Probability, Uncertainty and Quantitative Risk
Vol. 9, No. 4, 2024, 553–574

https://www.aimsciences.org/PUQR
DOI:10.3934/puqr.2024023

Doubly reflected BSDEs with quadratic growth and random terminal horizon

Mohammed Elhachemy[1,*], Mohamed El Jamali[2], Mohamed El Otmani[1]

[1] *Laboratory of Analysis, Mathematics and Applications (LAMA), Faculty of Sciences Agadir, Ibn Zohr University, Morocco*

[2] *National Institute of Statistics and Applied Economics, Rabat, B.P. 6217, Morocco*

Email: mohammed.elhachemy@edu.uiz.ac.ma, mohamed.eljamali@edu.uiz.ac.ma, m.elotmani@uiz.ac.ma

Abstract In this paper, we study one-dimensional backward stochastic differential equations featuring two reflecting barriers. When the terminal time is not necessarily bounded or finite and the generator $f(t,y,z)$ exhibits quadratic growth in z, we prove existence and uniqueness of solutions. In the Markovian case, we establish the link with an obstacle problem for quadratic elliptic partial differential equation with Dirichlet boundary conditions.

Keywords Double reflected BSDE, Random terminal time, Quadratic growth, Elliptic partial differential equations, Dirichlet boundary conditions

2020 Mathematics Subject Classification 60H05, 60H10, 60H30, 60J60, 35D40

1. Introduction

The concept of Backward Stochastic Differential Equations (commonly abbreviated as BSDEs) was initially introduced by Pardoux and Peng in their work referenced as [24]. In their groundbreaking research, they provided rigorous evidence for the existence and uniqueness of adapted processes denoted as (Y,Z) within the context of the following equation:

$$Y_t = \xi + \int_t^T f(s,Y_s,Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s, \quad t\in[0,T],$$

where $(W_t)_{t\geqslant 0}$ is the Wiener process. These BSDEs are characterized by the presence of a generator denoted as f and a terminal condition represented as ξ. Notably, this category of BSDEs finds extensive applications across various domains, including mathematical finance, stochastic control, and stochastic games. However, it's important to note that the original impetus behind BSDEs, as highlighted in sources like [25], was to provide a probabilistic framework for interpreting viscosity solutions of semi-linear Partial Differential Equations (PDE

Received 19 November 2023; Accepted 15 April 2024; Early access 27 June 2024
*Corresponding author

in short). Subsequent studies by Barles and Lesigne [5], followed by Bally and Matoussi [3], delved into the intricate relationship between BSDEs and solutions to semi-linear PDEs, particularly within Sobolev spaces. We also refer to [4, 23] for the link between BSDEs and Integral PDE.

Moreover, the realm of BSDEs has seen further expansion. El Karoui et al. introduced the concept of Reflected BSDEs in [14]. A solution to such an equation encompasses a triple of processes denoted as (Y, Z, K), which collectively satisfy the following conditions:

$$L_t \leqslant Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\mathrm{d}s + K_T - K_t - \int_t^T Z_s \mathrm{d}W_s, \quad t \in [0, T].$$

The inclusion of the increasing process K serves the purpose of ensuring that the solution Y of the BSDE remains greater than a given process L, which functions as an obstacle. This constraint is achieved with a minimum energy condition expressed as $\int_0^T (Y_t - L_t)\mathrm{d}K_t = 0$. Additionally, in their work [14], the authors demonstrated that the solution to a Reflected BSDE corresponds to the value function of an optimal stopping problem. They further provided a probabilistic interpretation for the viscosity solution associated with an obstacle problem linked to a nonlinear parabolic Partial Differential Equation.

Subsequently, Cvitanic and Karatzas, as discussed in [11], conducted an in-depth investigation into BSDEs featuring two reflecting barriers, often referred to as BSDE2RBs. In essence, a solution to these equations comprises a quadruple of processes, denoted as (Y, Z, K^+, K^-), which collectively satisfy the following conditions:

$$L_t \leqslant Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\mathrm{d}s + (K_T^+ - K_t^+) - (K_T^- - K_t^-) - \int_t^T Z_s \mathrm{d}W_s \leqslant U_t,$$

where

$$\int_0^T (Y_t - L_t)\mathrm{d}K_t^+ = \int_0^T (U_t - Y_t)\mathrm{d}K_t^- = 0.$$

In this framework, a solution is constrained to remain within the boundaries defined by a lower barrier, denoted as L, and an upper barrier, denoted as U. This constraint is maintained through the combined action of two continuous and monotonically increasing reflecting processes K^+ and K^-. Under the so-called Mokobodski's condition, the authors established the existence and uniqueness of solutions for cases where the generator function is Lipschitz with respect to the variables (y, z).

A natural question then arises: Are there any less stringent conditions under which the BSDE2RBs possesses a solution? This inquiry has piqued the interest of numerous researchers in this field. In their study referenced as [19], Hamadène et al. demonstrate the presence of a solution, not necessarily unique, in scenarios where f exhibits continuity with linear growth and only one of the barriers adheres to regularity criteria. Extending the same assumption regarding the generator, Lepeltier and San Martin [22] employ the penalization method to establish the existence of a solution for BSDE2RBs without imposing additional smoothness requirements on the barriers. They assume that the barriers are continuous and meet the conditions stipulated by Mokobodski. Meanwhile, Bahlali et al. [2] investigate BSDE2RBs under the framework of Mokobodski's condition, particularly when the generator displays continuous quadratic growth concerning z. In contrast, in the absence of Mokobodski's condition and without the constraints of barrier regularity, Hamadène et al. [17, 18] prove the existence of a solution, particularly when the barriers are entirely separate. To provide further specificity, Hamadène and Hassani [17] operate under the Lipschitz condition for the generator, while Hamadène and Hdiri [18] consider the generator's continuity with quadratic growth. For additional investigations into this variety of BSDE, we direct the reader to the significant contributions of Essaky et al. [15, 16].

In a different context, various authors have addressed BSDEs with random terminal times; see for example [7, 12, 31], and the references therein. However, most previous contributions to this kind of BSDEs have been obtained in the framework when the coefficient being Lipschitz, or alternatively, being monotone and exhibiting appropriate growth concerning y, in addition to maintaining Lipschitz continuity concerning z.

We aim to share some research that is truly inspiring concerning the random terminal value. Kobylanski's work, as documented in [21], centres around a real BSDE featuring a quadratic generator in terms of z, and this BSDE involves a random terminal time. In her study, she imposes the condition that the stopping time is either bounded or almost surely finite under the probability measure $\mathbb{P}$. Subsequently, Briand et al. [9] investigate BSDEs, aiming to establish both the existence and uniqueness of solutions for arbitrary stopping times. They achieve this result by requiring that the generator exhibits strict monotonicity concerning y.

It is well-known that, BSDEs with deterministic terminal times form the basis for the probabilistic interpretation of semi-linear second-order parabolic or elliptic PDEs through the Feynman-Kac formula. This concept extends to quasi-linear second-order PDE systems, as discussed by Peng, Pardoux, Tang, and others, as referenced in [28, 29]. The initial exploration of obstacle problems in PDEs utilized Reflected BSDEs, as documented in [14]. Meanwhile, BSDEs featuring random terminal times are closely linked to elliptic PDEs, with this connection being established in [12] for Lipschitz coefficients in (y, z), in [1, 21] for quadratic coefficients in z and in [31] for monotone coefficients in y with Lipschitz continuity in z.

In the current paper, we deal with one-dimensional BSDE2RBs with a random terminal time which is not necessarily bounded or finite, and a generator $f(t, y, z)$ exhibits quadratic growth in z. Assuming that the driver satisfies a condition on z coherent with the one of quadratic growth and employing linearization arguments, we exhibit a priori estimates that provides a control on the distance between two solutions based on the disparity between the associated bounded terminal conditions and generators. This implies in particular the uniqueness of solution for BSDE2RBs. Furthermore, instead of relying on a fixed point argument for establishing existence, we employ an approximation procedure wherein we assume a condition analogous to the so-called Mokobodski. In order to establish the convergence associated with the barriers, it is necessary to introduce an appropriate assumption regarding the lower barrier. Utilizing the flow property within the Markovian framework, we demonstrate that the solution to BSDE2RBs with random terminal times serves as a viscosity solution to a related obstacle problem for an elliptic PDE with Dirichlet boundary conditions. In this context, the non-linearity manifests as the square of the gradient.To the best of our knowledge, no previous work in the literature has addressed this specific issue, rendering our conclusion notably innovative.

The remainder of this paper is structured as follows: In Section 2, we formally introduce and analyze the problem of the quadratic BSDE2RBs, considering scenarios where the random time terminal is possibly unbounded. Section 3 is dedicated to establishing the intricate links between solutions to the quadratic BSDE2RBs and their counterparts in the realm of Partial Differential Equations. Finally, in the Appendix we give a comparison result for quadratic BSDE and we deal with the link between this BSDE and elliptic quadratic PDE.

2. BSDE2RBs with quadratic growth and random terminal time

Throughout this paper, let $(\Omega, \mathcal{F}, \mathbb{P})$ is a fixed probability space on which is defined a standard d-dimensional Brownian motion $(W_t)_{t\geqslant 0}$ whose natural filtration is $(\mathcal{F}_t)_{t\geqslant 0}$ and where $\mathcal{F}_0$ contains all $\mathbb{P}$−null sets of $\mathcal{F}$.

Given an $\mathcal{F}_t$-stopping time τ (not necessarily bounded or $\mathbb{P}$-a.s. finite) and, we define for $\lambda \in \mathbb{R}$ the following set of $\mathcal{F}_t$-progressively measurable process $(\psi_t)_{t\geqslant 0}$ taking values in $\mathbb{R}^k$

$$\mathcal{H}^{l,\lambda}_\tau(\mathbb{R}^k) = \left\{\psi : \mathbb{E}\left(\int_0^\tau e^{\lambda s}\, \|\psi_s\|^l_k \,\mathrm{d}s\right) < \infty\right\}$$

and for all $\mathcal{F}_t$-progressively measurable process $(\psi_t)_t$ taking values in $\mathbb{R}$, we define

$$\mathcal{S}^\infty_\tau = \left\{\psi : \sup_{0\leqslant t\leqslant \tau} |\psi_t| < \infty\right\}.$$

Furthermore, we denote by $\mathcal{A}_\tau$ the set of continuous $\mathcal{F}_t$-progressively measurable non-decreasing process $(K_t)_{t\geqslant 0}$ such that $K_0 = 0$ and $\mathbb{E}[|K_\tau|] < \infty$.

Now, let consider

($\boldsymbol{\mathcal{H}.0}$) A terminal value ξ, which is an $\mathcal{F}_\tau$-measurable random variable, is bounded by some real $\mathfrak{M}$.

($\boldsymbol{\mathcal{H}.1}$) A coefficient $f : \mathbb{R}^+ \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ such that, there exist $\kappa > 0$ and $\mu > 0$,

(1) $f(.,y,z)$ is progressively measurable for all $y \in \mathbb{R}$ and $z \in \mathbb{R}^d$.
(2) $(y,z) \longrightarrow f(t,w,y,z)$ is continuous.
(3) $\sup_{t\geqslant 0} |f(t,0,0)| \leqslant \mathfrak{M}$ a.s.
(4) $|f(t,y,z) - f(t,y,z')| \leqslant \kappa\|z - z'\|(\|z\| + \|z'\|), \quad \forall\, y \in \mathbb{R},\ z,z' \in \mathbb{R}^d$.
(5) $|f(t,y,z) - f(t,y',z)| \leqslant \mu|y - y'|, \quad \forall\, y,y' \in \mathbb{R}$ and $z \in \mathbb{R}^d$.

($\boldsymbol{\mathcal{H}.2}$) A pair of obstacles L and U satisfy

(1) $L_t < U_t$ for all $t \geqslant 0$ and $L_\tau \leqslant \xi \leqslant U_\tau$.
(2) $\mathbb{P}$-a.s. $\forall\, t \geqslant 0$, $|L_t| + |U_t| \leqslant \mathfrak{M}$.
(3) The Process $L = (L_t)_{t\geqslant 0}$ is a quasimartingale with canonical decomposition (see e.g. [30], Theorem 15, p.118):

$$L_t = L_0 + \mathcal{M}_t + \mathcal{P}_t,$$

where $\mathcal{M}$ is a uniformly integrable martingale and $\mathcal{P}$ is a predictable process of integrable variation with Jordan decomposition $\mathcal{P} = \mathcal{P}^+ - \mathcal{P}^-$.

($\boldsymbol{\mathcal{H}.3}$) There exist two progressively measurable processes η and $\zeta \in \mathcal{H}^{2,0}_\tau$ such that for $M_\tau := \xi\mathbb{I}_{\{\tau<\infty\}}$, we have for all $t \in \mathbb{R}^+$,

(1) $L_t \leqslant M_t \leqslant U_t$.
(2) $\mathrm{d}M_t = -\eta_t\mathbb{I}_{[0,\tau]}(t)\mathrm{d}t + \zeta_t\mathbb{I}_{[0,\tau]}(t)\mathrm{d}W_t$.
(3) $\mathbb{E}\int_0^\tau e^{\mu t}\,(1 + |\eta_t|)\,\mathrm{d}t < \infty$.

