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On the pricing and hedging of precipitation derivatives

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Abstract In this paper, we present a new precipitation model based on a multi-factor Ornstein-Uhlenbeck approach of pure-jump type. In this setup, we derive a representation for the related precipitation swap price process and infer its risk-neutral time dynamics. We further deduce a pricing formula for European options written on the precipitation swap and obtain the minimal variance hedging portfolio in the underlying weather market. In the second part of the paper, we provide a precipitation swap price representation under future information modeled by an initially enlarged filtration. We finally derive a formula for the associated information premium and investigate minimal variance hedging of precipitation derivatives under future information.

Keywords Precipitation model, Precipitation swap price, Minimal variance hedging, Option pricing, Information premium, Future information, Stochastic differential equation, Enlarged filtration, Stochastic maximum principle, Malliavin calculus, Fourier transform

2020 Mathematics Subject Classification 60G44, 60G51, 60H07, 60H10, 91B44, 91G10, 91G20, 93E20

1. Introduction

Within the last decades, competitive weather markets – like the Chicago Mercantile Exchange (CME) or the European Energy Exchange (EEX) – have been created all over the world. In such market places, numerous indices associated with different non-tradable/non-storable underlyings like outdoor temperature, precipitation amounts (rainfall, snowfall, hailstone), sunshine hours, wind speed, or even the number of frost days in a certain location are traded similarly to financial products in ordinary stock or interest rate markets. In Section 1.2.3 in [4] it is claimed that at the CME the market participants can, for example, trade in rain and snowfall swaps associated with the accumulated amount of precipitation over a month measured in various cities in the United States. In addition, the traders can invest into call and put options written on the mentioned swap contracts. With view on the ongoing climate change, weather derivatives play an important role for many companies and financial investors all over the world, as they constitute a useful hedging instrument against disadvantageous weather conditions.

The main motivation for the present article lies in the constant (i.e., time-independent)

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precipitation swap price derived in Eq. (8.31) in [4]. The authors notice this drawback and state that a time-varying (stochastic) price dynamics for precipitation derivatives is not possible in their model, though desirable. Moreover, the precipitation swap price presented in Eq. (8.30) in [4] is derived under a common backward-looking information filtration which neglects any weather forecasts on future precipitation amounts. In our opinion, it appears reasonable to take weather forecasts into account when it comes to precipitation derivatives pricing. In this context, we also refer to the argumentation given in [3] and [14]. In the present paper, we thus propose an innovative precipitation model in an enlarged filtration setup which models the anticipative information available to the traders. As a by-product, the resulting swap price process is time-dependent and follows a tractable stochastic dynamics with random fluctuations (recall Proposition 6.4).

In the literature, there seems to be only little theoretical work on the pricing and hedging of precipitation derivatives. For instance, in [6] precipitation derivatives under the utility indifference pricing approach are considered. The authors introduce a Markovian jump process to model the expected precipitation amounts and present a calibration procedure to empirical rainfall data. Also in [18] the authors pursue an indifference pricing approach for rainfall derivatives and assume the existence of a tradable financial instrument with a price process affected by the quantity of rainfall. In [20] the distributional properties of rainfall are analyzed and the underlying CARMA model is fitted to empirical data. We refer to the references in [20] for further literature on rainfall modeling and statistical properties. In [11] the authors investigate the pricing of basket options written on rainfall and temperature derivatives. Moreover, Sections 8.1 and 8.2 in [4] deal with the modeling of precipitation and the evaluation of related derivatives. In [14] a precipitation model based on an increasing compound Poisson process is considered and the approach is extended to different enlarged filtration frameworks accounting for weather forecasts. We stress that the precipitation model proposed in the present paper is fundamentally different from the approach in [14], as the present investigations are based on an arithmetic multi-factor mean-reverting Ornstein-Uhlenbeck approach of pure-jump type. In contrast to the present paper, minimal variance hedging of precipitation derivatives is neither considered in [14].

Diverse applications of the theory of enlarged filtrations in financial, commodity and weather markets can be found in [3, 5, 9, 10, 14–16]. To read more on classical portfolio optimization theory, the reader is referred to Sections 4.3 and 7.1 in [10], and Section 3.1 in [23]. Theoretical background and various applications of minimal variance hedging can be found in [1], Section 3 in [5], Section 10.4 in [8], Section 12.6 in [10], as well as Sections 1.2.3 and 6.1 in [23]. A survey of different quadratic hedging approaches is provided in [26], whereas hedging theory in a broader sense is treated in [23].

The paper is organized as follows. In Section 2 we briefly recall the precipitation model presented in [4] and discuss its limitations. In Section 3 we propose a new precipitation model which is based on an arithmetic multi-factor Ornstein-Uhlenbeck approach of pure-jump type. In this setup, we derive a representation for the related precipitation swap price process and infer its risk-neutral time dynamics. Section 4 is devoted to inverse Fourier pricing of European options written on the precipitation swap. In Section 5 we deduce the minimal variance hedging portfolio in the underlying precipitation swap market and present several practical examples. In Section 6 we infer a precipitation swap price formula under an initially enlarged filtration modeling future market information available to the traders. We also derive the time dynamics of the anticipative precipitation swap price process and obtain a formula for the associated information premium. In Section 7 we investigate minimal variance hedging of precipitation derivatives under future information by an application of a sufficient stochastic maximum principle.

2. Preliminaries

Let $\left(\Omega, \mathfrak{F}, (\mathcal{F}_t)_{t \geqslant 0}, \mathbb{P}\right)$ be a filtered probability space satisfying the usual hypotheses, i.e., $\mathfrak{F}$ denotes a sigma-algebra augmented by all $\mathbb{P}$-null sets, and $\mathcal{F} = (\mathcal{F}_t)_{t \geqslant 0}$ constitutes a complete and right-continuous filtration. Here, $\mathbb{P}$ is the physical probability measure. We assume that $\mathcal{F}_0$ is trivial. In the sequel, we briefly recall the precipitation model presented in Sections 8.1 and 8.2 in [4] on which all our further investigations will be based. First of all, in accordance to Section 8.1.2 in [4], we introduce the stochastic process

$$R_t := \int_0^t \theta(s) \, \mathrm{d}L_s$$

which models the cumulative amount of precipitation up to time $t \geqslant 0$ in a given location. More precisely, the value R_t represents the accumulated amount of rain, hail or snowfall etc. within the time period $[0, t]$. Here, $\theta : [0, \infty[\to [0, \infty[$ is a bounded, deterministic, continuous function of time, and L constitutes a square-integrable, increasing, pure-jump-type additive process (a so-called time-inhomogeneous subordinator) with finite variance and càdlàg paths. The function θ scales the jump amplitudes of L over time and thus, captures possible seasonality features. The model for R can be interpreted as follows: At random times there occurs precipitation of a particular quantity modeled by the jump sizes of L, which are rescaled by θ. Since L is increasing and θ is positive, the trajectory of R takes the shape of an increasing (random) step function. If $\theta(t) \equiv 1$, then it holds $R_t = L_t$ for all $t \geqslant 0$ yielding the simplest case. A very comprehensive description of the model for R can be found on the bottom of p. 186 in [4]. With reference to Eq. (8.21) in [4], let us further introduce the (non-negative) precipitation index

$$\mathfrak{I}(\tau_1, \tau_2) := R_{\tau_2} - R_{\tau_1} = \int_{\tau_1}^{\tau_2} \theta(s) \, \mathrm{d}L_s$$

with measurement period $[\tau_1, \tau_2]$, $0 < \tau_1 < \tau_2 < \infty$. If there are several locations considered during the time interval $[\tau_1, \tau_2]$, the accumulated precipitation index shall be defined via

$$\mathfrak{I}(\tau_1, \tau_2) := \sum_{k=1}^{n} \int_{\tau_1}^{\tau_2} \theta_k(s) \, \mathrm{d}L_s^k \tag{2.1}$$

with independent, càdlàg, square-integrable, increasing, $\mathcal{F}$-adapted, finite-activity, pure-jump, compound Poisson Lévy-type (additive, or independent increment, or non-stationary increment) processes $L^1, \ldots, L^n$, $n \in \mathbb{N}$, given by

$$L_t^k := \int_0^t \int_{\mathbb{R}^+} z \mathrm{d}N_k(s, z) \tag{2.2}$$

where $\mathbb{R}^+ =]0, \infty[$ and N_k constitutes a Poisson random measure (PRM) for every $k \in \{1, \ldots, n\}$. We refer to Section 8.1.1 in [4, 8, 10, 17, 24], to read more on independent increment processes and their properties. Moreover, for all $k \in \{1, \ldots, n\}$ and $(s, z) \in [0, \infty[\times \mathbb{R}^+$ we define the $\mathbb{P}$-compensated PRMs $\tilde{N}_k^{\mathbb{P}}$ via

$$\mathrm{d}\tilde{N}_k^{\mathbb{P}}(s, z) := \mathrm{d}N_k(s, z) - \varepsilon_k(s) \, \mathrm{d}\nu_k(z) \, \mathrm{d}s \tag{2.3}$$

which constitute $(\mathcal{F},\mathbb{P})$-martingale integrators. Herein, for all $k \in \{1,\ldots,n\}$ the positive and sigma-finite Lévy measures ν_k satisfy

$$\int_{\mathbb{R}^+} (1 \wedge z)\,\mathrm{d}\nu_k(z) < \infty$$

(cf. [8, 17, 23–25]), while $\varepsilon_k(s) > 0$ are deterministic functions modeling the time-dependent (seasonally varying) jump-intensities of the processes L^k. Diverse distributional choices for ν_k can be found in Appendix B.1.2 on p. 151 in [25]. We stress that the processes L^k given in (2.2) are not time-homogeneous Lévy processes, as the involved jump-intensities $\varepsilon_k(\cdot)$ are time-dependent. With reference to Eq. (8.30) in [4], for all $0 \leqslant t \leqslant \tau_1 < \tau_2$ we define the (non-negative) precipitation swap price process via

$$F_t(\tau_1,\tau_2) := F_t^{\mathcal{F}}(\tau_1,\tau_2) := \mathbb{E}_{\mathbb{Q}}(\Im(\tau_1,\tau_2)\,|\mathcal{F}_t) \tag{2.4}$$

where $\mathbb{Q}$ is a risk-neutral pricing measure. This price an investor has to pay at time $t \leqslant \tau_1$ in order to attain the right of receiving at time τ_2 a certain amount of money depending on the actually realized precipitation quantity $\Im(\tau_1,\tau_2)$ within the measurement period $[\tau_1,\tau_2]$. Parallel to [4], in the context of (2.4) we assume that entering the swap contract at time $t \leqslant \tau_1$ is costless. Further note that for any $t \leqslant \tau_1$ it holds

$$F_t(\tau_1,\tau_2) = \mathbb{E}_{\mathbb{Q}}(\mathbb{E}_{\mathbb{Q}}(\Im(\tau_1,\tau_2)\,|\mathcal{F}_{\tau_1})\,|\mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(\mathbb{E}_{\mathbb{Q}}[\Im(\tau_1,\tau_2)]\,|\mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}[\Im(\tau_1,\tau_2)] \tag{2.5}$$

due to the tower property of conditional expectations and the independent increment property of the processes L^k appearing in (2.1). Regarding (2.5), we observe that the precipitation swap price F is deterministic and independent of the trading time t in the current model setup.

Remark 2.1 *We stress that the precipitation swap is priced under an (with respect to $\mathbb{P}$) equivalent risk-neutral pricing measure $\mathbb{Q}$ which cannot be chosen uniquely, since precipitation cannot be traded (and neither stored) in the usual sense. Thus, similarly to the cases of electricity or temperature, the currently considered weather market is incomplete. Also, we refrain from calling $\mathbb{Q}$ an equivalent martingale measure, as it is not clear which discounted entity shall form a martingale under $\mathbb{Q}$ (recall the 'first fundamental theorem of asset pricing' provided in Proposition 9.2 in [8]).*

In the following, we establish a probability measure change from the physical measure $\mathbb{P}$ (under which the precipitation process R is observed) to an equivalent risk-neutral pricing measure $\mathbb{Q}$ (under which the precipitation swap is priced). To this end, for all $k \in \{1,\ldots,n\}$ and $t \geqslant 0$ we introduce the independent (strictly positive) Doléans-Dade exponentials

$$\Xi_t^k := \exp\left\{\int_0^t \int_{\mathbb{R}^+} h_k(s,z)\,\mathrm{d}\tilde{N}_k^{\mathbb{P}}(s,z) - \int_0^t \int_{\mathbb{R}^+} \left[e^{h_k(s,z)} - 1 - h_k(s,z)\right] \varepsilon_k(s)\,\mathrm{d}\nu_k(z)\,\mathrm{d}s\right\} \tag{2.6}$$

where $h_1,\ldots,h_n$ are continuous deterministic functions satisfying

$$\int_0^t \int_{\mathbb{R}^+} h_k(s,z)^2 \varepsilon_k(s)\,\mathrm{d}\nu_k(z)\,\mathrm{d}s < \infty$$

for all $t \geqslant 0$. We recall that $\Xi_0^k = 1$ and that

$$\mathrm{d}\Xi_t^k = \Xi_{t-}^k \int_{\mathbb{R}^+} \left[e^{h_k(t,z)} - 1\right] \mathrm{d}\tilde{N}_k^{\mathbb{P}}(t,z)$$

showing the local $\mathbb{P}$-martingale property of $\left(\Xi_t^k\right)_{t\geqslant 0}$ for every $k \in \{1, \ldots, n\}$. We further define the (strictly positive) Radon-Nikodym density process

$$Z_t := \left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} := \prod\nolimits_{k=1}^{n} \Xi_t^k$$

with Doléans-Dade exponentials Ξ_t^k as defined in (2.6). It holds

$$\mathrm{d}Z_t = Z_{t-} \sum_{k=1}^{n} \int_{\mathbb{R}^+} \left[e^{h_k(t,z)} - 1\right] \mathrm{d}\tilde{N}_k^{\mathbb{P}}(t,z)$$

such that $(Z_t)_{t\geqslant 0}$ is an $\mathcal{F}$-adapted local martingale under $\mathbb{P}$ satisfying $Z_0 = 1$. We further impose the Novikov condition claimed in Theorem 12.21 in [10] which in our setup translates into

$$\int_0^t \int_{\mathbb{R}^+} \left[1 - e^{h_k(s,z)} + h_k(s,z)\, e^{h_k(s,z)}\right] \varepsilon_k(s)\, \mathrm{d}\nu_k(z)\, \mathrm{d}s < \infty$$

where $t \geqslant 0$ and $k \in \{1, \ldots, n\}$. Assuming the Novikov condition to be in force, we obtain

$$\mathbb{E}_{\mathbb{P}}[Z_t] \equiv 1$$

for all $t \geqslant 0$ such that the Radon-Nikodym density process Z constitutes a strictly positive local $\mathbb{P}$-martingale with constant expectation and hence, a true martingale under $\mathbb{P}$. Applying Girsanov's theorem (see Theorem 12.21 in [10]), we state that $\mathbb{Q}$ constitutes a probability measure on $(\mathcal{F}_t)_{t\geqslant 0}$ which is equivalent to $\mathbb{P}$ and that the time-inhomogeneous $\mathbb{Q}$-compensated PRMs are of the form

$$\mathrm{d}\tilde{N}_k^{\mathbb{Q}}(s,z) := \mathrm{d}\tilde{N}_k^{\mathcal{F},\mathbb{Q}}(s,z) := \mathrm{d}N_k(s,z) - e^{h_k(s,z)}\varepsilon_k(s)\, \mathrm{d}\nu_k(z)\, \mathrm{d}s \tag{2.7}$$

where $(s,z) \in [0,\infty[\times \mathbb{R}^+$ and $k \in \{1, \ldots, n\}$. In the current setup, we get the following result which extends Eq. (8.31) in [4] to the multi-location case.

