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Asymptotic behavior of a weighted self-repelling diffusion driven by fractional Brownian motion

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Abstract Let B^H be a fractional Brownian motion with Hurst index $\frac{1}{2} \leqslant H < 1$. In this paper, we consider the self-repelling diffusion

$$X_t^H = B_t^H + \int_0^t \int_0^s g(u) \mathrm{d}X_u^H \mathrm{d}s + \nu t\,,$$

where $\nu \in \mathbb{R}$, and g is a nonnegative Borel function. The process is an analogue of linear self-interacting diffusion (M. Cranston and Y. Le Jan *Math. Ann.* **303** (1995), 87-93.). Based on the asymptotic behavior of the weight function g at infinity, we establish the large time behavior of the recursive convergence of the solution X^H. For example, when $g \in C^\infty(\mathbb{R}_+)$ and $0 < g(t) \to +\infty$ $(t \to +\infty)$, we demonstrate that there is a sequence $\{\lambda_n\}$ of positive real numbers such that $J_t^H(0; g) := g(t)e^{-G(t)}X_t^H \to \nu + \xi_\infty^H$ and

$$J_t^H(n; g) := G(t)\left(J_t^H(n-1; g) - \lambda_{n-1}\left(\xi_\infty^H + \nu\right)\right) \longrightarrow \lambda_n\left(\xi_\infty^H + \nu\right) \quad (t \to +\infty)$$

in L^2 and almost surely for every $n \in \{1, 2, \ldots\}$, where $G(t) = \int_0^t g(s)\mathrm{d}s$ and $\xi_\infty^H := \int_0^\infty g(r)e^{-G(r)}\mathrm{d}B_r^H$.

Keywords Fractional brownian motion, Self-repelling diffusion, Malliavin calculus, Recursive convergence

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1. Introduction

As a special case of path dependent stochastic differential equations, Cranston and Le Jan [6] introduced a linear self-attracting diffusion driven by the equation

$$X_t = B_t - \theta \int_0^t \int_0^s (X_s - X_r)\mathrm{d}r\mathrm{d}s + \nu t, \quad t \geqslant 0 \tag{1.1}$$

with $\theta > 0$ and $\nu \in \mathbb{R}$, where B is a one-dimensional standard Brownian motion. They showed that process X_t converges in L^2 and almost surely, as t tends to infinity. A more general path dependent stochastic differential equation was first developed by Durrett and Rogers [8] as a

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model for the shape of a growing polymer (Brownian polymer)

$$X_t = X_0 + B_t + \int_0^t \int_0^s f(X_s - X_r)\mathrm{d}r\mathrm{d}s, \tag{1.2}$$

where B is a d-dimensional standard Brownian motion and f is Lipschitz continuous. X_t corresponds to the location of the end of the polymer at time t. Under some conditions, they established an asymptotic behavior of the solution of the stochastic differential equation and gave some conjectures and questions. Let $\mathscr{L}^X(t,x)$ be the local time of the solution process X. Then, (1.2) can be rewritten as

$$X_t = X_0 + B_t + \int_0^t \mathrm{d}s \int_{\mathbb{R}} f(-x)\mathscr{L}^X(s, X_s + x)\mathrm{d}x$$

for all $t \geqslant 0$. This formulation shows how process X interacts with its own occupation density. We may call this solution a Brownian motion interacting with its own passed trajectory, i.e., a *self-interacting motion*. Equation (1.2) defines a self-interacting diffusion without any assumption on f. We will call it self-repelling (resp. self-attracting) if $x \cdot f(x) \geqslant 0$ (resp. $\leqslant 0$) for all $x \in \mathbb{R}^d$, in other words if it is more likely to stay away from (resp. come back to) the places it has already visited before. Benaïm et al. [2] considered a self-interacting diffusion with dependence on the (convolved) empirical measure as follows

$$\mathrm{d}X_t = \sqrt{2}\mathrm{d}B_t - \left(\frac{1}{t}\int_0^t \nabla W(X_t - X_s)\mathrm{d}s\right)\mathrm{d}t,$$

where W is an interaction potential function. A great difference between these diffusions and that of Brownian polymers is that the drift term is divided by t. In many cases of W, the interaction potential is attractive enough to compare the diffusion (a bit modified) to an Ornstein-Uhlenbeck process, which gives access to its ergodic behavior. More works can be found in Benaïm et al. [1], Cranston and Mountford [7], Gauthier [10], Herrmann and Roynette [11], Herrmann and Scheutzow [12], Mountford and Tarrés [16], Sun and Yan [20, 21], Yan et al. [24], and the references therein.

On the other hand, inspired by Chakravarti and Sebastian [4] and Cherayil and Biswas [5], Yan et al. [24] considered an analogue of the self-attracting diffusion driven by fractional Brownian motion and studied the next linear case:

$$X_t^H = B_t^H - \theta \int_0^t \int_0^s (X_s^H - X_r^H)\mathrm{d}r\mathrm{d}s + \nu t, \quad t \geqslant 0, \tag{1.3}$$

where B^H is a fractional Brownian motion (fBm, in short) with Hurst parameter $\frac{1}{2} \leqslant H < 1$. The solution of (1.3) is a Gaussian process, and when $\theta > 0$ Yan et al. [24] showed that the solution converges almost surely and in L^2, to a normal random variable, as t tends to infinity. Moreover, Gan and Yan [9] and Sun and Yan [21] considered the parameter estimation of (1.3), by using the least squares method due to Hu and Nualart [14].

In this study, our idea arises from the following simple analysis. Using integration by parts, Equation (1.1) can be rewritten as

$$X_t = x + B_t - \theta \int_0^t \int_0^s r\mathrm{d}X_r\mathrm{d}s + \nu t, \quad t \geqslant 0.$$

Inspired by this result, one can consider the equation

$$X_t = x + B_t + \int_0^t \int_0^s g(r, X_r)\mathrm{d}X_r\mathrm{d}s + \nu t.$$

As an attempt, in this paper we consider the equation

$$X_t^H = x + B_t^H + \int_0^t \int_0^s g(r) \mathrm{d}X_r^H \mathrm{d}s + \nu t \tag{1.4}$$

with $\nu \in \mathbb{R}$, where g is a Borel function and B^H is a fractional Brownian motion with Hurst index $0 < H < 1$. When $g(r) \equiv a \neq 0$, the equation is a classical Ornstein-Uhlenbeck process

$$X_t^H = x + B_t^H + a \int_0^t X_s^H \mathrm{d}s + \nu t.$$

When $g(r) = ar(a \neq 0)$, the equation is the linear self-interacting diffusion

$$X_t^H = x + B_t^H + a \int_0^t \int_0^s (X_s^H - X_r^H) \mathrm{d}r \mathrm{d}s + \nu t.$$

Equation (1.4) can be rewritten in the following form:

$$X_t^H = x + B_t^H + \int_0^t \int_0^s (X_s^H - X_r^H) \mathrm{d}g(r) \mathrm{d}s + \nu t \tag{1.5}$$

if $g(0) = 0$. This performance enables a good understanding of the physical explanation of Equation (1.4) and the meaning of the weight function $t \mapsto g(t)$ in the model. We will also find that the function $t \mapsto g(t)$ determines the rate of divergence and convergence of the solution process X^H.

Denote $G(t) = \int_0^t g(s)ds$ and $\xi_t^H = \int_0^t g(r)e^{-G(r)} \mathrm{d}B_r^H$ with $t \geqslant 0$. In this paper, we only consider the case where g is nonnegative, and the case where g is nonpositive is considered in Sun and Yan [22]. At this point, the solution to (1.4) is self-repelling and our main purposes are to expound and prove the following statements.

(i) Let $g \in C^\infty(\mathbb{R}_+)$. If as $t \to \infty$, either $g(t) \to +\infty$ or $g(t) \to 0$ and $tg(t) \to +\infty$; then, under some suitable conditions there is a sequence $\{\lambda_n, n \geqslant 1\}$ of nonzero real numbers such that

$$J_t^H(0; g) := g(t)e^{-G(t)} X_t^H \longrightarrow \xi_\infty^H + \nu$$

and

$$J_t^H(n; g) := G(t) \left(J_t^H(n-1; g) - \lambda_{n-1} \left(\xi_\infty^H + \nu \right) \right) \longrightarrow \lambda_n \left(\xi_\infty^H + \nu \right)$$

in L^2 and almost surely for every $n \geqslant 1$, as t tends to infinity, where $\lambda_0 = 1$.

(ii) Let $\lim_{t\to\infty} g(t) = \theta > 0$, $g \in C^\infty([0, \infty))$, and $g(t) > 0$ for all $t > 0$. Define the sequence of the functions $t \mapsto F_n(t)$, $n = 0, 1, 2, \dots$ by

$$F_0(t) = t, \quad F_1(t) = \frac{1}{g(t)}, \quad F_n(t) = \frac{1}{g(t)} \frac{\mathrm{d}}{\mathrm{d}t} F_{n-1}(t)$$

with $t > 0$. If

$$\lim_{t\to\infty} \frac{\mathrm{d}}{\mathrm{d}t} F_n(t) = 0, \quad \lim_{t\to\infty} \frac{F'_{n+1}(t)}{F'_n(t)} = 0$$

for all $n \geqslant 1$, then, as t tends to infinity we have

$$\hat{J}_t^H(n; g) := \frac{F'_{n-1}(t)}{F'_n(t)} \left(\hat{J}_t^H(n-1; g) - (-1)^{n-1} \left(\xi_\infty^H + \nu \right) \right) \longrightarrow (-1)^n \left(\xi_\infty^H + \nu \right)$$

in L^2 and almost surely for every $n \geqslant 1$, where $\hat{J}_t^H(0; g) = J_t^H(0; g)$.

(iii) Let $g \in C(\mathbb{R}_+)$ and let either $\lim_{t\to\infty} tg(t) = 0$, $G(+\infty) = \infty$ or $\int_0^\infty ug(u)\mathrm{d}u = \infty$, $G(+\infty) = K_0 \in (0, +\infty)$. Define the sequence of the functions

$$\beta_0(t) = t, \quad \beta_1(t) = \int_0^t sg(s)\mathrm{d}s, \quad \beta_{n+1}(t) = \int_0^t \beta_n(s)g(s)\mathrm{d}s.$$

If $\lim_{t\to\infty} \beta_n(t) = \infty$ for all $n \geqslant 2$, then, as t tends to infinity, the convergence

$$\tilde{J}_t^H(0; g) := \frac{1}{t} e^{-G(t)} X_t^H \longrightarrow \xi_\infty^H + \nu$$

and

$$\tilde{J}_t^H(n; g) := \frac{\beta_{n-1}(t)}{\beta_n(t)} \left(\tilde{J}_t^H(n-1; g) - (-1)^{n-1} \left(\xi_\infty^H + \nu \right) \right) \longrightarrow (-1)^n \left(\xi_\infty^H + \nu \right)$$

hold in L^2 and almost surely for every $n \geqslant 1$.

(iv) Let $g \in C^\infty(\mathbb{R}_+)$ with $g(t) > 0$ and $\lim_{t\to\infty} tg(t) = Q > 0$. Define the functions $\{\hat{F}_n(t), n \geqslant 1\}$ by the following recursion formulas

$$\hat{F}_0(t) = t, \quad \hat{F}_1(t) = \frac{tg(t) - Q}{g(t)}, \quad \hat{F}_2(t) = \frac{1}{g(t)} \frac{\mathrm{d}}{\mathrm{d}t} \hat{F}_1(t), \quad \hat{F}_n(t) = \frac{1}{g(t)} \frac{\mathrm{d}}{\mathrm{d}t} \hat{F}_{n-1}(t)$$

for $t \geqslant 0$ and $n \geqslant 2$. If

$$\lim_{t\to\infty} \frac{\hat{F}_{n+1}(t)}{\hat{F}_n(t)} = 0$$

for all $n \geqslant 1$, we then have that

$$\bar{J}_t^H(0; g) := \frac{1+Q}{t} e^{-G(t)} X_t^H \longrightarrow \xi_\infty^H + \nu$$

and

$$\bar{J}_t^H(n; g) := \frac{\hat{F}_{n-1}(t)}{\hat{F}_n(t)} \left(\bar{J}_t^H(n-1; g) - (-1)^{n-1} \left(\xi_\infty^H + \nu \right) \right) \longrightarrow (-1)^n \left(\xi_\infty^H + \nu \right)$$

in L^2 and almost surely for every $n \geqslant 1$, as t tends to infinity.

The convergences are new, even in the case of Brownian motion. Furthermore, when $g(t) \equiv \theta$ in the statement (ii), Equation (1.4) degenerates into the classical Ornstein-Uhlenbeck process; however, the convergence in the statement (ii) does not hold. Although many authors studying self-interaction diffusion have highlighted that it can be compared to the Ornstein-Uhlenbeck process, examining the limit characteristics of self-interaction diffusion does not appear in classical stochastic differential equations. This once again demonstrates the theoretical significance of studying self-interaction diffusion, even in linear cases. The rest of this paper is organized as follows: Section 2 presents some preliminaries. In Section 3, we prove the statement (i) and calculate the sequence $\{\lambda_n, n \geqslant 0\}$. In Section 4, we obtain the statement (ii). In Section 5 we check that the statements (iii) and (iv) hold, and consider the case $\int_0^\infty ug(u)\mathrm{d}u < \infty$ and $G(+\infty) = K_0 \in (0, +\infty)$.

2. Preliminaries

In this section, we describe some basic facts on the stochastic calculus with respect to a fractional Brownian motion with $\frac{1}{2} < H < 1$ and when $H = \frac{1}{2}$ we denote B^H by B. Some surveys and more details can be found in Biagini et al. [3], Hu [13], Mishura [15], Nourdin [17], Nualart [18] and Tudor [23].

2.1 Fractional Brownian motion

Throughout this paper we assume that $\frac{1}{2} \leqslant H < 1$ is arbitrary but fixed and we let $B^H = \{B_t^H, t \geqslant 0\}$ be a one-dimensional fBm with Hurst index H defined on $(\Omega, \mathscr{F}^H, P)$ such that $\mathscr{F}^H$ is the sigma field generated by B^H.