Remark 2.1 *The combination of* $(\mathcal{H}.1)(3-5)$ *leads to the following result:*

$$\begin{aligned} |f(t,y,z)| &\leqslant |f(t,0,0)| + |f(t,y,0) - f(t,0,0)| + |f(t,y,z) - f(t,y,0)| \\ &\leqslant \mathfrak{M} + \mu|y| + \kappa\|z\|^2 \\ &\leqslant \max(\mathfrak{M},\mu,\kappa)(1 + |y| + \|z\|^2); \quad \forall (t,y,z) \in (\mathbb{R}^+, \mathbb{R}, \mathbb{R}^d). \end{aligned}$$

Definition 2.2 *The process* (Y, Z, K^+, K^-) *with values in* $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}^+$ *is called a solution of BSDE2RB with data* (ξ, f, L, U) *if:*

$$\begin{cases} Y \in \mathcal{S}^\infty_\tau, \ Z \in \mathcal{H}^{2,-\mu}_\tau(\mathbb{R}^d) \text{ and } K^+, K^- \in \mathcal{A}_\tau, \\ L_t \leqslant Y_t \leqslant U_t \quad \text{on the set } [t \leqslant \tau], \\ Y_t = \xi, \ Z_t = 0, \ K^+_t = K^+_\tau \text{ and } K^-_t = K^-_\tau \text{ on the set } \{t > \tau\}, \\ \text{For all } T \geqslant t \geqslant 0, \\ Y_t = Y_T + \displaystyle\int_{\tau\wedge t}^{\tau\wedge T} f(s, Y_s, Z_s)\mathrm{d}s + (K^+_T - K^+_t) - (K^-_T - K^-_t) - \int_{\tau\wedge t}^{\tau\wedge T} Z_s \mathrm{d}W_s \ \ \mathbb{P}\text{-a.s.}, \\ \text{For all } T \geqslant 0, \quad \displaystyle\int_0^{\tau\wedge T} (U_s - Y_s)\mathrm{d}K^-_s = \int_0^{\tau\wedge T} (Y_s - L_s)\mathrm{d}K^+_s = 0 \ \ \mathbb{P}\text{-a.s.} \end{cases} \tag{2.1}$$

2.1 A priori estimate and uniqueness result

In this subsection, our goal is to demonstrate the uniqueness of solutions for the quadratic BSDE2RBs. To begin, we furnish an a priori estimate for the difference between solutions Y^1 and Y^2, which constitutes a pivotal element of our forthcoming analysis.

Proposition 2.3 *We suppose that* $(\mathcal{H}.0)$, $(\mathcal{H}.1)$, $(\mathcal{H}.2)(2)$ *and* $(\mathcal{H}.3)$ *hold and let* (Y, Z, K^+, K^-) *be the solution of the quadratic BSDE2RB (2.1). Then*

- Y *is bounded by* $\mathfrak{M}$.
- $\left(\int_0^t Z_s \mathrm{d}W_s\right)_{t\geqslant 0}$ *is a BMO-martingale.*

Proof It is clear that $|Y_t| \leqslant \max\{|L_t|, |U_t|\} \leqslant \mathfrak{M}$ for all $t \in \mathbb{R}^+$.

Next by applying Itô's formula to $Y - M$, we obtain for all stopping time $\nu \leqslant \tau$

$$\begin{aligned} |Y_\nu - M_\nu|^2 + \int_\nu^\tau \|Z_s - \zeta_s\|^2 \mathrm{d}s = |\xi - M_\tau|^2 &+ 2\int_\nu^\tau (Y_s - M_s)\left(f(s, Y_s, Z_s) - \eta_s\right)\mathrm{d}s \\ &+ 2\int_\nu^\tau (L_s - M_s)\mathrm{d}K^+_s - 2\int_\nu^\tau (U_s - M_s)\mathrm{d}K^-_s \\ &- 2\int_\nu^\tau (Y_s - M_s)\left(Z_s - \zeta_s\right)\mathrm{d}W_s. \end{aligned}$$

Taking the conditional expectation $\mathbb{E}_t[.] = \mathbb{E}[./\mathcal{F}_t]$ and using $(\mathcal{H}.0)$, $(\mathcal{H}.1)$ and $(\mathcal{H}.2)(2)$, we can write

$$\begin{aligned} \mathbb{E}_\nu\left[\int_\nu^\infty \|Z_s\|^2 \mathrm{d}s\right] &\leqslant 2\mathbb{E}_\nu\left[|\xi - M_\tau|^2 + 2\int_\nu^\tau |Y_s - M_s|\left(|f(s, Y_s, Z_s)| + |\eta_s|\right)\mathrm{d}s\right] \\ &\quad + 2\mathbb{E}_\nu\left[\int_\nu^\tau \|\zeta_s\|^2 \mathrm{d}s\right] \\ &\leqslant 2\mathbb{E}_\nu|\xi \mathbb{I}_{\{\tau=+\infty\}}|^2 + 2\mathbb{E}_\nu\left[\int_\nu^\tau \|\zeta_s\|^2 \mathrm{d}s\right] \\ &\quad + 4\mathbb{E}_\nu\left[\sup_{0\leqslant s\leqslant\tau} |Y_s - M_s| \int_\nu^\tau [C(1 + |Y_s| + \|Z_s\|^2) + |\eta_s|]\mathrm{d}s\right] \\ &\leqslant C\mathbb{E}_\nu[\Lambda_\tau] + 8\mathfrak{M}\kappa\mathbb{E}_\nu\left[\int_\nu^\infty \|Z_s\|^2 \mathrm{d}s\right], \end{aligned}$$

where $\Lambda_\tau = |\xi|^2 + \int_0^\tau \left(C(1 + \mathfrak{M}) + |\eta_s| + \|\zeta_s\|^2\right)\mathrm{d}s$. Choosing $1 > 8\mathfrak{M}\kappa$, we deduce that $\int_0^\cdot Z_s \mathrm{d}W_s$ is a BMO-martingale. □

Proposition 2.4 *Let* $(Y^i, Z^i, K^{i+}, K^{i-})_{i=1,2}$ *be solutions for the quadratic BSDE2RB*

$$\begin{cases} Y_t^i = \xi^i + \int_t^T \mathbb{I}_{\{s\leqslant\tau\}} f^i(s, Y_s^i, Z_s^i)\mathrm{d}s + (K_T^{i+} - K_t^{i+}) - (K_T^{i-} - K_t^{i-}) - \int_t^T Z_s^i \mathrm{d}W_s, \\ L_t \leqslant Y_t^i \leqslant U_t, \quad \forall\, t \leqslant T \text{ and } \int_0^T (U_s - Y_s^i)\mathrm{d}K_s^{i-} = \int_0^T (Y_s^i - L_s)\mathrm{d}K_s^{i+} = 0, \end{cases}$$

where ξ^i *are* $\mathcal{F}_\tau$*-measurable and bounded. We assume that the* Y^i *are bounded and the stochastic integral* $\left(\int_0^\cdot Z_t^i \mathrm{d}W_t\right)_{t\geqslant 0}$ *are BMO-martingales. We suppose also that there exists a deterministic function* ρ *with values in* $[0, +\infty)$ *such that*

$(\mathcal{H}_\rho)$ $|f^1(t, Y_t^2, Z_t^2) - f^2(t, Y_t^2, Z_t^2)| \leqslant \rho(t).$

Then,

$$|Y_t^1 - Y_t^2|^2 \leqslant e^{-\mu(T-t)}\|\xi^1 - \xi^2\|_\infty^2 + \frac{1}{\mu}\int_t^T e^{-\mu(s-t)}|\rho(s)|^2\mathrm{d}s.$$

Proof Denotes by $\bar{\varrho} = \varrho^1 - \varrho^2$ for $\varrho = \xi, Y, Z, K^+, K^-$ and defines the processes α, β in the following:

$$\alpha_s = \begin{cases} \dfrac{f^1(s, Y_s^1, Z_s^1) - f^1(s, Y_s^2, Z_s^1)}{\bar{Y}_s}, & \text{if } \bar{Y}_s \neq 0; \\ 0, & \text{otherwise;} \end{cases}$$

and

$$\beta_s = \begin{cases} \dfrac{\left(f^1(s, Y_s^2, Z_s^1) - f^1(s, Y_s^2, Z_s^2)\right)\bar{Z}_s}{\|\bar{Z}_s\|^2}, & \text{if } \bar{Z}_s \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Remark that

$$f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2) = \alpha_s\bar{Y}_s + \beta_s\bar{Z}_s + f^1(s, Y_s^2, Z_s^2) - f^2(s, Y_s^2, Z_s^2).$$

Fix $t \in \mathbb{R}^+$ and set for $s \geqslant t, A_s = \int_t^s (2\alpha_r - 3\mu)\mathrm{d}r$. From Itô's formula, we have

$$\begin{aligned} e^{A_T}|\bar{\xi}|^2 &= e^{A_t}|\bar{Y}_t|^2 + \int_t^T \mathbb{I}_{\{s\leqslant\tau\}} e^{A_s}(2\alpha_s - 3\mu)|\bar{Y}_s|^2\mathrm{d}s \\ &\quad - 2\int_t^T \mathbb{I}_{\{s\leqslant\tau\}} e^{A_s}\bar{Y}_s\left(f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2)\right)\mathrm{d}s \\ &\quad - 2\int_t^T e^{A_s}\bar{Y}_s\mathrm{d}\bar{K}_s^+ + 2\int_t^T e^{A_s}\bar{Y}_s\mathrm{d}\bar{K}_s^- + 2\int_t^T e^{A_s}\bar{Y}_s\bar{Z}_s\mathrm{d}W_s + \int_t^T e^{A_s}\|\bar{Z}_s\|^2\mathrm{d}s \\ &= |\bar{Y}_t|^2 - 3\mu\int_t^T \mathbb{I}_{\{s\leqslant\tau\}} e^{A_s}|\bar{Y}_s|^2\mathrm{d}s - 2\int_t^T \mathbb{I}_{\{s\leqslant\tau\}} e^{A_s}\bar{Y}_s\left(f^1(s, Y_s^2, Z_s^2) - f^2(s, Y_s^2, Z_s^2)\right)\mathrm{d}s \\ &\quad - 2\int_t^T e^{A_s}\bar{Y}_s\mathrm{d}\bar{K}_s^+ + 2\int_t^T e^{A_s}\bar{Y}_s\mathrm{d}\bar{K}_s^- + 2\int_t^T e^{A_s}\bar{Y}_s\bar{Z}_s(\mathrm{d}W_s - \beta_s\mathrm{d}s) + \int_t^T e^{A_s}\|\bar{Z}_s\|^2\mathrm{d}s. \end{aligned} \tag{2.2}$$

Let $\mathbb{Q}$ the probability measure on $(\Omega, \mathcal{F})$ whose density is

$$\mathcal{E}_\infty = \exp\left(\int_0^\infty \beta_s\mathrm{d}W_s - \frac{1}{2}\int_0^\infty |\beta_s|^2\mathrm{d}s\right).$$

By assumption $(\mathcal{H}.1)(4)$, it follows that $|\beta_s| \leqslant \kappa(\|Z_s^1\| + \|Z_s^2\|)$. Hence $\int b_s\mathrm{d}W_s$ is also a BMO-martingale. Theorem 2.3 in [20] implies that the probability measures $\mathbb{Q}$ and $\mathbb{P}$ are mutually absolutely continuous and $\widetilde{W}_t = W_t - \int_0^t \beta_r \mathrm{d}r$ for $t \geqslant 0$ is a Brownian motion under $\mathbb{Q}$. Combining Itô's formula, $(\mathcal{H}_\rho)$ and the fact that $\alpha_s \leqslant \mu$ to get

$$
\begin{aligned}
|\bar{Y}_t|^2 &\leqslant e^{A_T}|\bar{\xi}|^2 + 3\mu\int_t^T e^{A_s}|\bar{Y}_s|^2\mathrm{d}s + 2\int_t^T \mathbb{I}_{\{s\leqslant\tau\}} e^{A_s}\bar{Y}_s\left(f^1(s,Y_s^2,Z_s^2) - f^2(s,Y_s^2,Z_s^2)\right)\mathrm{d}s \\
&\quad + 2\int_t^T e^{A_s}\bar{Y}_s\mathrm{d}\bar{K}_s^+ - 2\int_t^T e^{A_s}\bar{Y}_s\mathrm{d}\bar{K}_s^- - 2\int_t^T e^{A_s}\bar{Y}_s\bar{Z}_s\mathrm{d}\widetilde{W}_s \\
&\leqslant e^{-\mu(T-t)}\|\bar{\xi}\|_\infty^2 + 4\mu\int_t^T e^{A_s}|\bar{Y}_s|^2\mathrm{d}s + \frac{1}{\mu}\int_t^T e^{-\mu(s-t)}|\rho(s)|^2\mathrm{d}s - 2\int_t^T e^{A_s}\bar{Y}_s\bar{Z}_s\mathrm{d}\widetilde{W}_s.
\end{aligned}
$$

Applying Gronwall lemma and taking the conditional expectation $\mathbb{E}_{\mathbb{Q}}[./\mathcal{F}_t^{\widetilde{W}}]$ to get the desired result. □

Corollary 2.5 *Let $(Y^i, Z^i, K^{i+}, K^{i-})_{i=1,2}$ be the solution of the quadratic BSDE2RB (2.1) with data (ξ, f, U, L). We suppose that the data verifies the same hypothesis in Proposition 2.4. Then $(Y_t^1, Z_t^1, K_t^{1+}, K_t^{1-}) = (Y_t^2, Z_t^2, K_t^{2+}, K_t^{2-})$ $\mathbb{P}$-a.s. for all $t \geqslant 0$.*