Proposition 2.2 *For all $0 \leqslant t \leqslant \tau_1 < \tau_2$ the precipitation swap price can be represented as*

$$F_t(\tau_1, \tau_2) = \sum_{k=1}^{n} \int_{\tau_1}^{\tau_2} \int_{\mathbb{R}^+} \theta_k(s)\, z e^{h_k(s,z)} \varepsilon_k(s)\, \mathrm{d}\nu_k(z)\, \mathrm{d}s. \tag{2.8}$$

Proof By substituting (2.1), (2.2) and (2.7) into (2.5), we obtain the claimed equality. □

Remark 2.3 *The latter result has already been stated in [4] for $n = 1$. Evidently, the precipitation swap price representation in (2.8) is not only deterministic, but even independent of the trading time t and hence, constant over time. This feature constitutes the main drawback of the modeling approach proposed in Section 8.2 in [4]. With view on reality, one would prefer a stochastic price dynamics for precipitation derivatives with random fluctuations varying over time. On p. 195 in [4] the authors mention this shortcoming and also propose a possible extension of their model by referring to a doubly stochastic Poisson process specification with stochastic jump intensities. However, the details are not worked out.*

The motivation for the present article is twofold. On the one hand, we want to get rid of the undesirable time-independency of the swap price (2.8), while, on the other hand, we aim at taking weather forecasts into account when pricing precipitation derivatives.

3. A new precipitation model

We now present a new approach to model the accumulated amount of precipitation $R = (R_t)_{t \geqslant 0}$ in a certain location. For this purpose, we introduce the non-negative and $\mathcal{F}$-adapted stochastic process $A = (A_t)_{t \geqslant 0}$ which is (for arbitrary $n \in \mathbb{N}$) defined via

$$A_t := \mu(t) + \sum_{k=1}^{n} w_k Y_t^k \tag{3.1}$$

where $\mu(t)$ is a bounded non-negative continuous and deterministic $\mathcal{L}^1$-function, $(w_k)_{k=1}^n$ is a sequence of non-negative deterministic weights, and the factor processes $Y^1, \ldots, Y^n$ constitute pure-jump zero-reverting Ornstein-Uhlenbeck (OU) processes satisfying stochastic differential equations (SDEs) of the form

$$\mathrm{d}Y_t^k = -\lambda_k Y_t^k \mathrm{d}t + \xi_k(t)\,\mathrm{d}L_t^k \tag{3.2}$$

with deterministic initial values $Y_0^k := y_k \geqslant 0$, constant mean-reversion velocities $\lambda_k > 0$, as well as bounded continuous and deterministic volatility functions $\xi_k(t) > 0$. Herein, the increasing Lévy-type processes $L^1, \ldots, L^n$ are such as specified in (2.2). We further assume that

$$\int_0^t \int_{\mathbb{R}^+} \xi_k(s)^2 z^2 e^{h_k(s,z)} \varepsilon_k(s)\,\mathrm{d}\nu_k(z)\,\mathrm{d}s < \infty$$

for all $t \geqslant 0$ and $k \in \{1, \ldots, n\}$. Then the solution of (3.2) is given by

$$Y_t^k = y_k e^{-\lambda_k t} + \int_0^t e^{-\lambda_k(t-s)} \xi_k(s)\,\mathrm{d}L_s^k = y_k e^{-\lambda_k t} + \int_0^t \int_{\mathbb{R}^+} e^{-\lambda_k(t-s)} \xi_k(s)\, z \mathrm{d}N_k(s,z) \tag{3.3}$$

where we used (2.2) for the second equality.

Remark 3.1 *Note that the processes (L_t^k), (Y_t^k) and (A_t) always jump simultaneously, while (Y_t^k) decays exponentially between its jumps due to the dampening linear drift term appearing in (3.2). Further recall that the processes (Y_t^k) are non-negative and thus, are zero-reverting from above. A typical trajectory of a Lévy-driven OU process is shown in Figure 15.1 in [8], whereas a typical trajectory of the process (A_t) defined in (3.1) is displayed in Figure 2 in [2]. The paths of (A_t) include both negative and positive fluctuations stemming from a combination of downward mean-reversion and upward jumps. Also note that the additive processes (L_t^k) constitute time-inhomogeneous subordinators. Since (L_t^k) is increasing and (Y_t^k) is zero-reverting from above, the function $\mu(t)$ constitutes the mean-reversion floor or lower bound of the process (A_t), i.e., it holds $A_t \geqslant \mu(t) \geqslant 0$ a.s. for all $t \geqslant 0$, while (A_t) is mean-reverting from above to $\mu(t)$. We further mention that the functions $\xi_k(t)$ control the seasonal variations of the jump amplitudes of (Y_t^k), whereas the functions $\varepsilon_k(t)$ control the seasonal variations of the jump intensities of (Y_t^k). We finally recall that a similar arithmetic multi-factor model as presented in (3.1)–(3.2) has previously been proposed in [2] in the context of modeling electricity spot prices.*

Further on, we define the $\mathcal{F}$-adapted and strictly positive stochastic process $H = (H_t)_{t \geqslant 0}$ via

$$H_t := \int_0^t A_s \mathrm{d}s$$

which is monotone increasing, since A is non-negative. It obviously holds

$$\mathrm{d}H_t = A_t \mathrm{d}t$$

for all $t \geqslant 0$. Having the definitions of A and H at hand, we propose to model the accumulated precipitation amount (R_t) during the time span $[0, t]$ for all $t \geqslant 0$ by the integral

$$R_t := \int_0^t \theta(s)\, \mathrm{d}H_s = \int_0^t \theta(s)\, A_s \mathrm{d}s \tag{3.4}$$

where $\theta : [0, \infty[\to [0, \infty[$ is the bounded deterministic and continuous function introduced in Section 2. Note that the process R claimed in (3.4) is $\mathcal{F}$-adapted, non-negative and monotone increasing. Thus, R may indeed serve as a precipitation model. In the next step, we substitute (3.1) and (3.3) into (3.4) and obtain

$$\begin{aligned} R_t = & \int_0^t \theta(s)\, \mu(s)\, \mathrm{d}s + \sum_{k=1}^n w_k y_k \int_0^t \theta(s)\, e^{-\lambda_k s} \mathrm{d}s \\ & + \sum_{k=1}^n w_k \int_0^t \int_0^s \int_{\mathbb{R}^+} e^{-\lambda_k (s-u)} \xi_k(u)\, z \mathrm{d}N_k(u, z)\, \theta(s)\, \mathrm{d}s \end{aligned}$$

for all $t \geqslant 0$. An application of Fubini's theorem yields the representation

$$\begin{aligned} R_t = & \int_0^t \theta(s)\, \mu(s)\, \mathrm{d}s + \sum_{k=1}^n w_k y_k \int_0^t \theta(s)\, e^{-\lambda_k s} \mathrm{d}s \\ & + \sum_{k=1}^n w_k \int_0^t \int_{\mathbb{R}^+} \int_u^t \theta(s)\, e^{-\lambda_k (s-u)} \mathrm{d}s \xi_k(u)\, z \mathrm{d}N_k(u, z) \end{aligned} \tag{3.5}$$

where the appearing ds-integrals can be computed further, as soon as the functions $\theta(\cdot)$ and $\mu(\cdot)$ have been determined concretely.

Remark 3.2 (multi-location case) *If we take $\mu(t) \equiv 0$, then we get*

$$R_t = \sum_{k=1}^n \Upsilon_t^k, \quad \Upsilon_t^k := w_k \int_0^t \theta(s)\, Y_s^k \mathrm{d}s,$$

where $\Upsilon^1, \ldots, \Upsilon^n$ are positive and monotone increasing stochastic processes, which can be interpreted as the respective precipitation amounts at n different locations, if desired. However, we stress that our original approach in (3.4) was designed to model the accumulated precipitation amount at a single location.

Under the modeling framework proposed in (3.4), the precipitation index $\mathfrak{I}(\tau_1, \tau_2)$ with measurement period $[\tau_1, \tau_2]$ can be expressed as

$$\mathfrak{I}(\tau_1, \tau_2) = R_{\tau_2} - R_{\tau_1} = \int_{\tau_1}^{\tau_2} \theta(s)\, A_s \mathrm{d}s = \int_{\tau_1}^{\tau_2} \theta(s)\, \mu(s)\, \mathrm{d}s + \sum_{k=1}^n w_k \int_{\tau_1}^{\tau_2} \theta(s)\, Y_s^k \mathrm{d}s \tag{3.6}$$

where we used (3.1) for the last equality. Taking (3.3) and Fubini's theorem into account, we further get

$$\int_{\tau_1}^{\tau_2} \theta(s)\, Y_s^k \mathrm{d}s = y_k \int_{\tau_1}^{\tau_2} \theta(s)\, e^{-\lambda_k s} \mathrm{d}s + \int_0^{\tau_2} \int_{\mathbb{R}^+} \left(\int_{u \vee \tau_1}^{\tau_2} \theta(s)\, e^{-\lambda_k (s-u)} \mathrm{d}s \right) \xi_k(u)\, z \mathrm{d}N_k(u, z)$$

where $u \vee \tau_1 := \max\{u, \tau_1\}$. Putting the latter equation into (3.6), we obtain

$$\mathfrak{I}(\tau_1,\tau_2) = c(\tau_1,\tau_2) + \sum_{k=1}^{n} \int_0^{\tau_2} \int_{\mathbb{R}^+} \vartheta_k(u,z,\tau_1,\tau_2)\,\mathrm{d}N_k(u,z) \tag{3.7}$$

with a non-negative constant

$$c(\tau_1,\tau_2) := \int_{\tau_1}^{\tau_2} \theta(s)\,\mu(s)\,\mathrm{d}s + \sum_{k=1}^{n} w_k y_k \int_{\tau_1}^{\tau_2} \theta(s)\,e^{-\lambda_k s}\mathrm{d}s \tag{3.8}$$

and non-negative deterministic functions

$$\vartheta_k(u,z,\tau_1,\tau_2) := w_k \left(\int_{u\vee\tau_1}^{\tau_2} \theta(s)\,e^{-\lambda_k(s-u)}\mathrm{d}s \right) \xi_k(u)\,z \tag{3.9}$$

where $k \in \{1,\ldots,n\}$. Note in passing that the precipitation index $\mathfrak{I}(\tau_1,\tau_2)$ given in (3.7) indeed is non-negative. In the current multi-factor setup, we get the subsequent result.

Proposition 3.3 (precipitation swap price) *For all $0 \leqslant t \leqslant \tau_1 < \tau_2$ the precipitation swap price $F_t(\tau_1,\tau_2)$ defined in (2.4) satisfies the non-negative $(\mathcal{F},\mathbb{Q})$-martingale representation*

$$F_t(\tau_1,\tau_2) = F_0(\tau_1,\tau_2) + \sum_{k=1}^{n} \int_0^{t} \int_{\mathbb{R}^+} \vartheta_k(u,z,\tau_1,\tau_2)\,\mathrm{d}\tilde{N}_k^{\mathbb{Q}}(u,z) \tag{3.10}$$

with non-negative and deterministic initial value

$$F_0(\tau_1,\tau_2) := c(\tau_1,\tau_2) + \sum_{k=1}^{n} \int_0^{\tau_2} \int_{\mathbb{R}^+} \vartheta_k(u,z,\tau_1,\tau_2)\,e^{h_k(u,z)}\varepsilon_k(u)\,\mathrm{d}\nu_k(z)\,\mathrm{d}u. \tag{3.11}$$

Here, the objects $c(\tau_1,\tau_2)$ and $\vartheta_k(u,z,\tau_1,\tau_2)$ are such as defined in (3.8) and (3.9), respectively.

Proof Merging (3.7) into (2.4) yields

$$F_t(\tau_1,\tau_2) = c(\tau_1,\tau_2) + \sum_{k=1}^{n} \mathbb{E}_{\mathbb{Q}} \left(\int_0^{\tau_2} \int_{\mathbb{R}^+} \vartheta_k(u,z,\tau_1,\tau_2)\,\mathrm{d}N_k(u,z)\,|\mathcal{F}_t \right)$$

where the appearing conditional expectations can be computed to

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}} \left(\int_0^{\tau_2} \int_{\mathbb{R}^+} \vartheta_k(u,z,\tau_1,\tau_2)\,\mathrm{d}N_k(u,z)\,|\mathcal{F}_t \right) \\ &= \int_0^{t} \int_{\mathbb{R}^+} \vartheta_k(u,z,\tau_1,\tau_2)\,\mathrm{d}\tilde{N}_k^{\mathbb{Q}}(u,z) + \int_0^{\tau_2} \int_{\mathbb{R}^+} \vartheta_k(u,z,\tau_1,\tau_2)\,e^{h_k(u,z)}\varepsilon_k(u)\,\mathrm{d}\nu_k(z)\,\mathrm{d}u \end{aligned}$$

due to (2.7) and the $(\mathcal{F},\mathbb{Q})$-martingale property of the $\mathrm{d}\tilde{N}_k^{\mathbb{Q}}$-integrals. Hence, the assertion follows. $\square$

We emphasize that – in contrast to the deterministic precipitation swap price formula claimed in (2.8) – the recently derived price representation in (3.10) is not constant over time, but instead follows a stochastic dynamics varying randomly as the trading time goes by. This fact constitutes the major advantage of our model over the corresponding result provided in Eq. (8.31) in [4].

4. Option pricing

The current section is devoted to the pricing of options written on the precipitation swap price process $(F_t(\tau_1,\tau_2))_{t\in[0,\tau_1]}$ with $\tau_1<\tau_2$. More precisely, for all $0\leqslant t\leqslant T\leqslant\tau_1<\tau_2$ we consider a European option with maturity time T and arbitrary (deterministic) payoff function $q:\mathbb{R}^+\to\mathbb{R}$ associated with the option price process $P=(P_t)_{t\in[0,T]}$ defined via

$$P_t := e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left(q\left(F_T\right)|\mathcal{F}_t\right) \tag{4.1}$$

where $r\geqslant 0$ is the constant interest rate, $\mathbb{Q}$ indicates the risk-neutral pricing measure introduced in Section 2, and F denotes the precipitation swap price process given in (3.10). We recall that T is the maturity time of the option, whereas τ_2 denotes the maturity time of the underlying precipitation swap. Hence, from an economical perspective, it makes sense to assume that $T\leqslant\tau_2$. In addition, in order to ensure that $t\leqslant T$ for all $t\in[0,\tau_1]$, we assume in the context of (4.1) that it holds $T\leqslant\tau_1<\tau_2$. Furthermore, in (4.1) we omitted the measurement period arguments τ_1 and τ_2 of the underlying precipitation swap F in order to keep the formula more compact. Since $q(\cdot)$ might not be an $\mathcal{L}^1$-function, we introduce for all $x\in\mathbb{R}^+$ the dampened payoff function

$$f(x) := e^{-ax}q(x) \tag{4.2}$$

with real-valued dampening parameter $a>0$ which is taken such that $f(\cdot)\in\mathcal{L}^1(\mathbb{R}^+)$. (If $q(\cdot)$ already is an $\mathcal{L}^1$-function, we reasonably take $a=0$.) As a consequence, the Fourier transform and inverse Fourier transform of $f(\cdot)$ exist and are given by, respectively,

$$\hat{f}(y) := \frac{1}{2\pi}\int_{\mathbb{R}^+} f(x)\,e^{-iyx}\mathrm{d}x,\quad f(x)=\int_{\mathbb{R}^+}\hat{f}(y)\,e^{iyx}\mathrm{d}y \tag{4.3}$$

where $i^2=-1$. In the present setup, we get the following result.