A fractional Brownian motion (fBm) $B^H = \{B_t^H, t \geqslant 0\}$ with Hurst index H is a mean zero Gaussian process such that

$$E\left[B_t^H B_s^H\right] = \frac{1}{2}\left[t^{2H} + s^{2H} - |t-s|^{2H}\right]$$

for all $t, s \geqslant 0$. For $H = 1/2$, B^H coincides with the standard Brownian motion B. B^H is neither a semimartingale nor a Markov process unless $H = 1/2$, therefore, many of the powerful techniques from stochastic analysis are unavailable when dealing with B^H. As a Gaussian process, we can construct the stochastic calculus of variations with respect to B^H. Let $\mathscr{H}$ be the completion of the linear space $\mathscr{E}$ generated by the indicator functions $1_{[0,t]}, t \in [0, T]$ with respect to the inner product

$$\langle 1_{[0,s]}, 1_{[0,t]} \rangle_{\mathscr{H}} = \frac{1}{2}\left[t^{2H} + s^{2H} - |t-s|^{2H}\right]$$

for all $s, t \in [0, T]$. When $\frac{1}{2} < H < 1$ we have

$$\mathscr{H} = \left\{ \varphi : [0,T] \to \mathbb{R} \;\middle|\; \|\varphi\|_{\mathscr{H}}^2 = \alpha_H \int_0^T \int_0^T \varphi(t)\varphi(s)|t-s|^{2H-2} \mathrm{d}s\mathrm{d}t < \infty \right\}$$

with $\alpha_H = H(2H-1)$. The elements of the Hilbert space $\mathscr{H}$ may not be functions but distributions of negative order (see, for instance, Pipiras and Taqqu [19]) provided $\frac{1}{2} < H < 1$. Therefore, to avoid unnecessary trouble, we introduce a subspace of $\mathscr{H}$ as follows

$$|\mathscr{H}| = \left\{\varphi : [0,T] \to \mathbb{R} \mid \|\varphi\|_{|\mathscr{H}|} < \infty \right\}$$

with

$$\|\varphi\|_{|\mathscr{H}|}^2 = \alpha_H \int_0^T \int_0^T |\varphi(t)||\varphi(s)||t-s|^{2H-2} \mathrm{d}s\mathrm{d}t$$

for all $\frac{1}{2} < H < 1$. It is not difficult to show that $|\mathscr{H}|$ is a Banach space with the norm $\|\cdot\|_{|\mathscr{H}|}$; $\mathscr{E}$ is dense in $|\mathscr{H}|$, and

$$L^2([0,T]) \subset L^{\frac{1}{H}}([0,T]) \subset |\mathscr{H}| \subset \mathscr{H}.$$

Define the linear mapping $\varphi \mapsto B^H(\varphi)$ on $\mathscr{E}$ by

$$1_{[0,t]} \mapsto B^H(1_{[0,t]}) := \int_0^T 1_{[0,t]} \mathrm{d}B_s^H = B_t^H, \quad t \in [0,T]$$

for every $T > 0$. Then, the linear mapping is an isometry from $\mathscr{E}$ to the Gaussian space generated by B^H and it can be extended to $\mathscr{H}$, which is called the Wiener integral with respect to B^H, denoted by

$$B_T^H(\varphi) = \int_0^T \varphi(t) \mathrm{d}B_t^H$$

for all $\varphi \in \mathscr{H}$. If the Wiener integral $\int_0^T \varphi(t)\mathrm{d}B_t^H$ is well-defined for every $T > 0$, we can define the integral

$$\int_0^\infty \varphi(t)\mathrm{d}B_t^H \tag{2.1}$$

for any φ satisfying $\|\varphi\|^2_{\mathscr{H}} < \infty$. Thus, we call (2.1) the indefinite Wiener integral. Moreover, we have

$$\int_0^t \varphi(s)\mathrm{d}B_s^H = \varphi(t)B_t^H - \int_0^t B_s^H \mathrm{d}\varphi(s) \tag{2.2}$$

for any $\varphi \in |\mathscr{H}|$.

B^H admits the integral representation of the form

$$B_t^H = \int_0^t K_H(t,s)\mathrm{d}B_s, \quad 0 \leqslant t \leqslant T, \tag{2.3}$$

where B is a standard Brownian motion and the kernel $K_H(t,s)$ satisfies

$$\frac{\partial K_H}{\partial t}(t,s) = \kappa_H \left(H - \frac{1}{2}\right)\left(\frac{s}{t}\right)^{\frac{1}{2}-H}(t-s)^{H-\frac{3}{2}}$$

with a normalizing constant $\kappa_H > 0$ given by

$$\kappa_H = \left(\frac{2H}{(1-2H)\beta(1-2H, H+\frac{1}{2})}\right)^{1/2}.$$

We have

$$K_H(t,s) = \left(H - \frac{1}{2}\right)\kappa_H s^{\frac{1}{2}-H}\int_s^t u^{H-\frac{1}{2}}(u-s)^{H-\frac{3}{2}}\mathrm{d}u$$

for $\frac{1}{2} < H < 1$. Define the operator $K^*_{H,T}\varphi$ on $\mathscr{H}$ as follows

$$(K^*_{H,T}\varphi)(u) = \int_0^T \frac{\partial K_H}{\partial t}(t,u)\varphi(t)\mathrm{d}t = \kappa_H\left(H - \frac{1}{2}\right)u^{\frac{1}{2}-H}\int_u^T t^{H-\frac{1}{2}}(t-u)^{H-\frac{3}{2}}\varphi(t)\mathrm{d}t$$

with $\frac{1}{2} < H < 1$. Then, we have

$$B_t^H(\varphi) = \int_0^T \varphi(s)\mathrm{d}B_s^H = \int_0^T K^*_{H,T}\varphi(s)\mathrm{d}B_s. \tag{2.4}$$

Using the relationship, we can show that the inequality (Mishura [15])

$$E\left[\sup_{T_1\leqslant t\leqslant T_2}\left|\int_t^T \varphi(s)\mathrm{d}B_s^H\right|^2\right] \leqslant C_H\int_{T_1}^T\int_{T_1}^T |\varphi(s)|\varphi(r)||s-r|^{2H-2}\mathrm{d}r\mathrm{d}s \tag{2.5}$$

holds for all $\frac{1}{2} \leqslant H < 1$, $0 \leqslant T_1 < T_2 \leqslant T \leqslant +\infty$ and $\varphi \in |\mathscr{H}|$. In fact, we have that

$$\begin{aligned}
\int_t^T \varphi(s)\mathrm{d}B_s^H &= \int_0^T \varphi(s)1_{\{t\leqslant s\leqslant T\}}\mathrm{d}B_s^H = \int_0^T \left(K^*_{H,T}\varphi 1_{[t,T]}\right)(s)\mathrm{d}B_s \\
&= \left(H - \frac{1}{2}\right)\kappa_H\int_0^T s^{\frac{1}{2}-H}\int_s^T r^{H-\frac{1}{2}}(r-s)^{H-\frac{3}{2}}\varphi(r)1_{\{t\leqslant r\leqslant T\}}\mathrm{d}r\mathrm{d}B_s \\
&= \left(H - \frac{1}{2}\right)\kappa_H\int_t^T r^{H-\frac{1}{2}}\varphi(r)\left(\int_0^r s^{\frac{1}{2}-H}(r-s)^{H-\frac{3}{2}}\mathrm{d}B_s\right)\mathrm{d}r
\end{aligned}$$

for all $t < T \leqslant +\infty$. Combining this with the fact

$$\int_0^{x\wedge y} s^{1-2H}[(x-s)(y-s)]^{H-\frac{3}{2}}\mathrm{d}s = \mathbb{B}(2-2H, H-\frac{1}{2})(xy)^{\frac{1}{2}-H}|x-y|^{2H-2}$$

for all $x, y > 0$ and $\frac{1}{2} < H < 1$, the left hand side in (2.5) becomes smaller than the following equation:

$$\begin{aligned}
&\left(H-\frac{1}{2}\right)^2 (\kappa_H)^2 E\left(\int_{T_1}^{T} r^{H-\frac{1}{2}}|\varphi(r)|\left|\int_0^r s^{\frac{1}{2}-H}(r-s)^{H-\frac{3}{2}}\mathrm{d}B_s\right|\mathrm{d}r\right)^2\\
&= C_H\int_{T_1}^{T}\int_{T_1}^{T}(r_1r_2)^{H-\frac{1}{2}}|\varphi(r_1)||\varphi(r_2)|\int_0^{r_1\wedge r_2} s^{1-2H}[(r_1-s)(r_2-s)]^{H-\frac{3}{2}}\mathrm{d}s\mathrm{d}r_1\mathrm{d}r_2\\
&= C_H\int_{T_1}^{T}\int_{T_1}^{T}|\varphi(r_1)||\varphi(r_2)||r_1-r_2|^{2H-2}\mathrm{d}r_1\mathrm{d}r_2.
\end{aligned}$$

2.2 The linear self-interacting diffusion with weight

Consider the equation

$$X_t^H = B_t^H + \int_0^t \int_0^s g(r)\mathrm{d}X_r^H\mathrm{d}s + \nu t \tag{2.6}$$

with $\nu \in \mathbb{R}$, where g is a Borel function and B^H is a fractional Brownian motion with Hurst index $0 < H < 1$. Denote $G(s) = \int_0^s g(r)\mathrm{d}r$ and define the kernel function

$$h_g(t,s) = 1 + g(s)e^{-G(s)}\int_s^t e^{G(u)}\mathrm{d}u$$

for $0 \leqslant s \leqslant t$. Then, using the method of variation of constants (of course, using the integration by parts also), we can show that the solution of (2.6) satisfies the representation

$$X_t^H = \int_0^t h_g(t,s)\mathrm{d}B_s^H + \nu\int_0^t h_g(t,s)\mathrm{d}s. \tag{2.7}$$

Throughout this paper we assume that $g \in C(\mathbb{R}_+)$ and $g(t) > 0$ for $t > 0$. Our discussion is decomposed into $G(+\infty) = \lim_{t\to\infty} G(t) = \infty$ and $G(+\infty) = K_0 \in (0, +\infty)$. Moreover, the case $G(+\infty) = +\infty$ can be decomposed into the following three cases:

- $g(t) \to +\infty$ as $t \to \infty$;
- $g(t) \to 0, \quad G(+\infty) = +\infty$ as $t \to \infty$;
- $g(t) \to \theta > 0$ as $t \to \infty$.

3. Recursive convergence I

Starting from this section, we will investigate the long-term behavior of the solution to (2.6). In this section, we obtain an asymptotic behavior with recursive convergence under the following situations:

$$\lim_{t\to\infty} g(t) = +\infty \tag{3.1}$$

and

$$\lim_{t\to\infty} g(t) = 0, \quad \lim_{t\to\infty} tg(t) = +\infty. \tag{3.2}$$

Lemma 3.1 *Let $g \in C^\infty(\mathbb{R})$ satisfy either the condition (3.1) or the condition (3.2). Define the sequence of the functions $t \mapsto F_n(t)$, $n = 0, 1, 2, \ldots$ by*

$$F_0(t) = t, \quad F_1(t) = \frac{1}{g(t)}, \quad F_n(t) = \frac{1}{g(t)}\frac{\mathrm{d}}{\mathrm{d}t}F_{n-1}(t)$$

with $t>0$ and suppose that there is a sequence $\{\lambda_n, n\geqslant 1\}$ of positive numbers such that

$$\alpha_{n,j}(t) := G^{n-j}(t)\left\{(-1)^j G^j(t)\frac{\mathrm{d}}{\mathrm{d}t}F_j(t)-\lambda_j\right\}\longrightarrow 0 \tag{3.3}$$

for all $n>j\geqslant 1$, as t tends to infinity. Consider the function

$$I_g(t)=g(t)e^{-G(t)}\int_0^t e^{G(u)}\mathrm{d}u-1$$

and the functions $t\mapsto I_g(t;n)$, $n=1,2,\ldots$ defined by the following recursion formulas:

$$I_g(t;1)=G(t)I_g(t),\quad I_g(t;n+1)=G(t)\left[I_g(t;n)-\lambda_n\right]$$

with $t\geqslant 0$. Then, we have

$$\lim_{t\to\infty} I_g(t;n)=\lambda_n$$

for every $n\geqslant 1$.

Almost all positively differentiable functions satisfy the above condition (3.3) if it tends to infinity and

$$\lim_{t\to\infty}\frac{g'(t)}{g^2(t)}=0.$$

For example, the functions $g(t)=(1+t)^{\alpha}$, $\log^{\alpha}(1+t)$, $e^{t^{\alpha}}$, $(1+t)^{\alpha}(1+\log(1+t))^p$ with $\alpha>0$ and $p\in\mathbb{R}$ satisfy the conditions in Lemma 3.1. In fact, for such functions we have

$$(-1)^m G^m(t)\frac{\mathrm{d}}{\mathrm{d}t}F_m(t)-\lambda_m=0,\quad \forall m\geqslant 1.$$

Proof of Lemma 3.1 The proof of the lemma is the same for functions that satisfy the conditions (3.1) and (3.2). We must only prove that this lemma holds for functions satisfying the condition (3.1). Denote

$$H_n(t)=e^{G(t)}-\sum_{j=0}^{n}\frac{1}{j!}G^j(t)$$

for every $n\in\{1,2,\ldots\}$. Because

$$\lim_{t\to 0}\frac{e^{G(t)}-1}{g(t)}=\lim_{t\to 0}\frac{G(t)}{g(t)}=0,$$

we have $H_n(t)\sim\frac{1}{(n+1)!}G^n(t)$ as $t\to 0$. It follows from Taylor's expansion and the definition of $F_{n+1}(t)$ that

$$\lim_{t\to 0}\left(e^{G(t)}-\sum_{j=0}^{n}\frac{1}{j!}G^j(t)\right)F_{n+1}(t)=0 \tag{3.4}$$

for every $n\in\{1,2,\ldots\}$. We can establish this lemma by expanding the function $t\mapsto I_g(t)$ into a polynomial of the form

$$\sum_{k=1}^{n}(-1)^k\frac{\mathrm{d}}{\mathrm{d}t}F_k(t)+\Delta_n(t)$$

for every $n\in\{1,2,\ldots\}$. Using integration by parts, we obtain

$$
\begin{aligned}
I_g(t) &= g(t)e^{-G(t)} \int_0^t e^{G(u)} \mathrm{d}u - 1 \\
&= g(t)e^{-G(t)} \int_0^t F_1(u) d(e^{G(u)} - 1) - 1 \\
&= g(t)e^{-G(t)} F_1(t)(e^{G(t)} - 1) - 1 - g(t)e^{-G(t)} \int_0^t (e^{G(u)} - 1)\mathrm{d}F_1(u) \\
&= \, - e^{-G(t)} - g(t)e^{-G(t)} \int_0^t (e^{G(u)} - 1)\mathrm{d}F_1(u) \\
&= \, - e^{-G(t)} - g(t)e^{-G(t)} \int_0^t F_2(u) d(e^{G(u)} - G(u) - 1) \\
&= \left(-1 + \frac{\mathrm{d}}{\mathrm{d}t}F_1(t)(G(t)+1)\right) e^{-G(t)} - \frac{\mathrm{d}}{\mathrm{d}t}F_1(t) + g(t)e^{-G(t)} \int_0^t (e^{G(u)} - G(u) - 1)\mathrm{d}F_2(u) \\
&= \left(-1 + \frac{\mathrm{d}}{\mathrm{d}t}F_1(t)(G(t)+1)\right) e^{-G(t)} \\
&\quad - \frac{\mathrm{d}}{\mathrm{d}t}F_1(t) + g(t)e^{-G(t)} \int_0^t F_3(u) d\left(e^{G(u)} - \frac{1}{2}G^2(u) - G(u) - 1\right) \\
&= \left(-\frac{\mathrm{d}}{\mathrm{d}t}F_0(t) + \frac{\mathrm{d}}{\mathrm{d}t}F_1(t)(G(t)+1) - \frac{\mathrm{d}}{\mathrm{d}t}F_2(t)\left(\frac{1}{2}G^2(t) + G(t) + 1\right)\right) e^{-G(t)} \\
&\quad - \frac{\mathrm{d}}{\mathrm{d}t}F_1(t) + \frac{\mathrm{d}}{\mathrm{d}t}F_2(t) - g(t)e^{-G(t)} \int_0^t \left(e^{G(u)} - \frac{1}{2}G^2(u) - G(u) - 1\right) \mathrm{d}F_3(u)
\end{aligned}
$$

for all $t > 0$. Then, by continuously applying integration by parts $m+1$ times, $I_g(t)$ can be expressed as

$$
I_g(t) = -\frac{\mathrm{d}}{\mathrm{d}t}F_1(t) + \ldots + (-1)^m \frac{\mathrm{d}}{\mathrm{d}t}F_m(t) + (C_m + \alpha_m(t))\, e^{-G(t)} + \Lambda_g(t,m) \tag{3.5}
$$

for all $t > 0$ and $m \geqslant 1$, where $\alpha_m(t)$ is an infinitesimal as $t \to \infty$ and

$$
\Lambda_g(t,m) = (-1)^{m+1} g(t)e^{-G(t)} \int_0^t \left(e^{G(u)} - \sum_{j=0}^m \frac{1}{j!}G^j(u)\right) \mathrm{d}F_{m+1}(u).
$$

