}_{Y_k}\{\varphi(S_k)-\varphi(S_{k-1})\}|\leqslant\frac{\lambda^*}{2n^2}\left[\sup_{\theta\in\Theta}E_\theta\{|X_k-E_\theta(X_k)|_2^2\}+\mathrm{diam}^2(\mathcal{P}_k)\right]. \tag{5}$$

We let $S_0=0$. As Y_n is independent of $S_{[n-1]}=\{S_1,\ldots,S_{n-1}\}$, using (5) recursively, we have

$$
\begin{aligned}
\mathbb{E}\{\varphi(S_n)\} - \varphi(0) &= \mathbb{E}\Big[\sum_{k=1}^{n}\{\varphi(S_k) - \varphi(S_{k-1})\}\Big] \\
&= \mathbb{E}\bigg(\mathbb{E}_{Y_n}\Big[\sum_{k=1}^{n-1}\{\varphi(S_k) - \varphi(S_{k-1})\} + \varphi(S_n) - \varphi(S_{n-1})\Big]\bigg) \\
&= \mathbb{E}\Big[\sum_{k=1}^{n-1}\{\varphi(S_k) - \varphi(S_{k-1})\} + \mathbb{E}_{Y_n}\{\varphi(S_n) - \varphi(S_{n-1})\}\Big] \\
&\leqslant \mathbb{E}\Big[\sum_{k=1}^{n-1}\{\varphi(S_k) - \varphi(S_{k-1})\}\Big] + \frac{\lambda^*}{2n^2}\Big[\sup_{\theta\in\Theta} E_\theta\{|X_n - E_\theta(X_n)|_2^2\} + \mathrm{diam}^2(\mathcal{P}_k)\Big] \\
&\leqslant \cdots \leqslant \frac{\lambda^*}{2n^2}\sum_{i=1}^{n}\Big[\sup_{\theta\in\Theta} E_\theta\{|X_i - E_\theta(X_i)|_2^2\} + \mathrm{diam}^2(\mathcal{P}_k)\Big]\,.
\end{aligned}
$$

However,

$$
\begin{aligned}
&\mathbb{E}\{\varphi(S_n)\} - \varphi(0) \\
= &\,\mathbb{E}\Big[\sum_{k=1}^{n-1}\{\varphi(S_k) - \varphi(S_{k-1})\} + \mathbb{E}_{Y_n}\{\varphi(S_n) - \varphi(S_{n-1})\}\Big] \\
\geqslant &\,\mathbb{E}\Big[\sum_{k=1}^{n-1}\{\varphi(S_k) - \varphi(S_{k-1})\}\Big] - \frac{\lambda^*}{2n^2}\Big[\sup_{\theta\in\Theta} E_\theta\{|X_n - E_\theta(X_n)|_2^2\} + \mathrm{diam}^2(\mathcal{P}_k)\Big] \\
\geqslant &\,\cdots \geqslant -\frac{\lambda^*}{2n^2}\sum_{i=1}^{n}\Big[\sup_{\theta\in\Theta} E_\theta\{|X_i - E_\theta(X_i)|_2^2\} + \mathrm{diam}^2(\mathcal{P}_k)\Big]\,.
\end{aligned}
$$

Thus, we complete the proof of Proposition 3.1. □

4.2 Proof of Proposition 3.2

We denote the set of vertices of the polytope $\mathcal{P}$ by $\mathcal{V}$. As $\mathcal{P}$ has K vertices, we have $|\mathcal{V}| = K$. For each $v \in \mathcal{V}$, we define

$$T_v = \{w \in \mathbb{R}^p : w - v = c(u - v) \text{ for some } u \in \mathcal{P} \text{ and } c \geqslant 0\}\,.$$

Let $\varphi(x) = d^2_{T_v - v}(x)$ with $T_v - v = \{u - v : u \in T_v\}$. By Lemma 5.1 of [8], it holds that $\lambda^* = \sup_{x\in\mathbb{R}^p}\|\mathrm{Hess}\varphi(x)\|_2 = 2$ and

$$v \cdot \nabla\varphi(x) = \sup_{\mu\in\mathcal{P}}\{\mu \cdot \nabla\varphi(x)\}, \quad \text{for all } x \in \mathbb{R}^p\,.$$