Proposition 4.1 (option price formula) *Let $F_\cdot(\tau_1,\tau_2)$ be the precipitation swap price process given in (3.10) and suppose that $f\in\mathcal{L}^1(\mathbb{R}^+)$ is the dampened payoff function defined in (4.2). Then, the option price P defined in (4.1) can for all $0\leqslant t\leqslant T\leqslant\tau_1<\tau_2$ be represented as*

$$P_t = e^{-r(T-t)}\int_{\mathbb{R}^+}\hat{f}(y)\exp\left\{(a+iy)\,F_t(\tau_1,\tau_2)+\phi(a,y,t,T,\tau_1,\tau_2)\right\}\mathrm{d}y \tag{4.4}$$

where the Fourier transform $\hat{f}(y)$ is such as defined in (4.3), and

$$\begin{aligned}\phi(a,y,t,T,\tau_1,\tau_2) := \sum_{k=1}^{n}\int_t^T\int_{\mathbb{R}^+}\Big[&e^{(a+iy)\vartheta_k(s,z,\tau_1,\tau_2)}-1\\ &-(a+iy)\,\vartheta_k(s,z,\tau_1,\tau_2)\Big]e^{h_k(s,z)}\varepsilon_k(s)\,\mathrm{d}\nu_k(z)\,\mathrm{d}s\end{aligned} \tag{4.5}$$

constitutes a complex-valued deterministic function.

Proof A combination of (4.1) and (4.2) yields

$$P_t = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left(e^{aF_T}f(F_T)\,|\mathcal{F}_t\right)$$

where $0\leqslant t\leqslant T$. By an application of the inverse Fourier transform, we get

$$P_t = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left(\int_{\mathbb{R}^+}\hat{f}(y)\,e^{(a+iy)F_T}\mathrm{d}y|\mathcal{F}_t\right)$$

where $\hat{f}(y)$ is such as defined in (4.3). In the next step, we use Fubini's theorem and obtain

$$P_t = e^{-r(T-t)} \int_{\mathbb{R}^+} \hat{f}(y)\, \mathbb{E}_{\mathbb{Q}} \left(e^{(a+iy)F_T} | \mathcal{F}_t \right) \mathrm{d}y$$

for all $0 \leqslant t \leqslant T$. What remains is the computation of the appearing conditional expectation. To this end, we take the independent increment property of the process F with respect to $\mathcal{F}$ and $\mathbb{Q}$ into account and infer

$$\mathbb{E}_{\mathbb{Q}} \left(e^{(a+iy)F_T} | \mathcal{F}_t \right) = e^{(a+iy)F_t} \mathbb{E}_{\mathbb{Q}} \left(e^{(a+iy)(F_T - F_t)} | \mathcal{F}_t \right) = e^{(a+iy)F_t} \mathbb{E}_{\mathbb{Q}} \left[e^{(a+iy)(F_T - F_t)} \right]$$

for all $0 \leqslant t \leqslant T$. From (3.10) we further deduce

$$\mathbb{E}_{\mathbb{Q}} \left(e^{(a+iy)F_T} | \mathcal{F}_t \right) = e^{(a+iy)F_t} \mathbb{E}_{\mathbb{Q}} \left[\exp \left\{ \sum_{k=1}^{n} \int_t^T \int_{\mathbb{R}^+} (a+iy)\, \vartheta_k (s, z, \tau_1, \tau_2)\, \mathrm{d}\tilde{N}_k^{\mathbb{Q}} (s, z) \right\} \right].$$

The resulting usual expectation can be computed by the Lévy-Khintchine formula for additive processes (cf., e.g., [8, 10, 17, 24]) which leads us to

$$\mathbb{E}_{\mathbb{Q}} \left(e^{(a+iy)F_T} | \mathcal{F}_t \right) = e^{(a+iy)F_t + \phi(a,y,t,T,\tau_1,\tau_2)} \tag{4.6}$$

where the deterministic function $\phi(a, y, t, T, \tau_1, \tau_2)$ is such as defined in (4.5). Hence, the proof is complete. □

To get more information on option pricing techniques with inverse Fourier transforms, we refer to [7, 12, 13], Section 8.2.2 in [4], and Section 11.1.3 in [8]. In the sequel, we apply the option price formula claimed in Proposition 4.1 to several practical examples.

Example 4.2 (call option) *For all $x \in \mathbb{R}^+$ we consider the call option payoff function*

$$q(x) := [x - K]^+$$

where $\kappa^+ := \max\{0, \kappa\}$ indicates the positive part of any real number κ, and $K > 0$ denotes the constant strike price. Since $q(\cdot) \notin \mathcal{L}^1(\mathbb{R}^+)$, we introduce the dampened payoff function

$$f(x) := e^{-ax}[x - K]^+ \in \mathcal{L}^1\left(\mathbb{R}^+\right)$$

[recall (4.2)] with constant dampening parameter $a > 0$. Hence, the price of a call option (written on the precipitation swap F) is given by (4.4), but now with Fourier transform

$$\hat{f}(y) = \frac{e^{-(a+iy)K}}{2\pi(a+iy)^2}.$$

Example 4.3 (put option) *For all $x \in \mathbb{R}^+$ we consider the put option payoff function*

$$q(x) := [K - x]^+$$

where $K > 0$ denotes the constant strike price. Since $q(\cdot) \in \mathcal{L}^1(\mathbb{R}^+)$, we may take $a = 0$ in (4.2) which implies $f(x) = q(x)$ for all $x \in \mathbb{R}^+$. Hence, the price of a put option (written on the precipitation swap F) is given by (4.4), but now with $a = 0$ therein and Fourier transform

$$\hat{f}(y) = \frac{1 - iyK - e^{-iyK}}{2\pi y^2}.$$

Example 4.4 (power option) *For all $x \in \mathbb{R}^+$ we consider the payoff function*

$$q(x) := x^m$$

with fixed power $m \in \mathbb{N}$. Since $q(\cdot) \notin \mathcal{L}^1(\mathbb{R}^+)$, we introduce the dampened payoff function

$$f(x) := e^{-ax} x^m \in \mathcal{L}^1\left(\mathbb{R}^+\right)$$

[recall (4.2)] with constant dampening parameter $a > 0$. Hence, the price of a power option (written on the precipitation swap F) is given by (4.4), but now with Fourier transform

$$\hat{f}(y) = (2\pi)^{-1} \int_0^\infty x^m e^{-(a+iy)x} \mathrm{d}x.$$

For instance, if we take $m = 2$, then we find

$$\hat{f}(y) = \frac{(a+iy)^{-3}}{\pi}.$$

Example 4.5 (digital option) *For all $x \in \mathbb{R}^+$ we consider the payoff function*

$$q(x) := \mathbb{1}_{[\delta_1, \delta_2]}(x)$$

where $\mathbb{1}$ denotes the indicator function and $0 \leqslant \delta_1 < \delta_2 < \infty$ are fixed constants. Since $q(\cdot) \in \mathcal{L}^1(\mathbb{R}^+)$, we may take $a = 0$ in (4.2) which implies $f(x) = q(x)$ for all $x \in \mathbb{R}^+$. Hence, the price of a digital option (written on the precipitation swap F) is given by (4.4), but now with $a = 0$ therein and Fourier transform

$$\hat{f}(y) = \frac{e^{-iy\delta_1} - e^{-iy\delta_2}}{2\pi i y}.$$

Remark 4.6 *Using similar methods as in the proof of Proposition 4.1, it is straightforward to compute the price Π of an option with exercise time τ_2 written on the precipitation index $\mathfrak{I}(\tau_1, \tau_2)$ satisfying (3.7). In accordance to Eq. (8.22) in [4], at any time $0 \leqslant t \leqslant \tau_1 < \tau_2$ we have*

$$\Pi_t = e^{-r(\tau_2 - t)} \mathbb{E}_{\mathbb{Q}}\left(\rho\left(\mathfrak{I}(\tau_1, \tau_2)\right) | \mathcal{F}_t\right)$$

where $\rho : \mathbb{R}^+ \to \mathbb{R}$ is an arbitrary payoff function.

In what follows, we derive a partial integro-differential equation (PIDE) associated with the option price P defined in (4.1). Suppose that the precipitation swap price F satisfies (3.10), and recall that for all $0 \leqslant t \leqslant T \leqslant \tau_1 < \tau_2$ it holds

$$\mathbb{E}_{\mathbb{Q}}\left(q(F_T) | \mathcal{F}_t\right) = \mathbb{E}_{\mathbb{Q}}\left(q(F_T) | F_t\right) \tag{4.7}$$

due to the time-inhomogeneous Markov process property of $(F_t)_{t \in [0,T]}$ with respect to $\mathcal{F}$ and $\mathbb{Q}$. A multiplication of (4.7) with the discount factor $e^{-r(T-t)}$ leads us to

$$P_t = M(t, F_t) \tag{4.8}$$

with a Markov function

$$M : [0, T] \times \mathbb{R}^+ \to \mathbb{R}, \quad M \in \mathcal{C}^{1,2}\left([0, T] \times \mathbb{R}^+\right)$$

defined via

$$M(t, F_t) := e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}\left(q(F_T) | F_t\right) \tag{4.9}$$

for all $0 \leqslant t \leqslant T \leqslant \tau_1 < \tau_2$. In accordance to no-arbitrage theory, the discounted option price process $\left(\hat{P}_t := e^{-rt} P_t\right)$ must form an $\mathcal{F}$-adapted martingale under $\mathbb{Q}$. Using integration by parts, Itô's formula, as well as (4.8), (3.10), and (2.7), we get the SDE

$$\mathrm{d}\hat{P}_t = e^{-rt}\left(D_t\mathrm{d}t + \sum_{k=1}^{n}\int_{\mathbb{R}^+}\left[M\left(t, F_{t-} + \vartheta_k\left(t,z\right)\right) - M\left(t, F_{t-}\right)\right]\mathrm{d}\tilde{N}_k^{\mathbb{Q}}\left(t,z\right)\right) \tag{4.10}$$

with stochastic drift process

$$\begin{aligned} D_t &:= \partial_t M\left(t, F_t\right) - rM\left(t, F_t\right) \\ &+ \sum_{k=1}^{n}\int_{\mathbb{R}^+}\left[M\left(t, F_t + \vartheta_k\left(t,z\right)\right) - M\left(t, F_t\right) - \vartheta_k\left(t,z\right)\partial_x M\left(t, F_t\right)\right]e^{h_k(t,z)}\varepsilon_k\left(t\right)\mathrm{d}\nu_k\left(z\right) \end{aligned}$$

where $\partial_t M\left(t,x\right)$ and $\partial_x M\left(t,x\right)$ denote the partial derivatives of M with respect to the first and second argument, respectively, and $\vartheta_k\left(t,z\right) := \vartheta_k\left(t,z,\tau_1,\tau_2\right)$ is the deterministic function defined in (3.9).

Proposition 4.7 (PIDE) *For all $k \in \{1,\ldots,n\}$ let $\vartheta_k\left(t,z\right) := \vartheta_k\left(t,z,\tau_1,\tau_2\right)$ be the deterministic function defined in (3.9). Denote the constant interest rate by $r \geqslant 0$, and let $q\left(\cdot\right)$ be the deterministic payoff function introduced in (4.1). Then, for all $(t,x) \in [0,T]\times\mathbb{R}^+$ the Markov function M defined in (4.9) satisfies the PIDE*

$$\begin{aligned} rM\left(t,x\right) &= \partial_t M\left(t,x\right) \\ &+ \sum_{k=1}^{n}\int_{\mathbb{R}^+}\left[M\left(t, x + \vartheta_k\left(t,z\right)\right) - M\left(t,x\right) - \vartheta_k\left(t,z\right)\partial_x M\left(t,x\right)\right]e^{h_k(t,z)}\varepsilon_k\left(t\right)\mathrm{d}\nu_k\left(z\right) \end{aligned}$$

with terminal condition $M\left(T,x\right) = q\left(x\right)$.

Proof As the discounted option price process $\hat{P}$ must form an $(\mathcal{F},\mathbb{Q})$-martingale, we set the drift process D appearing in (4.10) equal to zero and immediately obtain the assertion. □

5. Minimal variance hedging

The current section is dedicated to minimal variance hedging in a weather market wherein the traders can invest into the precipitation swap F with stochastic price dynamics given in (3.10). For notational convenience, we will often omit the fixed measurement period coefficients τ_1 and τ_2 in the sequel and simply write F_t instead of $F_t\left(\tau_1,\tau_2\right)$, similarly to Section 4. Under these conventions, we assume that the wealth process $X = (X_t)_{t\in[0,\tau_1]}$ satisfies

$$X_t = \varphi_t F_t \tag{5.1}$$

where (φ_t) is a predictable $\mathcal{F}$-adapted càdlàg stochastic portfolio process which denotes the amount invested at time t into the precipitation swap F. In other words, (X_t) describes the wealth at time t of a trader using the portfolio (φ_t). As X explicitly depends on the chosen portfolio φ, it appears reasonable to write $X = X^{(\varphi)}$ in order to emphasize that X is controlled by the portfolio process φ. We say that a portfolio φ is self-financing, if and only if the corresponding wealth $X = X^{(\varphi)}$ satisfies the SDE

$$\mathrm{d}X_t = \varphi_{t-}\mathrm{d}F_t \tag{5.2}$$

with deterministic initial wealth $X_0 = x_0 > 0$ (cf. Eq. (4.17) in [10], Section 2.4.2 in [17], Section 3.1 in [23]). Let us further denote the set of all admissible (i.e., predictable $\mathcal{F}$-adapted càdlàg and integrable) stochastic portfolio processes φ by $\mathcal{A}$. Then the minimal variance hedging problem can be formulated as follows: Given a τ_1-claim $C \in \mathcal{L}^2\left(\mathcal{F}_{\tau_1},\mathbb{Q}\right)$ (i.e., C constitutes an

$\mathcal{F}_{\tau_1}$-measurable square-integrable random variable under $\mathbb{Q}$), find the optimal portfolio $\hat{\varphi}$ in $\mathcal{A}$ which minimizes the objective functional

$$J(\varphi) := \mathbb{E}_{\mathbb{Q}}\left[(X_{\tau_1} - x_0 - C)^2\right] \tag{5.3}$$

where $X_{\tau_1} = X_{\tau_1}^{(\varphi)}$ denotes the terminal total wealth (cf. Eq. (12.29) in [10], Problem 3.1 in [5], Eq. (10.26) in [8], Eq. (6.2) in [23]). Hence, we consider the optimization problem

$$J(\hat{\varphi}) = \inf_{\varphi \in \mathcal{A}} J(\varphi) \tag{5.4}$$

where $J(\varphi)$ defined in (5.3) denotes the variance of the difference between the claim C and the wealth increment $X_{\tau_1} - x_0$ generated by trading with a portfolio $\varphi \in \mathcal{A}$ during the time interval $[0, \tau_1]$. Recall that the claim C is called hedgeable/replicable (in the minimal variance sense), if and only if there exists a unique optimal portfolio $\hat{\varphi} \in \mathcal{A}$ which fulfills

$$\hat{X}_{\tau_1} - x_0 = C \tag{5.5}$$

$\mathbb{Q}$-a.s. In this case, it holds $J(\hat{\varphi}) = 0$. Here, $\hat{X} = X^{(\hat{\varphi})}$ denotes the particular wealth process associated with the optimal portfolio $\hat{\varphi}$. If every claim is hedgeable, then the market is called complete, and incomplete otherwise (cf. Section 9.2 in [8], Section 2.1.5 in [17], Sections 1.1.2 and 3.2 in [23]). Integrating the SDE (5.2) from 0 to τ_1 yields

$$X_{\tau_1} = x_0 + \sum_{k=1}^{n} \int_0^{\tau_1} \int_{\mathbb{R}^+} \varphi_{t-} \vartheta_k(t, z, \tau_1, \tau_2)\, \mathrm{d}\tilde{N}_k^{\mathbb{Q}}(t, z) \tag{5.6}$$

where we used (3.10). Note in passing that (5.6) implies

$$x_0 = \mathbb{E}_{\mathbb{Q}}[X_{\tau_1}]. \tag{5.7}$$

Thus, by taking $\mathbb{Q}$-expectations in (5.5), we obtain $\mathbb{E}_{\mathbb{Q}}[C] = 0$ due to (5.7) [also recall (5.16) below]. From this observation we conclude that a necessary presumption for C to be hedgeable in the minimal variance sense is that it holds $\mathbb{E}_{\mathbb{Q}}[C] = 0$. In other words, if $\mathbb{E}_{\mathbb{Q}}[C] \neq 0$, then C cannot be (entirely) hedgeable in the minimal variance sense. This criterion is very useful to decide whether or not a given claim is hedgeable (i.e., fully replicable) in the minimal variance sense.