Combining this with the definitions of $I_g(t;n)$, we obtain

$$
\begin{aligned}
I_g(t;1) = G(t)I_g(t) &= -G(t)\frac{\mathrm{d}}{\mathrm{d}t}F_1(t) + G(t)\frac{\mathrm{d}}{\mathrm{d}t}F_2(t) \\
&\quad + G(t)\sum_{j=3}^m (-1)^j \frac{\mathrm{d}}{\mathrm{d}t}F_j(t) + G(t)\left(C_m + \alpha(t)\right) e^{-G(t)} + G(t)\Lambda_g(t,m), \\
I_g(t;2) = G(t)\left[I_g(t;1) - \lambda_1\right] &= G^2(t)\frac{\mathrm{d}}{\mathrm{d}t}F_2(t) - G^2(t)\frac{\mathrm{d}}{\mathrm{d}t}F_3(t) + G(t)\left\{-G(t)\frac{\mathrm{d}}{\mathrm{d}t}F_1(t) - \lambda_1\right\} \\
&\quad + G^2(t)\sum_{j=4}^m (-1)^j \frac{\mathrm{d}}{\mathrm{d}t}F_j(t) + G^2(t)\left(C_m + \alpha_m(t)\right) e^{-G(t)} + G^2(t)\Lambda_g(t,m),
\end{aligned}
$$

and

$$
\begin{aligned}
I_g(t;n) = (-1)^n G^n(t)\frac{\mathrm{d}}{\mathrm{d}t}F_n(t) + \sum_{j=1}^{n-1}\alpha_{n,j}(t) \\
+ G(t)^n \sum_{j=n+1}^{m}(-1)^j\frac{\mathrm{d}}{\mathrm{d}t}F_j(t) + (C_m+\alpha_m(t))\,G^n(t)e^{-G(t)} + G^n(t)\Lambda_g(t,m)
\end{aligned}
$$

for every integer $2 \leqslant n < m$. On the other hand, using L'Hôspital's rule and (3.3) we obtain

$$
\begin{aligned}
&\lim_{t\to\infty} G^k(t)g(t)e^{-G(t)}\int_0^t F'_{m+1}(u)e^{G(u)}\mathrm{d}u \\
&= \lim_{t\to\infty}\frac{1}{G^{-k}(t)g^{-1}(t)e^{G(t)}}\int_0^t F'_{m+1}(u)e^{G(u)}\mathrm{d}u \\
&= \lim_{t\to\infty}\frac{F'_{m+1}(t)}{-kG^{-k-1}(t)+F'_1(t)G^{-k}(t)+G^{-k}(t)} \\
&= \lim_{t\to\infty}\frac{G^{m+1}(t)F'_{m+1}(t)}{-kG^{m-k}(t)+F'_1(t)G^{m-k+1}(t)+G^{m-k+1}(t)} = 0
\end{aligned} \tag{3.6}
$$

and

$$
\lim_{t\to\infty} G^k(t)g(t)e^{-G(t)}\int_0^t G^j(u)\mathrm{d}F_{m+1}(u) = 0 \tag{3.7}
$$

for all integers $0 \leqslant j \leqslant n \leqslant m$ and $1 \leqslant k \leqslant n$, which means

$$
G^n(t)\Lambda_g(t,m) \longrightarrow 0 \quad (t\to\infty)
$$

for every integer $1 \leqslant n \leqslant m$. It follows from (3.3) that

$$
I_g(t;n) \longrightarrow \lambda_n \quad (t\to\infty)
$$

for every integer $n \geqslant 1$. This completes the proof. □

Lemma 3.2 *Let $\frac{1}{2} \leqslant H < 1$ and let g satisfy the conditions of Lemma 3.1. Then, the random variable $\int_0^\infty I_g(s)\mathrm{d}B_s^H$ is well-defined as an element in L^2, and*

$$
\int_0^t I_g(s)\mathrm{d}B_s^H \longrightarrow \int_0^\infty I_g(s)\mathrm{d}B_s^H \quad (t\to\infty) \tag{3.8}
$$

in L^2 and almost surely.

Proof We first consider the convergence in L^2. By Lemma 3.1 we find that $I_g(t)G(t) \to \lambda_1$, as t tends to infinity. It follows from the continuity of g and $g(t)\to\infty$, as t tends to infinity that

$$
E\left(\int_0^\infty I_g(s)\mathrm{d}B_s\right)^2 = \int_0^\infty I_g(s)^2\mathrm{d}s < \infty
$$

and

$$
E\left(\int_0^\infty I_g(s)\mathrm{d}B_s^H\right)^2 = \alpha_H\int_0^\infty\int_0^\infty I_g(s)I_g(r)|s-r|^{2H-2}\mathrm{d}r\mathrm{d}s < \infty
$$

with $\frac{1}{2} < H < 1$. Similarly, we also have

$$
\lim_{t\to\infty} E\left(\int_t^\infty I_g(s)\mathrm{d}B_s\right)^2 = \lim_{t\to\infty}\int_t^\infty I_g(s)^2\mathrm{d}s = 0,
$$

$$
\lim_{t\to\infty} E\left(\int_t^\infty I_g(s)\mathrm{d}B_s^H\right)^2 = \alpha_H\lim_{t\to\infty}\int_t^\infty\int_t^\infty I_g(s)I_g(r)|s-r|^{2H-2}\mathrm{d}r\mathrm{d}s = 0,
$$

which implies that the convergence (3.8) holds in L^2 for $\frac{1}{2} \leqslant H < 1$.

We now check that the convergence with probability one. We have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} I_g(t) &= g'(t) e^{-G(t)} \int_0^t e^{G(u)} \mathrm{d}u - g(t)^2 e^{-G(t)} \int_0^t e^{G(u)} \mathrm{d}u + g(t) \\
&= g'(t) e^{-G(t)} \int_0^t e^{G(u)} \mathrm{d}u - g(t) \left(g(t) e^{-G(t)} \int_0^t e^{G(u)} \mathrm{d}u - 1 \right) \\
&= \frac{g'(t)}{g(t)} [I_g(t) + 1] - g(t) I_g(t) = g(t) \left(\frac{g'(t)}{g(t)^2} [I_g(t) + 1] - I_g(t) \right) \\
&= \frac{g(t)}{G(t)} \left(G(t) \frac{g'(t)}{g(t)^2} [I_g(t) + 1] - G(t) I_g(t) \right) \\
&= \frac{g(t)}{G(t)} \left\{ -G(t) I_g(t) + \lambda_1 + G(t) \frac{g'(t)}{g(t)^2} [I_g(t) + 1] - \lambda_1 \right\} \\
&= -\frac{g(t)}{G(t)} \{ I_g(t;1) - \lambda_1 \} + \frac{g(t)}{G(t)} \left\{ G(t) \frac{g'(t)}{g(t)^2} [I_g(t) + 1] - \lambda_1 \right\} \\
&= -\frac{g(t)}{G(t)} \{ I_g(t;1) - \lambda_1 \} + \frac{g(t)}{G(t)} \left(G(t) \frac{g'(t)}{g(t)^2} - \lambda_1 \right) [I_g(t) + 1] + \frac{g(t)}{G(t)} I_g(t) \lambda_1 \\
&= \frac{g(t)}{G(t)^2} \left\{ -G(t) \{ I_g(t;1) - \lambda_1 \} + G(t) \left(G(t) \frac{g'(t)}{g(t)^2} - \lambda_1 \right) [I_g(t) + 1] + G(t) I_g(t) \lambda_1 \right\} \\
&= \frac{g(t)}{G(t)^2} \left\{ -G(t) \{ I_g(t;1) - \lambda_1 \} + G(t) I_g(t) \left(-G(t) \frac{\mathrm{d}}{\mathrm{d}t} F_t \right) - G(t) \left(G(t) \frac{\mathrm{d}}{\mathrm{d}t} F_t + \lambda_1 \right) \right\}
\end{aligned}$$

for all $t > 0$. It follows from the condition (3.3) that

$$\int_t^\infty B_s^H \frac{\mathrm{d}}{\mathrm{d}s} I_g(s) \mathrm{d}s \xrightarrow{a.s.} 0$$

as t tends to infinity. On the other hand, Lemma 3.1 and the mean value theorem give

$$I_g(t) B_t^H \overset{a.s.}{\sim} \frac{\lambda_1}{G(t)} B_t^H = \frac{\lambda_1}{t g(t) + t\alpha(t)} B_t^H \xrightarrow{a.s.} 0 \quad (t \to \infty)$$

for all $0 < H < 1$, where $\alpha(t)$ is an infinitesimal as t tends to infinity. Therefore, we obtain the desired convergence

$$\int_t^\infty I_g(s) \mathrm{d}B_s^H = -I_g(t) B_t^H - \int_t^\infty B_s^H \frac{\mathrm{d}}{\mathrm{d}s} I_g(s) \mathrm{d}s \xrightarrow{a.s.} 0$$

for all $0 < H < 1$, as t tends to infinity. □

Lemma 3.3 *Let $\frac{1}{2} \leqslant H < 1$ and let g satisfy the conditions of Lemma 3.1. Define the process*

$$\xi_t^H := \int_0^t g(s) e^{-G(s)} \mathrm{d}B_s^H, \quad t \geqslant 0.$$

Then, the random variable $\xi_\infty^H := \int_0^\infty g(s) e^{-G(s)} \mathrm{d}B_s^H$ is well-defined as an element in L^2 and

$$G(t)^n \Psi(t) \left(\xi_\infty^H - \xi_t^H \right) \longrightarrow 0$$

in L^2 and almost surely, as t tends to infinity, for all $n \geqslant 0$, where $\Psi(t) = g(t) e^{-G(t)} \int_0^t e^{G(u)} \mathrm{d}u$.

Proof When $H=\frac{1}{2}$ the lemma is clear. Now, let $\frac{1}{2}<H<1$. The random variable ξ_∞^H is well-defined as an element in L^2, and from the mean value theorem and the continuity we obtain

$$e^{-G(t)}\int_t^{t+x} g(y)\mathrm{d}y = xe^{-G(t)}g(t+\theta x)\leqslant C_H x$$

for all $x\in[0,1]$ and some $\theta\in[0,1]$. It follows that

$$\begin{aligned}
&\int_t^{+\infty} g(s)e^{-G(s)}(s-t)^{2H-2}\mathrm{d}s = \int_0^{+\infty} g(t+x)e^{-G(t+x)}x^{2H-2}\mathrm{d}x\\
&= \int_0^{+\infty} x^{2H-2}d_x\left(e^{-G(t)}-e^{-G(t+x)}\right)\\
&=(2-2H)\int_0^{+\infty} x^{2H-3}\left(e^{-G(t)}-e^{-G(t+x)}\right)\mathrm{d}x\\
&=(2-2H)\int_1^{+\infty} x^{2H-3}\left(e^{-G(t)}-e^{-G(t+x)}\right)\mathrm{d}x\\
&\quad+(2-2H)\int_0^{1} x^{2H-3}\left(e^{-G(t)}-e^{-G(t+x)}\right)\mathrm{d}x\\
&\leqslant 1+(2-2H)\int_0^{1} x^{2H-3}\left(e^{-G(t)}-e^{-G(t+x)}\right)\mathrm{d}x\\
&\leqslant 1+(2-2H)\int_0^{1} x^{2H-3}e^{-G(t)}\left(1-e^{-G(t+x)+G(t)}\right)\mathrm{d}x\\
&\leqslant 1+(2-2H)\int_0^{1} x^{2H-3}e^{-G(t)}\int_t^{t+x} g(y)\mathrm{d}y\mathrm{d}x\leqslant C_H
\end{aligned}\tag{3.9}$$

for all $t\geqslant 0$. Combining this with L'Hôspital's rule, we obtain

$$\begin{aligned}
\Lambda_g(H) &:= \lim_{t\to\infty} e^{G(t)}\int_t^{+\infty}\int_s^{+\infty} g(s)g(r)e^{-G(s)-G(r)}(r-s)^{2H-2}\mathrm{d}r\mathrm{d}s\\
&= \lim_{t\to\infty}\int_t^{+\infty} g(s)e^{-G(s)}(s-t)^{2H-2}\mathrm{d}s\leqslant C_H
\end{aligned}$$

for all $\frac{1}{2}<H<1$, which implies that

$$\begin{aligned}
\lim_{t\to\infty} G(t)^{2n}E\left(\xi_\infty^H-\xi_t^H\right)^2 &= 2\alpha_H\lim_{t\to\infty} G(t)^{2n}\int_t^{+\infty}\int_s^{+\infty} g(s)g(r)e^{-G(s)-G(r)}(r-s)^{2H-2}\mathrm{d}r\mathrm{d}s\\
&= 2\alpha_H\lim_{t\to\infty} G(t)^{2n}e^{-G(t)}\Lambda_g(H)=0
\end{aligned}$$

for all $\frac{1}{2}<H<1$ and $n\geqslant 0$. This gives the convergence in L^2 because $\Psi(t)\to 1$ as t tends to infinity.