Proof Let $(Y^1, Z^1, K^{1+}, K^{1-})$ and $(Y^2, Z^2, K^{2+}, K^{2-})$ be two solutions of the quadratic BSDE2RB (2.1) associated with data (ξ, f, U, L). First of all, under Proposition 2.4, we have immediately $Y^1 = Y^2$ $\mathbb{P}$-a.s. for all $t \geqslant 0$. Again in view of (2.2), we get for all $t \geqslant 0$ that $Z^1 = Z^2$ $\mathbb{P}$-a.s. Additionally, it can be shown that $K^{1-} = K^{2-}$. Indeed

$$
\begin{aligned}
&\int_0^t (U_s - L_s)\mathrm{d}K_s^{1-} = \int_0^t (Y_s - L_s)\mathrm{d}K_s^{1-} \\
&= -\int_0^t (Y_s - L_s)\mathrm{d}(K_s^{1+} - K_s^{1-}) = -\int_0^t (Y_s - L_s)\mathrm{d}(K_s^{2+} - K_s^{2-}) \\
&= \int_0^t (Y_s^2 - L_s)\mathrm{d}K_s^{2-} = \int_0^t (U_s - L_s)\mathrm{d}K_s^{2-}.
\end{aligned}
$$

Since $L < U$, we conclude that $K^{1-} = K^{2-}$ $\mathbb{P}$-a.s. for all $t \geqslant 0$. Similarly, we can prove that $K^{1+} = K^{2+}$ $\mathbb{P}$-a.s. for all $t \geqslant 0$. □

2.2 Existence result

In this context, we demonstrate the presence of a solution for quadratic BSDE2RB. The generator f is a continuous function that conforms to a quadratic growth condition with respect to the variable z. Before proceeding, it is essential to introduce the following result: for each $n \in \mathbb{N}$, in view of ([2], Theorem 3.1) and Corollary 2.5, there exists a unique solution $(\bar{Y}_t^n, \bar{Z}_t^n, \bar{K}_t^{n+}, \bar{K}_t^{n-})_{t\in[0,n]}$ to the following quadratic BSDE2RB

$$
\begin{cases}
\bar{Y}_t^n = M_n + \displaystyle\int_t^n \mathbb{I}_{\{s\leqslant\tau\}} f(s, \bar{Y}_s^n, \bar{Z}_s^n)\mathrm{d}s + (\bar{K}_{n\wedge\tau}^{n+} - \bar{K}_t^{n+}) - (\bar{K}_{n\wedge\tau}^{n-} - \bar{K}_t^{n-}) - \displaystyle\int_t^n \bar{Z}_s^n\mathrm{d}W_s, \\
L_t \leqslant \bar{Y}_t^n \leqslant U_t, \ \forall\, t \leqslant n \text{ and } \displaystyle\int_0^n (U_s - \bar{Y}_s^n)\mathrm{d}\bar{K}_s^{n-} = \int_0^n (\bar{Y}_s^n - L_s)\mathrm{d}\bar{K}_s^{n+} = 0.
\end{cases}
$$

Furthermore, it is subsequently established that the existence of $(\bar{Y}_t^n, \bar{Z}_t^n, \bar{K}_t^{n+}, \bar{K}_t^{n-})$ is also valid in the set $\in \mathcal{S}_n^\infty \times \mathcal{H}_n^{2,-\mu} \times (\mathcal{A}_n)^2$.

Remark 2.6 *Note that $\left\|\sup_{t\leqslant n\wedge\tau}|\bar{Y}_t^n|\right\|_\infty \leqslant \mathfrak{M}$ and there exists a constant $C(n) > 0$ such that*

$$\left\|\int_0^{\cdot} \bar{Z}_s^n \mathrm{d}W_s\right\|_{BMO_2} \leqslant C(n).$$

Moreover, we have the following lemma:

Lemma 2.7 *For all $\epsilon > 0$, we have*

$$\sup_{n\geqslant 0} \mathbb{E}\int_0^{\tau} e^{-\epsilon s}\|\bar{Z}_s^n\|^2 \mathrm{d}s < \infty. \tag{2.3}$$

Proof Indeed, if we apply Itô's formula for $\varphi(x) = e^{-5Cx}$ with $(\bar{Y}^n - M)$ and using the fact that

$$\frac{1}{4}\|\bar{Z}_s^n\|^2 \leqslant \frac{1}{2}\|\bar{Z}_s^n - \zeta_s\|^2 + \frac{1}{2}\|\zeta_s\|^2,$$

we obtain

$$\begin{aligned}
&\varphi(\bar{Y}_0^n - M_0) - \epsilon\mathbb{E}\int_0^{n\wedge\tau} e^{-\epsilon s}\varphi(\bar{Y}_s^n - M_s)\mathrm{d}s + \frac{1}{4}\mathbb{E}\int_0^{n\wedge\tau} e^{-\epsilon s}\varphi''(\bar{Y}_s^n - M_s)\|\bar{Z}_s^n\|^2\mathrm{d}s\\
\leqslant\ & \mathbb{E}\int_0^{n\wedge\tau} e^{-\epsilon s}\varphi'(\bar{Y}_s^n - M_s)\left(f(s,\bar{Y}_s^n,\bar{Z}_s^n) - \eta_s\right)\mathrm{d}s + \mathbb{E}\int_0^{n\wedge\tau} e^{-\epsilon s}\varphi'(\bar{Y}_s^n - M_s)\mathrm{d}\bar{K}^{n+}\\
&- \mathbb{E}\int_0^{n\wedge\tau} e^{-\epsilon s}\varphi'(\bar{Y}_s^n - M_s)\mathrm{d}\bar{K}^{n-} + \frac{1}{2}\mathbb{E}\int_0^{n\wedge\tau} e^{-\epsilon s}\varphi''(\bar{Y}_s^n - M_s)\|\zeta_s\|^2\mathrm{d}s\\
\leqslant\ & -C\mathbb{E}\int_0^{n\wedge\tau} e^{-\epsilon s}\varphi'(\bar{Y}_s^n - M_s)\left(1 + |\bar{Y}_s^n| + \|\bar{Z}_s^n\|^2 + |\eta_s|\right)\mathrm{d}s + \mathbb{E}\int_0^{n\wedge\tau} e^{-\epsilon s}\varphi'(\bar{Y}_s^n - M_s)\mathrm{d}\bar{K}^{n+}\\
&- \mathbb{E}\int_0^{n\wedge\tau} e^{-\epsilon s}\varphi'(\bar{Y}_s^n - M_s)\mathrm{d}\bar{K}^{n-} + \frac{1}{2}\mathbb{E}\int_0^{n\wedge\tau} e^{-\epsilon s}\varphi''(\bar{Y}_s^n - M_s)\|\zeta_s\|^2\mathrm{d}s.
\end{aligned}$$

Remark that

$$-\mathbb{E}\int_0^{n\wedge\tau} e^{-\epsilon s}\varphi'(\bar{Y}_s^n - M_s)\mathrm{d}\bar{K}_s^{n-} \leqslant C_{\varphi,\mathfrak{M},\epsilon}\mathbb{E}(\bar{K}_n^{n-}),$$

and since $\frac{1}{4}\varphi''(x) + C\varphi'(x) = \frac{5}{4}C^2\varphi(x)$ with $\varphi' < 0$ and $\bar{K}^{n+}$ is increasing, we get

$$\begin{aligned}
&\mathbb{E}\int_0^{n\wedge\tau} e^{-\epsilon s}\frac{5}{4}C^2\varphi(\bar{Y}_s^n - M_s)\|\bar{Z}_s^n\|^2\mathrm{d}s\\
\leqslant\ & \epsilon\mathbb{E}\int_0^{n\wedge\tau} e^{-\epsilon s}\varphi(\bar{Y}_s^n - M_s)\mathrm{d}s + C_{\varphi,\mathfrak{M},\epsilon}\mathbb{E}(\bar{K}^{n-}) - C\mathbb{E}\int_0^{n\wedge\tau} e^{-\epsilon s}\varphi'(\bar{Y}_s^n - M_s)\left(1 + |\bar{Y}_s^n| + |\eta_s|\right)\mathrm{d}s\\
&+ \frac{1}{2}\mathbb{E}\int_0^{n\wedge\tau} e^{-\epsilon s}\varphi''(\bar{Y}_s^n - M_s)\|\zeta_s\|^2\mathrm{d}s.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{5}{4}C^2C_{\varphi,\mathfrak{M}}\mathbb{E}\int_0^{n\wedge\tau} e^{-\epsilon s}\|\bar{Z}_s^n\|^2\mathrm{d}s \leqslant\ & C_{\varphi,\mathfrak{M},\epsilon} + C_{C,\varphi,\mathfrak{M}}\mathbb{E}\int_0^{n\wedge\tau} e^{-\epsilon s}\left(1 + |\bar{Y}_s^n| + |\eta_s|\right)\mathrm{d}s\\
&+ C_{\varphi,\mathfrak{M}}\mathbb{E}\int_0^{n\wedge\tau} e^{-\epsilon s}\|\zeta_s\|^2\mathrm{d}s + C_{\varphi,\mathfrak{M},\epsilon}\mathbb{E}(\bar{K}^{n-}),
\end{aligned}$$

which implies (2.3). □

Subsequently, our focus will be on presenting the principal outcome of this section. To commence, we shall define the process $(Y^n, Z^n, K^{n+}, K^{n-})$ as follows:

$$(Y_t^n, Z_t^n, K_t^{n+}, K_t^{n-}) := \begin{cases} (\bar{Y}_t^n, \bar{Z}_t^n, \bar{K}_t^{n+}, \bar{K}_t^{n-}), & \text{on the set } [t \leqslant n \wedge \tau]; \\ (M_t, \zeta_t \mathbb{I}_{[0,\tau]}, \bar{K}_{n\wedge\tau}^{n+}, \bar{K}_{n\wedge\tau}^{n-}), & \text{on the set } [t \geqslant n \wedge \tau]. \end{cases}$$

Step 1 $(Y^n, Z^n, K^n)_{n\geqslant 0}$ is a Cauchy sequence on the space $\mathcal{S}_\infty^\infty \times \mathcal{H}_\infty^{2,-\mu}(\mathbb{R}^d) \times \mathcal{A}_\infty$

Fix $t \leqslant n \leqslant m$ and set $\varrho^{m,n} = \varrho^m - \varrho^n$ for $\varrho = Y, Z, K^+, K^-$. Remark that

$$Y_t^n = M_m + \int_t^m \mathbb{I}_{\{s\leqslant\tau\}} \tilde{f}(s, Y_s^n, Z_s^n) \mathrm{d}s + \int_t^m \mathrm{d}K_s^{n+} - \int_t^m \mathrm{d}K_s^{n-} - \int_t^m Z_s^n \mathrm{d}W_s,$$

where $\tilde{f}(s,y,z) = f(s,y,z)\mathbb{I}_{\{s\leqslant n\}} - \eta_s \mathbb{I}_{\{s>n\}}$. In view of this notation, we have

$$|f(s, Y_s^n, Z_s^n) - \tilde{f}(s, Y_s^n, Z_s^n)| = |\left(f(s, M_s, \zeta_s) + \eta_s\right) \mathbb{I}_{\{s>n\}}| \leqslant C\mathbb{I}_{\{s>n\}}.$$

Hence, can apply Proposition 2.4 to get for all $t \in [0,m]$

$$\begin{aligned} |Y_t^m - Y_t^n|^2 &\leqslant \frac{1}{\mu} \int_n^m C e^{-\mu(s-t)} \mathrm{d}s \\ &\leqslant \frac{C}{\mu^2} \left(e^{-\mu(n-t)} - e^{-\mu(m-t)}\right) \quad \mathbb{P}\text{-a.s.} \end{aligned} \tag{2.4}$$

It follows that the sequence of continuous processes Y^n converges to a limit which we denote Y. If m goes to infinity in the last inequality, it comes that

$$|Y_t^n - Y_t|^2 \leqslant \frac{C}{\mu^2} e^{-\mu(n-t)} \quad \mathbb{P}\text{-a.s. for all } t \leqslant n.$$

This implies that Y^n converges almost surely to Y uniformly with respect to t on compact sets. The limit process Y is also continuous and bounded. Furthermore,

$$\begin{aligned} \mathbb{E}\int_0^\tau e^{-\mu s}|Y_s^n - Y_s|^2 \mathrm{d}s &= \mathbb{E}\int_0^{n\wedge\tau} e^{-\mu s}|Y_s^n - Y_s|^2 \mathrm{d}s + \mathbb{E}\int_{n\wedge\tau}^\tau e^{-\mu s}|M_s - Y_s|^2 \mathrm{d}s \\ &\leqslant Cne^{-\mu n} + \mathbb{E}\left(\mathbb{I}_{\{\tau\geqslant n\}} \int_n^\tau e^{-\mu s}|M_s - Y_s|^2 \mathrm{d}s\right) \xrightarrow[n\to\infty]{} 0. \end{aligned}$$

Now, combining Itô's formula and the fact that $Y_s^{m,n}\mathrm{d}K_s^{m,n+}$ and $Y_s^{m,n}\mathrm{d}K_s^{m,n-}$ remain non-positive, we obtain

$$\begin{aligned} &\mathbb{E}|Y_0^{m,n}|^2 - \mu\mathbb{E}\int_0^{m\wedge\tau} e^{-\mu s}|Y_s^{m,n}|^2 \mathrm{d}s + \mathbb{E}\int_0^{m\wedge\tau} e^{-\mu s}\|Z_s^{m,n}\|^2 \mathrm{d}s \\ &\leqslant 2\mathbb{E}\int_0^{m\wedge\tau} e^{-\mu s} Y_s^{m,n}\left(f(s, Y_s^m, Z_s^m) - f(s, Y_s^n, Z_s^n)\right) \mathrm{d}s + 2\mathbb{E}\int_{n\wedge\tau}^{m\wedge\tau} e^{-\mu s} Y_s^{m,n}\left(f(s, M_s, \zeta_s) + \eta_s\right) \mathrm{d}s. \end{aligned}$$

Using the hypothesis of the function f, we find

$$\begin{aligned} 2Y_s^{m,n}\left(f(s, Y_s^m, Z_s^m) - f(s, Y_s^n, Z_s^n)\right) &\leqslant 2\mu|Y_s^{m,n}|^2 + 2|Y_s^{m,n}||f(s, Y_s^n, Z_s^n) - f(s, Y_s^n, Z_s^m)| \\ &\leqslant 2\mu|Y_s^{m,n}|^2 + 4\kappa^2|Y_s^{m,n}|^2(\|Z_s^m\|^2 + \|Z_s^n\|^2) + \frac{1}{2}\|Z_s^{m,n}\|^2. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \frac{1}{2}\mathbb{E}\int_0^\tau e^{-\mu s}\|Z_s^{m,n}\|^2 \mathrm{d}s &\leqslant 3\mu\mathbb{E}\int_0^\tau e^{-\mu s}|Y_s^{m,n}|^2 \mathrm{d}s \\ &\quad + C\frac{4\kappa^2}{\mu^2} e^{-\mu n}\left(1 - e^{-\mu(m-n)}\right)\mathbb{E}\int_0^\tau (\|Z_s^m\|^2 + \|Z_s^n\|^2) \mathrm{d}s \\ &\quad + C\mathbb{E}\int_n^m e^{-\mu s}\left(1 + |M_s| + \|\zeta_s\|^2 + |\eta_s|\right) \mathrm{d}s, \end{aligned}$$

which converge to 0 as n goes to ∞. Hence, Z^n is a Cauchy sequence in $\mathcal{H}^{2,-\mu}_{\tau}(\mathbb{R}^d)$ and converges to a process Z.