In the next step, we recall a well-known result from Malliavin calculus, namely the Clark-Ocone formula, which will be applied to the minimal variance hedging problem (5.3)–(5.4) later. For this purpose, we introduce the Malliavin derivative operator $\mathcal{D}_{k,t,z}(\cdot)$ associated with the PRM $\mathrm{d}\tilde{N}_k^{\mathbb{Q}}(t, z)$ defined in (2.7). More precisely, for all $k \in \{1, \ldots, n\}$ and $(t, z) \in [0, \tau_1] \times \mathbb{R}^+$ we assume that $\mathcal{D}_{k,t,z}(\cdot)$ is defined as in Sections 12 and 13 in [10], i.e., we particularly understand $\mathcal{D}_{k,t,z}(\cdot)$ as the Malliavin derivative operator extended from the space $\mathbb{D}_{1,2}$ to $\mathcal{L}^2(\mathcal{F}_{\tau_1}, \mathbb{Q})$. Under these presumptions, the following version of the Clark-Ocone formula holds.

Theorem 5.1 (Clark-Ocone formula; multi-factor version) *In the mathematical framework introduced in Section 2, any claim $C \in \mathcal{L}^2(\mathcal{F}_{\tau_1}, \mathbb{Q})$ can be represented as*

$$C = \mathbb{E}_{\mathbb{Q}}[C] + \sum_{k=1}^{n} \int_0^{\tau_1} \int_{\mathbb{R}^+} \mathbb{E}_{\mathbb{Q}}(\mathcal{D}_{k,t,z}(C) \,|\, \mathcal{F}_t)\, \mathrm{d}\tilde{N}_k^{\mathbb{Q}}(t, z)$$

where $\tilde{N}_1^{\mathbb{Q}}, \ldots, \tilde{N}_n^{\mathbb{Q}}$ are the $\mathbb{Q}$-compensated PRMs defined in (2.7). In the latter equation, the integrands $\mathbb{E}_{\mathbb{Q}}(\mathcal{D}_{k,t,z}(C) \,|\, \mathcal{F}_t)$ are for every $k \in \{1, \ldots, n\}$ understood as the corresponding predictable versions.

Proof See Theorems 12.20 and 13.28 in [10]. □

Evidently, the Clark-Ocone formula can be interpreted as an extension of Itô's representation theorem provided in Theorem 9.10 in [10]. To read more on Malliavin calculus and its applications, we refer – besides [10] – to [19] and [21]. Let us now return to the minimal variance hedging problem (5.3)–(5.4).

Proposition 5.2 (optimal portfolio) *For any τ_1-claim $C \in \mathcal{L}^2(\mathcal{F}_{\tau_1}, \mathbb{Q})$ the minimal variance hedging portfolio $\hat{\varphi} = (\hat{\varphi}_t)_{t\in[0,\tau_1]}$ related to the optimization problem (5.3)–(5.4) is given by*

$$\hat{\varphi}_t = \frac{\sum_{k=1}^{n} \int_{\mathbb{R}^+} \mathbb{E}_{\mathbb{Q}}\left(\mathcal{D}_{k,t,z}(C) \,|\mathcal{F}_t\right) \vartheta_k(t,z,\tau_1,\tau_2)\, e^{h_k(t,z)} \varepsilon_k(t)\, \mathrm{d}\nu_k(z)}{\sum_{k=1}^{n} \int_{\mathbb{R}^+} \vartheta_k(t,z,\tau_1,\tau_2)^2 e^{h_k(t,z)} \varepsilon_k(t)\, \mathrm{d}\nu_k(z)} \tag{5.8}$$

where $\mathcal{D}_{k,t,z}(\cdot)$ denotes the Malliavin derivative operator with respect to $\mathrm{d}\tilde{N}_k^{\mathbb{Q}}(t,z)$, and $\vartheta_k(t,z,\tau_1,\tau_2)$ is the deterministic function defined in (3.9).

Proof Suppose that $\hat{\varphi} = (\hat{\varphi}_t)_{t\in[0,\tau_1]}$ is such as given in (5.8). Therewith, we define the claim

$$\hat{C} := \mathbb{E}_{\mathbb{Q}}[C] + \sum_{k=1}^{n} \int_0^{\tau_1} \int_{\mathbb{R}^+} \hat{\varphi}_{t-} \vartheta_k(t,z,\tau_1,\tau_2)\, \mathrm{d}\tilde{N}_k^{\mathbb{Q}}(t,z). \tag{5.9}$$

Similarly to the proof of Theorem 12.24 in [10] (also recall Eq. (3.11) in [5]), in order to prove the optimality of $\hat{\varphi}$ for problem (5.3)–(5.4), it suffices to show that it holds

$$\mathbb{E}_{\mathbb{Q}}\left[\left(C - \hat{C}\right) \Gamma\right] = 0 \tag{5.10}$$

for all claims $\Gamma \in \mathcal{L}^2(\mathcal{F}_{\tau_1}, \mathbb{Q})$ which are of the particular form

$$\Gamma := \mathbb{E}_{\mathbb{Q}}[\Gamma] + \sum_{k=1}^{n} \int_0^{\tau_1} \int_{\mathbb{R}^+} \psi_{t-} \vartheta_k(t,z,\tau_1,\tau_2)\, \mathrm{d}\tilde{N}_k^{\mathbb{Q}}(t,z) \tag{5.11}$$

where the involved stochastic process $\psi = (\psi_t)_{t\in[0,\tau_1]}$ constitutes the $\mathcal{F}$-adapted self-financing portfolio associated with Γ. Equation (5.10) constitutes the so-called "orthogonality condition" (cf. [5, 10]). Using Theorem 5.1 as well as (5.9), we get

$$\mathbb{E}_{\mathbb{Q}}\left[\left(C - \hat{C}\right) \Gamma\right] = \mathbb{E}_{\mathbb{Q}}\left[\left(\sum_{k=1}^{n} \int_0^{\tau_1} \int_{\mathbb{R}^+} G_{t-}^k(z)\, \mathrm{d}\tilde{N}_k^{\mathbb{Q}}(t,z)\right) \Gamma\right] \tag{5.12}$$

wherein for all $k \in \{1,\ldots,n\}$ we introduced the $\mathcal{F}$-adapted stochastic processes

$$G_t^k(z) := G_t^k(z,\tau_1,\tau_2) := \mathbb{E}_{\mathbb{Q}}\left(\mathcal{D}_{k,t,z}(C) \,|\mathcal{F}_t\right) - \hat{\varphi}_t \vartheta_k(t,z,\tau_1,\tau_2) \tag{5.13}$$

for notational convenience. Next, we insert (5.11) into (5.12) and obtain

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}}\left[\left(C - \hat{C}\right) \Gamma\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[\left(\sum_{k=1}^{n} \int_0^{\tau_1} \int_{\mathbb{R}^+} G_{t-}^k(z)\, \mathrm{d}\tilde{N}_k^{\mathbb{Q}}(t,z)\right) \left(\sum_{k=1}^{n} \int_0^{\tau_1} \int_{\mathbb{R}^+} \psi_{t-} \vartheta_k(t,z,\tau_1,\tau_2)\, \mathrm{d}\tilde{N}_k^{\mathbb{Q}}(t,z)\right)\right] \end{aligned}$$

while Itô's isometry yields

$$\mathbb{E}_{\mathbb{Q}}\left[\left(C-\hat{C}\right)\Gamma\right]=\mathbb{E}_{\mathbb{Q}}\left[\int_0^{\tau_1}\psi_t\Lambda_t\left(\tau_1,\tau_2\right)\mathrm{d}t\right]$$

with an $\mathcal{F}$-adapted stochastic process

$$\Lambda_t\left(\tau_1,\tau_2\right):=\sum_{k=1}^{n}\int_{\mathbb{R}^+}G_t^k\left(z\right)\vartheta_k\left(t,z,\tau_1,\tau_2\right)e^{h_k(t,z)}\varepsilon_k\left(t\right)\mathrm{d}\nu_k\left(z\right). \tag{5.14}$$

Merging (5.13) and (5.8) into (5.14), we find

$$\Lambda_t\left(\tau_1,\tau_2\right)\equiv 0 \tag{5.15}$$

$\mathbb{Q}$-a.s. for all $t\in[0,\tau_1]$, which in turn implies

$$\mathbb{E}_{\mathbb{Q}}\left[\left(C-\hat{C}\right)\Gamma\right]=0.$$

Hence, in accordance to (5.10), we conclude that the optimal portfolio process claimed in (5.8) indeed constitutes the minimizer of the minimal variance hedging problem (5.3)–(5.4). The explicit representation provided in (5.8) can be inferred from (5.13), (5.14), and (5.15) by a straightforward algebraic transformation. □

Example 5.3 *As an introductory example, let us consider the claim*

$$C:=\sum_{k=1}^{n}\int_0^{\tau_1}\int_{\mathbb{R}^+}\eta_k\left(s,z\right)\mathrm{d}\tilde{N}_k^{\mathbb{Q}}\left(s,z\right)$$

where $\eta_k:[0,\tau_1]\times\mathbb{R}^+\to\mathbb{R}$ are square-integrable deterministic functions for all $k\in\{1,\ldots,n\}$. From Example 12.3 in [10] we deduce that for all $k\in\{1,\ldots,n\}$ and $(t,z)\in[0,\tau_1]\times\mathbb{R}^+$ the Malliavin derivative of C reads as $\mathcal{D}_{k,t,z}\left(C\right)=\eta_k\left(t,z\right)$ which is deterministic and thus implies $\mathbb{E}_{\mathbb{Q}}\left(\mathcal{D}_{k,t,z}\left(C\right)|\mathcal{F}_t\right)=\eta_k\left(t,z\right)$. Hence, the minimal variance hedging portfolio associated with the claim C is given by (5.8), but now with $\mathbb{E}_{\mathbb{Q}}\left(\mathcal{D}_{k,t,z}\left(C\right)|\mathcal{F}_t\right)$ therein replaced by $\eta_k\left(t,z\right)$.

Example 5.4 *Consider the claim*

$$C:=F_{\tau_1}\left(\tau_1,\tau_2\right)$$

where the precipitation swap price process F satisfies (3.10). Then, for all $k\in\{1,\ldots,n\}$ and $(t,z)\in[0,\tau_1]\times\mathbb{R}^+$ we obtain the deterministic Malliavin derivative $\mathcal{D}_{k,t,z}\left(C\right)=\vartheta_k\left(t,z,\tau_1,\tau_2\right)$ which implies $\mathbb{E}_{\mathbb{Q}}\left(\mathcal{D}_{k,t,z}\left(C\right)|\mathcal{F}_t\right)=\vartheta_k\left(t,z,\tau_1,\tau_2\right)$. Thus, the minimal variance hedging portfolio given in (5.8) presently simplifies to $\hat{\varphi}_t\equiv 1$ for all $t\in[0,\tau_1]$. We further recall that any claim C is hedgeable in the minimal variance sense, if and only if there exists a unique optimal portfolio $\hat{\varphi}\in\mathcal{A}$ which fulfills

$$C=\hat{X}_{\tau_1}-x_0=\sum_{k=1}^{n}\int_0^{\tau_1}\int_{\mathbb{R}^+}\hat{\varphi}_{t-}\vartheta_k\left(t,z,\tau_1,\tau_2\right)\mathrm{d}\tilde{N}_k^{\mathbb{Q}}\left(t,z\right) \tag{5.16}$$

being a consequence of (5.5) and (5.6). Due to (3.10), we observe in the current example

$$\hat{X}_{\tau_1}-x_0=F_{\tau_1}\left(\tau_1,\tau_2\right)-F_0\left(\tau_1,\tau_2\right)$$

which is different from C, unless $F_0\left(\tau_1,\tau_2\right)\equiv 0$. With view on the definition of $F_0\left(\tau_1,\tau_2\right)$ claimed in (3.11), we state that it holds $F_0\left(\tau_1,\tau_2\right)>0$ for every reasonable choice of the model parameters. From this observation we conclude that the presently considered claim is non-

hedgeable in the minimal variance sense, while the derived optimal portfolio $\hat{\varphi}_t \equiv 1$ merely constitutes the closest hedge of C in terms of minimal variance.

Example 5.5 *Consider the claim*

$$C := \int_0^{\tau_1} \chi(s)\,\mathrm{d}F_s(\tau_1,\tau_2) = \sum_{k=1}^n \int_0^{\tau_1} \int_{\mathbb{R}^+} \chi(s)\,\vartheta_k(s,z,\tau_1,\tau_2)\,\mathrm{d}\tilde{N}_k^{\mathbb{Q}}(s,z)$$

where F satisfies (3.10), and $\chi(\cdot)$ is an arbitrary deterministic function of time. Then, for all $k \in \{1,\dots,n\}$ and $(t,z) \in [0,\tau_1] \times \mathbb{R}^+$ we obtain

$$\mathbb{E}_{\mathbb{Q}}(\mathcal{D}_{k,t,z}(C)\,|\mathcal{F}_t) = \chi(t)\,\vartheta_k(t,z,\tau_1,\tau_2).$$

Thus, the minimal variance hedging portfolio provided in (5.8) presently simplifies to $\hat{\varphi}_t = \chi(t)$ for all $t \in [0,\tau_1]$. In the current example, it holds $\hat{X}_{\tau_1} - x_0 = C$ such that the considered claim is entirely hedgeable in the minimal variance sense.

Example 5.6 *Consider the claim*

$$C := \int_0^{\tau_1} F_s(\tau_1,\tau_2)\,\mathrm{d}s$$

where F satisfies (3.10). Taking (3.10) and Fubini's theorem into account, we obtain

$$C = \tau_1 F_0(\tau_1,\tau_2) + \sum_{k=1}^n \int_0^{\tau_1} \int_{\mathbb{R}^+} (\tau_1 - u)\,\vartheta_k(u,z,\tau_1,\tau_2)\,\mathrm{d}\tilde{N}_k^{\mathbb{Q}}(u,z)$$

which implies

$$\mathbb{E}_{\mathbb{Q}}(\mathcal{D}_{k,t,z}(C)\,|\mathcal{F}_t) = (\tau_1 - t)\,\vartheta_k(t,z,\tau_1,\tau_2)$$

for all $(k,t,z) \in \{1,\dots,n\} \times [0,\tau_1] \times \mathbb{R}^+$. As a consequence, the minimal variance hedging portfolio provided in (5.8) presently simplifies to a linear function reading $\hat{\varphi}_t = \tau_1 - t$ for all $t \in [0,\tau_1]$. With respect to (5.16), we observe

$$\hat{X}_{\tau_1} - x_0 = C - \tau_1 F_0(\tau_1,\tau_2).$$

From this equation we conclude that the currently investigated claim C is non-hedgeable in the minimal variance sense, unless it either holds $\tau_1 = 0$ or $F_0(\tau_1,\tau_2) = 0$ (both of which constitute unrealistic scenarios). Hence, the optimal portfolio $\hat{\varphi}_t = \tau_1 - t$ merely constitutes the closest hedge of C in terms of minimal variance.