We now verify the convergence with probability one. Using integration by parts and L'Hôspital's rule we have

$$\begin{aligned}
G(t)^n\left(\xi_\infty^H-\xi_t^H\right) &= G(t)^n\int_t^\infty g(s)e^{-G(s)}\mathrm{d}B_s^H\\
&= -g(t)G(t)^n e^{-G(t)}B_t^H - G(t)^n\int_t^\infty B_s^H\left\{g'(s)-g(s)^2\right\}e^{-G(s)}\mathrm{d}s\xrightarrow{a.s.} 0
\end{aligned}$$

for all $\frac{1}{2}\leqslant H<1$, as t tends to infinity. □

Lemma 3.4 *Let g satisfy the conditions of Lemma 3.1. Then, the convergence*

$$J_t^H(0;g):=g(t)e^{-G(t)}X_t^H\longrightarrow \xi_\infty^H+\nu\quad(t\to\infty)\tag{3.10}$$

holds in L^2 and almost surely.

Proof We have

$$
\begin{aligned}
J_t^H(0;g) &= g(t)e^{-G(t)}X_t^H = g(t)e^{-G(t)}\int_0^t h_g(t,s)\mathrm{d}B_s^H + \nu g(t)e^{-G(t)}\int_0^t h_g(t,s)\mathrm{d}s \\
&= g(t)e^{-G(t)}B_t^H + g(t)e^{-G(t)}\int_0^t \xi_u^H e^{G(u)}\mathrm{d}u + \nu g(t)e^{-G(t)}\int_0^t e^{G(u)}\mathrm{d}u \\
&= g(t)e^{-G(t)}B_t^H - g(t)e^{-G(t)}\int_0^t \left(\xi_\infty^H - \xi_u^H\right)e^{G(u)}\mathrm{d}u + \left(\xi_\infty^H + \nu\right)g(t)e^{-G(t)}\int_0^t e^{G(u)}\mathrm{d}u \\
&= g(t)e^{-G(t)}\left\{B_t^H - \int_0^t e^{G(u)}\mathrm{d}u\int_u^t g(s)e^{-G(s)}\mathrm{d}B_s^H\right\} \\
&\quad - g(t)e^{-G(t)}\left(\int_0^t e^{G(u)}\mathrm{d}u\right)\int_t^\infty g(s)e^{-G(s)}\mathrm{d}B_s^H + \left(\xi_\infty^H + \nu\right)g(t)e^{-G(t)}\int_0^t e^{G(u)}\mathrm{d}u \\
&= -g(t)e^{-G(t)}\int_0^t I_g(s)\mathrm{d}B_s^H - \Psi(t)\left(\xi_\infty^H - \xi_t^H\right) + \left(\xi_\infty^H + \nu\right)g(t)e^{-G(t)}\int_0^t e^{G(u)}\mathrm{d}u
\end{aligned}
$$

for all $t > 0$. It follows that

$$
J_t^H(0;g) - \left(\xi_\infty^H + \nu\right) = \left(\xi_\infty^H + \nu\right)I_g(t) - g(t)e^{-G(t)}\int_0^t I_g(s)\mathrm{d}B_s^H - \Psi(t)\left(\xi_\infty^H - \xi_t^H\right) \tag{3.11}
$$

for all $t > 0$. On the other hand, using Lemma 3.1 and the proof in Lemma 3.2 we find that

$$
g(t)e^{-G(t)}\int_0^t I_g(s)\mathrm{d}B_s^H = g(t)e^{-G(t)}I_g(t)B_t^H - g(t)e^{-G(t)}\int_0^t B_s^H\frac{\mathrm{d}}{\mathrm{d}s}I_g(s)\mathrm{d}s \longrightarrow 0
$$

in L^2 and almost surely, as t tends to infinity. Combining this with Lemma 3.1 with $n = 1$ and Lemma 3.3, we show that

$$
J_t^H(0;g) \longrightarrow \xi_\infty^H + \nu
$$

in L^2 and almost surely, as t tends to infinity. □

Theorem 3.5 *Let $\frac{1}{2} \leqslant H < 1$ and let the conditions in Lemma 3.1 hold. Recursively define the processes $J^H(n;g) = \{J_t^H(n;g), t \geqslant 0\}$, $n = 1, 2, \ldots$ by*

$$
J_t^H(n;g) := G(t)\left(J_t^H(n-1;g) - \lambda_{n-1}\left(\xi_\infty^H + \nu\right)\right).
$$

Then, we have

$$
J_t^H(n;g) \longrightarrow \lambda_n\left(\xi_\infty^H + \nu\right) \quad (t \to \infty) \tag{3.12}
$$

in L^2 and almost surely for every $n \geqslant 1$.

Proof Clearly, using (3.11) we obtain

$$
\begin{aligned}
J_t^H(1;g) &= G(t)\left(J_t^H(0;g) - \left(\xi_\infty^H + \nu\right)\right) \\
&= \left(\xi_\infty^H + \nu\right)G(t)I_g(t) - G(t)g(t)e^{-G(t)}\int_0^t I_g(s)\mathrm{d}B_s^H - G(t)\Psi(t)\left(\xi_\infty^H - \xi_t^H\right) \\
&= \left(\xi_\infty^H + \nu\right)I_g(t;1) - G(t)g(t)e^{-G(t)}\int_0^t I_g(s)\mathrm{d}B_s^H - G(t)\Psi(t)\left(\xi_\infty^H - \xi_t^H\right)
\end{aligned}
$$

and

$$
\begin{aligned}
J_t^H(n;g) &= G(t)\left(J_t^H(n-1;g) - \lambda_{n-1}\left(\xi_\infty^H + \nu\right)\right) \\
&= \left(\xi_\infty^H + \nu\right)I_g(t;n) - G(t)^n g(t)e^{-G(t)}\int_0^t I_g(s)\mathrm{d}B_s^H - G(t)^n\Psi(t)\left(\xi_\infty^H - \xi_t^H\right)
\end{aligned}
$$

for all $t > 0$ and $n \geqslant 1$, where $I_g(t,n)$ is defined in Lemma 3.1. On the other hand, using Lemma 3.1 and the proof in Lemma 3.2 we obtain

$$G(t)^n g(t) e^{-G(t)} \int_0^t I_g(s) \mathrm{d}B_s^H = G(t)^n g(t) e^{-G(t)} I_g(t) B_t^H - G(t)^n g(t) e^{-G(t)} \int_0^t B_s^H \frac{\mathrm{d}}{\mathrm{d}s} I_g(s) \mathrm{d}s \longrightarrow 0$$

in L^2 and almost surely, as t tends to infinity. Thus, the theorem follows from Lemma 3.1, Lemma 3.3 and Lemma 3.4. □

Remark 3.6 *From the expansion (3.5) of the function $t \mapsto I_g(t)$, we can also show that*

$$J_t^H(n;g) := \frac{F'_{n-1}(t)}{F'_n(t)} \left(J_t^H(n-1;g) - (-1)^{n-1} \left(\xi_\infty^H + \nu \right) \right) \longrightarrow (-1)^n \left(\xi_\infty^H + \nu \right) \quad (t \to \infty)$$

in L^2 and almost surely for all $n \geqslant 1$ if

$$F'_1(t), \quad \frac{F''_n(t)}{g(t) F'_n(t)}, \quad \frac{F'_n(t)}{F'_{n-1}(t)} \longrightarrow 0,$$

as t tends to infinity for all $n \geqslant 1$.

Example 3.7 *Let $g(t) = t^\alpha$ with $\alpha > 0$. Then, we have*

$$F_1(t) = t^{-\alpha}, \;\; F_n(t) = (-1)^{n+1} \alpha (1+2\alpha)(2+3\alpha) \cdots (n-2+(n-1)\alpha) t^{-n-1+n\alpha}$$

with $n \geqslant 2$, and

$$\lambda_1 = \frac{\alpha}{1+\alpha}, \;\; \lambda_n = \frac{1}{(1+\alpha)^n} \alpha (1+2\alpha)(2+3\alpha) \cdots (n-1+n\alpha).$$

Moreover, we have

$$(-1)^j G^j(t) \frac{\mathrm{d}}{\mathrm{d}t} F_j(t) - \lambda_j = 0$$

for all $t \geqslant 0$ and $j \geqslant 1$. It follows from Theorem 3.5 that

$$J_t^H(0;g) := t^\alpha e^{-\frac{1}{1+\alpha} t^{1+\alpha}} X_t^H \longrightarrow \xi_\infty^H + \nu$$

and

$$J_t^H(n;g) := \frac{t^{1+\alpha}}{1+\alpha} \left(J_t^H(n-1;g) - \lambda_{n-1} \left(\xi_\infty^H + \nu \right) \right) \longrightarrow \lambda_n \left(\xi_\infty^H + \nu \right)$$

in L^2 and almost surely for every $n \geqslant 1$, as t tends to infinity, where $\lambda_0 = 1$. When $\alpha = 1$, we obtain

$$\lambda_0 = 1, \quad \lambda_n = \frac{1}{2^n} (2n-1)!!.$$

4. Recursive convergence II

In this section, we obtain the second recursive convergence on the solution to (2.6) under the following situation:

$$\lim_{t \to \infty} g(t) = \theta > 0, \quad g(t) \not\equiv \theta \quad (t \geqslant 0). \tag{4.1}$$

The asymptotic structure of this situation depends entirely on the structure of infinitesimal $g(t) - \theta$ and the velocity tending to zero. Thus, the process (2.6) does not coincide with the (non-ergodic) Ornstein-Uhlenbeck process

$$\mathrm{d}X_t^H = \mathrm{d}B_t^H + \theta X_t^H \mathrm{d}t + \nu \mathrm{d}t.$$

The asymptotic behavior obtained in this section is ineffective for the Ornstein-Uhlenbeck process. Keep the notations in Section 3.

Lemma 4.1 *Let g satisfy the condition (4.1) and $g \in C^{\infty}(\mathbb{R})$, and let the sequence $\{F_n(t), n \geqslant 0\}$ be defined in Lemma 3.1 such that*

$$\lim_{t\to\infty} \frac{\mathrm{d}}{\mathrm{d}t} F_n(t) = 0, \quad \lim_{t\to\infty} \frac{F'_{n+1}(t)}{F'_n(t)} = 0 \tag{4.2}$$

for all $n \geqslant 1$. Define the functions $\hat{I}_g(t;n) : \mathbb{R}_+ \to \mathbb{R}$, $n = 1, 2, \ldots$ using the following recursion formulas:

$$\hat{I}_g(t;1) = \frac{1}{F'_1(t)} I_g(t), \quad \hat{I}_g(t;n) = \frac{F'_{n-1}(t)}{F'_n(t)} \left[\hat{I}_g(t;n-1) - (-1)^{n-1}\right]$$

for $n \geqslant 2$. Then, we have

$$\lim_{t\to\infty} \hat{I}_g(t;n) = (-1)^n \tag{4.3}$$

for every $n \geqslant 1$.

Remark 4.2 *When $g(t) - \theta \equiv 0$, the condition (4.2) does not hold. The condition (4.2) gives the ratio of $g'(t)$ tending to zero. There are many differentiable functions $t \mapsto g(t)$ that satisfy the condition (4.2). For example, the functions satisfying one of the following conditions:*

$$g(t) - \theta = O\left((1+t)^{-\alpha}\right), \quad g(t) - \theta = O\left(\log^{-\alpha}(2+t)\right) \quad (t \to \infty)$$

with $\alpha > 0$. However, such a function $g(t) = \theta + O\left(e^{-t}\right)$ $(t \to \infty)$ does not satisfy the condition (4.2).

Proof Similar to Lemma 3.1 we can obtain the lemma. In fact, using the proof of Lemma 3.1 we obtain

$$I_g(t) = -\frac{\mathrm{d}}{\mathrm{d}t} F_1(t) + \ldots + (-1)^m \frac{\mathrm{d}}{\mathrm{d}t} F_m(t) + (C_m + \alpha_m(t))\, e^{-G(t)} + \Lambda_g(t,m)$$

for all $t > 0$ and $m \geqslant 1$, where $\alpha_m(t)$ is an infinitesimal as $t \to \infty$ and

$$\Lambda_g(t,m) = (-1)^{m+1} g(t) e^{-G(t)} \int_0^t \left(e^{G(u)} - \sum_{j=0}^{m} \frac{1}{j!} G^j(u) \right) \mathrm{d}F_{m+1}(u).$$

Combining this with the definitions of $\hat{I}_g(t;n)$, we obtain

$$\hat{I}_g(t;1) = \frac{1}{F'_1(t)} I_g(t) = -1 + \frac{F'_2(t)}{F'_1(t)} + \frac{1}{F'_1(t)} \left(C_2 + \alpha_2(t)\right) e^{-G(t)} + \frac{1}{F'_1(t)} \Lambda_g(t,2)$$

and

$$\hat{I}_g(t;n) = (-1)^n + (-1)^{n+1} \frac{F'_{n+1}(t)}{F'_n(t)} + \left(C_{n+1} + \alpha_{n+1}(t)\right) \frac{1}{F'_n(t)} e^{-G(t)} + \frac{1}{F'_n(t)} \Lambda_g(t,n+1)$$

for every integer $n \geqslant 1$.