We next state the convergence of $K^n = K^{n+} - K^{n-}$. Firstly, we have

$$K^{m,n}_m = Y^{m,n}_0 - \int_0^m \mathbb{I}_{\{s\leqslant\tau\}}\left(\tilde{f}(s,Y^m_s,Z^m_s) - \tilde{f}(s,Y^n_s,Z^n_s)\right)\mathrm{d}s + \int_0^m Z^{m,n}_s \mathrm{d}W_s.$$

But before we start, using the Cauchy-Schwarz inequality twice results in:

$$\begin{aligned}&\mathbb{E}\int_0^\tau \|Z^{m,n}_s\|(\|Z^m_s\| + \|Z^n_s\|)\mathrm{d}s\\ &\leqslant \sqrt{2}\left(\mathbb{E}\int_0^\tau e^{-\mu s}\|Z^{m,n}_s\|^2\mathrm{d}s\right)^{1/2}\left(\mathbb{E}\int_0^\tau e^{\mu s}(\|Z^m_s\|^2 + \|Z^n_s\|^2)\mathrm{d}s\right)^{1/2}.\end{aligned}$$

Afterward, we have

$$\begin{aligned}\mathbb{E}[|K^{m,n}_m|] &\leqslant \mathbb{E}|Y^{m,n}_0| + \mathbb{E}\int_0^{m\wedge\tau} |\tilde{f}(s,Y^m_s,Z^m_s) - \tilde{f}(s,Y^n_s,Z^n_s)|\mathrm{d}s\\ &\leqslant \mathbb{E}|Y^{m,n}_0| + \mathbb{E}\int_0^{n\wedge\tau} |f(s,Y^m_s,Z^m_s) - f(s,Y^n_s,Z^n_s)|\,\mathrm{d}s + \mathbb{E}\int_{n\wedge\tau}^{m\wedge\tau} |f(s,M_s,\zeta_s) + \eta_s|\mathrm{d}s.\end{aligned}$$

Subsequently, using the formula (2.4) and assumption $(\mathcal{H}.3).3$ yields

$$\begin{aligned}\mathbb{E}[|K^{m,n}_m|] &\leqslant \mathbb{E}|Y^{m,n}_0| + \mu\mathbb{E}\int_0^\tau |Y^{m,n}_s|\mathrm{d}s + \kappa\mathbb{E}\int_0^\tau \|Z^{m,n}_s\|(\|Z^m_s\| + \|Z^n_s\|)\mathrm{d}s\\ &\quad + \mathbb{E}\int_n^m e^{-\mu s}|f(s,M_s,\zeta_s) + \eta_s|\mathrm{d}s\\ &\leqslant Ce^{-\mu n/2} + Ce^{-\mu n/2}\mathbb{E}\int_0^\tau e^{\mu s/2}\mathrm{d}s\\ &\quad + \sqrt{2}\kappa\left(\mathbb{E}\int_0^\tau e^{-\mu s}\|Z^{m,n}_s\|^2\mathrm{d}s\right)^{1/2}\left(\mathbb{E}\int_0^\tau e^{\mu s}(\|Z^m_s\|^2 + \|Z^n_s\|^2)\mathrm{d}s\right)^{1/2}\\ &\quad + C\mathbb{E}\int_n^m \left(1 + |M_s| + \|\zeta_s\|^2 + |\eta_s|\right)\mathrm{d}s.\end{aligned}$$

Consequently, one can show

$$\mathbb{E}[|K^{m,n}_m|] \xrightarrow[n,m\to\infty]{} 0.$$

Then, there exists $K \in \mathcal{A}_\infty$ such that K^n converges almost surely to K.

Step 2 There exists a constant $\mathbf{C} > 0$ such that

$$\mathbb{E}|K^{n+}_{T\wedge\tau}| + \mathbb{E}|K^{n-}_{T\wedge\tau}| \leqslant \mathbf{C}.$$

Firstly, we obtain by considering the hypothesis $(\mathcal{H}.2)$

$$\mathrm{d}(Y^n_t - L_t) = -\mathbb{I}_{\{t\leqslant\tau\}}\tilde{f}(t,Y^n_t,Z^n_t)\mathrm{d}t - \mathrm{d}K^{n+}_t + \mathrm{d}K^{n-}_t + Z^n_t\mathrm{d}W_t - \mathrm{d}\mathcal{P}_t - \mathrm{d}\mathcal{M}_t.$$

Utilizing Theorem 68 from reference [30] on the convex function $(Y^n - L)^+$ leads to:

$$\begin{aligned}\mathrm{d}(Y^n_t - L_t)^+ &= \mathbb{I}_{\{Y^n_t > L_t\}}\left(-\mathbb{I}_{\{t\leqslant\tau\}}\tilde{f}(t,Y^n_t,Z^n_t)\mathrm{d}t - \mathrm{d}K^{n+}_t + \mathrm{d}K^{n-}_t + Z^n_t\mathrm{d}W_t - \mathrm{d}\mathcal{P}_t - \mathrm{d}\mathcal{M}_t\right)\\ &\quad + \mathbb{I}_{\{Y^n_t > L_t\}}(Y^n_t - L_t)^- + \mathbb{I}_{\{Y^n_t \leqslant L_t\}}(Y^n_t - L_t)^+ + \frac{1}{2}\mathrm{d}\mathbf{L}^0_t,\end{aligned}$$

where $\mathbf{L}^0$ is the local time of $Y^n - L$ at 0. Whence

$$
\begin{aligned}
&\mathbb{I}_{\{Y_t^n=L_t\}}\left(Z_t^n \mathrm{d}W_t - \mathrm{d}\mathcal{M}_t\right)\\
&= \mathbb{I}_{\{Y_t^n=L_t\}}\Big(\mathrm{d}(Y_t^n - L_t) + \mathbb{I}_{\{t\leqslant\tau\}}\tilde{f}(t,Y_t^n,Z_t^n)\mathrm{d}t + \mathrm{d}K_t^{n+} - \mathrm{d}K_t^{n-} + \mathrm{d}\mathcal{P}_t\Big)\\
&= \frac{1}{2}\mathrm{d}\mathbf{L}_t^0 + \mathbb{I}_{\{Y_t^n=L_t\}}\Big(\mathbb{I}_{\{t\leqslant\tau\}}\tilde{f}^+(t,Y_t^n,Z_t^n) + \mathrm{d}\mathcal{P}_t^+\Big) + \mathrm{d}K_t^{n+}\\
&\quad - \mathbb{I}_{\{Y_t^n=L_t\}}\Big(\mathbb{I}_{\{t\leqslant\tau\}}\tilde{f}^-(t,Y_t^n,Z_t^n) + \mathrm{d}K_t^{n-} + \mathrm{d}\mathcal{P}_t^-\Big).
\end{aligned}
$$

Since $\mathcal{M}$ is integrable, the process $\left(\frac{1}{2}\mathrm{d}\mathbf{L}_t^0 + \mathbb{I}_{\{Y_t^n=L_t\}}\Big(\mathbb{I}_{\{t\leqslant\tau\}}\tilde{f}^+(t,Y_t^n,Z_t^n) + \mathrm{d}\mathcal{P}_t^+\Big)\right)_t$ is a nondecreasing integrable process. Denoting its compensator as $\mathcal{C}$ and its compensatrix as $\overline{\mathcal{C}}$. Therefore

$$
\begin{aligned}
&\mathbb{I}_{\{Y_t^n=L_t\}}\left(Z_t^n \mathrm{d}W_t - \mathrm{d}\mathcal{M}_t\right) - \mathrm{d}\overline{\mathcal{C}}_t\\
&= \mathrm{d}\mathcal{C}_t - \mathbb{I}_{\{Y_t^n=L_t\}}\Big(\mathbb{I}_{\{t\leqslant\tau\}}\tilde{f}^-(t,Y_t^n,Z_t^n) + \mathrm{d}K_t^{n-} + \mathrm{d}\mathcal{P}_t^-\Big) + \mathrm{d}K_t^{n+}.
\end{aligned}
$$

As all the terms in the last line are predictable, the equality yields zero. Consequently

$$
\mathrm{d}\mathcal{C}_t + \mathrm{d}K_t^{n+} = \mathbb{I}_{\{Y_t^n=L_t\}}\Big(\mathbb{I}_{\{t\leqslant\tau\}}\tilde{f}^-(t,Y_t^n,Z_t^n) + \mathrm{d}K_t^{n-} + \mathrm{d}\mathcal{P}_t^-\Big).
$$

Thus

$$
\mathrm{d}K_t^{n+} \leqslant \mathbb{I}_{\{Y_t^n=L_t\}}\Big(\mathbb{I}_{\{t\leqslant\tau\}}\tilde{f}^-(t,Y_t^n,Z_t^n) + \mathrm{d}K_t^{n-} + \mathrm{d}\mathcal{P}_t^-\Big).
$$

By employing the mutual singularity of K^{n+} and K^{n-}, we derive that

$$
\mathrm{d}K_t^{n+} \leqslant \mathbb{I}_{\{Y_t^n=L_t\}}\Big(\mathbb{I}_{\{t\leqslant\tau\}}\tilde{f}^-(t,Y_t^n,Z_t^n) + \mathrm{d}\mathcal{P}_t^-\Big).
$$

Upon taking the expectation, we obtain that

$$
\mathbb{E}|K_{T\wedge\tau}^{n+}| \leqslant C\mathbb{E}\int_0^{T\wedge\tau}\left(1 + |Y_t^n| + \|Z_t^n\|^2 + |\eta_s|\right)\mathrm{d}s + \mathbb{E}|\mathcal{P}_{T\wedge\tau}| \leqslant \mathbf{C}.
$$

On the other hand, we have

$$
\mathbb{E}[|K_{T\wedge\tau}^{n-}|] - \mathbb{E}[|K_{T\wedge\tau}^{n+}|] \leqslant \mathbb{E}[|K_{T\wedge\tau}^{n}|] \leqslant C.
$$

That implies that

$$
\mathbb{E}|K_{T\wedge\tau}^{n-}| \leqslant C + \mathbb{E}[|K_{T\wedge\tau}^{n+}|] \leqslant \mathbf{C}.
$$

Hence, the proof of this step is now complete.