Example 5.7 *Consider the claim*

$$C := (F_{\tau_1}(\tau_1,\tau_2))^2$$

where F satisfies (3.10). Then, for all $k \in \{1,\dots,n\}$ and $(t,z) \in [0,\tau_1] \times \mathbb{R}^+$ we obtain

$$\mathcal{D}_{k,t,z}(C) = \vartheta_k(t,z,\tau_1,\tau_2)\,[2F_{\tau_1}(\tau_1,\tau_2) + \vartheta_k(t,z,\tau_1,\tau_2)]$$

where we applied the chain rule for Malliavin derivatives provided in Theorem 12.8 in [10], and also used the equality $\mathcal{D}_{k,t,z}(F_{\tau_1}(\tau_1,\tau_2)) = \vartheta_k(t,z,\tau_1,\tau_2)$. Consequently, we find

$$\mathbb{E}_{\mathbb{Q}}(\mathcal{D}_{k,t,z}(C)\,|\mathcal{F}_t) = \vartheta_k(t,z,\tau_1,\tau_2)\,[2F_t(\tau_1,\tau_2) + \vartheta_k(t,z,\tau_1,\tau_2)]$$

due to the $(\mathcal{F},\mathbb{Q})$-martingale property of the precipitation swap price process F. Putting the *latter equation into (5.8), we are led to the optimal hedging portfolio*

$$\hat{\varphi}_t = 2F_t(\tau_1,\tau_2) + \frac{\sum_{k=1}^{n}\int_{\mathbb{R}^+}\vartheta_k(t,z,\tau_1,\tau_2)^3 e^{h_k(t,z)}\varepsilon_k(t)\,\mathrm{d}\nu_k(z)}{\sum_{k=1}^{n}\int_{\mathbb{R}^+}\vartheta_k(t,z,\tau_1,\tau_2)^2 e^{h_k(t,z)}\varepsilon_k(t)\,\mathrm{d}\nu_k(z)}$$

which constitutes the closest hedge of C in terms of minimal variance.

Example 5.8 *Consider the claim*

$$C := q\left(F_{\tau_1}(\tau_1,\tau_2)\right)$$

where the precipitation swap price F satisfies (3.10), and $q(\cdot)\in\mathcal{L}^1(\mathbb{R}^+)$ constitutes an arbitrary deterministic payoff function. For notational reasons, we will omit the measurement period arguments and simply write $F_.$ instead of $F_.(\tau_1,\tau_2)$ in the remainder of the current example. Since it holds $\mathcal{D}_{k,t,z}(F_{\tau_1})=\vartheta_k(t,z,\tau_1,\tau_2)$, we deduce from Theorem 12.8 in [10] that

$$\mathcal{D}_{k,t,z}(C) = q\left(F_{\tau_1}+\vartheta_k(t,z,\tau_1,\tau_2)\right) - q\left(F_{\tau_1}\right)$$

for all $k\in\{1,\ldots,n\}$ and $(t,z)\in[0,\tau_1]\times\mathbb{R}^+$. We now represent the function $q(\cdot)$ in the latter equation twice by the inverse Fourier transform [recall (4.3)] and arrive at the expression

$$\mathcal{D}_{k,t,z}(C) = \int_{\mathbb{R}^+}\hat{q}(y)\left[e^{iy\vartheta_k(t,z,\tau_1,\tau_2)}-1\right]e^{iyF_{\tau_1}}\mathrm{d}y$$

with complex-valued and deterministic Fourier transform $\hat{q}(y)$. As a consequence, we get

$$\mathbb{E}_{\mathbb{Q}}\left(\mathcal{D}_{k,t,z}(C)\,|\mathcal{F}_t\right) = \int_{\mathbb{R}^+}\hat{q}(y)\left[e^{iy\vartheta_k(t,z,\tau_1,\tau_2)}-1\right]\mathbb{E}_{\mathbb{Q}}\left(e^{iyF_{\tau_1}}|\mathcal{F}_t\right)\mathrm{d}y.$$

Taking $a:=0$ and $T:=\tau_1$ in (4.6), we further obtain

$$\mathbb{E}_{\mathbb{Q}}\left(e^{iyF_{\tau_1}}|\mathcal{F}_t\right) = e^{iyF_t+\phi(0,y,t,\tau_1,\tau_1,\tau_2)}$$

where the function ϕ is such as defined in (4.5). Hence, the minimal variance hedging portfolio associated with the presently investigated claim C is given by (5.8), but now with

$$\mathbb{E}_{\mathbb{Q}}\left(\mathcal{D}_{k,t,z}(C)\,|\mathcal{F}_t\right) = \int_{\mathbb{R}^+}\hat{q}(y)\left[e^{iy\vartheta_k(t,z,\tau_1,\tau_2)}-1\right]e^{iyF_t(\tau_1,\tau_2)+\phi(0,y,t,\tau_1,\tau_1,\tau_2)}\mathrm{d}y$$

therein. A practically relevant choice of the payoff function would be $q(x):=[K-x]^+\in\mathcal{L}^1(\mathbb{R}^+)$ which is associated with the put option investigated in Example 4.3. In this case, we find

$$\hat{q}(y) = \frac{1-iyK-e^{-iyK}}{2\pi y^2}.$$

Remark 5.9 *We recall that the minimal variance hedging problem has been formulated under a risk-neutral measure $\mathbb{Q}$ in (5.3)–(5.4), and not under the physical measure $\mathbb{P}$. This procedure stands in accordance to the argumentation given on p. 103 in [23] and in Section 1.2.3 in [23], wherein the authors argue in favor for a minimal variance hedging approach under $\mathbb{Q}$. Moreover, as the weather market considered in this paper is incomplete, we cannot expect that every claim is (entirely) hedgeable/replicable by trading with the underlying precipitation swap. In this regard, it is not surprising that we had to classify some claims in the previous examples to be non-hedgeable in the minimal variance sense.*

5.1 Minimal variance hedging with a stochastic maximum principle

In this section, we show that the minimal variance hedging portfolio $\hat{\varphi}$ claimed in (5.8) can alternatively be derived by an application of the stochastic maximum principle provided in Section 3.2 in [22]. First of all, we note that the minimization problem (5.3)–(5.4) is equivalent to the maximization problem

$$J(\hat{\varphi}) = \sup_{\varphi \in \mathcal{A}} \mathbb{E}_{\mathbb{Q}} \left[-(X_{\tau_1} - x_0 - C)^2 \right]. \tag{5.17}$$

Moreover, by substituting (3.10) into (5.2), we get the SDE

$$\mathrm{d}X_t = \sum_{k=1}^{n} \int_{\mathbb{R}^+} \gamma_k^1 (t-, z) \, \mathrm{d}\tilde{N}_k^{\mathbb{Q}} (t, z) \tag{5.18}$$

where we introduced the abbreviation

$$\gamma_k^1 (t, z) := \varphi_t \vartheta_k (t, z, \tau_1, \tau_2) \tag{5.19}$$

for notational convenience. Note that the SDE (5.18) can alternatively be expressed as

$$\mathrm{d}X_t = \int_{\mathbb{R}^+} \gamma^1 (t-, z) * \mathrm{d}\tilde{N}^{\mathbb{Q}} (t, z) \tag{5.20}$$

where the symbol $*$ indicates the inner product in the space $\mathbb{R}^n$ and

$$\gamma^1 (t, z) = \begin{bmatrix} \gamma_1^1 (t, z) \\ \vdots \\ \gamma_n^1 (t, z) \end{bmatrix} \in \mathbb{R}^n, \quad \mathrm{d}\tilde{N}^{\mathbb{Q}} (t, z) = \begin{bmatrix} \mathrm{d}\tilde{N}_1^{\mathbb{Q}} (t, z) \\ \vdots \\ \mathrm{d}\tilde{N}_n^{\mathbb{Q}} (t, z) \end{bmatrix} \in \mathbb{R}^n$$

are n-dimensional vectors. From (5.20) we immediately deduce the integral representation

$$X_{\tau_1} = x_0 + \int_0^{\tau_1} \int_{\mathbb{R}^+} \gamma^1 (t-, z) * \mathrm{d}\tilde{N}^{\mathbb{Q}} (t, z). \tag{5.21}$$

In accordance to Theorem 5.1, we assume that the claim $C \in \mathcal{L}^2 (\mathcal{F}_{\tau_1}, \mathbb{Q})$ can be represented as

$$C = \mathbb{E}_{\mathbb{Q}} [C] + \int_0^{\tau_1} \int_{\mathbb{R}^+} \gamma^2 (t-, z) * \mathrm{d}\tilde{N}^{\mathbb{Q}} (t, z) \tag{5.22}$$

with an n-dimensional $\mathcal{F}$-adapted càdlàg coefficient process

$$\gamma^2 (t, z) := \begin{bmatrix} \gamma_1^2 (t, z) \\ \vdots \\ \gamma_n^2 (t, z) \end{bmatrix} := \begin{bmatrix} \mathbb{E}_{\mathbb{Q}} (\mathcal{D}_{1,t,z} (C) | \mathcal{F}_t) \\ \vdots \\ \mathbb{E}_{\mathbb{Q}} (\mathcal{D}_{n,t,z} (C) | \mathcal{F}_t) \end{bmatrix} \in \mathbb{R}^n. \tag{5.23}$$

Plugging (5.21) and (5.22) into (5.17), we obtain

$$J(\hat{\varphi}) = \sup_{\varphi \in \mathcal{A}} \mathbb{E}_{\mathbb{Q}} \left[-\left(X_{\tau_1}^1 - X_{\tau_1}^2 \right)^2 \right] \tag{5.24}$$

wherein for all $t \in [0, \tau_1]$ we introduced the stochastic processes

$$X_t^1 := \int_0^t \int_{\mathbb{R}^+} \gamma^1 (s-, z) * \mathrm{d}\tilde{N}^{\mathbb{Q}} (s, z), \quad X_t^2 := \mathbb{E}_{\mathbb{Q}} [C] + \int_0^t \int_{\mathbb{R}^+} \gamma^2 (s-, z) * \mathrm{d}\tilde{N}^{\mathbb{Q}} (s, z). \tag{5.25}$$

Note that the processes (X_t) in (5.20) and $\left(X_t^1\right)$ in (5.25) do not coincide, as they possess different initial values, i.e. $X_0 = x_0 > 0$ and $X_0^1 = 0$. We are now prepared to apply the stochastic maximum principle provided in Section 3.2 in [22] to our optimization problem (5.24). To this end, from (5.25) we infer the SDEs

$$\mathrm{d}X_t^1 = \int_{\mathbb{R}^+} \gamma^1(t-,z) * \mathrm{d}\tilde{N}^{\mathbb{Q}}(t,z), \quad X_0^1 = 0,$$

$$\mathrm{d}X_t^2 = \int_{\mathbb{R}^+} \gamma^2(t-,z) * \mathrm{d}\tilde{N}^{\mathbb{Q}}(t,z), \quad X_0^2 = \mathbb{E}_{\mathbb{Q}}[C],$$

where $t \in [0,\tau_1]$. Comparing the latter SDEs with Eq. (3.2.1) in [22], we can read off $n=2$ and $k=1$ while $u_t := \varphi_t$ constitutes the one-dimensional $\mathcal{F}$-adapted càdlàg control process. Comparing (5.24) with Eq. (3.2.2) in [22], we further observe

$$f(t,X_t,u_t) \equiv 0, \quad g(x_1,x_2) := -(x_1-x_2)^2, \quad \nabla g(x_1,x_2) = 2(x_2-x_1, x_1-x_2)$$

where $g : \mathbb{R}^2 \to \,]-\infty,0]$ is a concave $\mathcal{C}^1$-function. In accordance to Eq. (3.2.3) in [22], the Hamiltonian presently takes the form

$$H\left(t,x_1,x_2,\varphi,p^1,p^2,\varrho^1,\varrho^2\right) = \sum_{k=1}^{n} \int_{\mathbb{R}^+} \left[\gamma_k^1(t,z)\,\varrho_k^1(t,z) + \gamma_k^2(t,z)\,\varrho_k^2(t,z)\right] e^{h_k(t,z)} \varepsilon_k(t)\,\mathrm{d}\nu_k(z) \tag{5.26}$$

where

$$\begin{bmatrix} p^1 \\ p^2 \end{bmatrix} \in \mathbb{R}^2, \quad \varrho^1(t,\cdot) = \begin{bmatrix} \varrho_1^1(t,\cdot) \\ \vdots \\ \varrho_n^1(t,\cdot) \end{bmatrix} \in \mathbb{R}^n, \quad \varrho^2(t,\cdot) = \begin{bmatrix} \varrho_1^2(t,\cdot) \\ \vdots \\ \varrho_n^2(t,\cdot) \end{bmatrix} \in \mathbb{R}^n$$

are unknown $\mathcal{F}$-adapted càdlàg adjoint processes. From (5.26) we deduce the partial derivatives $\partial_{x_1} H = \partial_{x_2} H = 0$ which imply a vanishing gradient vector $\nabla_x H = 0 \in \mathbb{R}^2$. Hence, taking Eq. (3.2.4) in [22] into account, we arrive at the adjoint equations

$$\mathrm{d}p_t^1 = \int_{\mathbb{R}^+} \varrho^1(t-,z) * \mathrm{d}\tilde{N}^{\mathbb{Q}}(t,z), \quad \mathrm{d}p_t^2 = \int_{\mathbb{R}^+} \varrho^2(t-,z) * \mathrm{d}\tilde{N}^{\mathbb{Q}}(t,z) \tag{5.27}$$

where $t \in [0,\tau_1[$ and the respective terminal values are given by

$$p_{\tau_1}^1 = -2\left(X_{\tau_1}^1 - X_{\tau_1}^2\right), \quad p_{\tau_1}^2 = 2\left(X_{\tau_1}^1 - X_{\tau_1}^2\right). \tag{5.28}$$

Thus, p^1 and p^2 fulfill backward stochastic differential equations (BSDEs). From (5.28) we infer

$$X_{\tau_1}^1 - X_{\tau_1}^2 = -\frac{p_{\tau_1}^1}{2} = \frac{p_{\tau_1}^2}{2}$$

which implies

$$\begin{aligned} X_{\tau_1}^1 - X_{\tau_1}^2 &= -\frac{1}{2}\left[p_0^1 + \int_0^{\tau_1}\int_{\mathbb{R}^+} \varrho^1(t-,z) * \mathrm{d}\tilde{N}^{\mathbb{Q}}(t,z)\right] \\ &= \frac{1}{2}\left[p_0^2 + \int_0^{\tau_1}\int_{\mathbb{R}^+} \varrho^2(t-,z) * \mathrm{d}\tilde{N}^{\mathbb{Q}}(t,z)\right] \end{aligned} \tag{5.29}$$

due to (5.27). On the other hand, from (5.25) we deduce the representation

$$X_{\tau_1}^1 - X_{\tau_1}^2 = -X_0^2 + \int_0^{\tau_1}\int_{\mathbb{R}^+} \left[\gamma^1(t-,z) - \gamma^2(t-,z)\right] * \mathrm{d}\tilde{N}^{\mathbb{Q}}(t,z). \tag{5.30}$$