By applying Proposition 3.1 to $\{X_i\}_{i=1}^{\infty}$ with $\lambda^* = 2$ and $\mu_i = v$, we have

$$\left|\mathbb{E}\left[\varphi\left\{\frac{\sum_{i=1}^n (X_i - v)}{n}\right\}\right] - \varphi(0)\right| \leqslant \frac{1}{n^2}\sum_{i=1}^{n}\sup_{\theta\in\Theta} E_\theta\{|X_i - E_\theta(X_i)|_2^2\} + \frac{\mathrm{diam}^2(\mathcal{P})}{n}\,.$$

Here,

$$\varphi\left\{\frac{\sum_{i=1}^n (X_i - v)}{n}\right\} = d^2_{T_v - v}(\bar{X} - v) = d^2_{T_v}(\bar{X})\,.$$

Owing to $0 \in T_v - v$, we know $\varphi(0) = d^2_{T_v - v}(0) = 0$. Then

$$\mathbb{E}\{d^2_{T_v}(\bar{X})\} \leqslant \frac{1}{n^2}\sum_{i=1}^n \sup_{\theta\in\Theta} E_\theta\{|X_i - E_\theta(X_i)|_2^2\} + \frac{\mathrm{diam}^2(\mathcal{P})}{n}\,.$$

As $\mathcal{P} = \cap_{v\in\mathcal{V}} T_v$ and $|\mathcal{V}| = K$, we have

$$\mathbb{E}\{d^2_{\mathcal{P}}(\bar{X})\} \leqslant \sum_{v\in\mathcal{V}} \mathbb{E}\{d^2_{T_v}(\bar{X})\} \leqslant \frac{K}{n^2}\sum_{i=1}^n \sup_{\theta\in\Theta} E_\theta\{|X_i - E_\theta(X_i)|_2^2\} + \frac{K\mathrm{diam}^2(\mathcal{P})}{n}\,.$$

Thus, we complete the proof of Proposition 3.2. □

4.3 Proof of Theorem 3.1

In this section, based on Proposition 3.2, we use the *large-and-small-blocks* technique to derive Theorem 3.1.

First, we deal with the case of $m \geqslant 1$. Let $Q = o(n)$ be a positive integer that diverges with n. We first decompose the sequence $\{X_i\}_{i=1}^n$ to $L+1$ blocks with $L = \lfloor n/Q \rfloor$: $\mathcal{G}_\ell = \{(\ell-1)Q+1,\ldots,\ell Q\}$ for $\ell \in [L]$ and $\mathcal{G}_{L+1} = \{LQ+1,\ldots,n\}$. Let $b \gg m$ be a nonnegative integer such that $Q = b + m$. We decompose each $\mathcal{G}_\ell$ for $\ell\in[L]$ to a "large" block with length b and a " small" block with length m. Specifically, $\mathcal{I}_\ell = \{(\ell-1)Q+1,\ldots,(\ell-1)Q+b\}$ and $\mathcal{J}_\ell = \{(\ell-1)Q+b+1,\ldots,\ell Q\}$ for $\ell\in[L]$, and $\mathcal{J}_{L+1} = \mathcal{G}_{L+1}$. Define $\tilde{X}_\ell = b^{-1}\sum_{i\in\mathcal{I}_\ell} X_i$ and $\check{X}_\ell = m^{-1}\sum_{i\in\mathcal{J}_\ell} X_i$ for $\ell\in[L]$, and $\check{X}_{L+1} = (n-LQ)^{-1}\sum_{i\in\mathcal{J}_{L+1}} X_i$. As $\{X_i\}_{i=1}^n$ is an m-dependent sequence under sublinear expectation $\mathbb{E}$, we know that $\{\tilde{X}_\ell\}_{\ell=1}^L$ is an independent sequence under $\mathbb{E}$. Let $\bar{X}^* = L^{-1}\sum_{\ell=1}^L \tilde{X}_\ell$ and $\mu_\theta = E_\theta(X_1)$. As $E_\theta(X_i) = \mu_\theta$ for all $1\leqslant i\leqslant n$, we know that $E_\theta(\tilde{X}_\ell) = \mu_\theta$ for all $1\leqslant \ell\leqslant n$. Recall that the convex hull of the closure of $\{\mu_\theta : \theta\in\Theta\}$ is bounded convex polytope $\mathcal{P}$ with K vertices. By applying Proposition 3.2 to the sequence $\{\tilde{X}_\ell\}_{\ell=1}^L$, we have