Step 3 The process (Y, Z, K^+, K^-) satisfies the quadratic BSDE2RB (2.1)

Fix t and T. We shall pass to the limit in the following equality:

$$
Y_{t\wedge\tau}^n = Y_{T\wedge\tau}^n + \int_{t\wedge\tau}^{T\wedge\tau} f(s,Y_s^n,Z_s^n)\mathrm{d}s + \int_{t\wedge\tau}^{T\wedge\tau}\mathrm{d}K_s^n - \int_{t\wedge\tau}^{T\wedge\tau} Z_s^n \mathrm{d}W_s. \tag{2.5}
$$

The sequence $Y_{t\wedge\tau}^n$ converges almost surely to Y_t. In addition, it is bounded since $\|\sup_{t\leqslant n\wedge\tau}|Y_t^n|\|_\infty \leqslant \mathfrak{M}$. We obtain that the sequence converges to $Y_{t\wedge\tau}$ in $\mathbb{L}^1$ from Lebesgue Dominated Convergence Theorem. Moreover, we have

$$
\mathbb{E}\left(\int_{t\wedge\tau}^{T\wedge\tau} Z_s^n \mathrm{d}W_s - \int_{t\wedge\tau}^{T\wedge\tau} Z_s \mathrm{d}W_s\right)^2 \leqslant e^{\mu T}\mathbb{E}\int_0^{T\wedge\tau} e^{-\mu s}|Z_s^n - Z_s|^2\mathrm{d}s.
$$

Then, $\int_{t\wedge\tau}^{T\wedge\tau} Z_s^n \mathrm{d}W_s$ converges in $\mathbb{L}^2$ to $\int_{t\wedge\tau}^{T\wedge\tau} Z_s \mathrm{d}W_s$. Additionally, $\int_{t\wedge\tau}^{T\wedge\tau} f(s,Y_s^n,Z_s^n)\mathrm{d}s$ converges to $\int_{t\wedge\tau}^{T\wedge\tau} f(s,Y_s,Z_s)\mathrm{d}s$ in $\mathbb{L}^1$ since

$$
\begin{aligned}
&\mathbb{E}\left|\int_{t\wedge\tau}^{T\wedge\tau} f(s,Y_s^n,Z_s^n)\mathrm{d}s-\int_{t\wedge\tau}^{T\wedge\tau} f(s,Y_s,Z_s)\mathrm{d}s\right|\\
&\leqslant \mathbb{E}\int_0^T |f(s,Y_s^n,Z_s^n)-f(s,Y_s,Z_s)|\,\mathrm{d}s\\
&\leqslant \mu e^{\mu T}\mathbb{E}\int_0^T e^{-\mu s}|Y_s^n-Y_s|\mathrm{d}s\\
&\quad+\kappa\sqrt{2}e^{\mu T}\left(\mathbb{E}\int_0^T e^{-\mu s}\|Z_s^n-Z_s\|^2\mathrm{d}s\right)^{1/2}\left(\mathbb{E}\int_0^T e^{-\mu s}(\|Z_s^n\|^2+\|Z_s\|^2)\mathrm{d}s\right)^{1/2}.
\end{aligned}
$$

Furthermore, for the last integral with respect to K, we get

$$
\begin{aligned}
\mathbb{E}\left|\int_{t\wedge\tau}^{T\wedge\tau}(\mathrm{d}K_s^n-\mathrm{d}K_s)\right| &\leqslant \mathbb{E}|Y_{t\wedge\tau}^n-Y_{t\wedge\tau}|+\mathbb{E}|Y_{T\wedge\tau}^n-Y_{T\wedge\tau}|\\
&\quad+\mathbb{E}\int_0^T|f(s,Y_s^n,Z_s^n)-f(s,Y_s,Z_s)|\mathrm{d}s+\mathbb{E}\left|\int_{t\wedge\tau}^{T\wedge\tau}(Z_s^n-Z_s)\mathrm{d}W_s\right|.
\end{aligned}
$$

Then, passing to the limit as $n\to\infty$ in (2.5), we find

$$
Y_{t\wedge\tau}=Y_{T\wedge\tau}+\int_{t\wedge\tau}^{T\wedge\tau} f(s,Y_s,Z_s)\mathrm{d}s+\int_{t\wedge\tau}^{T\wedge\tau}\mathrm{d}K_s-\int_{t\wedge\tau}^{T\wedge\tau}Z_s\mathrm{d}W_s.
$$

On the other hand, let $\nu\leqslant T$ be a stopping time. Step 2 shows that the sequences $K_{\nu\wedge\tau}^{n\pm}$ are bounded in $\mathbb{L}^1$. Consequently, there exists $\mathcal{F}_{\nu\wedge\tau}$-measurable random variables $K_{\nu\wedge\tau}^{\pm}\in\mathbb{L}^1$ such that there exists the subsequences of $K_{\nu\wedge\tau}^{n\pm}$ weakly converging in $K_{\nu\wedge\tau}^{\pm}$.

We are now putting $K_{\nu\wedge\tau}=K_{\nu\wedge\tau}^{+}-K_{\nu\wedge\tau}^{-}$. By Mazur's lemma (see e.g. [33], p.120), for all $n\in\mathbb{N}$, there exists an integer $N\geqslant n$ and weights $\delta_j^{\nu\wedge\tau,n}\geqslant 0$ with $j=n,\dots,N$ and $\sum_{j=n}^N\delta_j^{\nu\wedge\tau,n}=1$ such that

$$
\widetilde{K}_{\nu\wedge\tau}^{n\pm}:=\sum_{j=n}^N\delta_j^{\nu\wedge\tau,n}(K_{\nu\wedge\tau}^{\pm})_j\xrightarrow[n\to\infty]{}K_{\nu\wedge\tau}^{\pm}. \tag{2.6}
$$

Denoting $\widetilde{K}_{\nu\wedge\tau}^{n}=\widetilde{K}_{\nu\wedge\tau}^{n+}-\widetilde{K}_{\nu\wedge\tau}^{n-}$, as a result of this

$$
\mathbb{E}|\widetilde{K}_{\nu\wedge\tau}^{n}-\widetilde{K}_{\nu\wedge\tau}|\xrightarrow[n\to\infty]{}0. \tag{2.7}
$$

Noting that, we have for all $\epsilon>0$, $\|K_{\nu\wedge\tau}^n-K_{\nu\wedge\tau}\|_{\mathbb{L}^1}<\epsilon$. Hence

$$
\|\widetilde{K}_{\nu\wedge\tau}^{n}-K_{\nu\wedge\tau}\|_{\mathbb{L}^1}=\left\|\sum_{j=n}^N\delta_j^{\nu\wedge\tau,n}(K_{\nu\wedge\tau}^{\pm})_j-K_{\nu\wedge\tau}\right\|_{\mathbb{L}^1}\leqslant\sum_{j=n}^N\delta_j^{\nu\wedge\tau,n}\|(K_{\nu\wedge\tau}^{\pm})_j-K_{\nu\wedge\tau}\|_{\mathbb{L}^1}<\epsilon.
$$

Therefore

$$
\mathbb{E}|\widetilde{K}_{\nu\wedge\tau}^{n}-K_{\nu\wedge\tau}|\xrightarrow[n\to\infty]{}0. \tag{2.8}
$$

Mixing (2.7) and (2.8) together, we obtain $K_{\nu\wedge\tau}=\widetilde{K}_{\nu\wedge\tau}$ a.s. As a result of the optional section theorem ([13], Theorem 86), we obtain $\widetilde{K}_{t\wedge\tau}=K_{t\wedge\tau}$, for all $t\geqslant 0$.

For $\nu=T$, the formula (2.6) implies the existence of two subsequences

$$
\widetilde{K}_{T\wedge\tau}^{n+}=\sum_{j=n}^N\delta_j^{T\wedge\tau,n}(K_{T\wedge\tau}^{+})_j\ \text{ and }\ \widetilde{K}_{T\wedge\tau}^{n-}=\sum_{j=n}^N\delta_j^{T\wedge\tau,n}(K_{T\wedge\tau}^{-})_j,
$$

which converge almost surely to $K^+_{T\wedge\tau}$ and $K^-_{T\wedge\tau}$ respectively. Then, for $\mathbb{P}$-a.s. $\omega\in\Omega$, the subsequences $\widetilde{K}^{n+}_{T\wedge\tau}(\omega)$ and $\widetilde{K}^{n-}_{T\wedge\tau}(\omega)$ are bounded. Subsequently, by Helly's selection theorem ([8], p.88), we can obtain that, for $\mathbb{P}$-a.s. $\omega\in\Omega$ and $t\geqslant 0$, there exist two subsequences of $\widetilde{K}^{n+}_{t\wedge\tau}(\omega)$ and $\widetilde{K}^{n-}_{t\wedge\tau}(\omega)$ tending weakly to $K^+_{t\wedge\tau}(\omega)$ and $K^-_{t\wedge\tau}(\omega)$ respectively.

On the other hand, we are now in possession of $L_t\leqslant Y^n_t\leqslant U_t$ for all $t\geqslant 0$. Indeed, by the definition of Y^n_t, we have

- $L_t\leqslant Y^n_t=\bar{Y}^n_t\leqslant U_t$ on the set $[t\leqslant n\wedge\tau]$;
- $L_t\leqslant Y^n_t=M_t\leqslant U_t$ on the set $[t\geqslant n\wedge\tau]$.

By passing to limit in n, we can obtain $L_t\leqslant Y_t\leqslant U_t$ for all $t\geqslant 0$.

Now we are going to show the Skorokhod's conditions. In fact, due to $\widetilde{K}^{n+}_t$ tends to $K^+_t(\omega)$, we can write the following using the result presented on p. 465 of [32]:

$$\int_0^{T\wedge\tau}(Y^n_t(\omega)-L_t(\omega))\mathrm{d}\widetilde{K}^{n+}_t(\omega)\xrightarrow[n\to\infty]{}\int_0^{T\wedge\tau}(Y_t(\omega)-L_t(\omega))\mathrm{d}K^+_t(\omega). \tag{2.9}$$

Using the fact that $\int_0^{T\wedge\tau}(Y^n_t-L_t)\mathrm{d}K^{n+}_t\leqslant 0$ a.s. for all $n\in\mathbb{N}$. Then for all $n,m\geqslant 0$, the formula (2.4) and Step 2 yield that

$$\begin{aligned}\mathbb{E}\left|\int_0^{T\wedge\tau}(Y^n_t-Y^m_t)\mathrm{d}K^{m+}_t\right| &\leqslant e^{\mu T/2}\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}e^{-\mu t/2}\left|Y^n_t-Y^m_t\right|\left|K^{m+}_T\right|\right]\\ &\leqslant Ce^{\mu T/2}e^{-\mu n/2}\mathbb{E}\left[\left|K^{m+}_T\right|\right]\xrightarrow[n,m\to\infty]{}0.\end{aligned}$$

Regarding that

$$\int_0^{T\wedge\tau}(Y^n_t-L_t)\mathrm{d}K^{m+}_t=\int_0^{T\wedge\tau}(Y^n_t-Y^m_t)\mathrm{d}K^{m+}_t+\int_0^{T\wedge\tau}(Y^m_t-L_t)\mathrm{d}K^{m+}_t,$$

we get

$$\limsup_{n\to\infty}\int_0^{T\wedge\tau}(Y^n_t-L_t)\mathrm{d}\widetilde{K}^{n+}_t\leqslant 0\ \ \mathbb{P}\text{-a.s.} \tag{2.10}$$

Combining (2.9) and (2.10), we get $\int_0^{T\wedge\tau}(Y_t-L_t)\mathrm{d}K^+_t\leqslant 0$ $\mathbb{P}$-a.s. Noting that $Y\geqslant L$, then $\int_0^{T\wedge\tau}(Y_t-L_t)\mathrm{d}K^+_t=0$ $\mathbb{P}$-a.s.

The same logic allows us to demonstrate $\int_0^{T\wedge\tau}(U_t-Y_t)\mathrm{d}K^-_t=0$ $\mathbb{P}$-a.s.

The demonstration is coming to an end. In conclusion, the existence result of the solution has been well-proven.

3. Viscosity solution to an obstacle problem for quadratic elliptic PDE with Dirichlet boundary condition

The main result of this section is to prove that in the Markovian case, the solution of the quadratic BSDE with two reflecting barriers with random terminal time (2.1) is a viscosity solution for an obstacle problem for elliptic quadratic PDE, in which the non-linearity appears as the square of the gradient.

3.1 Preliminaries

Let E be an open set of $\mathbb{R}^n$ endowed with its sigma-algebra $\mathcal{E}$, and that it contains a bounded set D of the form $D = \{x : \Psi(x) > 0\}$, where Ψ is a some function in $\mathcal{C}_b^2(\mathbb{R}^n)$. We require that $|\nabla\Psi(x)| \neq 0$ for any $x \in \partial D$. Let

$$X_s^x = x + \int_0^s b(X_r^x)\mathrm{d}r + \int_0^s \sigma(X_r^x)\mathrm{d}W_r, \quad s \geqslant 0, \tag{3.1}$$

be a diffusion process with the infinitesimal generator

$$\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^{n} (\sigma(x)\sigma^*(x))_{i,j}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i}$$

and we denote by $\{\mathcal{T}_t, t \geqslant 0\}$ the semigroup generated by $\mathcal{L}$ and its resolvent by $\mathcal{U} = (U_\gamma)_{\gamma>0}$. We assume that $b : \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ be continuous mapping and satisfy for some constant $\kappa > 0$ and $\mathcal{K} > 0$ and for any $(x, x') \in \bar{D}$,

$$\begin{cases} \text{(i)} \;\; |b(x) - b(x')| + |\sigma(x) - \sigma(x')| \leqslant \kappa|x - x'|, \\ \text{(ii)} \;\; \sup\limits_{x\in\bar{D}} |b(x)| + \sup\limits_{x\in\bar{D}} |\sigma(x)| \leqslant \mathcal{K}. \end{cases} \tag{3.2}$$

The assumptions (3.2) on b and σ ensure existence and uniqueness of a strong solution $(X_s^x)_{s\geqslant 0}$ to (3.1) at least up to the stopping times

$$\tau_x(\omega) = \inf\left\{t \geqslant 0 \;\middle|\; X_t^x(\omega) \notin \bar{D}\right\}.$$

We suppose that $\mathbb{P}(\tau_x < \infty) = 1$, $\forall x \in \bar{D}$. In other words, if a diffusion starts at x, then it exits from the compact in a finite time. Moreover, we assume that the set Γ of the points on the boundary that immediately exits is a closed set. Precisely,

$$\Gamma = \left\{x \in \partial D \middle| \mathbb{P}(\tau_x > 0) = 0\right\}.$$

Fix $x \in \bar{D}$ and let us set

$$\xi(\omega) := H(X_{\tau_x}^x), \quad L_s(\omega) := \ell(X_s^x), \quad U_s(\omega) := h(X_s^x), \quad f(s,\omega,y,z) := f\big(X_s^x, y, z\big).$$

Assume that $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, and $H, \ell, h : \mathbb{R}^n \to \mathbb{R}$ are $\mathcal{E}$-measurable and satisfy, for some constants $\mu, \kappa > 0$ and $\forall x, x'$ fixed in $\bar{D}$, the following conditions:

(1) $|f(x, y, z) - f(x', y', z)| \leqslant \kappa|x - x'| + \mu|y - y'|$;

(2) $|f(x, y, z) - f(x, y, z')| \leqslant \kappa\|z - z'\|(\|z\| + \|z'\|)$;

(3) $|f(x, 0, 0)| + |H(x)| + |\ell(x)| + |h(x)| \leqslant \kappa\big(1 + |x|\big)$;

(4) $\ell(x) < h(x)$ and $\ell(X_{\tau_x}^x) \leqslant H(X_{\tau_x}^x) \leqslant h(X_{\tau_x}^x)$, $\forall x \in \bar{D}$. Moreover, we suppose there exist two real-valued functions $\ell_1, \ell_2 \in D(\mathcal{U})$ such that $\ell = \ell_1 - \ell_2$.