Comparing (5.29) with (5.30), we are led (by the uniqueness of the Lévy-Itô decomposition of stochastic processes; cf. [8, 10, 17, 24]) to the equality system

$$-\frac{p_0^1}{2} = \frac{p_0^2}{2} = -X_0^2, \quad -\frac{\varrho^1(t,z)}{2} = \frac{\varrho^2(t,z)}{2} = \gamma^1(t,z) - \gamma^2(t,z)$$

which is equivalent to

$$p_0^1 = -p_0^2 = 2X_0^2, \quad \varrho^1(t,z) = -\varrho^2(t,z) = 2\left[\gamma^2(t,z) - \gamma^1(t,z)\right]. \tag{5.31}$$

Putting (5.31) and (5.19) into (5.26), we obtain

$$\begin{aligned} & H\left(t, X_t^1, X_t^2, \varphi_t, p_t^1, p_t^2, \varrho^1(t,\cdot), \varrho^2(t,\cdot)\right) \\ = & -2\sum_{k=1}^n \int_{\mathbb{R}^+} \left[\gamma_k^1(t,z) - \gamma_k^2(t,z)\right]^2 e^{h_k(t,z)} \varepsilon_k(t)\,\mathrm{d}\nu_k(z) \\ = & -2\sum_{k=1}^n \int_{\mathbb{R}^+} \left[\varphi_t \vartheta_k(t,z,\tau_1,\tau_2) - \gamma_k^2(t,z)\right]^2 e^{h_k(t,z)} \varepsilon_k(t)\,\mathrm{d}\nu_k(z) \end{aligned}$$

for all $t \in [0,\tau_1]$. Since the process γ^2 defined in (5.23) is independent of φ, we find the derivatives

$$\partial_\varphi H = -4\sum_{k=1}^n \int_{\mathbb{R}^+} \left[\varphi_t \vartheta_k(t,z,\tau_1,\tau_2) - \gamma_k^2(t,z)\right] \vartheta_k(t,z,\tau_1,\tau_2)\, e^{h_k(t,z)} \varepsilon_k(t)\,\mathrm{d}\nu_k(z),$$

$$\partial_{\varphi\varphi} H = -4\sum_{k=1}^n \int_{\mathbb{R}^+} \vartheta_k(t,z,\tau_1,\tau_2)^2 e^{h_k(t,z)} \varepsilon_k(t)\,\mathrm{d}\nu_k(z),$$

where $t \in [0,\tau_1]$. As it holds $\partial_{\varphi\varphi} H < 0$, the map

$$\varphi \mapsto H\left(t, X_t^1, X_t^2, \varphi, p_t^1, p_t^2, \varrho^1(t,\cdot), \varrho^2(t,\cdot)\right)$$

is concave for all $t \in [0,\tau_1]$. Thanks to Theorem 3.4 in [22] we thus know that the optimal control $\hat{\varphi} \in \mathcal{A}$ for the optimization problem (5.24) can be found by maximizing the Hamiltonian H with respect to φ. In this regard, we require the optimality condition $\partial_\varphi H = 0$ which is equivalent to (5.8). Hence, we have found an alternative way to prove Proposition 5.2. For more information on the theory and related applications of stochastic maximum principles we refer to [15] and [22].

6. Pricing precipitation swaps under future information

In this section, we derive a pricing formula for a precipitation swap under an enlarged filtration modeling future weather information available to the traders. More precisely, we now assume that a trader has knowledge (respectively, an idea/a notion/a guess/an estimate) about the future behavior of the noises $L^1, \ldots, L^p$ driving the factor processes $Y^1, \ldots, Y^p$ in (3.2), where $p \in \mathbb{N}$ satisfies $0 \leqslant p \leqslant n$. In this regard, we introduce the initially enlarged filtration

$$\mathcal{G}_t := \mathcal{F}_t \vee \sigma\left\{L_\tau^1, \ldots, L_\tau^p\right\} \tag{6.1}$$

where $\mathcal{G}_t \supset \mathcal{F}_t$ for all $0 \leqslant t < \tau$, and $\mathcal{G}_t = \mathcal{F}_t$ whenever $t \geqslant \tau$. Here, the letter σ denotes the sigma algebra generator, while $\tau > 0$ is a fixed and finite deterministic future time. Enlarged filtrations of this kind have formerly been dealt with in [3, 5, 9, 10, 14, 15], and [16]. Further note that for $p = 0$ we get $\mathcal{G}_t = \mathcal{F}_t$ for all $t \geqslant 0$, which is associated with the "no information case", whereas $p = n$ corresponds to the "full information case" with future information on all noises appearing in (3.2). Moreover, we assume from now on that for all $k \in \{1, \ldots, p\}$ and $z \in \mathbb{R}^+$ the functions $h_k(t, z) := h_k(z)$ and $\varepsilon_k(t) \equiv \varepsilon_k$ are time-independent, such that the noises $L^1, \ldots, L^p$ constitute time-homogeneous Lévy processes under the risk-neutral measure $\mathbb{Q}$. Under this assumption, the following key result holds.

Lemma 6.1 (a) *For an initially enlarged filtration $\mathcal{G}$ as defined in (6.1) the stochastic processes*

$$\left(\tilde{L}_t^k := L_t^k - \int_0^t \frac{L_\tau^k - L_s^k}{\tau - s} \mathrm{d}s\right)_{t \in [0,\tau[} \tag{6.2}$$

constitute $\mathcal{G}$-adapted martingales under $\mathbb{Q}$ for every $k \in \{1, \ldots, p\}$.

(b) *For all $k \in \{1, \ldots, p\}$ and $0 \leqslant t \leqslant s \leqslant \tau$ (with $t \neq \tau$) it holds*

$$\mathbb{E}_{\mathbb{Q}}\left(L_\tau^k - L_s^k | \mathcal{G}_t\right) = \frac{\tau - s}{\tau - t}\left(L_\tau^k - L_t^k\right).$$

Proof Part (a) is a direct consequence of Theorem 2.6 in [16] or the proof of Proposition 16.52 in [10]. Part (b) follows from Proposition A.3 in [3] combined with Remark A.4 in [3]. □

Remark 6.2 *We stress that – for one fixed $j \in \{1, \ldots, p\}$ – the stochastic process $\left(\tilde{L}_t^j\right)$ actually constitutes an $\left(\mathcal{F}_t \vee \sigma\left\{L_\tau^j\right\}\right)$-adapted martingale, the latter being a stronger conclusion than the one given in Lemma 6.1 (a), as the filtration $\mathcal{F}_t \vee \sigma\left\{L_\tau^j\right\}$ is contained in $\mathcal{G}_t$ for every $t < \tau$.*

For all $k \in \{1, \ldots, p\}$ and $t \in [0, \tau[$ we further introduce the stochastic compensator processes (also called information yield processes)

$$U_t^k := U_t^k(\tau) := \frac{L_\tau^k - L_t^k}{\tau - t} \tag{6.3}$$

satisfying the $(\mathcal{G}, \mathbb{Q})$-martingale dynamics

$$\mathrm{d}U_t^k = \frac{1}{t - \tau} \mathrm{d}\tilde{L}_t^k \tag{6.4}$$

where $\left(\tilde{L}_t^k\right)$ is such as defined in (6.2). From (6.2) and (6.3) we infer the semi-martingale decomposition

$$\mathrm{d}L_t^k = \mathrm{d}\tilde{L}_t^k + U_t^k \mathrm{d}t \tag{6.5}$$

for all $k \in \{1, \ldots, p\}$ and $t \in [0, \tau[$. Also recall that for all $0 \leqslant t \leqslant s < \tau$ and $k \in \{1, \ldots, p\}$ we find

$$\mathbb{E}_{\mathbb{Q}}\left(U_s^k | \mathcal{G}_t\right) = U_t^k \tag{6.6}$$

due to Lemma 6.1 (b), such that the processes $U^1, \ldots, U^p$ not only serve as compensators, but constitute $(\mathcal{G}, \mathbb{Q})$-martingales themselves. We next substitute (2.2) into (6.2) and obtain the representation

$$\tilde{L}_t^k = \int_0^t \int_{\mathbb{R}^+} z \mathrm{d}\tilde{N}_k^{\mathcal{G},\mathbb{Q}}(s, z) \tag{6.7}$$

where for every $k \in \{1, \ldots, p\}$ the compensated random measure

$$\mathrm{d}\tilde{N}_k^{\mathcal{G},\mathbb{Q}}(s, z) := \mathrm{d}N_k(s, z) - \mathrm{d}\nu_{k,s}^{\mathcal{G}}(z)\,\mathrm{d}s \tag{6.8}$$

constitutes a $(\mathcal{G}, \mathbb{Q})$-martingale integrator on $[0, \tau[\times \mathbb{R}^+$. Herein, for all $k \in \{1, \ldots, p\}$, $s \in [0, \tau[$ and $z \in \mathbb{R}^+$ the $(\mathcal{G}, \mathbb{Q})$-compensator is given by

$$\mathrm{d}\nu_{k,s}^{\mathcal{G}}(z) := \frac{1}{\tau - s} \int_s^\tau \mathrm{d}N_k(u, z) \tag{6.9}$$

which is stochastic and time-dependent and thus, does not constitute a Lévy measure in the common sense. In the context of (6.7)–(6.9), we refer to Proposition 16.53 in [10] and Theorem 2.9 (ii) in [16]. Moreover, merging (2.2) into (6.3), we arrive at the representation

$$U_t^k = \frac{1}{\tau - t} \int_t^\tau \int_{\mathbb{R}^+} z \mathrm{d}N_k(s, z) \tag{6.10}$$

where $k \in \{1, \ldots, p\}$ and $t \in [0, \tau[$. We stress that the processes $\tilde{L}^1, \ldots, \tilde{L}^p$ given by (6.7) indeed are well-defined $\mathcal{G}$-martingales on $[0, \tau[$, since in our case the condition

$$\mathbb{E}_{\mathbb{Q}}\left[\sum_{k=1}^p \int_0^\tau \int_{\mathbb{R}^+} z^2 e^{h_k(z)} \varepsilon_k \mathrm{d}\nu_k(z)\,\mathrm{d}t\right] = \tau \sum_{k=1}^p \varepsilon_k \int_{\mathbb{R}^+} z^2 e^{h_k(z)} \mathrm{d}\nu_k(z) < \infty$$

required in Eq. (2.17) in [16] is in force. Furthermore, for all $k \in \{1, \ldots, n\}$ let us introduce the non-negative deterministic functions

$$\alpha_k(u, \tau_1, \tau_2) := w_k \xi_k(u) \int_{u \vee \tau_1}^{\tau_2} \theta(s) e^{-\lambda_k(s-u)} \mathrm{d}s \tag{6.11}$$

such that we may write

$$\vartheta_k(u, z, \tau_1, \tau_2) = \alpha_k(u, \tau_1, \tau_2) z \tag{6.12}$$

due to (3.9). Consequently, the precipitation index $\mathfrak{I}(\tau_1, \tau_2)$ claimed in (3.7) can be expressed as

$$\mathfrak{I}(\tau_1, \tau_2) = c(\tau_1, \tau_2) + \sum_{k=1}^n \int_0^{\tau_2} \alpha_k(s, \tau_1, \tau_2) \mathrm{d}L_s^k \tag{6.13}$$

where $c(\tau_1, \tau_2)$ is the non-negative constant defined in (3.8). With reference to (2.4), we define the forward-looking precipitation swap price process under the enlarged filtration $\mathcal{G}$ for all $0 \leqslant t \leqslant \tau_1 < \tau_2 = \tau$ via

$$F_t^{\mathcal{G}}(\tau_1, \tau_2) := \mathbb{E}_{\mathbb{Q}}(\mathfrak{I}(\tau_1, \tau_2) | \mathcal{G}_t) \tag{6.14}$$

where $\mathbb{Q}$ denotes the risk-neutral pricing measure, and $\mathcal{G}$ is such as defined in (6.1). Here, we assumed that $\tau_2 = \tau$ in order to ensure that $t < \tau$ which helps to avoid several technical case distinctions in the sequel. In the current enlarged filtration setup, we get the following result.

Proposition 6.3 *Let $\mathcal{G}$ be the initially enlarged filtration defined in (6.1) and $c(\tau_1, \tau_2)$ be the non-negative constant introduced in (3.8). Then, for all $0 \leqslant t \leqslant \tau_1 < \tau_2 = \tau$ the precipitation swap price process $F_\cdot^{\mathcal{G}}(\tau_1, \tau_2)$ defined in (6.14) can be represented as*

$$F_t^{\mathcal{G}}(\tau_1, \tau_2) = c(\tau_1, \tau_2) + \sum_{k=1}^p \Phi_t^k(\tau_1, \tau_2) + \sum_{k=p+1}^n \Psi_t^k(\tau_1, \tau_2) \tag{6.15}$$

where

$$\Phi_t^k(\tau_1, \tau_2) = \int_0^t \alpha_k(s, \tau_1, \tau_2) \mathrm{d}L_s^k + U_t^k \int_t^{\tau_2} \alpha_k(s, \tau_1, \tau_2) \mathrm{d}s \tag{6.16}$$

constitutes a $\mathcal{G}$-adapted stochastic process for every $k \in \{1, \ldots, p\}$, and

$$\Psi_t^k(\tau_1,\tau_2) = \int_0^t \int_{\mathbb{R}+} \vartheta_k(s,z,\tau_1,\tau_2)\,\mathrm{d}\tilde{N}_k^{\mathcal{F},\mathbb{Q}}(s,z) + \int_0^{\tau_2} \int_{\mathbb{R}+} \vartheta_k(s,z,\tau_1,\tau_2)\,e^{h_k(s,z)}\varepsilon_k(s)\,\mathrm{d}\nu_k(z)\,\mathrm{d}s \tag{6.17}$$

constitutes an $\mathcal{F}$-adapted stochastic process for every $k \in \{p+1,\ldots,n\}$. Herein, the deterministic functions ϑ_k and α_k are such as defined in (3.9) and (6.11), respectively, while the stochastic processes L^k and U^k are such as claimed in (2.2) and (6.3), respectively.