On the other hand, by using L'Hôspital's rule and (4.2) we obtain

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{F'_n(t)} e^{-G(t)} \int_0^t F'_{n+1}(u) e^{G(u)} \mathrm{d}u = \lim_{t\to\infty} \frac{1}{F'_n(t) e^{G(t)}} \int_0^t F'_{n+1}(u) e^{G(u)} \mathrm{d}u \\
= &\lim_{t\to\infty} \frac{F'_{n+1}(t)}{F''_n(t) + F'_n(t) g(t)} = 0
\end{aligned} \tag{4.4}$$

and

$$\lim_{t\to\infty} \frac{1}{F'_n(t)} e^{-G(t)} \int_0^t G^j(u) \mathrm{d}F_{n+1}(u) = 0 \tag{4.5}$$

for all integers $0 \leqslant j \leqslant n < m$ and $1 \leqslant k \leqslant n < m$, which means

$$\frac{1}{F_n'(t)} \Lambda_g(t, n+1) \longrightarrow 0 \quad (t \to \infty)$$

for every integer $n \geqslant 1$. It follows from (4.2) that

$$\hat{I}_g(t; n) \longrightarrow (-1)^n \quad (t \to \infty)$$

for every integer $n \geqslant 1$. This completes the proof. □

Theorem 4.3 *Let $\frac{1}{2} \leqslant H < 1$, and let the conditions in Lemma 4.1 hold. Recursively define the processes $\hat{J}^H(n; g) = \{\hat{J}_t^H(n; g), t \geqslant 0\}$, $n = 0, 1, 2, \dots$ by*

$$\hat{J}_t^H(1; g) := \frac{1}{F_1'(t)} \left(\hat{J}_t^H(0; g) - \xi_\infty^H + \nu \right)$$

and

$$\hat{J}_t^H(n; g) := \frac{F_{n-1}'(t)}{F_n'(t)} \left(\hat{J}_t^H(n-1; g) - (-1)^{n-1} \left(\xi_\infty^H + \nu \right) \right),$$

where $\hat{J}_t^H(0; g) = g(t) e^{-G(t)} X_t^H$. Then, we have

$$\hat{J}_t^H(n; g) \longrightarrow (-1)^n \left(\xi_\infty^H + \nu \right) \quad (t \to \infty) \tag{4.6}$$

in L^2 and almost surely for every $n \geqslant 1$.

Proof Using (3.11) we obtain

$$\begin{aligned}
\hat{J}_t^H(1; g) &= \frac{1}{F_1'(t)} \left(J_t^H(0; g) - \left(\xi_\infty^H + \nu \right) \right) \\
&= \left(\xi_\infty^H + \nu \right) \frac{1}{F_1'(t)} I_g(t) - \frac{g(t)}{F_1'(t)} e^{-G(t)} \int_0^t I_g(s) \mathrm{d}B_s^H - \frac{\Psi(t)}{F_1'(t)} \left(\xi_\infty^H - \xi_t^H \right) \\
&= \left(\xi_\infty^H + \nu \right) \hat{I}_g(t; 1) - \frac{g(t)}{F_1'(t)} e^{-G(t)} \int_0^t I_g(s) \mathrm{d}B_s^H - \frac{\Psi(t)}{F_1'(t)} \left(\xi_\infty^H - \xi_t^H \right)
\end{aligned}$$

and

$$\begin{aligned}
\hat{J}_t^H(n; g) &= \frac{F_{n-1}'(t)}{F_n'(t)} \left(\hat{J}_t^H(n-1; g) - (-1)^{n-1} \left(\xi_\infty^H + \nu \right) \right) \\
&= \left(\xi_\infty^H + \nu \right) \hat{I}_g(t; n) - \frac{g(t)}{F_n'(t)} e^{-G(t)} \int_0^t I_g(s) \mathrm{d}B_s^H - \frac{\Psi(t)}{F_n'(t)} \left(\xi_\infty^H - \xi_t^H \right)
\end{aligned}$$

for all $t > 0$ and $n \geqslant 2$, where $\Psi(t) = g(t) e^{-G(t)} \int_0^t e^{G(u)} \mathrm{d}u$.

On the other hand, (4.3) implies that

$$\int_0^t I_g(s) \mathrm{d}B_s^H \longrightarrow \int_0^\infty I_g(s) \mathrm{d}B_s^H \quad (t \to \infty)$$

in L^2 and almost surely. Using L'Hôspital's rule and (4.2), we obtain

$$\lim_{t \to \infty} \frac{1}{(F_n'(t))^2} E \left(\xi_\infty^H - \xi_t^H \right)^2 = \lim_{t \to \infty} \frac{1}{(F_n'(t))^2} \int_t^\infty g(s)^2 e^{-2G(s)} \mathrm{d}s = 0$$

for $H = \frac{1}{2}$ and

$$\lim_{t \to \infty} \frac{1}{(F_n'(t))^2} E \left(\xi_\infty^H - \xi_t^H \right)^2 = \lim_{t \to \infty} \frac{1}{(F_n'(t))^2} \int_t^\infty \int_t^\infty g(r) g(s) e^{-G(r) - G(s)} |s - r|^{2H-2} \mathrm{d}r \mathrm{d}s = 0$$

for $\frac{1}{2} < H < 1$, which shows that

$$\frac{1}{F_n'(t)}\Psi(t)\left(\xi_\infty^H-\xi_t^H\right)\longrightarrow 0\quad(t\to\infty)$$

in L^2 for all $\frac{1}{2}\leqslant H<1$ and $n\geqslant 1$. Moreover, by (4.2) we also obtain

$$\begin{aligned}\frac{1}{F_n'(t)}\Psi(t)\left(\xi_\infty^H-\xi_t^H\right)&\sim\frac{1}{F_n'(t)}\int_t^\infty g(s)e^{-G(s)}\mathrm{d}B_s^H\\ &=-\frac{g(t)}{F_n'(t)}e^{-G(t)}B_t^H-\frac{1}{F_n'(t)}\int_t^\infty B_s^H\left[g'(s)-g(s)^2\right]e^{-G(s)}\mathrm{d}s\longrightarrow 0\quad(t\to\infty)\end{aligned}$$

almost surely for all $\frac{1}{2}\leqslant H<1$ and $n\geqslant 1$. Thus, the theorem follows from Lemma 4.1. □

Remark 4.4 *In this section, to give asymptotic behaviors with recursive convergence, we assume that g is differentiable. When g is not differentiable we also can obtain the diverged speed of the solution X^H ; however, we cannot obtain such recursive asymptotic behavior.*

Example 4.5 *Let $g(t)=\frac{1+t}{2+t}=1-\frac{1}{2+t}$. Then, we have*

$$F_1'(t)=-\frac{1}{(1+t)^2},F_2'(t)\sim\frac{2}{(1+t)^3},\ldots,F_n'(t)\sim\frac{n!}{(1+t)^{n+1}},$$

as t tends to infinity for all $n\geqslant 2$, (4.2) holds. It follows from Theorem 4.3 that

$$\hat{J}_t^H(0;g)=\left(1-\frac{1}{2+t}\right)e^{-t+\log(2+t)-\log 2}X_t^H\longrightarrow\xi_\infty^H+\nu$$

and

$$\frac{1}{n}(1+t)\left\{\hat{J}_t^H(n-1;g)-(-1)^{n-1}\left(\xi_\infty^H+\nu\right)\right\}\longrightarrow(-1)^n\left(\xi_\infty^H+\nu\right)$$

in L^2 and almost surely for every $n\geqslant 1$, as t tends to infinity.

5. Recursive convergence III

In this section, we obtain the third recursive convergence on the solution to (2.6) under the following conditions:

$$\int_0^\infty ug(u)\mathrm{d}u=\infty,\quad G(+\infty)=K_0>0\tag{5.1}$$

and

$$\lim_{t\to\infty}tg(t)=0,\quad G(+\infty)=\infty.\tag{5.2}$$

We keep the notations in Sections 3 and 4 and define function

$$\tilde{I}_g(t)=1-\frac{1}{t}e^{-G(t)}\int_0^t e^{G(s)}\mathrm{d}s$$

with $t>0$.

Lemma 5.1 *Let $g\in C(\mathbb{R}_+)$ satisfy either the condition (5.1) or (5.2). We define the functions $\{\beta_n(t),t\geqslant 1\}$ and $\{\tilde{I}_g(t;n),n\geqslant 1\}$ by the following recursion formulas*

$$\beta_0(t)=t,\quad\beta_1(t)=\int_0^t sg(s)\mathrm{d}s,\quad\beta_{n+1}(t)=\int_0^t\beta_n(s)g(s)\mathrm{d}s$$

and

$$\tilde{I}_g(t;0)=\frac{t}{\beta_1(t)}\tilde{I}_g(t),\quad\tilde{I}_g(t;n)=\frac{\beta_n(t)}{\beta_{n+1}(t)}\left[\tilde{I}_g(t;n-1)-(-1)^{n-1}\right]$$

for $t\geqslant 0$ and $n\geqslant 1$. If $\beta_n(t)\longrightarrow+\infty$ as t tends to infinity, for all $n\geqslant 2$, we then have, for all $n\geqslant 0$

$$\lim_{t\to\infty} \frac{\beta_{n+1}(t)}{\beta_n(t)} = 0 \tag{5.3}$$

and

$$\lim_{t\to\infty} \tilde{I}_g(t;n) = (-1)^n. \tag{5.4}$$

Remark 5.2 *The condition (5.2) implies that* $\lim\limits_{t\to\infty} \beta_n(t) = \infty$ *for* $n \geqslant 2$*. However, when the condition (5.1) holds, the limit* $\lim\limits_{t\to\infty} \beta_n(t)$ *may converge, for example,* $g(t) = \frac{1}{(1+t)^2}$*. Thus,*

$$\beta_1(t) = \int_0^t sg(s)\mathrm{d}s = \log(1+t) - 1 + \frac{1}{1+t}$$

and

$$0 < \beta_2(t) = \int_0^t \beta_1(s)g(s)\mathrm{d}s \leqslant \int_0^t \log(1+s)\frac{\mathrm{d}s}{(1+s)^2} \longrightarrow \int_0^\infty \log(1+s)\frac{\mathrm{d}s}{(1+s)^2} < \infty,$$

as t tends to infinity.

Proof of Lemma 5.1 The convergence (5.3) exists owing to

$$\lim_{t\to\infty} \frac{\beta_1(t)}{\beta_0(t)} = \lim_{t\to\infty} \frac{\int_0^t sg(s)\mathrm{d}s}{t} = 0$$

and the inductive method. For the convergence (5.4), integration by parts is applied continuously to obtain

$$\begin{aligned}
\tilde{I}_g(t) &= 1 - \frac{1}{t}e^{-G(t)}\int_0^t e^{G(s)}\mathrm{d}s = \frac{1}{t}e^{-G(t)}\int_0^t sg(s)e^{G(s)}\mathrm{d}s \\
&= \frac{1}{t}e^{-G(t)}\int_0^t e^{G(s)}d\beta_1(s) \\
&= \frac{1}{t}\beta_1(t) - \frac{1}{t}e^{-G(t)}\int_0^t g(s)\beta_1(s)e^{G(s)}\mathrm{d}s \\
&= \frac{1}{t}\beta_1(t) - \frac{1}{t}e^{-G(t)}\int_0^t e^{G(s)}d\beta_2(s) \\
&= \frac{1}{t}\beta_1(t) - \frac{1}{t}\beta_2(t) + \frac{1}{t}e^{-G(t)}\int_0^t g(s)\beta_2(s)e^{G(s)}\mathrm{d}s \\
&= \frac{1}{t}\sum_{k=1}^m (-1)^{k+1}\beta_k(t) + (-1)^m\frac{1}{t}e^{-G(t)}\int_0^t g(s)\beta_m(s)e^{G(s)}\mathrm{d}s
\end{aligned}$$

for all $t > 0$ and $m \geqslant 1$. It follows that

$$\tilde{I}_g(t;0) = \frac{t}{\beta_1(t)}\tilde{I}_g(t) = 1 - \frac{1}{\beta_1(t)e^{G(t)}}\int_0^t g(s)\beta_1(s)e^{G(s)}\mathrm{d}s \longrightarrow 1, \tag{5.5}$$

and

$$\tilde{I}_g(t;1) = \frac{\beta_1(t)}{\beta_2(t)}\left\{\tilde{I}_g(t;0) - 1\right\} = -1 + \frac{1}{\beta_2(t)e^{G(t)}}\int_0^t g(s)\beta_2(s)e^{G(s)}\mathrm{d}s \longrightarrow -1$$

as t tends to infinity. Moreover, by taking $m = n$

$$\begin{aligned}
\tilde{I}_g(t;n) &= \frac{\beta_n(t)}{\beta_{n+1}(t)}\left\{\tilde{I}_g(t;n-1) - (-1)^{n+1}\right\} \\
&= (-1)^n + (-1)^{n+1}\frac{1}{\beta_{n+1}(t)e^{G(t)}}\int_0^t g(s)\beta_{n+1}(s)e^{G(s)}\mathrm{d}s \longrightarrow (-1)^n
\end{aligned}$$

as t tends to infinity. This completes the proof. $\square$

Lemma 5.3 *Let the conditions in Lemma 5.1 hold. Then, for all $n \geqslant 2$ and $\gamma \in (0,1)$ we obtain*

$$\lim_{t\to\infty} \frac{t^{-\gamma}\beta_1(t)}{\beta_n(t)} = 0. \tag{5.6}$$

Proof Using the principle of induction one can establish the convergence (5.6). By applying the induction method, we first obtain the convergence

$$\lim_{t\to\infty} \frac{t^{\alpha}}{\beta_n(t)} = 0 \tag{5.7}$$

for all $n \geqslant 1$ and any $\alpha \in (0,1)$. L'Hôspital's rule and $\int_0^t g(s)\mathrm{d}s \to \infty$ $(t\to\infty)$ imply that

$$\lim_{t\to\infty} \frac{t^{\alpha}}{\beta_1(t)} = \lim_{t\to\infty} \frac{\alpha t^{\alpha-1}}{tg(t)} = \alpha \lim_{t\to\infty} \frac{1}{t^{2-\alpha}g(t)} = 0 \tag{5.8}$$

for any $\alpha \in (0,1)$. According to L'Hôspital's rule and the arbitrariness of $\alpha \in (0,1)$, we get

$$\lim_{t\to\infty} \frac{t^{\alpha}}{\beta_n(t)} = \lim_{t\to\infty} \frac{\alpha t^{\alpha-1}}{g(t)\beta_{n-1}(t)} = \alpha \lim_{t\to\infty} \frac{1}{t^{1+\kappa}g(t)} \cdot \frac{t^{\kappa+\alpha}}{\beta_{n-1}(t)} = 0$$

for all $\kappa \in (0,1)$ and $0 < \kappa + \alpha < 1$, if (5.7) holds for $n-1$, and the convergence (5.7) holds for all $n \geqslant 1$ and any $\alpha \in (0,1)$.

(5.6) is an immediate result of (5.7). Therefore, the arbitrariness of $\alpha \in (0,1)$ implies that

$$\frac{t^{-\gamma}\beta_1(t)}{\beta_n(t)} = \frac{t^{\alpha}}{\beta_n(t)} \cdot \frac{\beta_1(t)}{t^{\alpha+\gamma}} \longrightarrow 0 \quad (t\to\infty)$$

by taking $\alpha = 1-\gamma$. This completes the proof. $\square$

Lemma 5.4 *Let the conditions in Lemma 5.1 hold. Then, we have*

$$\tilde{J}_t(0,g) := \frac{1}{t}e^{-G(t)}X_t^H \longrightarrow \xi_\infty^H + \nu \tag{5.9}$$

in L^2 and almost surely, as t tends to infinity.