Remark 3.1 *The assumption* (4) *on* ℓ *implies that* $\ell(X^x)$ *becomes a quasi-martingale where* $D(\mathcal{U})$ *denotes the convex cone of all excessive functions* $g : D \to [0, \infty]$ *i.e.* $\mathcal{T}_t g \leqslant g$ *and* $\lim\limits_{t\to 0} \mathcal{T}_t g = g$. *For more details about this assumption, we direct the reader to paper [6] (pp. 7764 and 7765).*

Also, assume that

(5) $\sup\limits_{x\in\bar{D}} \mathbb{E}\left[e^{\rho\tau_x}\right] < \infty$, for some $\rho \in \mathbb{R}$;

(6) For any fixed $x \in \bar{D}$, there exist two progressively measurable processes η^x and $\zeta^x \in \mathcal{H}_{\tau_x}^{2,0}$ such that for $M_{\tau_x}^x = H(X_{\tau_x}^x)\mathbb{I}_{\{\tau_x<\infty\}}$, we have for all $t \in \mathbb{R}^+$:

(a) $\ell(X_t^x) \leqslant M_t^x \leqslant h(X_t^x)$.

(b) $\mathrm{d}M_t^x = -\eta_t^x \mathbb{I}_{[0,\tau_x]}(t)\mathrm{d}t + \zeta_t^x \mathbb{I}_{[0,\tau_x]}(t)\mathrm{d}W_t$.

(c) $\mathbb{E}\int_0^{\tau_x} e^{\mu t}\, |\eta_t^x|\, \mathrm{d}t < \infty$.

Remark 3.2 *In particular* H, ℓ, h *and* $f(.,0,0)$ *are bounded on the compact set* D. *We denote by* $\mathfrak{M}$ *an upper bound.*

It follows that for all $x \in \bar{D}$ there exists a unique process $(Y_s^x, Z_s^x, K_s^{x,+}, K_s^{x,-})_{s\geqslant 0}$ solution of the Markovian quadratic BSDE2RB:

$$\begin{cases} \text{(i)} \quad Y^x \in \mathcal{S}_{\tau_x}^{\infty},\ Z^x \in \mathcal{H}_{\tau_x}^{2,-\mu}(\mathbb{R}^d) \text{ and } K^{x,+}, K^{x,-} \in (\mathcal{A}_{\tau_x})^2, \\ \text{(ii)} \ Y_s^x = H(X_{\tau_x}^x) + \displaystyle\int_{s\wedge\tau_x}^{\tau_x} f(X_r^x, Y_r^x, Z_r^x)\mathrm{d}r + (K_{\tau_x}^{x,+} - K_s^{x,+}) - (K_{\tau_x}^{x,-} - K_s^{x,-}) - \int_{s\wedge\tau_x}^{\tau_x} Z_r^x \mathrm{d}W_r, \\ \text{(iii)} \ \ell(X_s^x) \leqslant Y_s^x \leqslant h(X_s^x), \quad \forall s \geqslant 0, \\ \text{(iv)} \ \displaystyle\int_0^{\tau_x} (Y_s^x - \ell(X_s^x))\mathrm{d}K_s^{x,+} = \int_0^{\tau_x} (h(X_s^x) - Y_s^x)\mathrm{d}K_s^{x,-} = 0, \quad \mathbb{P}\text{-a.s.} \end{cases} \tag{3.3}$$

Now, let us examine the following obstacle problem involving an elliptic quadratic PDE:

$$\begin{cases} \big(u(x) - \ell(x)\big) \wedge \Big\{ \big(u(x) - h(x)\big) \vee \Big[-\mathcal{L}u(x) - f\left(x, u(x), (\nabla u \sigma^*)(x)\right) \Big] \Big\} = 0, \quad \forall x \in D; \\ u(x) = H(x), \quad \forall x \in \partial D. \end{cases} \tag{3.4}$$

Define

$$u(x) = Y_0^x, \quad x \in \bar{D}, \tag{3.5}$$

which is a deterministic quantity since Y_0^x is measurable with respect to the σ-algebra $\sigma\left(W_r : 0 \leqslant r \leqslant \tau_x\right)$. By the uniqueness of the solution of (3.3) on the interval $[t \wedge \tau_x, \tau_x]$, the Markov property of the diffusion process $(X_t^x)_{t\geqslant 0}$ solution of (3.1) and with the same argument used in [31] (Lemma 4.4), one has

$$Y_t^x = u(X_t^x), \quad 0 \leqslant t \leqslant \tau_x \quad \text{a.s.}$$

Furthermore, it is evident on the domain $\bar{D}$, the function u is bounded, and

$$\ell(x) \leqslant u(x) \leqslant h(x).$$

Subsequently, following the steps outlined in Proposition 2.3, we establish that

Lemma 3.3 *When a constant* $\lambda \in \mathbb{R}$ *and* $T \geqslant 0$ *are considered, we find that*

$$\sup_{x\in\bar{D}} \left(\mathbb{E} \int_0^T e^{\lambda s} \|Z_s^x\|^2 \mathrm{d}s \right) \leqslant \mathcal{C}.$$

With this, we are ready to demonstrate the continuity of the function u as defined in equation (3.5).

Proposition 3.4 *The function* $x \to u(x)$ *is continuous.*

Proof To begin, under the given assumptions on τ_x, D, and Γ, it can be established that the function mapping $x \to \tau_x$ is continuous on the domain $\bar{D}$ (as discussed in [27], Proposition 5.76, pp. 484). This, in turn, implies, due to the well-established spatial continuity properties of stochastic flows, the almost sure continuity of the function

$$x \to \left(\tau_x, X_{\tau_x}^x\right).$$

Fix $x' \in \bar{D}$. Our main objective is to prove that

$$|u(x)-u(x')| = \left|Y_0^x - Y_0^{x'}\right| \xrightarrow[x\to x']{} 0.$$

Indeed, as in Section 2.2 and for any fixed $x\in\bar{D}$ let $(Y_t^{x,n}, Z_t^{x,n}, K_t^{x,n+}, K_t^{x,n+})_{t\geqslant 0}$ the solution of the Markovian BSDE2RB $\big(f, H(X_{\tau_x}^x)\mathbb{I}_{\{\tau_x\leqslant n\}}, \ell, h, n\wedge\tau_x\big)$. Obviously, we have $\forall t\leqslant n\wedge\tau_x$, $|Y_t^{x,n}|\leqslant\mathfrak{M}$. Moreover, the solution satisfies the following equation

$$\begin{cases} \text{(i) } Y_t^{x,n} = H(X_{\tau_x}^x)\mathbb{I}_{\{\tau_x\leqslant n\}} + \displaystyle\int_{t\wedge\tau_x}^{n\wedge\tau_x} f(s, Y_s^{x,n}, Z_s^{x,n})\mathrm{d}s + (K_{n\wedge\tau}^{x,n+} - K_t^{x,n+}) - (K_{n\wedge\tau}^{x,n-} - K_t^{x,n-}) \\ \qquad\qquad - \displaystyle\int_{t\wedge\tau_x}^{n\wedge\tau_x} Z_s^{x,n}\mathrm{d}W_s; \\ \text{(ii) } \quad \ell(X_t^x)\leqslant Y_t^{x,n}\leqslant h(X_t^x),\ \forall\, t\leqslant n \text{ and} \\ \quad \displaystyle\int_0^n (h(X_s^x) - Y_s^{x,n})\mathrm{d}K_s^{x,n-} = \int_0^n (Y_s^{x,n} - \ell(X_s^x))\mathrm{d}K_s^{x,n+} = 0. \end{cases} \tag{3.6}$$

We establish that the process $(Y^{x,n}, Z^{x,n}, K^{x,n+}, K^{x,n-})_{n\geqslant 0}$ converges to $(Y^x, Z^x, K^{x,+}, K^{x,-})$ on the space $\mathcal{S}_{\tau_x}^\infty\times\mathcal{H}_{\tau_x}^{2,-\mu}(\mathbb{R}^d)\times(\mathcal{A}_{\tau_x})^2$. Besides, we can prove that

$$|Y_t^{x,n} - Y_t^x|\leqslant\mathfrak{C}e^{-\frac{\mu}{2}(n-t)} \quad \mathbb{P}\text{-a.s. for all } t\leqslant n \text{ where } \mathfrak{C}>0.$$

Let $\varepsilon>0$, there exists n_0 such that $\forall n\geqslant n_0$ we have $2\mathfrak{C}e^{-\frac{\mu}{2}n}\leqslant\varepsilon$. Then

$$|Y_0^x - Y_0^{x'}|\leqslant\varepsilon + |Y_0^{x,n_0} - Y_0^{x',n_0}|.$$

It is enough to check that $\forall\delta>0$ such that

$$|x-x'|\leqslant\delta\Longrightarrow|Y_0^{x,n_0} - Y_0^{x',n_0}|\leqslant\varepsilon.$$

For simplicity, denote $q=n_0$ and $T=\max(\tau_x,\tau_{x'})$ and let us set

$$\tilde{f}(X_t^x, y, z) := \begin{cases} f(X_t^x, y, z), & \text{if } \quad 0\leqslant t\leqslant\tau_x\wedge q; \\ 0, & \text{otherwise;} \end{cases}$$

and also

$$(\tilde{Y}_t^{x,q}, \tilde{Z}_t^{x,q}, \tilde{K}_t^{x,q+}, \tilde{K}_t^{x,q-}) := \begin{cases} (Y_t^{x,q}, Z_t^{x,q}, K_t^{x,q+}, K_t^{x,q-}), & \text{if } 0\leqslant t\leqslant q\wedge\tau_x; \\ (M_{\tau_x}^x\mathbb{I}_{\{\tau_x\leqslant q\}}, 0, K_{q\wedge\tau_x}^{x,q+}, K_{q\wedge\tau_x}^{x,q-}), & \text{otherwise.} \end{cases}$$

Then equation (3.6) becomes with parameters $(\tilde{f}, M_{\tau_x}^x\mathbb{I}_{\{\tau_x\leqslant q\}}, \ell, h, q\wedge T)$ (resp. $(\tilde{f}', M_{\tau_{x'}}^{x'}\mathbb{I}_{\{\tau_{x'}\leqslant q\}}, \ell, h, q\wedge T)$ for x'). Furthermore, let us establish $\hat{\mathcal{R}} = \tilde{\mathcal{R}}^{x,q} - \tilde{\mathcal{R}}^{x',q}$ for $\mathcal{R} = Y, Z, K^+, K^-$. This leads us to the following equation:

$$\begin{aligned} \hat{Y}_{t\wedge T} &= M_{\tau_x}^x\mathbb{I}_{\{\tau_x\leqslant q\}} - M_{\tau_{x'}}^{x'}\mathbb{I}_{\{\tau_{x'}\leqslant q\}} + \int_{t\wedge T}^{q\wedge T}\Big(\tilde{f}(X_s^x, \tilde{Y}_s^{x,q}, \tilde{Z}_s^{x,q}) - \tilde{f}'(X_s^{x'}, \tilde{Y}_s^{x',q}, \tilde{Z}_s^{x',q})\Big)\mathrm{d}s \\ &\quad + \int_{t\wedge T}^{q\wedge T}\mathrm{d}\hat{K}_s^+ - \int_{t\wedge T}^{q\wedge T}\mathrm{d}\hat{K}_s^- - \int_{t\wedge T}^{q\wedge T}\hat{Z}_s\mathrm{d}W_s \\ &= M_{\tau_x}^x\mathbb{I}_{\{\tau_x\leqslant q\}} - M_{\tau_{x'}}^{x'}\mathbb{I}_{\{\tau_{x'}\leqslant q\}} + \int_{t\wedge T}^{q\wedge T}\Big(\tilde{f}(X_s^x, \tilde{Y}_s^{x,q}, \tilde{Z}_s^{x,q}) - \tilde{f}(X_s^x, \tilde{Y}_s^{x',q}, \tilde{Z}_s^{x',q})\Big)\mathrm{d}s \\ &\quad + \int_{t\wedge T}^{q\wedge T}\Big(\tilde{f}(X_s^x, \tilde{Y}_s^{x',q}, \tilde{Z}_s^{x',q}) - \tilde{f}'(X_s^{x'}, \tilde{Y}_s^{x',q}, \tilde{Z}_s^{x',q})\Big)\mathrm{d}s \\ &\quad + \int_{t\wedge T}^{q\wedge T}\mathrm{d}\hat{K}_s^+ - \int_{t\wedge T}^{q\wedge T}\mathrm{d}\hat{K}_s^- - \int_{t\wedge T}^{q\wedge T}\hat{Z}_s\mathrm{d}W_s \\ &= M_{\tau_x}^x\mathbb{I}_{\{\tau_x\leqslant q\}} - M_{\tau_{x'}}^{x'}\mathbb{I}_{\{\tau_{x'}\leqslant q\}} + \int_{t\wedge T}^{q\wedge T}(\bar{\alpha}_s + \bar{\beta}_s)\,\mathrm{d}s + \int_{t\wedge T}^{q\wedge T}\Big(\mathrm{d}\hat{K}_s^+ - \mathrm{d}\hat{K}_s^-\Big) - \int_{t\wedge T}^{q\wedge T}\hat{Z}_s\mathrm{d}W_s, \end{aligned}$$

where

$$\bar{\alpha}_s = \tilde{f}(X_s^x, \tilde{Y}_s^{x,q}, \tilde{Z}_s^{x,q}) - \tilde{f}(X_s^x, \tilde{Y}_s^{x',q}, \tilde{Z}_s^{x',q})$$

and

$$\bar{\beta}_s = \begin{cases} f(X_s^x, M_{\tau_{x'}}^{x'} \mathbb{I}_{\{\tau_{x'} \leqslant q\}}, 0), & \text{if } q \wedge \tau_{x'} < s \leqslant q \wedge \tau_x; \\ -f(X_s^{x'}, \tilde{Y}_s^{x'}, \tilde{Z}_s^{x'}), & \text{if } q \wedge \tau_x < s \leqslant q \wedge \tau_{x'}; \\ f(X_s^x, \tilde{Y}_s^{x'}, \tilde{Z}_s^{x'}) - f(X_s^{x'}, \tilde{Y}_s^{x'}, \tilde{Z}_s^{x'}), & \text{if } 0 \leqslant s \leqslant q \wedge \tau_x \wedge \tau_{x'}; \\ 0, & \text{if } s \geqslant q \wedge \tau_x \text{ and } s \geqslant q \wedge \tau_{x'}. \end{cases}$$