$$\mathbb{E}\{d^2_{\mathcal{P}}(\bar{X}^*)\} \leqslant \frac{K}{L^2}\sum_{\ell=1}^L \sup_{\theta\in\Theta} E_\theta(|\tilde{X}_\ell - \mu_\theta|_2^2) + \frac{K\mathrm{diam}^2(\mathcal{P})}{n}\,.$$

Notice that

$$\begin{aligned}
E_\theta(|\tilde{X}_\ell - \mu_\theta|_2^2) &= E_\theta\bigg\{\bigg|\frac{1}{b}\sum_{i\in I_\ell}(X_i-\mu_\theta)\bigg|_2^2\bigg\} \\
&= \frac{1}{b^2}E_\theta\bigg\{\sum_{i\in\mathcal{I}_\ell}(X_i-\mu_\theta)\bigg\}^\top\bigg\{\sum_{i\in\mathcal{I}_\ell}(X_i-\mu_\theta)\bigg\} \\
&= \frac{1}{b^2}\sum_{i_i,i_2\in\mathcal{I}_\ell} E_\theta\{(X_{i_1}-\mu_\theta)^\top(X_{i_2}-\mu_\theta)\} \\
&\leqslant \frac{1}{b}\sum_{i\in\mathcal{I}_\ell} E_\theta(|X_i-\mu_\theta|_2^2) \leqslant \max_{1\leqslant i\leqslant n} E_\theta(|X_i-\mu_\theta|_2^2)\,,
\end{aligned}$$

where the fourth step is owing to the Cauchy–Schwarz inequality and the fact that $2ab \leqslant a^2+b^2$. Then, we have

$$\mathbb{E}\{d^2_{\mathcal{P}}(\bar{X}^*)\} \leqslant \frac{K}{L}\sup_{\theta\in\Theta}\max_{1\leqslant i\leqslant n} E_\theta(|X_i-\mu_\theta|_2^2) + \frac{K\mathrm{diam}^2(\mathcal{P})}{n}\,. \tag{6}$$

As $\bar{X}^* = L^{-1}\sum_{\ell=1}^{L}\tilde{X}_\ell = (Lb)^{-1}\sum_{\ell=1}^{L}\sum_{i\in\mathcal{I}_\ell}X_i$, we have

$$\begin{aligned}\bar{X}-\bar{X}^* &= \frac{1}{n}\sum_{i=1}^{n}X_i - \frac{1}{Lb}\sum_{\ell=1}^{L}\sum_{i\in\mathcal{I}_\ell}X_i\\ &= \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu_\theta) - \frac{1}{Lb}\sum_{\ell=1}^{L}\sum_{i\in\mathcal{I}_\ell}(X_i-\mu_\theta)\\ &= \frac{1}{n}\bigg\{\sum_{\ell=1}^{L}\sum_{i\in\mathcal{I}_\ell}(X_i-\mu_\theta)+\sum_{\ell=1}^{L+1}\sum_{i\in\mathcal{J}_\ell}(X_i-\mu_\theta)\bigg\} - \frac{1}{Lb}\sum_{\ell=1}^{L}\sum_{i\in\mathcal{I}_\ell}(X_i-\mu_\theta)\\ &= \bigg(\frac{1}{n}-\frac{1}{Lb}\bigg)\sum_{\ell=1}^{L}\sum_{i\in\mathcal{I}_\ell}(X_i-\mu_\theta) + \frac{1}{n}\sum_{\ell=1}^{L+1}\sum_{i\in\mathcal{J}_\ell}(X_i-\mu_\theta)\,.\end{aligned}$$