Proof By substituting (6.13) into (6.14), we immediately obtain (6.15) with

$$\Phi_t^k(\tau_1,\tau_2) := \mathbb{E}_\mathbb{Q}\left(\int_0^{\tau_2} \alpha_k(s,\tau_1,\tau_2)\,\mathrm{d}L_s^k|\mathcal{G}_t\right), \Psi_t^k(\tau_1,\tau_2) := \mathbb{E}_\mathbb{Q}\left(\int_0^{\tau_2} \alpha_k(s,\tau_1,\tau_2)\,\mathrm{d}L_s^k|\mathcal{F}_t\right)$$

for all $0 \leqslant t \leqslant \tau_1 < \tau_2 = \tau$. We now compute these conditional expectations separately. Taking (6.5) into account, we deduce

$$\Phi_t^k(\tau_1,\tau_2) = \int_0^t \alpha_k(s,\tau_1,\tau_2)\,\mathrm{d}\tilde{L}_s^k + \int_0^{\tau_2} \alpha_k(s,\tau_1,\tau_2)\,\mathbb{E}_\mathbb{Q}\left(U_s^k|\mathcal{G}_t\right)\mathrm{d}s$$

where we used Fubini's theorem along with the $(\mathcal{G},\mathbb{Q})$-martingale property of the processes $\tilde{L}^1,\ldots,\tilde{L}^p$ claimed in Lemma 6.1 (a). By splitting up the last integral, we get

$$\int_0^{\tau_2} \alpha_k(s,\tau_1,\tau_2)\,\mathbb{E}_\mathbb{Q}\left(U_s^k|\mathcal{G}_t\right)\mathrm{d}s = \int_0^t \alpha_k(s,\tau_1,\tau_2)\,U_s^k\mathrm{d}s + \int_t^{\tau_2} \alpha_k(s,\tau_1,\tau_2)\,\mathbb{E}_\mathbb{Q}\left(U_s^k|\mathcal{G}_t\right)\mathrm{d}s$$

since the information yield process $\left(U_s^k\right)_{0\leqslant s\leqslant t<\tau}$ is $(\mathcal{G}_s \subseteq \mathcal{G}_t)$-adapted for every $k \in \{1,\ldots,p\}$. As it holds $0 \leqslant t \leqslant s \leqslant \tau_2$, $t < \tau$, inside the last ds-integral, we can apply Eq. (6.6) which yields

$$\int_0^{\tau_2} \alpha_k(s,\tau_1,\tau_2)\,\mathbb{E}_\mathbb{Q}\left(U_s^k|\mathcal{G}_t\right)\mathrm{d}s = \int_0^t \alpha_k(s,\tau_1,\tau_2)\,U_s^k\mathrm{d}s + U_t^k\int_t^{\tau_2} \alpha_k(s,\tau_1,\tau_2)\,\mathrm{d}s$$

for all $k \in \{1,\ldots,p\}$. Putting the latter equations together, we finally arrive at the representation

$$\Phi_t^k(\tau_1,\tau_2) = \int_0^t \alpha_k(s,\tau_1,\tau_2)\,\mathrm{d}L_s^k + U_t^k\int_t^{\tau_2} \alpha_k(s,\tau_1,\tau_2)\,\mathrm{d}s$$

where we used (6.5). Hence, we have proven (6.16). On the other hand, we take (2.2), (6.12), as well as (2.7) into account, and obtain for all $k \in \{p+1,\ldots,n\}$

$$\Psi_t^k(\tau_1,\tau_2) = \int_0^t \int_{\mathbb{R}+} \vartheta_k(s,z,\tau_1,\tau_2)\,\mathrm{d}\tilde{N}_k^{\mathbb{Q}}(s,z) + \int_0^{\tau_2} \int_{\mathbb{R}+} \vartheta_k(s,z,\tau_1,\tau_2)\,e^{h_k(s,z)}\varepsilon_k(s)\,\mathrm{d}\nu_k(z)\,\mathrm{d}s$$

due to the $(\mathcal{F},\mathbb{Q})$-martingale property of the appearing $\mathrm{d}\tilde{N}_k^{\mathbb{Q}}$-integral. Thus, we have also proven (6.17) which completes the proof. □

Proposition 6.4 *Let $\mathcal{G}$ be the enlarged filtration defined in (6.1) and suppose that $\tilde{N}_k^{\mathcal{G},\mathbb{Q}}$ is the compensated random measure introduced in (6.8). Then, for all $0 \leqslant t \leqslant \tau_1 < \tau_2 = \tau$ the precipitation swap price process $F_\cdot^{\mathcal{G}}(\tau_1,\tau_2)$ satisfies the $(\mathcal{G},\mathbb{Q})$-martingale dynamics*

$$\mathrm{d}F_t^{\mathcal{G}}(\tau_1,\tau_2) = \sum_{k=1}^{p} \int_{\mathbb{R}+} \beta_k(t,z,\tau_1,\tau_2)\,\mathrm{d}\tilde{N}_k^{\mathcal{G},\mathbb{Q}}(t,z) + \sum_{k=p+1}^{n} \int_{\mathbb{R}+} \vartheta_k(t,z,\tau_1,\tau_2)\,\mathrm{d}\tilde{N}_k^{\mathcal{F},\mathbb{Q}}(t,z) \tag{6.18}$$

where $\vartheta_k(t,z,\tau_1,\tau_2)$ and

$$\beta_k(t,z,\tau_1,\tau_2) := \vartheta_k(t,z,\tau_1,\tau_2) - \frac{1}{\tau_2 - t}\int_t^{\tau_2} \vartheta_k(s,z,\tau_1,\tau_2)\,\mathrm{d}s \tag{6.19}$$

are deterministic functions. The initial value of the SDE (6.18) reads as

$$F_0^{\mathcal{G}}(\tau_1,\tau_2) = c(\tau_1,\tau_2) + \sum_{k=1}^{p} U_0^k \int_0^{\tau_2} \alpha_k(s,\tau_1,\tau_2)\,\mathrm{d}s + \sum_{k=p+1}^{n} \int_0^{\tau_2}\int_{\mathbb{R}^+} \vartheta_k(s,z,\tau_1,\tau_2)\, e^{h_k(s,z)} \varepsilon_k(s)\,\mathrm{d}\nu_k(z)\,\mathrm{d}s. \tag{6.20}$$

Proof Taking t-differentials in (6.15) yields

$$\mathrm{d}F_t^{\mathcal{G}}(\tau_1,\tau_2) = \sum_{k=1}^{p} \mathrm{d}\Phi_t^k(\tau_1,\tau_2) + \sum_{k=p+1}^{n} \mathrm{d}\Psi_t^k(\tau_1,\tau_2)$$

where $0 \leqslant t \leqslant \tau_1 < \tau_2 = \tau$. Using (6.16) as well as integration by parts, we obtain

$$\mathrm{d}\Phi_t^k(\tau_1,\tau_2) = \alpha_k(t,\tau_1,\tau_2)\,\mathrm{d}L_t^k - U_t^k \alpha_k(t,\tau_1,\tau_2)\,\mathrm{d}t + \left(\int_t^{\tau_2} \alpha_k(s,\tau_1,\tau_2)\,\mathrm{d}s\right)\mathrm{d}U_t^k$$

for all $k \in \{1,\ldots,p\}$. With reference to (6.5) and (6.4), the latter equation can be rewritten as

$$\mathrm{d}\Phi_t^k(\tau_1,\tau_2) = \left(\alpha_k(t,\tau_1,\tau_2) + \frac{1}{t-\tau}\int_t^{\tau_2} \alpha_k(s,\tau_1,\tau_2)\,\mathrm{d}s\right)\mathrm{d}\tilde{L}_t^k$$

where $\tilde{L}^k$ is the $(\mathcal{G},\mathbb{Q})$-martingale defined in (6.2). Due to (6.7) and (6.12), we further find

$$\mathrm{d}\Phi_t^k(\tau_1,\tau_2) = \int_{\mathbb{R}^+} \beta_k(t,z,\tau_1,\tau_2)\,\mathrm{d}\tilde{N}_k^{\mathcal{G},\mathbb{Q}}(t,z)$$

where we identified the deterministic function β_k defined in (6.19). On the other hand, we take (6.17) into account and infer the equality

$$\mathrm{d}\Psi_t^k(\tau_1,\tau_2) = \int_{\mathbb{R}^+} \vartheta_k(t,z,\tau_1,\tau_2)\,\mathrm{d}\tilde{N}_k^{\mathbb{Q}}(t,z)$$

for all $k \in \{p+1,\ldots,n\}$. Putting the latter equations together, we ultimately end up with (6.18). The representation of the initial value claimed in (6.20) is an immediate consequence of (6.15)–(6.17). □

Corollary 6.5 (a) *In the "full information case" with $p = n$ the precipitation swap price process satisfies the $(\mathcal{G},\mathbb{Q})$-martingale dynamics*

$$\mathrm{d}F_t^{\mathcal{G}}(\tau_1,\tau_2) = \sum_{k=1}^{n} \int_{\mathbb{R}^+} \beta_k(t,z,\tau_1,\tau_2)\,\mathrm{d}\tilde{N}_k^{\mathcal{G},\mathbb{Q}}(t,z).$$

(b) *In the "no information case" with $p = 0$ we find $\mathcal{G} = \mathcal{F}$ due to (6.1), and the precipitation swap price process fulfills the $(\mathcal{F},\mathbb{Q})$-martingale dynamics*

$$\mathrm{d}F_t^{\mathcal{F}}(\tau_1,\tau_2) = \sum_{k=1}^{n} \int_{\mathbb{R}^+} \vartheta_k(t,z,\tau_1,\tau_2)\,\mathrm{d}\tilde{N}_k^{\mathcal{F},\mathbb{Q}}(t,z)$$

which coincides with (3.10).

Proof Both results are direct consequences of Proposition 6.4. □

6.1 The information premium

In [3] the authors introduce a stochastic process called *information premium* which measures the difference between electricity futures prices once derived under the historical filtration and once under an enlarged insider filtration. Inspired by this concept, we now define the precipitation swap price information premium process $I_t^{\mathcal{F},\mathcal{G}}(\tau_1,\tau_2)$ for all $0 \leqslant t \leqslant \tau_1 < \tau_2 = \tau$ via

$$I_t^{\mathcal{F},\mathcal{G}}(\tau_1,\tau_2) := F_t^{\mathcal{G}}(\tau_1,\tau_2) - F_t^{\mathcal{F}}(\tau_1,\tau_2) \tag{6.21}$$

where $F^{\mathcal{G}}$ and $F^{\mathcal{F}}$ are the precipitation swap price processes introduced in (6.14) and (2.4), respectively. Note in passing that it holds

$$I_t^{\mathcal{F},\mathcal{G}}(\tau_1,\tau_2) = \mathbb{E}_{\mathbb{Q}}\left(\mathfrak{I}(\tau_1,\tau_2)\,|\mathcal{G}_t\right) - \mathbb{E}_{\mathbb{Q}}\left(\mathfrak{I}(\tau_1,\tau_2)\,|\mathcal{F}_t\right)$$

for all $0 \leqslant t \leqslant \tau_1 < \tau_2 = \tau$, which implies $I_t^{\mathcal{F},\mathcal{G}}(\tau_1,\tau_2) \equiv 0$ whenever $t \geqslant \tau$. In the latter equation, $\mathfrak{I}(\tau_1,\tau_2)$ denotes the precipitation index claimed in (3.6). In the remainder of the current section, we assume that $h_k(t,z) := h_k(z)$ and $\varepsilon_k(t) \equiv \varepsilon_k$ for all $k \in \{1,\ldots,n\}$.

Corollary 6.6 *Let the deterministic functions ϑ_k and α_k be such as defined in (3.9) and (6.11), respectively. Further suppose that the information yield processes U^k are like defined in (6.3). Then, the precipitation swap price information premium process can be expressed as*

$$I_t^{\mathcal{F},\mathcal{G}}(\tau_1,\tau_2) = \sum_{k=1}^{p}\int_t^{\tau_2}\left(U_t^k\alpha_k(s,\tau_1,\tau_2) - \varepsilon_k\int_{\mathbb{R}^+}\vartheta_k(s,z,\tau_1,\tau_2)\,e^{h_k(z)}\mathrm{d}\nu_k(z)\right)\mathrm{d}s$$

where $0 \leqslant t \leqslant \tau_1 < \tau_2 = \tau$.

Proof We substitute (6.15)–(6.17) as well as (3.10)–(3.11) into (6.21) and hereafter, use (2.2), (6.12), and (2.7). This procedure immediately yields the assertion. □

Remark 6.7 (a) *Note that the information premium process can for all $0 \leqslant t \leqslant \tau_1 < \tau_2 = \tau$ be expressed as a linear combination of the information yield processes $U^1,\ldots,U^p$ as follows*

$$I_t^{\mathcal{F},\mathcal{G}}(\tau_1,\tau_2) = \sum_{k=1}^{p}\left[v_k^1(t,\tau_1,\tau_2)\,U_t^k - v_k^0(t,\tau_1,\tau_2)\right] = \sum_{k=1}^{p}\left[v_k^1(t,\tau_1,\tau_2)\,\frac{L_{\tau_2}^k - L_t^k}{\tau_2 - t} - v_k^0(t,\tau_1,\tau_2)\right]$$

with deterministic coefficient functions

$$v_k^1(t,\tau_1,\tau_2) := \int_t^{\tau_2}\alpha_k(s,\tau_1,\tau_2)\,\mathrm{d}s,$$
$$v_k^0(t,\tau_1,\tau_2) := \varepsilon_k\int_t^{\tau_2}\int_{\mathbb{R}^+}\vartheta_k(s,z,\tau_1,\tau_2)\,e^{h_k(z)}\mathrm{d}\nu_k(z)\,\mathrm{d}s.$$

(b) *The information premium has the following economical interpretation: The object $I_t^{\mathcal{F},\mathcal{G}}(\tau_1,\tau_2)$ measures the premium that should be charged on the non-anticipative precipitation swap price $F_t^{\mathcal{F}}(\tau_1,\tau_2)$ by the seller of the swap contract at time t, if the noise processes $L^1,\ldots,L^p$ (driving the factor processes $Y^1,\ldots,Y^p$) take the values $L_\tau^1,\ldots,L_\tau^p$ at the future time τ. In other words, the difference $F_t^{\mathcal{G}}(\tau_1,\tau_2) - F_t^{\mathcal{F}}(\tau_1,\tau_2)$ describes the "cost of information" stemming from knowledge about the future behavior of the first p noise processes driving the underlying precipitation index $\mathfrak{I}(\tau_1,\tau_2)$ [cf. (6.13)]. We eventually recall that the information premium has initially been introduced in the context of energy, commodity and weather markets in [3].*