Applying Itô's formula, we get

$$\begin{aligned} \left|\hat{Y}_0\right|^2 &= e^{\mu q} \left|\hat{Y}_q\right|^2 - 2\int_0^q e^{\mu s} \hat{Y}_s \hat{Z}_s \mathrm{d}W_s - \int_0^q e^{\mu s} \|\hat{Z}_s\|^2 \mathrm{d}s - \mu \int_0^q e^{\mu s} |\hat{Y}_s|^2 \mathrm{d}s \\ &+ 2\int_0^q e^{\mu s} \hat{Y}_s \left(\tilde{f}(X_s^x, \tilde{Y}_s^{x,q}, \tilde{Z}_s^{x,q}) - \tilde{f}'(X_s^{x'}, \tilde{Y}_s^{x',q}, \tilde{Z}_s^{x',q})\right) \mathrm{d}s + 2\int_0^q e^{\mu s} \hat{Y}_s \left(\mathrm{d}\hat{K}_s^+ - \mathrm{d}\hat{K}_s^-\right). \end{aligned}$$

Upon taking the expectation and combining it with the Skorokhod's condition, we obtain:

$$\begin{aligned} \left|\hat{Y}_0\right|^2 &= \mathbb{E}\left(e^{\mu q} \left|\hat{Y}_q\right|^2\right) - \mathbb{E}\int_0^q e^{\mu s} \|\hat{Z}_s\|^2 \mathrm{d}s - \mu \mathbb{E}\int_0^q e^{\mu s} |\hat{Y}_s|^2 \mathrm{d}s \\ &+ 2\mathbb{E}\int_0^q e^{\mu s} \hat{Y}_s \left(\tilde{f}(X_s^x, \tilde{Y}_s^{x,q}, \tilde{Z}_s^{x,q}) - \tilde{f}'(X_s^{x'}, \tilde{Y}_s^{x',q}, \tilde{Z}_s^{x',q})\right) \mathrm{d}s \\ &\leqslant \mathbb{E}\left(e^{\mu q} \left|\hat{Y}_q\right|^2\right) - \mathbb{E}\int_0^q e^{\mu s} \|\hat{Z}_s\|^2 \mathrm{d}s + 4\mathfrak{M}\mathbb{E}\int_0^q e^{\mu s} |\bar{\beta}_s| \mathrm{d}s + 4\mathfrak{M}\kappa C \left(\mathbb{E}\int_0^q e^{\mu s} \|\hat{Z}_s\|^2 \mathrm{d}s\right)^{\frac{1}{2}}. \end{aligned}$$

Suppose that $4\mathfrak{M}\kappa C \leqslant 1$, we end up with

$$\left|\hat{Y}_0\right|^2 \leqslant e^{\mu q} \mathbb{E}\left(\left|\hat{Y}_q\right|^2\right) + 4\mathfrak{M}\mathbb{E}\int_0^q e^{\mu s} |\bar{\beta}_s| \mathrm{d}s.$$

Since

$$\hat{Y}_q = M_{\tau_x}^x \mathbb{I}_{\{\tau_x \leqslant q\}} - M_{\tau_{x'}}^{x'} \mathbb{I}_{\{\tau_{x'} \leqslant q\}} = H(X_{\tau_x}^x) \mathbb{I}_{\{\tau_x \leqslant q\}} - H(X_{\tau_{x'}}^{x'}) \mathbb{I}_{\{\tau_{x'} \leqslant q\}}$$

and H is supposed to be bounded by $\mathfrak{M}$ in $\bar{D}$, then

$$\mathbb{E}\left(\left|\hat{Y}_q\right|^2\right) \leqslant 3\mathbb{E}\left(\left|H(X_{\tau_x}^x) - H(X_{\tau_{x'}}^{x'})\right|^2\right) + 3\mathfrak{M}^2 \mathbb{E}\left(\mathbb{I}_{\{\tau_x \leqslant q < \tau_{x'}\}} - \mathbb{I}_{\{\tau_{x'} \leqslant q < \tau_x\}}\right).$$

The continuity in $\bar{D}$ of the functions $x \to \tau_x$ and $x \to H(X_{\tau_x}^x)$ combined with Lebesgue dominated convergence's theorem implies that

$$\mathbb{E}\left(\left|\hat{Y}_q\right|^2\right) \underset{x \to x'}{\longrightarrow} 0.$$

Moreover, by definition of $\bar{\beta}_s$ we have

$$\begin{aligned} \left|\bar{\beta}_s\right| \leqslant& \ (\mathfrak{M} + \mu\mathfrak{M}) \mathbb{I}_{\{q \wedge \tau_{x'} < s \leqslant q \wedge \tau_x\}} + \left(\mathfrak{M} + \mu\mathfrak{M} + \kappa \|\tilde{Z}_s^{x'}\|^2\right) \mathbb{I}_{\{q \wedge \tau_x < s \leqslant q \wedge \tau_{x'}\}} \\ &+ \left|f(X_s^x, \tilde{Y}_s^{x'}, \tilde{Z}_s^{x'}) - f(X_s^{x'}, \tilde{Y}_s^{x'}, \tilde{Z}_s^{x'})\right| \mathbb{I}_{\{0 \leqslant s \leqslant q \wedge \tau_x \wedge \tau_{x'}\}}. \end{aligned}$$

Then, once more, by applying Lebesgue's dominated convergence theorem, we can conclude that

$$\mathbb{E}\int_0^q e^{\mu s}|\bar{\beta}_s|\mathrm{d}s \leqslant \mathbb{E}\int_0^q e^{\mu s}\left|f(X_s^x,\tilde{Y}_s^{x'},\tilde{Z}_s^{x'}) - f(X_s^{x'},\tilde{Y}_s^{x'},\tilde{Z}_s^{x'})\right|\mathrm{d}s$$
$$+ C\mathbb{E}\int_0^q e^{\mu s}\left(1+\|\tilde{Z}_s^{x'}\|^2\right)\left(\mathbb{I}_{\{q\wedge\tau_{x'}<s\leqslant q\wedge\tau_x\}} + \mathbb{I}_{\{q\wedge\tau_x<s\leqslant q\wedge\tau_{x'}\}}\right)\mathrm{d}s \underset{x\to x'}{\longrightarrow} 0.$$

In conclusion, we have successfully demonstrated that for any fixed x', and for all $\delta \geqslant 0$ and $x \in \bar{D}$ such that $|x-x'| \leqslant \delta$, the following inequality holds:

$$\left|Y_0^x - Y_0^{x'}\right|^2 \leqslant 2\varepsilon.$$

This concludes the proof of the proposition. □

Now, let's introduce the notation:

$$F(x,u,p,Q) = \frac{1}{2}trace[\sigma\sigma^*(x)]Q + \langle b(x),p\rangle + f(x,u,\sigma^*(x)p).$$

We proceed to define a viscosity solution for equation (3.4). For a more comprehensive understanding of viscosity solutions, please refer to [10].

Definition 3.5 *Let u be a function which belongs to $\mathcal{C}(\bar{D})$. It is called a viscosity:*

(a) *Sub-solution of (3.4), if for any $\varphi \in \mathcal{C}^2(\bar{D})$ such that whenever $x \in \bar{D}$ is a local maximum of $u-\varphi$, we have, suppressing dependence on x,*

$$\begin{cases} (u-\ell)\wedge\Big\{(u-h)\vee\Big[-F(x,u,D\varphi,D^2\varphi)\Big]\Big\} \leqslant 0, & x\in D \\ \Big[u-H\Big]\wedge\Bigg\{(u-\ell)\wedge\Big\{(u-h)\vee\Big[-F(x,u,D\varphi,D^2\varphi)\Big]\Big\}\Bigg\} \leqslant 0, & x\in\partial D \end{cases}$$

(b) *Super-solution of (3.4), if for any $\varphi \in \mathcal{C}^2(\bar{D})$ such that whenever $x \in \bar{D}$ is a local minimum of $u-\varphi$, we have, suppressing dependence on x,*

$$\begin{cases} (u-\ell)\vee\Big\{(u-h)\wedge\Big[-F(x,u,D\varphi,D^2\varphi)\Big]\Big\} \geqslant 0, & x\in D \\ \Big[u-H\Big]\vee\Bigg\{(u-\ell)\vee\Big\{(u-h)\wedge\Big[-F(x,u,D\varphi,D^2\varphi)\Big]\Big\}\Bigg\} \geqslant 0, & x\in\partial D \end{cases}$$

(c) *$u \in \mathcal{C}(\bar{D})$ is said to be a viscosity solution of (3.4), if it is both sub and super-solution.*

3.2 Existence of a viscosity solution

In this paragraph, our focus turns to the existence of the viscosity solution for equation (3.4). To begin, we contemplate the following BSDE:

$$Y_s^{x,n} = H(X_{\tau_x}^x) + \int_{s\wedge\tau_x}^{\tau_x} f(X_r^x,Y_r^{x,n},Z_r^{x,n})\mathrm{d}r + n\int_{s\wedge\tau_x}^{\tau_x}\left(Y_r^{x,n}-\ell(X_r^x)\right)^-\mathrm{d}r$$
$$- n\int_{s\wedge\tau_x}^{\tau_x}\left(Y_r^{x,n}-h(X_r^x)\right)^+\mathrm{d}r - \int_{s\wedge\tau_x}^{\tau_x} Z_r^{x,n}\mathrm{d}W_r, \quad 0\leqslant s\leqslant\tau_x.$$

The existence and uniqueness of the solution $(Y^{x,n},Z^{x,n})$ is also guaranteed by section 2.2 (Without reflection). If we define the deterministic function

$$u^n(x) = Y_0^{x,n},$$

we can find the following results:

Proposition 3.6 (1) u^n *is a viscosity solution of the following PDE:*

$$\begin{cases} -\mathcal{L}u^n - f\big(x, u^n, \nabla u^n\sigma\big) - n(u^n-\ell)^- + n(u^n-h)^+ = 0, \quad \forall x \in D; \\ u^n(x) = H(x), \quad \forall x \in \partial D. \end{cases} \tag{3.7}$$

(2) $|u^n(x)-u(x)|^2 \leqslant \mathbb{E}\Big[\sup_{0\leqslant s\leqslant \tau_x} \big|Y_s^{x,n} - Y_s^x\big|^2\Big] \xrightarrow[n\to\infty]{0}$.

In particular, u^n *converges uniformly to* u *on compact sets.*

Proof From Theorem A.3 in Appendix we have the first claim. For the proof of (2), it's the same one used in the proof of Proposition 2.3 in [26]. □

Now, we are ready to demonstrate the following theorem:

Theorem 3.7 *The function* u *defined in* (3.5) *is a viscosity solution of* (3.4).

Proof We are only going to demonstrate that u is the viscosity sub-solution of (3.4). A similar argument would show that is a viscosity super-solution. Let $\varphi \in \mathcal{C}^2(\bar{D})$ and $x \in \bar{D}$ be a local maximum of $u-\varphi$. If $x \in \Gamma$ then $\tau_x = 0$, and hence $u(x) = Y_0^x = H(X_0^x) = H(x)$. We now consider the case $x \neq \Gamma$. Then, $\tau_x > 0$. We know that

$$\ell(x) \leqslant u(x) \leqslant h(x).$$

Then, it is sufficient to prove that for any $\varphi \in \mathcal{C}^2(D)$ and for any local maximum $x \in D$ of $u-\varphi$ such that $u(x) > \ell(x)$, we have

$$-\mathcal{L}\varphi(x) - f\big(x, u(x), (\nabla\varphi\sigma)(x)\big) \leqslant 0, \quad x \in D.$$

So for $n \geqslant 0$, let $(x_n)_{n\geqslant 0}$ be a sequence of local maximum points of $u^n - \varphi$ such that $\lim_n x_n \to x$ (the existence of such sequence follows from the uniform convergence of u^n to u). Note that for n large enough we have $u^n(x_n) > \ell(x_n)$, then using the fact that u^n is a viscosity solution of (3.7) we have

$$\mathcal{L}\varphi(x_n) - f\big(x_n, u^n(x_n), (\nabla\varphi\sigma)(x_n)\big) - n\left(u^n(x_n) - \ell(x_n)\right)^- + n\left(u^n(x_n) - h(x_n)\right)^+ \leqslant 0, \quad x \in D.$$

Passing to the limit as $n \to \infty$ in the above inequalities, and using the continuity of the functions and the uniform convergence we obtain that u is a viscosity sub-solution of (3.4). □

Appendix

First, we are going to establish a comparison Theorem for solutions of BSDEs exhibiting quadratic growth with respect to a random terminal time.