If we let $b \leqslant (mn/6)^{1/2}$, it holds that $n - Lb \leqslant 2Lm$, which implies $|n^{-1}-(Lb)^{-1}| = |(n-Lb)(nLb)^{-1}| \leqslant 2m(nb)^{-1}$. Then, we have

$$\begin{aligned}E_\theta(|\bar{X}-\bar{X}^*|_2) &\leqslant \bigg|\frac{1}{n}-\frac{1}{Lb}\bigg|\sum_{\ell=1}^{L}\sum_{i\in\mathcal{I}_\ell}E_\theta(|X_i-\mu_\theta|_2) + \frac{1}{n}\sum_{\ell=1}^{L+1}\sum_{i\in\mathcal{J}_\ell}E_\theta(|X_i-\mu_\theta|_2)\\ &\leqslant \frac{4Lm}{n}\max_{1\leqslant i\leqslant n}E_\theta(|X_i-\mu_\theta|_2)\,.\end{aligned}$$

Taking the supremum over Θ, we have

$$\mathbb{E}(|\bar{X}-\bar{X}^*|_2) \leqslant \frac{4Lm}{n}\sup_{\theta\in\Theta}\max_{1\leqslant i\leqslant n}E_\theta(|X_i-\mu_\theta|_2)\,. \tag{7}$$

Based on the triangle inequality, we obtain

$$d_{\mathcal{P}}(\bar{X}) = \inf_{w\in\mathcal{P}}|w-\bar{X}|_2 \leqslant \inf_{w\in\mathcal{P}}|w-\bar{X}^*|_2 + |\bar{X}^*-\bar{X}|_2 = d_{\mathcal{P}}(\bar{X}^*) + |\bar{X}-\bar{X}^*|_2\,.$$

Combined with (6) and (7), and noticing that $4^{-1}nb^{-1} \leqslant L \leqslant 2nb^{-1}$, it holds that

$$\begin{aligned}&\mathbb{E}\{d_{\mathcal{P}}^2(\bar{X})\} \leqslant \mathbb{E}\{d_{\mathcal{P}}^2(\bar{X}^*)\} + \mathbb{E}(|\bar{X}-\bar{X}^*|_2)\\ &\leqslant \frac{4Kb}{n}\sup_{\theta\in\Theta}\max_{1\leqslant i\leqslant n}E_\theta(|X_i-\mu_\theta|_2^2) + \frac{K\mathrm{diam}^2(\mathcal{P})}{n} + \frac{8m}{b}\sup_{\theta\in\Theta}\max_{1\leqslant i\leqslant n}E_\theta(|X_i-\mu_\theta|_2)\\ &\leqslant \frac{4Kb\bar{\sigma}_n^2}{n} + \frac{8m\bar{\sigma}_n}{b} + \frac{K\mathrm{diam}^2(\mathcal{P})}{n}\,,\end{aligned} \tag{8}$$

where the last step is owing to the Cauchy–Schwarz inequality and $E_\theta(X_i) = \mu_\theta$. By balancing the first and the second terms, we obtain $b = (2K^{-1}\bar{\sigma}_n^{-1}mn)^{1/2}$. If $\bar{\sigma}_n^2 \geqslant 1$. As $K \geqslant 2$, we know that $b \leqslant (mn)^{1/2}$. Then, we have

$$\mathbb{E}\{d_{\mathcal{P}}^2(\bar{X})\} \leqslant \frac{8\sqrt{2}(Km)^{1/2}\bar{\sigma}_n^{3/2}}{n^{1/2}} + \frac{K\mathrm{diam}^2(\mathcal{P})}{n}\,. \tag{9}$$

If $\bar{\sigma}_n^2 < 1$, by (8), it holds that

$$\mathbb{E}\{d_{\mathcal{P}}^2(\bar{X})\} \leqslant \frac{4Kb}{n} + \frac{8m}{b} + \frac{K\mathrm{diam}^2(\mathcal{P})}{n}\,.$$