7. Minimal variance hedging under future information

In this section, we investigate minimal variance hedging of precipitation derivatives under future information modeled by the enlarged filtration $\mathcal{G}$ introduced in (6.1). As in Corollary 6.5 (a) we assume that it holds $p = n$ in the following, and thus work with the $\mathcal{G}$-anticipative precipitation swap price representation

$$F_t^{\mathcal{G}}(\tau_1, \tau_2) = F_0^{\mathcal{G}}(\tau_1, \tau_2) + \sum_{k=1}^{n} \int_0^t \int_{\mathbb{R}^+} \beta_k(s, z, \tau_1, \tau_2)\, \mathrm{d}\tilde{N}_k^{\mathcal{G},\mathbb{Q}}(s, z) \tag{7.1}$$

where $0 \leqslant t \leqslant \tau_1 < \tau_2 = \tau$ and $\mathcal{G}_t = \mathcal{F}_t \vee \sigma\left\{L_\tau^1, \dots, L_\tau^n\right\}$. Herein, the entities $F_0^{\mathcal{G}}$, β_k and $\tilde{N}_k^{\mathcal{G},\mathbb{Q}}$ are such as specified in Proposition 6.4. In the sequel, we work in a similar setup as introduced in Section 5, but now with $F^{\mathcal{F}}$ therein replaced by $F^{\mathcal{G}}$. In addition, we presently allow the portfolio process φ to be $\mathcal{G}$-adapted and the involved τ_1-claim C to be an element of $\mathcal{L}^2(\mathcal{G}_{\tau_1}, \mathbb{Q})$. Under these presumptions, we assume that the wealth process $X = (X_t)_{t\in[0,\tau_1]}$ satisfies

$$\mathrm{d}X_t = \varphi_{t-}\mathrm{d}F_t^{\mathcal{G}} \tag{7.2}$$

with deterministic initial wealth $X_0 = x_0 > 0$ [cf. (5.2)]. Integrating (7.2) from 0 to τ_1 yields

$$X_{\tau_1} = x_0 + \sum_{k=1}^{n} \int_0^{\tau_1} \int_{\mathbb{R}^+} \varphi_{t-}\beta_k(t, z, \tau_1, \tau_2)\, \mathrm{d}\tilde{N}_k^{\mathcal{G},\mathbb{Q}}(t, z) \tag{7.3}$$

where we used (7.1). With respect to (5.3)–(5.4), we now consider the optimization problem

$$J(\hat{\varphi}) = \inf_{\varphi\in\mathcal{A}(\mathcal{G})} \mathbb{E}_{\mathbb{Q}}\left[(X_{\tau_1} - x_0 - C)^2\right] \iff J(\hat{\varphi}) = \sup_{\varphi\in\mathcal{A}(\mathcal{G})} \mathbb{E}_{\mathbb{Q}}\left[-(X_{\tau_1} - x_0 - C)^2\right] \tag{7.4}$$

where $\mathcal{A}(\mathcal{G})$ denotes the set of all admissible (i.e. predictable, $\mathcal{G}$-adapted, càdlàg, and integrable) portfolio processes. In what follows, we restrict ourselves to $\mathcal{L}^2(\mathcal{G}_{\tau_1}, \mathbb{Q})$-claims of the form

$$C = \mathbb{E}_{\mathbb{Q}}[C] + \sum_{k=1}^{n} \int_0^{\tau_1} \int_{\mathbb{R}^+} \eta_k(t, z)\, \mathrm{d}\tilde{N}_k^{\mathcal{G},\mathbb{Q}}(t, z) \tag{7.5}$$

with arbitrary deterministic functions $\eta_k : [0, \tau_1] \times \mathbb{R}^+ \to \mathbb{R}$ satisfying

$$\int_0^{\tau_1} \int_{\mathbb{R}^+} \eta_k(t, z)^2 e^{h_k(z)} \varepsilon_k \mathrm{d}\nu_k(z)\, \mathrm{d}t < \infty$$

for all $k \in \{1, \dots, n\}$. Further on, we recall that in Section 3 in [15] it was shown that the sufficient stochastic maximum principle provided in Section 3.2 in [22] likewise applies to an initially enlarged filtration setup of the type (6.1). Hence, also in the current enlarged filtration framework, we can derive the minimal variance hedging portfolio $\hat{\varphi}$ associated with the optimization problems claimed in (7.4) by an application of Theorem 3.4 in [22], similarly to our previous argumentation in Section 5.1. [Of course, it would also be possible to apply Theorem 3.2 in [15] and Corollary 3.4 in [15] to (7.4).] In the sequel, we essentially make the same steps as previously in Section 5.1 and thus do not work out all details. In the current enlarged filtration framework, we find

$$\mathrm{d}X_t = \sum_{k=1}^{n} \int_{\mathbb{R}^+} \gamma_k^1(t-, z)\, \mathrm{d}\tilde{N}_k^{\mathcal{G},\mathbb{Q}}(t, z), \quad X_0 = x_0, \quad \gamma_k^1(t, z) := \varphi_t \beta_k(t, z, \tau_1, \tau_2)$$

instead of (5.18)–(5.19). Comparing (5.22)–(5.23) with (7.5), we see that

$$\gamma^2(t,z) := \begin{bmatrix} \gamma_1^2(t,z) \\ \vdots \\ \gamma_n^2(t,z) \end{bmatrix} := \begin{bmatrix} \eta_1(t,z) \\ \vdots \\ \eta_n(t,z) \end{bmatrix} \in \mathbb{R}^n.$$

Hence, from (7.4) we deduce

$$J(\hat{\varphi}) = \sup_{\varphi \in \mathcal{A}(\mathcal{G})} \mathbb{E}_{\mathbb{Q}} \left[-\left(X_{\tau_1}^1 - X_{\tau_1}^2 \right)^2 \right]$$

with $\mathcal{G}$-adapted stochastic processes

$$X_t^1 := \int_0^t \int_{\mathbb{R}^+} \gamma^1(s-,z) * \mathrm{d}\tilde{N}^{\mathcal{G},\mathbb{Q}}(s,z), \quad X_t^2 := \mathbb{E}_{\mathbb{Q}}[C] + \int_0^t \int_{\mathbb{R}^+} \gamma^2(s-,z) * \mathrm{d}\tilde{N}^{\mathcal{G},\mathbb{Q}}(s,z).$$

Here, it holds $X_0^1 = 0$ and $X_0^2 = \mathbb{E}_{\mathbb{Q}}[C]$, while $*$ indicates the inner product in the space $\mathbb{R}^n$ and

$$\gamma^1(t,z) = \begin{bmatrix} \gamma_1^1(t,z) \\ \vdots \\ \gamma_n^1(t,z) \end{bmatrix} \in \mathbb{R}^n, \quad \mathrm{d}\tilde{N}^{\mathcal{G},\mathbb{Q}}(t,z) = \begin{bmatrix} \mathrm{d}\tilde{N}_1^{\mathcal{G},\mathbb{Q}}(t,z) \\ \vdots \\ \mathrm{d}\tilde{N}_n^{\mathcal{G},\mathbb{Q}}(t,z) \end{bmatrix} \in \mathbb{R}^n$$

are n-dimensional vectors. Taking Eq. (3.2.2) in [22] into account, we further observe

$$f(t, X_t, u_t) \equiv 0, \quad g(x_1, x_2) := -(x_1 - x_2)^2, \quad \nabla g(x_1, x_2) = 2(x_2 - x_1, x_1 - x_2)$$

where $g : \mathbb{R}^2 \to \,]-\infty, 0]$ is a concave $\mathcal{C}^1$-function. In accordance to Eq. (3.2.3) in [22] (also see Eq. (3.6) in [15]), the Hamiltonian presently takes the form

$$H\left(t, x_1, x_2, \varphi, p^1, p^2, \varrho^1, \varrho^2\right) = \sum_{k=1}^n \int_{\mathbb{R}^+} \left[\gamma_k^1(t,z)\, \varrho_k^1(t,z) + \gamma_k^2(t,z)\, \varrho_k^2(t,z) \right] \mathrm{d}\nu_{k,t}^{\mathcal{G}}(z)$$

where $\mathrm{d}\nu_{k,t}^{\mathcal{G}}(z)$ is such as defined in (6.9), and

$$\begin{bmatrix} p^1 \\ p^2 \end{bmatrix} \in \mathbb{R}^2, \quad \varrho^1(t,\cdot) = \begin{bmatrix} \varrho_1^1(t,\cdot) \\ \vdots \\ \varrho_n^1(t,\cdot) \end{bmatrix} \in \mathbb{R}^n, \quad \varrho^2(t,\cdot) = \begin{bmatrix} \varrho_1^2(t,\cdot) \\ \vdots \\ \varrho_n^2(t,\cdot) \end{bmatrix} \in \mathbb{R}^n$$

are unknown $\mathcal{G}$-adapted càdlàg adjoint processes. As in Section 5.1, we again obtain a vanishing gradient vector $\nabla_x H = 0 \in \mathbb{R}^2$. Hence, taking Eq. (3.2.4) in [22] into account (also see Eq. (3.18) in [15]), we arrive at the adjoint equations

$$\mathrm{d}p_t^1 = \int_{\mathbb{R}^+} \varrho^1(t-,z) * \mathrm{d}\tilde{N}^{\mathcal{G},\mathbb{Q}}(t,z), \quad \mathrm{d}p_t^2 = \int_{\mathbb{R}^+} \varrho^2(t-,z) * \mathrm{d}\tilde{N}^{\mathcal{G},\mathbb{Q}}(t,z),$$

where $t \in [0, \tau_1[$ and the respective terminal values are like given in (5.28). Parallel to (5.29), we obtain

$$\begin{aligned} X_{\tau_1}^1 - X_{\tau_1}^2 &= -\frac{1}{2} \left[p_0^1 + \int_0^{\tau_1} \int_{\mathbb{R}^+} \varrho^1(t-,z) * \mathrm{d}\tilde{N}^{\mathcal{G},\mathbb{Q}}(t,z) \right] \\ &= \frac{1}{2} \left[p_0^2 + \int_0^{\tau_1} \int_{\mathbb{R}^+} \varrho^2(t-,z) * \mathrm{d}\tilde{N}^{\mathcal{G},\mathbb{Q}}(t,z) \right] \end{aligned}$$

whereas, on the other hand, we deduce the representation

$$X_{\tau_1}^1 - X_{\tau_1}^2 = -X_0^2 + \int_0^{\tau_1} \int_{\mathbb{R}^+} \left[\gamma^1(t-,z) - \gamma^2(t-,z) \right] * \mathrm{d}\tilde{N}^{\mathcal{G},\mathbb{Q}}(t,z).$$

Comparing the two latter representations, we are led to the equality system

$$p_0^1 = -p_0^2 = 2X_0^2, \quad \varrho^1(t,z) = -\varrho^2(t,z) = 2\left[\gamma^2(t,z) - \gamma^1(t,z)\right].$$

As a consequence of the second equality chain, the Hamiltonian can be expressed as

$$\begin{aligned} H\left(t, X_t^1, X_t^2, \varphi_t, p_t^1, p_t^2, \varrho^1(t,\cdot), \varrho^2(t,\cdot)\right) &= -2\sum_{k=1}^n \int_{\mathbb{R}^+} \left[\gamma_k^1(t,z) - \gamma_k^2(t,z)\right]^2 \mathrm{d}\nu_{k,t}^{\mathcal{G}}(z) \\ &= -2\sum_{k=1}^n \int_{\mathbb{R}^+} \left[\varphi_t \beta_k(t,z,\tau_1,\tau_2) - \eta_k(t,z)\right]^2 \mathrm{d}\nu_{k,t}^{\mathcal{G}}(z) \end{aligned}$$

for all $t \in [0,\tau_1]$. As the deterministic functions $\eta_1, \ldots, \eta_n$ are independent of φ, we find the derivatives

$$\begin{aligned} \partial_\varphi H &= -4\sum_{k=1}^n \int_{\mathbb{R}^+} \left[\varphi_t \beta_k(t,z,\tau_1,\tau_2) - \eta_k(t,z)\right] \beta_k(t,z,\tau_1,\tau_2)\, \mathrm{d}\nu_{k,t}^{\mathcal{G}}(z), \\ \partial_{\varphi\varphi} H &= -4\sum_{k=1}^n \int_{\mathbb{R}^+} \beta_k(t,z,\tau_1,\tau_2)^2 \mathrm{d}\nu_{k,t}^{\mathcal{G}}(z), \end{aligned}$$

where $t \in [0,\tau_1]$. Since $\partial_{\varphi\varphi} H < 0$, the map $\varphi \mapsto H\left(t, X_t^1, X_t^2, \varphi, p_t^1, p_t^2, \varrho^1(t,\cdot), \varrho^2(t,\cdot)\right)$ is concave for all $t \in [0,\tau_1]$. Due to Theorem 3.4 in [22], we thus know that the optimal control $\hat{\varphi} \in \mathcal{A}(\mathcal{G})$ for the optimization problem (7.4) can be found by maximizing the Hamiltonian H with respect to φ. In this regard, we require the optimality condition $\partial_\varphi H = 0$ which immediately leads us to the subsequent proposition.

Proposition 7.1 *For $k \in \{1, \ldots, n\}$ let $\eta_k(t,z)$ be the deterministic function introduced in (7.5), and $\beta_k(t,z,\tau_1,\tau_2)$ be the deterministic function defined in (6.19). Then, under the enlarged filtration $\mathcal{G}_t = \mathcal{F}_t \vee \sigma\left\{L_\tau^1, \ldots, L_\tau^n\right\}$, the minimal variance hedging portfolio process $\hat{\varphi} \in \mathcal{A}(\mathcal{G})$ is for all $0 \leqslant t \leqslant \tau_1 < \tau_2 = \tau$ given by*

$$\hat{\varphi}_t = \frac{\sum_{k=1}^n \int_{\mathbb{R}^+} \eta_k(t,z)\, \beta_k(t,z,\tau_1,\tau_2)\, \mathrm{d}\nu_{k,t}^{\mathcal{G}}(z)}{\sum_{k=1}^n \int_{\mathbb{R}^+} \beta_k(t,z,\tau_1,\tau_2)^2 \mathrm{d}\nu_{k,t}^{\mathcal{G}}(z)} \tag{7.6}$$

where $\mathrm{d}\nu_{k,t}^{\mathcal{G}}(z)$ constitutes the stochastic $(\mathcal{G},\mathbb{Q})$-compensator defined in (6.9).

Remark 7.2 *If we substitute (6.9) into (7.6), we obtain the alternative representation*

$$\hat{\varphi}_t = \frac{\sum_{k=1}^n \int_t^\tau \int_{\mathbb{R}^+} \eta_k(t,z)\, \beta_k(t,z,\tau_1,\tau_2)\, \mathrm{d}N_k(u,z)}{\sum_{k=1}^n \int_t^\tau \int_{\mathbb{R}^+} \beta_k(t,z,\tau_1,\tau_2)^2 \mathrm{d}N_k(u,z)} \tag{7.7}$$

where $N_1, \ldots, N_n$ are the Poisson random measures introduced in (2.2). Note that all integrands appearing in (7.7) are independent of the integrating time parameter u. We further stress that the anticipative optimal portfolio process $\hat{\varphi}_t$ given in (7.6) corresponds to the non-anticipative portfolio provided in (5.8).

Example 7.3 *Consider the claim*

$$C := \int_0^{\tau_1} \chi(t)\, \mathrm{d}F_t^{\mathcal{G}}(\tau_1, \tau_2) = \sum_{k=1}^{n} \int_0^{\tau_1} \int_{\mathbb{R}+} \chi(t)\, \beta_k(t, z, \tau_1, \tau_2)\, \mathrm{d}\tilde{N}_k^{\mathcal{G},\mathbb{Q}}(t, z)$$

where $F^{\mathcal{G}}$ satisfies (7.1), and $\chi(\cdot)$ is an arbitrary deterministic function of time. Comparing the latter equation with (7.5), we see that $\eta_k(t, z) := \chi(t)\, \beta_k(t, z, \tau_1, \tau_2)$ and $\mathbb{E}_{\mathbb{Q}}[C] = 0$. Thus, the minimal variance hedging portfolio provided in (7.6) presently simplifies to $\hat{\varphi}_t = \chi(t)$ for all $t \in [0, \tau_1]$. In the special case where $\chi(t) \equiv 1$, we find

$$C = F_{\tau_1}^{\mathcal{G}}(\tau_1, \tau_2) - F_0^{\mathcal{G}}(\tau_1, \tau_2)$$

and the related minimal variance hedging portfolio is given by $\hat{\varphi}_t \equiv 1$.

Example 7.4 *Consider the claim*

$$C := \int_0^{\tau_1} F_t^{\mathcal{G}}(\tau_1, \tau_2)\, \mathrm{d}t$$

where $F^{\mathcal{G}}$ satisfies (7.1). Taking (7.1) and Fubini's theorem into account, we obtain

$$C = \tau_1 F_0^{\mathcal{G}}(\tau_1, \tau_2) + \sum_{k=1}^{n} \int_0^{\tau_1} \int_{\mathbb{R}+} (\tau_1 - t)\, \beta_k(t, z, \tau_1, \tau_2)\, \mathrm{d}\tilde{N}_k^{\mathcal{G},\mathbb{Q}}(t, z).$$

Comparing the latter equation with (7.5), we observe

$$\eta_k(t, z) := (\tau_1 - t)\, \beta_k(t, z, \tau_1, \tau_2), \quad \mathbb{E}_{\mathbb{Q}}[C] = \tau_1 F_0^{\mathcal{G}}(\tau_1, \tau_2).$$

As a consequence, the minimal variance hedging portfolio provided in (7.6) presently simplifies to a linear function reading $\hat{\varphi}_t = \tau_1 - t$ for all $t \in [0, \tau_1]$.

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