Proof Consider the decomposition

$$\begin{aligned}
\tilde{J}_t^H(0;g) &= \frac{1}{t}e^{-G(t)}X_t^H = \frac{1}{t}e^{-G(t)}\int_0^t h_g(t,s)\mathrm{d}B_s^H + \nu\frac{1}{t}e^{-G(t)}\int_0^t h_g(t,s)\mathrm{d}s \\
&= \frac{1}{t}e^{-G(t)}B_t^H + \frac{1}{t}e^{-G(t)}\int_0^t \xi_u^H e^{G(u)}\mathrm{d}u + \nu\frac{1}{t}e^{-G(t)}\int_0^t e^{G(u)}\mathrm{d}u \\
&= \frac{1}{t}e^{-G(t)}B_t^H - \frac{1}{t}e^{-G(t)}\int_0^t \left(\xi_\infty^H - \xi_u^H\right)e^{G(u)}\mathrm{d}u + \left(\xi_\infty^H + \nu\right)\frac{1}{t}e^{-G(t)}\int_0^t e^{G(u)}\mathrm{d}u \\
&= -\frac{1}{t}e^{-G(t)}\int_0^t I_g(s)\mathrm{d}B_s^H - \frac{1}{t}e^{-G(t)}\int_0^t e^{G(u)}\mathrm{d}u\left(\xi_\infty^H - \xi_t^H\right) + \left(\xi_\infty^H + \nu\right)\frac{1}{t}e^{-G(t)}\int_0^t e^{G(u)}\mathrm{d}u
\end{aligned}$$

and

$$\begin{aligned}
\tilde{J}_t^H(0;g) - \left(\xi_\infty^H + \nu\right) = &-\frac{1}{t}e^{-G(t)}\int_0^t I_g(s)\mathrm{d}B_s^H \\
&-\frac{1}{t}e^{-G(t)}\int_0^t e^{G(u)}\mathrm{d}u\left(\xi_\infty^H - \xi_t^H\right) - \left(\xi_\infty^H + \nu\right)\tilde{I}_g(t)
\end{aligned} \tag{5.10}$$

for all $t > 0$.

We first obtain the convergence with probability one. By Lemma 5.1 we can find that

$$\int_0^t I_g(s)\mathrm{d}B_s^H$$

does not converge as t tends to infinity, because

$$I_g(t) = g(t)e^{-G(t)}\int_0^t e^{G(u)}\mathrm{d}u - 1 = tg(t)\cdot\frac{1}{t}e^{-G(t)}\int_0^t e^{G(u)}\mathrm{d}u - 1 \longrightarrow -1,$$

as t tends to infinity. However, using $\lim_{t\to\infty} tg(t) = 0$ and (2.5) for all $n \geqslant 1$ and $0 \leqslant \tau < 1$ we obtain

$$\begin{aligned}
&E\left[\sup_{T_n\leqslant t<T_{n+1}}\left|t^{\tau-1}e^{-G(t)}\int_0^t I_g(s)\mathrm{d}B_s^H\right|^2\right]\\
&\leqslant \frac{1}{T_n^{2-2\tau}}e^{-2G(T_n)}E\left[\sup_{T_n\leqslant t<T_{n+1}}\left|\int_0^t I_g(s)\mathrm{d}B_s^H\right|^2\right]\\
&\leqslant \frac{C_H}{T_n^{2-2\tau}}e^{-2G(T_n)}\int_0^{T_{n+1}}(I_g(s))^2\mathrm{d}s \leqslant C_H\frac{T_{n+1}}{T_n^{2-2\tau}}e^{-2G(T_n)}
\end{aligned}\tag{5.11}$$

with $H = \frac{1}{2}$ and

$$\begin{aligned}
&E\left[\sup_{T_n\leqslant t<T_{n+1}}\left|t^{\tau-1}e^{-G(t)}\int_0^t I_g(s)\mathrm{d}B_s^H\right|^2\right]\\
&\leqslant \frac{1}{T_n^{2-2\tau}}e^{-2G(T_n)}E\left[\sup_{T_n\leqslant t<T_{n+1}}\left|\int_0^t I_g(s)\mathrm{d}B_s^H\right|^2\right]\\
&\leqslant \frac{C_H}{T_n^{2-2\tau}}e^{-2G(T_n)}\int_0^{T_{n+1}}\int_0^{T_{n+1}}|I_g(r)I_g(s)||s-r|^{2H-2}\mathrm{d}r\mathrm{d}s\\
&\leqslant \frac{C_H}{T_n^{2-2\tau}}e^{-2G(T_n)}\int_0^{T_{n+1}}\int_0^{T_{n+1}}|s-r|^{2H-2}\mathrm{d}r\mathrm{d}s \leqslant C_H\frac{T_{n+1}^{2H}}{T_n^{2-2\tau}}e^{-2G(T_n)}
\end{aligned}\tag{5.12}$$

with $\frac{1}{2} < H < 1$, where $\{T_n, n \geqslant 1\}$ is a sequence of positive numbers such that $T_n \uparrow \infty$, which shows that the series

$$\sum_{n=0}^{\infty} P\left(\sup_{T_n\leqslant t<T_{n+1}}\left|t^{\tau-1}e^{-G(t)}\int_0^t I_g(s)\mathrm{d}B_s^H\right| \geqslant \varepsilon\right)\tag{5.13}$$

converges for all $\frac{1}{2} \leqslant H < 1$ by taking $T_n = n^\kappa$ with $\kappa > \frac{1}{2(1-H-\tau)}$ and $0 \leqslant \tau < 1 - H$, and so

$$t^{\tau-1}\int_0^t I_g(s)\mathrm{d}B_s^H \longrightarrow 0 \quad (t\to\infty)\tag{5.14}$$

almost surely for all $0 \leqslant \tau < 1 - H$ and $\frac{1}{2} \leqslant H < 1$. Similarly, for all $n \geqslant 1$ and $0 \leqslant \tau < 1$ we also obtain

$$\begin{aligned}
E\left[\sup_{T_n\leqslant t<T_{n+1}} t^{2\tau}\left|\xi_\infty^H - \xi_t^H\right|^2\right] &\leqslant T_{n+1}^{2\tau}E\left[\sup_{T_n\leqslant t<T_{n+1}}\left|\int_t^{+\infty} g(s)e^{-G(s)}\mathrm{d}B_s^H\right|^2\right]\\
&\leqslant C_H T_{n+1}^{2\tau}\int_{T_n}^{+\infty}|g(s)|^2e^{-2G(s)}\mathrm{d}s \leqslant CT_{n+1}^{2\tau}(T_n)^{-1}e^{-2G(T_n)}
\end{aligned}\tag{5.15}$$

with $H = \frac{1}{2}$ and

$$
\begin{aligned}
&E\left[\sup_{T_n\leqslant t<T_{n+1}} t^{2\tau}\left|\xi_\infty^H-\xi_t^H\right|^2\right]\leqslant T_{n+1}^{2\tau}E\left[\sup_{T_n\leqslant t<T_{n+1}}\left|\int_t^{+\infty} g(s)e^{-G(s)}\mathrm{d}B_s^H\right|^2\right]\\
&\leqslant C_H T_{n+1}^{2\tau}\int_{T_n}^{+\infty}\int_{T_n}^{+\infty} g(r)g(s)e^{-G(r)-G(s)}|s-r|^{2H-2}\mathrm{d}r\mathrm{d}s\\
&\leqslant C\alpha_H T_{n+1}^{2\tau}T_n^{2H-2}e^{-2G(T_n)}\int_1^\infty\int_1^\infty |x-y|^{2H-2}\frac{\mathrm{d}xdy}{xy}
\end{aligned}\tag{5.16}
$$

with $\frac{1}{2}<H<1$, which shows that the series

$$
\sum_{n=0}^\infty P\left(\sup_{T_n\leqslant t<T_{n+1}} t^\tau|\xi_\infty^H-\xi_t^H|\geqslant\varepsilon\right)
$$

converges for all $\frac{1}{2}\leqslant H<1$ by taking $T_n=n^\kappa$ with $\kappa>\frac{1}{2(1-H-\tau)}$ and $0\leqslant\tau<1-H$, and so

$$
t^\tau\left(\xi_\infty^H-\xi_t^H\right)\longrightarrow 0\quad(t\to\infty)\tag{5.17}
$$

almost surely for all $0\leqslant\tau<1-H$ and $\frac{1}{2}\leqslant H<1$. Combining these with (5.10), (5.13) and Lemma 5.1, we obtain the convergence (5.9) with probability one.

Finally, the convergence in L^2 follows from (5.11), (5.12), (5.15), (5.16) and (5.10). □

Theorem 5.5 *Let $\frac{1}{2}\leqslant H<1$, and let the conditions in Lemma 5.1 hold. We recursively define the processes $\tilde{J}^H(n;g)=\{\tilde{J}_t^H(n;g),t\geqslant 0\}$, $n=0,1,2,\ldots$ by*

$$
\tilde{J}_t^H(1;g):=\frac{t}{\beta_1(t)}\left(\tilde{J}_t^H(0;g)-\xi_\infty^H-\nu\right)
$$

and

$$
\tilde{J}_t^H(n;g):=\frac{\beta_{n-1}(t)}{\beta_n(t)}\left(\tilde{J}_t^H(n-1;g)-(-1)^{n-1}\left(\xi_\infty^H+\nu\right)\right).
$$

Then, the convergence

$$
\tilde{J}_t^H(n;g)\longrightarrow(-1)^n\left(\xi_\infty^H+\nu\right)\quad(t\to\infty)\tag{5.18}
$$

holds in L^2 and almost surely for every $n\geqslant 1$.

Proof We first obtain the convergence (5.18) with probability one. For $n=1$, using (5.7), (5.14) and (5.17), we obtain

$$
\frac{1}{\beta_1(t)}e^{-G(t)}\int_0^t I_g(s)\mathrm{d}B_s^H=\frac{t^{1-\tau}}{\beta_1(t)}\cdot t^{\tau-1}e^{-G(t)}\int_0^t I_g(s)\mathrm{d}B_s^H\longrightarrow 0\quad(t\to\infty)\tag{5.19}
$$

and

$$
\frac{1}{\beta_1(t)}e^{-G(t)}\int_0^t e^{G(u)}\mathrm{d}u\left(\xi_\infty^H-\xi_t^H\right)=\frac{1}{t}e^{-G(t)}\int_0^t e^{G(u)}\mathrm{d}u\cdot\frac{t^{1-\tau}}{\beta_1(t)}\cdot t^\tau\left(\xi_\infty^H-\xi_t^H\right)\longrightarrow 0\tag{5.20}
$$

for any $0<\tau<1-H$ almost surely, as t tends to infinity. It follows from Lemma 5.1 that

$$
\begin{aligned}
\tilde{J}_t^H(1;g)=&-\left(\xi_\infty^H+\nu\right)\cdot\tilde{I}_g(t;0)\\
&-\frac{1}{\beta_1(t)}e^{-G(t)}\int_0^t I_g(s)\mathrm{d}B_s^H-\frac{1}{\beta_1(t)}e^{-G(t)}\int_0^t e^{G(u)}\mathrm{d}u\left(\xi_\infty^H-\xi_t^H\right)\\
\longrightarrow&-\left(\xi_\infty^H+\nu\right)
\end{aligned}\tag{5.21}
$$

almost surely, as t tends to infinity.

For any $n\geqslant 2$, by (5.7), (5.14) and (5.17) we obtain

$$\frac{1}{\beta_n(t)}e^{-G(t)}\int_0^t I_g(s)\mathrm{d}B_s^H = \frac{t^{1-\tau}}{\beta_n(t)}\cdot t^{\tau-1}e^{-G(t)}\int_0^t I_g(s)\mathrm{d}B_s^H \longrightarrow 0 \tag{5.22}$$

and

$$\frac{1}{\beta_n(t)}e^{-G(t)}\int_0^t e^{G(u)}\mathrm{d}u\left(\xi_\infty^H-\xi_t^H\right) = \frac{1}{t}e^{-G(t)}\int_0^t e^{G(u)}\mathrm{d}u\cdot\frac{t^{1-\tau}}{\beta_n(t)}\cdot t^\tau\left(\xi_\infty^H-\xi_t^H\right)\longrightarrow 0 \tag{5.23}$$

almost surely for any $0<\tau<1-H$, as t tends to infinity. Combining this with Lemma 5.1 we obtain

$$\begin{aligned}\tilde{J}_t^H(n;g) = &-\left(\xi_\infty^H+\nu\right)\cdot\tilde{I}_g(t;n-1)\\ &-\frac{1}{\beta_n(t)}e^{-G(t)}\int_0^t I_g(s)\mathrm{d}B_s^H-\frac{1}{\beta_n(t)}e^{-G(t)}\int_0^t e^{G(u)}\mathrm{d}u\left(\xi_\infty^H-\xi_t^H\right)\\ \longrightarrow\ &(-1)^n(\xi_\infty^H+\nu)\end{aligned} \tag{5.24}$$

almost surely, as t tends to infinity, and the convergence (5.18) follows with probability one.