By balancing the first and the second terms, we have $b = (2K^{-1}mn)^{1/2} \leqslant (mn)^{1/2}$. Consequently, we obtain

$$\mathbb{E}\{d_{\mathcal{P}}^2(\bar{X})\} \leqslant \frac{8\sqrt{2}(Km)^{1/2}}{n^{1/2}} + \frac{K\mathrm{diam}^2(\mathcal{P})}{n}\,. \tag{10}$$

Combining (9) and (10), it holds that

$$\mathbb{E}\{d_{\mathcal{P}}^2(\bar{X})\} \leqslant \frac{8\sqrt{2}(Km)^{1/2}(\bar{\sigma}_n^2 \vee 1)^{3/4}}{n^{1/2}} + \frac{K\mathrm{diam}^2(\mathcal{P})}{n}\,.$$

When $m=0$, we know that $\{X_i\}_{i=1}^{\infty}$ is an independent sequence. By applying Proposition 3.2 directly, we have

$$\mathbb{E}\{d_{\mathcal{P}}^2(\bar{X})\} \leqslant \frac{K\bar{\sigma}_n^2}{n} + \frac{K\mathrm{diam}^2(\mathcal{P})}{n}\,.$$

Thus, we complete the proof of Theorem 3.1. □

4.4 Proof of Corollary 3.1

For any integer $k \geqslant 3$, denote by $\mathcal{P}_k$ a k-sided polytope with $\mathcal{P}$ as the inscribed circle. We write R_k as the radius of $\mathcal{P}_k$. As shown in the proof of Remark 2.3 of [8], $R_k = R/\cos(\pi/k) \leqslant 2R$ and $R_k - R \leqslant 7\pi^2 R/k^2$. Then, $\mathrm{diam}(\mathcal{P}_k) = 2R_k \leqslant 4R$. Consequently, Θ_k such that $\mathcal{P}_k = \{E_\theta(X_1) : \theta \in \Theta_k\}$ and

$$\sup_{\theta\in\Theta_k} \max_{1\leqslant i\leqslant n} E_\theta\{|X_i - E_\theta(X_i)|_2^2\} = \sup_{\theta\in\Theta} \max_{1\leqslant i\leqslant n} E_\theta\{|X_i - E_\theta(X_i)|_2^2\}\,.$$

We write $\mathbb{E}_k = \sup_{\theta\in\Theta_k} E_\theta$. As $d_{\mathcal{P}}(\bar{X}) \leqslant d_{\mathcal{P}_k}(\bar{X}) + R_k - R$, by Theorem 3.1, we have

$$\begin{aligned}\mathbb{E}\{d_{\mathcal{P}}^2(\bar{X})\} \leqslant& \mathbb{E}_k\{d_{\mathcal{P}}^2(\bar{X})\} \leqslant 2\mathbb{E}_k\{d_{\mathcal{P}_k}^2(\bar{X})\} + 2(R_k - R)^2 \\ \leqslant& \frac{2k^{1/2}\kappa_m}{n^{1/2}} + \frac{2k\mathrm{diam}^2(\mathcal{P})}{n} + 2(R_k - R)^2 \\ \leqslant& \frac{2k^{1/2}\kappa_m}{n^{1/2}} + \frac{32kR^2}{n} + 2\left(\frac{7\pi^2 R}{k^2}\right)^2,\end{aligned}$$

where $\kappa_m = 8\sqrt{2}m^{1/2}(\bar{\sigma}_n^2 \vee 1)^{3/4} I(m \geqslant 1) + (k/n)^{1/2}\bar{\sigma}_n^2 I(m=0)$. By balancing the second and the third terms, we have $k = (49n/16)^{1/5}\pi \geqslant 3$. Then, we have

$$\mathbb{E}\{d_{\mathcal{P}}^2(\bar{X})\} \leqslant \left(\frac{7}{4}\right)^{1/5} \frac{2\sqrt{\pi}\kappa_m}{n^{2/5}} + \left(\frac{49}{16}\right)^{1/5} \frac{64\pi R^2}{n^{4/5}}\,,$$

with $\kappa_m = 8\sqrt{2}m^{1/2}(\bar{\sigma}_n^2 \vee 1)^{3/4} I(m \geqslant 1) + \{7/(4n^2)\}^{1/5}\sqrt{\pi}\bar{\sigma}_n^2 I(m=0)$. Thus, we complete the proof of Corollary 3.1. □

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References

[1] Artzner, P., Delbaen, F., Eber, J. and Heath, D., Coherent measures of risk, Mathematical Finance, 1999, 9: 203–228.