Theorem A.1 *Let* $(Y^i, Z^i, 0, 0)_{i=1,2}$ *be the solution of the quadratic BSDE2RB (2.1) with data* $(\xi^i, f^i, 0, 0)_{i=1,2}$. *Assume that for any* $t \geqslant 0$ *one has* $f^1(t, Y_t^2, Z_t^2) \leqslant f^2(t, Y_t^2, Z_t^2)$ *and* $\xi^1 \leqslant \xi^2$. *Then*

$$Y_t^1 \leqslant Y_t^2 \ \mathbb{P}\text{-a.s.} \ for\ all\ t \in \mathbb{R}^+.$$

Next, taking account of the SDE (3.1), we consider the following one dimensional quadratic BSDE:

$$\begin{cases} Y^x \in \mathcal{S}_{\tau_x}^\infty, \ Z^x \in \mathcal{H}_{\tau_x}^{2,-\mu}(\mathbb{R}^d), \\ Y_s^x = H(X_{\tau_x}^x) + \displaystyle\int_{s\wedge\tau_x}^{\tau_x} f(X_r^x, Y_r^x, Z_r^x)\mathrm{d}r - \int_{s\wedge\tau_x}^{\tau_x} Z_r^x \mathrm{d}W_r. \end{cases} \tag{A.1}$$

Since H and f satisfy assumptions (1)–(3) (see p.14), it follows that for all $x \in \bar{D}$ there exists a unique pair of processes $(Y_s^x, Z_s^x)_{s\geqslant 0}$ solution of (A.1).

Moreover, we consider the following elliptic quadratic PDE

$$\begin{cases} -\mathcal{L}v(x) - f\big(x, v(x), (\nabla v\sigma^T)(x)\big) = 0, \quad \forall x \in D, \\ v(x) = H(x), \quad \forall x \in \partial D, \end{cases} \tag{A.2}$$

We define

$$F(x,v,p,Q) = \frac{1}{2}trace[\sigma\sigma^*(x)]Q + \langle b(x), p\rangle + f(x,v,\sigma^*(x)p).$$

Let us give a definition of viscosity solution of (A.2).

Definition A.2 *Let v be a function which belongs to $\mathcal{C}(\bar{D})$. It is called a viscosity sub-solution (resp. super-solution) of (A.2), if for any $\varphi \in \mathcal{C}^2(\bar{D})$* such that whenever $x \in \bar{D}$ *is a local maximum (resp. minimum) of $v - \varphi$, we have, suppressing dependence on x,*

$$\begin{cases} -F(x,v,D\varphi,D^2\varphi) \leqslant 0, \quad x \in D, \\ \Big[v - H\Big] \wedge \Big[-F(x,v,D\varphi,D^2\varphi)\Big] \leqslant 0, \quad x \in \partial D, \end{cases}$$

$\Big($ *respectively,*

$$\begin{cases} -F(x,v,D\varphi,D^2\varphi) \geqslant 0, \quad x \in D, \\ \Big[v - H\Big] \vee \Big[-F(x,v,D\varphi,D^2\varphi)\Big] \geqslant 0, \quad x \in \partial D\Big). \end{cases}$$

Moreover, $v \in \mathcal{C}(\bar{D})$ is said to be a viscosity solution of (A.2), if it is both sub and super-solution.

Define

$$v(x) = Y_0^x, \quad x \in \bar{D}, \tag{A.3}$$

and let us focus to prove that v is a viscosity solution of (A.2). Then

Theorem A.3 *The deterministic function defined in* (A.3) *is a continuous viscosity solution of* (A.2).

Proof The continuity can be demonstrated in a manner similar to the approach outlined in Proposition 3.4. Let us focus to the viscosity solution proof. We show only that u is viscosity sub-solution of (3.4). A similar argument would show that is a viscosity super-solution. Let $\varphi \in C^2(\bar{D})$ and $x \in \bar{D}$ be a local maximum of $u - \varphi$. If $x \in \Gamma$ then $\tau_x = 0$, and hence $u(x) = Y_0^x = H(X_0^x) = H(x)$. We now consider the case $x \in D \cup (\partial D \setminus \Gamma)$. Then, $\tau_x > 0$. Without loss of generality, we suppose that $u(x) = \varphi(x)$. Hence $u(\bar{x}) \leqslant \varphi(\bar{x}), \forall \bar{x} \in \bar{D}$.

Suppose for contradiction that

$$\mathcal{L}\varphi(x) + f\big(x, u, (\nabla\varphi\sigma)(x)\big) < 0.$$

Let $\varepsilon > 0$ be such that whenever $|x_0 - x| \leqslant \varepsilon$, then

$$\begin{cases} u(x_0) \leqslant \varphi(x_0) \\ \mathcal{L}\varphi(x_0) + f\big(x_0, u, (\nabla\varphi\sigma)(x_0)\big) < 0. \end{cases}$$

We define a stopping time

$$\bar{\tau} = \inf\{t > 0; |X_t^x - x| \geqslant \varepsilon\} \wedge \tau_x \wedge \varepsilon.$$

Let us define

$$(\bar{Y}_t, \bar{Z}_t) = \big((Y_{t\wedge\bar{\tau}}^x), \mathbb{I}_{[0,\bar{\tau}]}(Z_t^x)\big), \quad t \in [0, \varepsilon].$$

These processes $(\bar{Y}_t, \bar{Z}_t)$, satisfy the following BSDE

$$\bar{Y}_t = u(X^x_{\bar{\tau}}) + \int_t^\varepsilon \mathbb{I}_{[0,\bar{\tau}]}(r) f(X^x_r, u(X^x_r), \bar{Z}_r)\mathrm{d}r - \int_t^\varepsilon \bar{Z}_r \mathrm{d}W_r.$$

On the other hand, according Itô's formula, the pair

$$(\underline{Y}_t, \underline{Z}_t) = \left(\varphi(X^x_{t\wedge\bar{\tau}}), \mathbb{I}_{[0,\bar{\tau}]}(t)(\nabla\varphi\sigma)(X^x_t)\right), \quad t \in [0,\varepsilon],$$

satisfies the following BSDE:

$$\underline{Y}_t = \varphi(X^x_{\bar{\tau}}) - \int_t^\varepsilon \mathbb{I}_{[0,\bar{\tau}]}(r)\mathcal{L}\varphi(X^x_r)\mathrm{d}r - \int_t^\varepsilon \underline{Z}_r \mathrm{d}W_r, \quad t \in [0,\varepsilon].$$

Then, by the assumption that $u \leqslant \varphi$ and the choice of ε and $\bar{\tau}$ and with the help of Theorem A.1, we deduce that $\bar{Y}_0 < \underline{Y}_0$, i.e. $u(x) < \varphi(x)$, which is a contradiction. □

References

[1] Bahlali, K., Eddahbi, M. and Ouknine, Y., Quadratic BSDE with $\mathbb{L}^2$-terminal data: Krylov's estimate, Itô's-Krylov's formula and existence results, The Annals of Probability, 2017, 45(4): 2377–2397.

[2] Bahlali, K., Hamadène, S. and Mezerdi, B., Backward stochastic differential equations with two reflecting barriers and continuous with quadratic growth coefficient, Stochastic Processes and their Applications, 2005, 115(7): 1107–1129.

[3] Bally, V. and Matoussi, A., Weak solutions for SPDEs and backward doubly stochastic differential equations, Journal of Theoretical Probability, 2001, 14: 125–164.

[4] Barles, G., Buckdahn, R. and Pardoux, E., Backward stochastic differential equations and integral-partial differential equations, Stochastics, 1997, 60(1-2): 57–83.

[5] Barles, G. and Lesigne, E., SDE, BSDE and PDE, Pitman Research Notes in Mathematics Series, 1997: 47–82.

[6] Beznea, L. and Cîmpean, I., Quasimartingales associated to Markov processes, Transactions of the American Mathematical Society, 2018, 370(11): 7761–7787.

[7] Briand, P. and Hu, Y., Stability of BSDEs with random terminal time and homogenization of semilinear elliptic PDEs, Journal of Functional Analysis, 1998, 155(2): 455–494.

[8] Chung, K. L., A Course in Probability Theory, Academic Press, 3rd edn., 2001.

[9] Confortola, F. and Briand, P., Quadratic BSDEs with random terminal time and elliptic pdes in infinite dimension, Electronic Journal of Probability, 2008, 13: 1529–1561.

[10] Crandall, M. G. and Lions, P.-L., Viscosity solutions of Hamilton-Jacobi equations, Transactions of the American Mathematical Society, 1983, 277(1): 1–42.

[11] Cvitanić, J. and Karatzas, I., Backward stochastic differential equations with reflection and Dynkin games, The Annals of Probability, 1996, 24(4): 2024–2056.

[12] Darling, R. W. and Pardoux, E., Backwards SDE with random terminal time and applications to semilinear elliptic PDE, The Annals of Probability, 1997, 25(3): 1135–1159.

[13] Dellacherie, C. and Meyer, P.-A., Probabilités et Potentiel, Hermann, 1975.

[14] El Karoui, N., Kapoudjian, C., Pardoux, E., Peng, S. and Quenez, M. C., Reflected solutions of backward SDE's, and related obstacle problems for PDE's, The Annals of Probability, 1997, 25(2): 702–737.

[15] Essaky, E. and Hassani, M., Generalized BSDE with 2-reflecting barriers and stochastic quadratic growth, Journal of Differential Equations, 2013, 254(3): 1500–1528.

[16] Essaky, E., Hassani, M. and Rhazlane, C., Doubly reflected BSDEs with stochastic quadratic growth: Around the predictable obstacles, Stochastic Processes and their Applications, 2023, 163: 473–497.

[17] Hamadène, S. and Hassani, M., BSDEs with two reflecting barriers: The general result, Probability Theory and Related Fields, 2005, 132: 237–264.

[18] Hamadène, S. and Hdhiri, I., Backward stochastic differential equations with two distinct reflecting barriers and quadratic growth generator, Journal of Applied Mathematics and Stochastic Analysis, 2006, 2006: 095818.

[19] Hamadène, S., Lepeltier, J. P. and Matoussi, A., Double Barrier Backward SDEs with Continuous Coefficient, Pitman Research Notes in Mathematics Series, 1997: 161–176.

[20] Kazamaki, N., Continuous Exponential Martingales and BMO, Springer, 2006.

[21] Kobylanski, M., Backward stochastic differential equations and partial differential equations with quadratic growth, The Annals of Probability, 2000, 28(2): 558–602.

[22] Lepeltier, J.-P. and San Martín, J., Backward SDEs with two barriers and continuous coefficient: An existence result, Journal of Applied Probability, 2004, 41(1): 162–175.

[23] Madoui, I., Eddahbi, M. and Khelfallah, N., Quadratic BSDEs with jumps and related pides, Stochastics, 2022, 94(3): 386–414.

[24] Pardoux, E. and Peng, S., Adapted solution of a backward stochastic differential equation, Systems & Control Letters, 1990, 14(1): 55–61.

[25] Pardoux, E. and Peng, S., Backward stochastic differential equations and quasilinear parabolic partial differential equations, In: Rozovskii, B. L. and Sowers, R. B. (eds.), Stochastic Partial Differential Equations and Their Applications, Lecture Notes in Control and Information Sciences, Springer, Berlin, Heidelberg, 2002, 176: 200–217.

[26] Pardoux, E. and Răşcanu, A., Backward stochastic differential equations with subdifferential operator and related variational inequalities, Stochastic Processes and their Applications, 1998, 76(2): 191–215.

[27] Pardoux, E. and Răşcanu, A., Stochastic Differential Equations, Backward SDEs, Partial Differential Equations, Stochastic Modelling and Applied Probability 69, 1st edn., Springer, 2014.

[28] Pardoux, E. and Tang, S., Forward-backward stochastic differential equations and quasilinear parabolic PDEs, Probability Theory and Related Fields, 1999, 114: 123–150.

[29] Peng, S., Probabilistic interpretation for systems of quasilinear parabolic partial differential equations, Stochastics and Stochastics Reports, 1991, 37(1-2): 61–74.

[30] Protter, P. E., Stochastic Integration and Differential Equations, 2nd edn., Springer, 2004.

[31] Royer, M., BSDEs with a random terminal time driven by a monotone generator and their links with PDEs, Stochastics and Stochastic Reports, 2004, 76(4): 281–307.

[32] Saisho, Y., Stochastic differential equations for multi-dimensional domain with reflecting boundary, Probability Theory and Related Fields, 1987, 74(3): 455–477.

[33] Yosida, K., Functional Analysis, Grundlehren der Mathematischen Wissenschaften, 6th edn., Springer-Verlag, 1980.