Now, we consider the convergence (5.18) in L^2. Let $n\geqslant 1$. Then,

$$\frac{e^{-2G(t)}}{\beta_n(t)^2}E\left|\int_0^t I_g(s)\mathrm{d}B_s^H\right|^2 = \frac{e^{-2G(t)}}{\beta_n(t)^2}\int_0^t I_g(s)^2\mathrm{d}s\leqslant C\frac{t}{\beta_n(t)^2}e^{-2G(t)}\longrightarrow 0$$

with $H=\frac{1}{2}$ and

$$\frac{e^{-2G(t)}}{\beta_n(t)^2}E\left|\int_0^t I_g(s)\mathrm{d}B_s^H\right|^2\leqslant\frac{e^{-2G(t)}}{\beta_n(t)^2}\int_0^t\int_0^t|I_g(r)I_g(s)||s-r|^{2H-2}\mathrm{d}r\mathrm{d}s\leqslant C\frac{t^{2H}}{\beta_n(t)^2}e^{-2G(t)}\longrightarrow 0$$

with $\frac{1}{2}\leqslant H<1$, as t tends to infinity, which proves

$$\frac{1}{\beta_n(t)}e^{-G(t)}\int_0^t I_g(s)\mathrm{d}B_s^H\longrightarrow 0$$

in L^2 for all $\frac{1}{2}\leqslant H<1$, as t tends to infinity. Similarly, we obtain

$$\frac{1}{\beta_n(t)}e^{-G(t)}\int_0^t e^{G(u)}\mathrm{d}u\left(\xi_\infty^H-\xi_t^H\right)\longrightarrow 0$$

in L^2 for all $\frac{1}{2}\leqslant H<1$, as t tends to infinity. Combining this with Lemma 5.1, we show that the convergence (5.24) holds in L^2, and the theorem follows. □

Example 5.6 *Let $g(t)=\frac{1}{(2+t)\log^p(2+t)}$ with $p>0$. Then, when $0<p\leqslant 1$, the function g satisfies the condition (5.2) and when $p>1$, the function g satisfies the condition (5.1). Furthermore, we obtain*

$$\beta_1(t)\sim\frac{2+t}{\log^p(2+t)},\quad\beta_2(t)\sim\frac{2+t}{\log^{p+1}(2+t)},\quad\ldots,\quad\beta_n(t)\sim\frac{2+t}{\log^{p+n}(2+t)}$$

for all $n\geqslant 2$. Then, we obtain

$$\frac{\beta_n(t)}{\beta_{n-1}(t)}\sim\frac{1}{\log(2+t)}\longrightarrow 0\quad(t\to\infty)$$

and

$$\frac{t^{-\gamma}\beta_1(t)}{\beta_n(t)}\sim\frac{\log^n(2+t)}{t^\gamma}\longrightarrow 0\quad(t\to\infty)$$

for all $n\geqslant 1$ and $\gamma\in(0,1)$. It follows from Theorem 5.5 that

$$\tilde{J}_t^H(0,g) := a_p(t)X_t^H \longrightarrow \xi_\infty^H + \nu$$

and

$$\tilde{J}_t^H(n,g) = \log(2+t)\left(\tilde{J}_t^H(n-1;g) - (-1)^{n-1}\left(\xi_\infty^H + \nu\right)\right) \longrightarrow (-1)^n(\xi_\infty^H + \nu)$$

in L^2 and almost surely for every $n \geqslant 1$ and $p > 0$, as t tends to infinity, where

$$a_p(t) = \begin{cases} \frac{1}{t}\exp\left\{\frac{1}{1-p}(\log 2)^{1-p} - \frac{1}{1-p}(\log(2+t))^{1-p}\right\}, & \text{if } 0 < p < 1, \\ \frac{\log 2}{t\log(2+t)}, & \text{if } p = 1, \\ \frac{1}{t}\exp\left\{\frac{1}{1-p}(\log 2)^{1-p} - \frac{1}{1-p}(\log(2+t))^{1-p}\right\}, & \text{if } p > 1. \end{cases}$$

We consider the case $\lim_{t\to\infty} tg(t) = Q > 0$. The commonality between the condition (5.1) and (5.2) is that $\lim_{t\to\infty} tg(t) = 0$, but the difference is that the speed at which the function $tg(t)$ tends to zero is different as $t \to \infty$. However, when $\lim_{t\to\infty} tg(t) = Q > 0$, the solution to (2.6) does not satisfy the similar recursive convergence as (5.18). The main difference between the conditions $\lim_{t\to\infty} tg(t) = 0$ and $\lim_{t\to\infty} tg(t) = Q \neq 0$ is that when $Q \neq 0$, the limit $\lim_{t\to\infty} \frac{tg(t)-Q}{g(t)}$ can be equal to zero, a nonzero constant, or infinity, and even the function $\frac{tg(t)-Q}{g(t)}$ may be identical to a non zero constant. Let X^H be the solution of (2.6), such that $\lim_{t\to\infty} tg(t) = Q > 0$. It follows that

$$\begin{aligned}\bar{J}_t^H(0;g) &:= \frac{1+Q}{t}e^{-G(t)}X_t^H \\ &= \left(\xi_\infty^H + \nu\right) - \frac{1+Q}{t}e^{-G(t)}\int_0^t I_g(s)\mathrm{d}B_s^H \\ &\quad - \frac{1+Q}{t}e^{-G(t)}\int_0^t e^{G(u)}\mathrm{d}u\left(\xi_\infty^H - \xi_t^H\right) - \left(\xi_\infty^H + \nu\right)\check{I}_g(t;Q) \end{aligned} \tag{5.25}$$

for all $t > 0$, where

$$\check{I}_g(t;Q) = 1 - \frac{1+Q}{t}e^{-G(t)}\int_0^t e^{G(s)}\mathrm{d}s.$$

Noting that the convergence (5.14) and (5.17) hold also for $Q \neq 0$, and $\lim_{t\to\infty} \check{I}_g(t;Q) = 0$,

$$\bar{J}_t^H(0;g) \longrightarrow \xi_\infty^H + \nu$$

in L^2 and almost surely, as t tends to infinity. Notably,

$$\begin{aligned}\check{I}_g(t;Q) &= -Q + \frac{1+Q}{t}e^{-G(t)}\int_0^t sg(s)e^{G(s)}\mathrm{d}s \\ &= -Q + \frac{1+Q}{t}e^{-G(t)}\int_0^t [sg(s) - Q]e^{G(s)}\mathrm{d}s + Q\frac{1+Q}{t}e^{-G(t)}\int_0^t e^{G(s)}\mathrm{d}s \\ &= -Q\left(1 - \frac{1+Q}{t}e^{-G(t)}\int_0^t e^{G(s)}\mathrm{d}s\right) + \frac{1+Q}{t}e^{-G(t)}\int_0^t [sg(s) - Q]e^{G(s)}\mathrm{d}s \\ &= -Q\check{I}_g(t;Q) + \frac{1+Q}{t}e^{-G(t)}\int_0^t [sg(s) - Q]e^{G(s)}\mathrm{d}s\end{aligned}$$

for all $t>0$ by the integration by parts, which implies that

$$\check{I}_g(t;Q)=\frac{1}{t}e^{-G(t)}\int_0^t[sg(s)-Q]e^{G(s)}\mathrm{d}s \tag{5.26}$$

for all $t>0$. On the other hand, noting that the convergence

$$\frac{t}{\frac{tg(t)-Q}{g(t)}}=\frac{tg(t)}{tg(t)-Q}\longrightarrow\infty \tag{5.27}$$

holds, as t tends to infinity, we can obtain the following asymptotic behavior.

Lemma 5.7 *Let $g\in C^\infty(\mathbb{R}_+)$ with $g(t)>0$ for $t>0$, and let $\lim_{t\to\infty}tg(t)=Q>0$. We define the functions $\{\hat{F}_n(t),n\geqslant 1\}$ using the following recursion formulas*

$$\hat{F}_0(t)=t,\quad \hat{F}_1(t)=\frac{tg(t)-Q}{g(t)},\quad \hat{F}_2(t)=\frac{1}{g(t)}\frac{\mathrm{d}}{\mathrm{d}t}\hat{F}_1(t),\quad \hat{F}_n(t)=\frac{1}{g(t)}\frac{\mathrm{d}}{\mathrm{d}t}\hat{F}_{n-1}(t)$$

for $t\geqslant 0$ and $n\geqslant 2$. Consider the sequence $\{L_g(t;n),n\geqslant 1\}$ of functions on $\mathbb{R}_+$ defined by the following recursion formulas:

$$L_g(t;0)=\frac{t}{\hat{F}_1(t)}\check{I}_g(t;Q),\quad L_g(t;n)=\frac{\hat{F}_{n-1}(t)}{\hat{F}_n(t)}\left[L_g(t;n-1)-(-1)^{n-1}\right]$$

for $n\geqslant 1$. If

$$\lim_{t\to\infty}\frac{\hat{F}_{n+1}(t)}{\hat{F}_n(t)}=0 \tag{5.28}$$

for all $n\geqslant 1$, then

$$\lim_{t\to\infty}L_g(t;n)=(-1)^n \tag{5.29}$$

for all $n\geqslant 0$.

The condition (5.28) in Lemma 5.7 is general, and many functions satisfy it, such as $g(t)=\frac{1}{t}+\frac{1}{t^2}$ for $t>1$.

Proof of Lemma 5.7 By (5.26) we have that

$$\begin{aligned}
\check{I}_g(t;Q)&=\frac{1}{t}e^{-G(t)}\int_0^t[sg(s)-Q]e^{G(s)}\mathrm{d}s=\frac{1}{t}e^{-G(t)}\int_0^t\frac{sg(s)-Q}{g(s)}d\left(e^{G(s)}-1\right)\\
&=\frac{1}{t}e^{-G(t)}\left(e^{G(t)}-1\right)\hat{F}_1(t)-\frac{1}{t}e^{-G(t)}\int_0^t\left(e^{G(s)}-1\right)\frac{\mathrm{d}}{\mathrm{d}s}\hat{F}_1(s)\mathrm{d}s\\
&=\frac{1}{t}\left(1-e^{-G(t)}\right)\hat{F}_1(t)-\frac{1}{t}e^{-G(t)}\int_0^t\frac{1}{g(s)}\frac{\mathrm{d}}{\mathrm{d}s}\hat{F}_1(s)d\left(e^{G(s)}-G(s)-1\right)\\
&=\frac{1}{t}\left(1-e^{-G(t)}\right)\hat{F}_1(t)-\frac{1}{t}\left(1-G(t)e^{-G(t)}-e^{-G(t)}\right)\hat{F}_2(t)\\
&\quad+\frac{1}{t}e^{-G(t)}\int_0^t\left(e^{G(s)}-G(s)-1\right)d\hat{F}_2(s)
\end{aligned}$$

for all $t>0$. Then, by continuously applying integration by parts m times, $\check{I}_g(t;Q)$ can be expressed as

$$\begin{aligned}\check{I}_g(t;Q) = &\left(1-e^{-G(t)}\right)\hat{F}_1(t)\frac{1}{t}-\left(1-G(t)e^{-G(t)}-e^{-G(t)}\right)\hat{F}_2(t)\frac{1}{t}+(-1)^m\tilde{F}_m(t)\frac{1}{t}\\ &+\ldots+(-1)^{m+1}\left(1-e^{-G(t)}\sum_{j=0}^{m-1}\frac{1}{j!}G^j(t)\right)\hat{F}_m(t)\frac{1}{t}+\Theta_g(t,m)\frac{1}{t}\\ =&\left(\hat{F}_1(t)-\hat{F}_2(t)+\ldots+(-1)^{m+1}\hat{F}_m(t)\right)\frac{1}{t}+(-1)^m\tilde{F}_m(t)\frac{1}{t}+(-1)^{m+1}\Theta_g(t,m)\frac{1}{t}\\ &-\frac{1}{te^{G(t)}}\left(\hat{F}_1(t)-(1+G(t))\hat{F}_2(t)+\ldots+(-1)^{m+1}\sum_{j=0}^{m-1}\frac{1}{j!}G^j(t)\hat{F}_m(t)\right)\end{aligned}$$

for all $t>0$ and $m\geqslant 1$, where

$$\tilde{F}_m(t)=e^{-G(t)}\int_0^t\hat{F}'_m(u)e^{G(u)}\mathrm{d}u$$

and

$$\Theta_g(t,m)=e^{-G(t)}\sum_{j=0}^{m-1}\frac{1}{j!}\int_0^t G^j(u)d\hat{F}_m(u).$$

Combining this with the definitions of $L_g(t;n)$, we obtain

$$L_g(t;0)=\frac{t}{\hat{F}_1(t)}\check{I}_g(t;Q)=1-\frac{\hat{F}_2(t)}{\hat{F}_1(t)}+\frac{\tilde{F}_2(t)}{\hat{F}_1(t)}-\frac{1}{\hat{F}_1(t)e^{G(t)}}\left(\hat{F}_1(t)-(1+G(t))\hat{F}_2(t)\right)-\frac{\Theta_g(t,2)}{\hat{F}_1(t)}$$

and

$$\begin{aligned}L_g(t;n)=&\frac{\hat{F}_{n-1}(t)}{\hat{F}_n(t)}\left\{L_g(t;n-1)-(-1)^{n-1}\right\}=(-1)^n+(-1)^n\frac{\hat{F}_{n+1}(t)}{\hat{F}_n(t)}+(-1)^{n+1}\frac{\tilde{F}_{n+1}(t)}{\hat{F}_n(t)}\\ &-\frac{1}{\hat{F}_n(t)e^{G(t)}}\left(F_1(t)-(1+G(t))F_2(t)+\ldots+(-1)^{n+1}\sum_{j=0}^{n-1}\frac{1}{j!}G^j(t)\hat{F}_n(t)\right)\\ &+\Theta_g(t,n+1)\frac{1}{\hat{F}_n(t)}\end{aligned}$$

for all $n\geqslant 1$.

Using L'Hôspital's rule and (5.28), we show that

$$\begin{aligned}\lim_{t\to\infty}\frac{\tilde{F}_{n+1}(t)}{\hat{F}_n(t)}&=\lim_{t\to\infty}\frac{1}{\hat{F}_n(t)e^{G(t)}}\int_0^t\hat{F}'_{n+1}(u)e^{G(u)}\mathrm{d}u\\ &=\lim_{t\to\infty}\frac{\hat{F}'_{n+1}(t)}{\hat{F}'_n(t)+\hat{F}_n(t)g(t)}=\lim_{t\to\infty}\frac{\hat{F}_{n+2}(t)}{\hat{F}_{n+1}(t)+\hat{F}_n(t)}=0\end{aligned}$$

and

$$\lim_{t\to\infty}\Theta_g(t,n+1)\frac{1}{\hat{F}_n(t)}=0$$

for all integers $n\geqslant 1$. It follows from (5.28) that

$$L_g(t;n)\longrightarrow(-1)^n\quad(t\to\infty)$$

for every integer $n\geqslant 0$. This completes the proof. □

Theorem 5.8 *Let $\frac{1}{2} \leqslant H < 1$, and let g satisfy the conditions in Lemma 5.7. We recursively define the processes $\bar{J}^H(n;g) = \{\bar{J}_t^H(n;g), t \geqslant 0\}$, $n = 1,2,\dots$ by*

$$\bar{J}_t^H(1;g) := \frac{t}{\hat{F}_1(t)}\left(\bar{J}_t^H(0;g) - \xi_\infty^H - \nu\right)$$

and

$$\bar{J}_t^H(n;g) := \frac{\hat{F}_{n-1}(t)}{\hat{F}_n(t)}\left(\bar{J}_t^H(n-1;g) - (-1)^n\left(\xi_\infty^H + \nu\right)\right)$$

with $n \geqslant 2$. Then, the convergence

$$\bar{J}_t^H(n;g) \longrightarrow (-1)^n\left(\xi_\infty^H + \nu\right) \quad (t \to \infty) \tag{5.30}$$

holds in L^2 and almost surely for every $n \geqslant 1$.