[2] Berk, K. N., A central limit theorem for m-dependent random variables with unbounded m, The Annals of

Probability, 1973, 1: 352–354.

[3] Chang, J., Chen, X. and Wu, M., Central limit theorems for high dimensional dependent data, Bernoulli, 2024, 30: 712–742.

[4] Chen, Z. and Epstein, L., Ambiguity, risk, and asset returns in continuous time, Econometrica, 2002, 70: 1403–1443.

[5] Denis, L. and Martini, C., A theoretical framework for the pricing of contingent claims in the presence of model uncertainty, The Annals of Applied Probability, 2006, 16: 827–852.

[6] Diananda, P. H., The central limit theorem for m-dependent variables, Proceedings of the Cambridge Philosophical Society, 1955, 51: 92–95.

[7] El Karoui, N., Peng, S. and Quenez, M. C., Backward stochastic differential equations in finance, Mathematical Finance, 1997, 7: 1–71.

[8] Fang, X., Peng, S., Shao, Q.-M. and Song, Y., Limit theorems with rate of convergence under sublinear expectations, Bernoulli, 2019, 25: 2564–2596.

[9] Föllmer, H. and Schied, A., Statistic Finance: An Introduction in Discrete Time, De Gruyter, 2004.

[10] Gu, W. Y. and Zhang, L. X., Central limit theorem for m-dependent random variables under sub-linear expectations, arXiv: 2306.04355, 2023.

[11] Guo, S. and Zhang, Y., Central limit theorem for linear processes generated by m-dependent random variables under the sub-linear expectation, Communications in Statistics Theory and Methods, 2023, 52: 6407–6419.

[12] Hoeffding, W. and Robbins, H., The central limit theorem for dependent random variables, Duke Mathematical Journal, 1948, 15: 773–780.

[13] Hu, M., Li, X. and Li, X., Convergence rate of Peng's law of large numbers under sublinear expectations, Probability, Uncertainty and Quantitative Risk, 2021, 6: 261–266.

[14] Hu, Z.-C. and Zhou, L., Multi-dimensional central limit theorems and laws of large numbers under sublinear expectations, Acta Mathematica Sinica, 2015, 31: 305–318.

[15] Huang, S. and Liang, G., A monotone scheme for G-equations with application to the explicit convergence rate of robust central limit theorem, Journal of Differential Equations, 2024, 398: 1–37.

[16] Li, X. P., A central limit theorem for m-dependent random variables under sublinear expectations, Acta Mathematicae Applicatae Sinica, English Series, 2015, 31: 435–444.

[17] Orey, S. A., Central limit theorem form-dependent random variables, Duke Mathematical Journal, 1958, 25: 543–546.

[18] Peng, S., Law of large numbers and central limit theorem under nonlinear expectations, arXiv: math/0702358, 2007.

[19] Peng, S., A new central limit theorems under sublinear expectations, arXiv: 0803.2656, 2008.

[20] Peng, S., Survey on normal distributions, central limit theorem, Brownian motion and the related stochastic calculus under sublinear expectations, Science in China Series A: Mathematics, 2009, 52: 1391–1411.

[21] Peng, S., Nonlinear expectations and stochastic calculus under uncertainty, arXiv: 1002.4546, 2010.

[22] Peng, S., Nonlinear Expectations and Stochastic Calculus under Uncertainty, Springer, Berlin, 2019.

[23] Romano, J. P. and Wolf, M., A more general central limit theorem for m-dependent random variables with unbounded m, Statistics & Probability Letters, 2000, 47: 115–124.

[24] Song, Y., Stein's method for law of large numbers under sublinear expectations, Probability, Uncertainty and Quantitative Risk, 2021, 6: 199–212.

[25] Zhang, L.-X., Functional central limit theorems for random vectors under sub-linear expectations, arXiv: 1912.04715, 2019.