Proof Using (5.25), the condition (5.28), and Lemma 5.7 and noting that the convergence (5.14) and (5.17) hold for $Q > 0$, we obtain

$$\begin{aligned}\bar{J}_t^H(1;g) &:= \frac{t}{\hat{F}_1(t)}\left(\bar{J}_t^H(0;g) - \xi_\infty^H - \nu\right) = -\frac{1+Q}{\hat{F}_1(t)}e^{-G(t)}\int_0^t I_g(s)\mathrm{d}B_s^H\\ &\quad - \frac{1+Q}{\hat{F}_1(t)}e^{-G(t)}\int_0^t e^{G(u)}\mathrm{d}u\left(\xi_\infty^H - \xi_t^H\right) - \left(\xi_\infty^H + \nu\right)\frac{t}{\hat{F}_1(t)}\check{I}_g(t;Q)\\ &\longrightarrow -\left(\xi_\infty^H + \nu\right)\end{aligned}$$

and for $n \geqslant 2$

$$\begin{aligned}\bar{J}_t^H(n;g) &= \frac{\hat{F}_{n-1}(t)}{\hat{F}_n(t)}\left\{\bar{J}_t^H(n-1;g) - (-1)^{n-1}(\xi_\infty^H + \nu)\right\} = -\frac{1+Q}{\hat{F}_n(t)}e^{-G(t)}\int_0^t I_g(s)\mathrm{d}B_s^H\\ &\quad - \frac{1+Q}{\hat{F}_n(t)}e^{-G(t)}\int_0^t e^{G(u)}\mathrm{d}u\left(\xi_\infty^H - \xi_t^H\right) - \left(\xi_\infty^H + \nu\right)\frac{\hat{F}_{n-1}(t)}{\hat{F}_n(t)}\bar{I}_g(t;n)\\ &\longrightarrow (-1)^n\left(\xi_\infty^H + \nu\right)\end{aligned}$$

in L^2 and almost surely, as t tends to infinity. This completes the proof. □

Example 5.9 *Let $g(t) = \frac{\rho}{1+t}$ with $t \geqslant 0$ and $\rho > 0$. Then, the condition (5.28) in Lemma 5.7 does not hold because $Q = \rho$ and $\frac{tg(t)-Q}{g(t)} = -1$ for all $t \geqslant 0$. It follows that $G(t) = \rho\log(1+t)$,*

$$\begin{aligned}t\check{I}_g(t;Q) &= e^{-G(t)}\int_0^t [sg(s) - Q]e^{G(s)}\mathrm{d}s\\ &= -e^{-G(t)}\left(e^{G(t)} - 1\right) = -1 + e^{-G(t)} = -1 + \frac{1}{(1+t)^\rho} \longrightarrow -1,\end{aligned}$$

and

$$\begin{aligned}\ell_g(t;0) &:= t^\rho\left\{t\check{I}_g(t;Q) + 1\right\} = \frac{t^\rho}{(1+t)^\rho} = \left(1 - \frac{1}{1+t}\right)^\rho\\ &= 1 - \frac{\rho}{1+t} + \sum_{m=2}^\infty (-1)^m\frac{1}{m!}\rho(\rho-1)\cdots(\rho-m+1)\frac{1}{(1+t)^m} \longrightarrow 1,\end{aligned}$$

as t tends to infinity. When ρ is not an integer,

$$\begin{aligned}\ell_g(t;1) &:= (1+t)\left\{\ell_g(t;0)-1\right\} = -\rho + \frac{\rho(\rho-1)}{2(1+t)} \\ &\quad + (1+t)\sum_{m=3}^{\infty}(-1)^m \frac{1}{m!}\rho(\rho-1)\cdots(\rho-m+1)\frac{1}{(1+t)^m} \longrightarrow -\rho, \\ \ell_g(t;2) &:= (1+t)\left\{\ell_g(t;1)+\rho\right\} = \frac{1}{2}\rho(\rho-1) - \frac{\rho(\rho-1)(\rho-2)}{3!(1+t)} \\ &\quad + (1+t)^2\sum_{m=4}^{\infty}(-1)^m \frac{1}{m!}\rho(\rho-1)\cdots(\rho-m+1)\frac{1}{(1+t)^m} \longrightarrow \frac{1}{2}\rho(\rho-1),\end{aligned}$$

as t tends to infinity, and for all $n \geqslant 3$,

$$\begin{aligned}\ell_g(t;n) &:= (1+t)\left\{\ell_g(t;n-1) - (-1)^{n-1}\frac{1}{(n-1)!}\rho(\rho-1)\cdots(\rho-n+2)\right\} \\ &= (-1)^n\frac{1}{n!}\rho(\rho-1)\cdots(\rho-n+1) + (-1)^{n+1}\frac{\rho(\rho-1)\cdots(\rho-n)}{(n+1)!(1+t)} \\ &\quad + (1+t)^n\sum_{m=n+2}^{\infty}(-1)^m\frac{1}{m!}\rho(\rho-1)\cdots(\rho-m+1)\frac{1}{(1+t)^m} \\ &\longrightarrow (-1)^n\frac{1}{n!}\rho(\rho-1)\cdots(\rho-n+1),\end{aligned}$$

as t tends to infinity. Similarly, when $\rho = m$ is an integer, we can obtain the corresponding asymptotic behavior as follows:

$$\ell_g(t;n) = (1+t)\left\{\ell_g(t;n-1) - (-1)^{n-1}\binom{m}{n-1}\right\} \longrightarrow (-1)^n\binom{m}{n}$$

for all $1 \leqslant n \leqslant m$, as t tends to infinity, and $\ell_g(t;n) = 0$ for all $n > m$. Furthermore, we can also obtain the asymptotic convergence by replacing the weight $1+t$ in $\ell_g(t;n)$ with t. For example, when $\rho = 1$,

$$\begin{aligned}\hat{\ell}_g(t;1) &= t\left\{\ell_g(t;0)-1\right\} = -\frac{t}{1+t} = -1 + \frac{1}{1+t} \longrightarrow -1, \\ \hat{\ell}_g(t;n) &= t\left\{\hat{\ell}_g(t;n-1) - (-1)^{n-1}\right\} \longrightarrow (-1)^n\end{aligned}$$

for all $n \geqslant 2$, as t tends to infinity. However, the solution of (2.6) with $g(t) = \frac{\rho}{1+t}$ does not satisfy such recursive convergence like before. When $\rho > H$, similar to (5.14) and (5.17), we obtain

$$\frac{1}{(1+t)^\rho}\int_0^t I_g(s)\mathrm{d}B_s^H, \quad t\left(\xi_\infty^H - \xi_t^H\right) \longrightarrow 0$$

in L^2 and almost surely, as t tends to infinity. It follows from (5.25) and Example 5.9 that

$$\begin{aligned}\bar{J}_t^H(1;g) &= t\left\{\bar{J}_t^H(0;g) - \left(\xi_\infty^H + \nu\right)\right\} \\ &= -(1+\rho)e^{-G(t)}\int_0^t I_g(s)\mathrm{d}B_s^H - (1+\rho)e^{-G(t)}\int_0^t e^{G(u)}\mathrm{d}u\left(\xi_\infty^H - \xi_t^H\right) - \left(\xi_\infty^H + \nu\right)\cdot t\check{I}_g(t;Q) \\ &= -\frac{1+\rho}{(1+t)^\rho}\int_0^t I_g(s)\mathrm{d}B_s^H - \frac{1+\rho}{(1+t)^\rho}\int_0^t e^{G(u)}\mathrm{d}u\left(\xi_\infty^H - \xi_t^H\right) - \left(\xi_\infty^H + \nu\right)\cdot t\check{I}_g(t;Q) \\ &\longrightarrow \xi_\infty^H + \nu\end{aligned}$$

in L^2 and almost surely, as t tends to infinity. But, we have

$$\begin{aligned}\bar{J}_t^H(2;g) &= t^{\rho-H}\left\{\bar{J}_t^H(1;g) - \left(\xi_\infty^H+\nu\right)\right\}\\ &= -(1+\rho)\frac{t^{\rho-H}}{(1+t)^\rho}\left\{\int_0^t I_g(s)\mathrm{d}B_s^H + \int_0^t e^{G(u)}\mathrm{d}u\left(\xi_\infty^H-\xi_t^H\right)\right\}\\ &\quad - \left(\xi_\infty^H+\nu\right)\cdot t^{\rho-H}\left\{t\check{I}_g(t;Q)+1\right\}\end{aligned}$$

converges in distribution to a normal random variable, as t tends to infinity.

Finally, at the end of this section, we consider large time behaviors of the solution X^H to (2.6) under the remaining case, namely

$$G(+\infty)=K_0>0,\qquad \int_0^\infty ug(u)\mathrm{d}u<\infty. \tag{5.31}$$

We will find that there is no recursive convergence like before in this situation. The main reason, in our opinion, is that $g(t)=o\left(\frac{1}{t^2}\right)$ as t tends infinity. We only assume that g is continuous and $g(t)\geqslant 0$ for all $t\geqslant 0$. Then, the random variables $\xi_\infty^H=\int_0^\infty g(s)e^{-G(s)}\mathrm{d}B_s^H$ are well-defined as elements in L^2 because $g\in C(\mathbb{R})$ and $g(t)=o(t^{-2})$ as t tends to infinity.

Proposition 5.10 *Let $\frac{1}{2}\leqslant H<1$ and let g satisfy the condition* (5.31)*. Then, the convergence*

$$\xi_t^H\longrightarrow\xi_\infty^H \tag{5.32}$$

and

$$\tilde{J}_t^H(0;g)=\frac{1}{t}e^{-G(t)}X_t^H\longrightarrow\xi_\infty^H+\nu \tag{5.33}$$

holds in L^2 and almost surely, as t tends to infinity. Moreover, the convergence

$$t^{1-H}\left(\tilde{J}_t^H(0;g)-(\xi_\infty^H+\nu)\right)\longrightarrow N\left(0,e^{-2K_0}\right) \tag{5.34}$$

holds in distribution, as t tends to infinity, where $N(0,\sigma^2)$ denotes a normal distribution with mean 0 and variance σ^2 .

Proof The convergence (5.32) and (5.33) follows from (5.10) and the proof of Lemma 5.4. Now, we obtain the convergence (5.34).

Noting that the condition (5.31) implies that

$$E\left(\xi_\infty^H-\xi_t^H\right)^2=E\left(\int_t^\infty g(s)e^{-G(s)}\mathrm{d}B_s^H\right)^2=\int_t^\infty g(s)^2e^{-2G(s)}\mathrm{d}s=o\left(\frac{1}{t^3}\right)$$

for $H=\frac{1}{2}$ and

$$\begin{aligned}E\left(\xi_\infty^H-\xi_t^H\right)^2&=E\left(\int_t^\infty g(s)e^{-G(s)}\mathrm{d}B_s^H\right)^2\\&=\int_0^t\int_0^s g(s)g(r)e^{-G(s)-G(r)}(s-r)^{2H-2}\mathrm{d}r\mathrm{d}s=o\left(\frac{1}{t^{4-2H}}\right)\end{aligned}$$

for $\frac{1}{2}<H<1$ as $t\to\infty$, we obtain

$$t^\kappa\left(\xi_\infty^H-\xi_t^H\right)\longrightarrow 0 \tag{5.35}$$

in L^2 for all $\kappa<2-H$ and $\frac{1}{2}\leqslant H<1$, as $t\to\infty$.

On the other hand, considering the condition (5.31), we realize that

$$I_g(t)=g(t)e^{-G(t)}\int_0^t e^{G(u)}\mathrm{d}u-1\longrightarrow -1\quad(t\to\infty)$$

and

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{t^{2H-1}}\int_0^t g(r)e^{-G(r)}\left(\int_0^r e^{G(u)}\mathrm{d}u\right)(t-r)^{2H-2}\mathrm{d}r\\
\leqslant\ &\lim_{t\to\infty}\frac{1}{t^{2H-1}}\int_0^t g(r)r(t-r)^{2H-2}\mathrm{d}r\\
=\ &\lim_{t\to\infty}\frac{1}{t}\int_0^t g(r)r\left(1-\frac{r}{t}\right)^{2H-2}\mathrm{d}r=0
\end{aligned}$$

for all $\frac{1}{2}<H<1$. It follows that

$$\lim_{t\to\infty}E\left(t^{-\frac{1}{2}}\int_0^t I_g(s)\mathrm{d}B_s^H\right)^2=\lim_{t\to\infty}\frac{1}{t}\int_0^t I_g(s)^2\mathrm{d}s=1$$

for $H=\frac{1}{2}$ and

$$\begin{aligned}
\lim_{t\to\infty}E\left(t^{-H}\int_0^t I_g(s)\mathrm{d}B_s^H\right)^2&=\lim_{t\to\infty}\frac{2H(2H-1)}{t^{2H}}\int_0^t\int_0^s I_g(s)I_g(r)(s-r)^{2H-2}\mathrm{d}r\mathrm{d}s\\
&=-\lim_{t\to\infty}\frac{2H-1}{t^{2H-1}}\int_0^t I_g(r)(t-r)^{2H-2}\mathrm{d}r\\
&=1-\lim_{t\to\infty}\frac{2H-1}{t^{2H-1}}\int_0^t g(r)e^{-G(r)}\left(\int_0^r e^{G(u)}\mathrm{d}u\right)(t-r)^{2H-2}\mathrm{d}r=1
\end{aligned}$$

for all $\frac{1}{2}<H<1$. Thus, the normality of $t^{-H}\int_0^t I_g(s)\mathrm{d}B_s^H$ with $t>0$ implies that

$$t^{-H}\int_0^t I_g(s)\mathrm{d}B_s^H\longrightarrow N(0,1)\quad(t\to\infty)\tag{5.36}$$

in distribution for all $\frac{1}{2}\leqslant H<1$. Combining this with (5.10), (5.31), (5.35), Slutsky's theorem and

$$t^{1-H}\tilde{I}_g(t)=t^{1-H}\left(1-\frac{1}{t}e^{-G(t)}\int_0^t e^{G(u)}\mathrm{d}u\right)=t^{-H}e^{-G(t)}\int_0^t ug(u)e^{G(u)}\mathrm{d}u\longrightarrow 0\quad(t\to\infty),$$

we obtain

$$\begin{aligned}
t^{1-H}\left(\tilde{J}_t^H(0;g)-\left(\xi_\infty^H+\nu\right)\right)&=e^{-G(t)}\cdot t^{-H}\int_0^t I_g(s)\mathrm{d}B_s^H-\left(\xi_\infty^H+\nu\right)\cdot t^{1-H}\tilde{I}_g(t)\\
&\quad-\frac{1}{t}e^{-G(t)}\int_0^t e^{G(u)}\mathrm{d}u\cdot t^{1-H}\left(\xi_\infty^H-\xi_t^H\right)\\
&\longrightarrow N\left(0,e^{-2K_0}\right)
\end{aligned}$$

in distribution, as t tends to infinity. □

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