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Invariance times transfer properties

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Abstract Invariance times are stopping times τ such that local martingales with respect to some reduced filtration and an equivalently changed probability measure, stopped before τ, are local martingales with respect to the original model filtration and probability measure. They arise naturally for modeling the default time of a dealer bank, in the mathematical finance context of counterparty credit risk. Assuming an invariance time endowed with an intensity and a positive Azéma supermartingale, this work establishes a dictionary relating the semimartingale calculi in the original and reduced stochastic bases, regarding conditional expectations, martingales, stochastic integrals, random measure stochastic integrals, martingale representation properties, semimartingale characteristics, Markov properties, transition semigroups and infinitesimal generators, and solutions of backward stochastic differential equations.

Keywords Progressive enlargement of filtration, Invariance time, Semimartingale calculus, Markov process, Backward stochastic differential equation, Counterparty risk, Credit risk

2020 Mathematics Subject Classification 60G07, 60G44, 60H10, 91G20, 91G40

1. Introduction

This paper is about a concept in progressive enlargement of filtrations called invariance times introduced by Crépey and Song [15]. Progressive enlargement of filtrations refers to a situation where two filtrations are involved, a smaller and a bigger one, the bigger one making a certain random time in the small one, τ, a stopping time. τ is called an invariance time when local martingales X in the small filtration, once stopped before τ, so $X^{\tau-} := X\mathbb{1}_{[0,\tau)} + X_{\tau-}\mathbb{1}_{[\tau,+\infty)}$, are local martingales in the large one. However, this is not necessarily required to hold under the original probability measure, but only under a possibly modified one. In other words, there exists a measure change that "compensates" the change of filtration. The basic situation, called immersion, is when local martingales in the small filtration do not jump at τ and are local martingales in the large one, for the original probability measure (so no measure change is required). But immersion is tantamount to a certain form of independence between τ and the small filtration [4, Lemma 3.2.1(ii)−(iii)]. This is too restrictive for applications to credit risk in finance, which is

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a lot about the dependence between the market risk represented by the small filtration and the default time τ (especially adverse dependence, dubbed "wrong way risk"). [13, 14] show that invariance times offer a much more flexible framework in this regard.

The "stopping before τ" feature in the above resonates particularly well with a particular application, namely the pricing of the implications for a bank of its own default time τ. Indeed, for the shareholders of the bank, only the pre-default cash flows matter, hence the corresponding pricing equations are stopped before τ. These equations are known as the XVA equations, where X is a catch-up letter to be replaced by C for credit, F for funding or K for capital, while VA stands for valuation adjustment [10, Eqns. (2.12), (2.13), and (2.17)]. The setup of invariance times arises as one which is together flexible enough in terms of credit-market dependence, whilst being amenable to an elegant solution of the XVA equations, by reduction to simpler equations stated with respect to a smaller filtration in which the default risk of the bank only appears through its intensity. However, to perform this reduction rigorously, one needs to relate the stochastic calculi in the small and the large filtrations, regarding conditional expectations, martingales, stochastic integrals, random measure stochastic integrals, martingale representation properties, semimartingale characteristics, Markov properties, transition semigroups and infinitesimal generators, and (eventually) solutions of backward stochastic differential equations (BSDEs). The elaboration of the corresponding transfer properties is the contribution of this paper. Section 2 sets the stage. The conditional expectations transfer formulas of Section 3 underlie most of the subsequent developments of Sections 4–9 that, apart from this common base, are quite independent from each other. Hence the reader can cherrypick freely among these. The BSDE Section 10 puts more or less everything together. Section 11 concludes. Our conditional expectations transfer formulas present a resemblance with earlier formulas stated in terms of a singular measure change but no change of filtration [9, Theorem 1]. The connection between the two approaches is illustrated in the end of the paper.

1.1 Standing notation and terminology

The real line and half-line are denoted by $\mathbb{R}$ and $\mathbb{R}_+$; $|\cdot|$ denotes any Euclidean norm (in the dimension of its argument), $\cdot^\top$ means vector transposition; $\mathcal{B}(E)$ denotes the Borel σ algebra on a metrizable space E; $\boldsymbol{\lambda}$ is the Lebesgue measure on $\mathbb{R}_+$, $\boldsymbol{\delta}_a$ represents a Dirac measure at a point a.

Unless otherwise stated, a function (or process) is real valued; order relationships between random variables (respectively processes) are meant almost surely (respectively in the indistinguishable sense); a time interval is random (in particular, the graph of a random time θ is simply written $[\theta]$). We do not explicitly mention the domain of definition of a function when it is implied by the measurability, e.g. we write "a $\mathcal{B}(\mathbb{R})$ measurable function h (or $h(x)$)" rather than "a $\mathcal{B}(\mathbb{R})$ measurable function h defined on $\mathbb{R}$". For a function $h(\omega, x)$ defined on a product space $\Omega \times E$, we write $h(x)$ (or h_t in the case of a stochastic process), without ω.

We use the terminology of the general theory of processes and of filtrations as given in the books by Dellacherie and Meyer [16] and He, Wang, and Yan [23]. For any semimartingale, always taken in a càdlàg version in this work, X, and for any predictable X integrable process L, the corresponding stochastic integral is denoted by $\int_0^\cdot L_t \mathrm{d}X_t = \int_{(0,\cdot]} L_t \mathrm{d}X_t = L \centerdot X$, with the precedence convention $KL \centerdot X = (KL) \centerdot X$ if K is another predictable process such that KL is X integrable. The stochastic exponential of a semimartingale X is denoted by $\mathcal{E}(X)$. By drift of a special semimartingale (i.e. a semimartingale with locally integrable jumps), we mean the finite variation predictable part of its canonical Doob-Meyer decomposition. Stochastic integrals of random functions with respect to jump measures and their compensations are meant in the sense of [24], to which we also borrow the usage of including the optionality with respect to a reference filtration in the definition of an integer valued random measure. Random measure stochastic integrals and transform of measures by densities are respectively denoted by " $*$ " and " $\cdot$ ". We denote by $\mathcal{P}(\mathfrak{H})$ and $\mathcal{O}(\mathfrak{H})$ the predictable and optional σ fields with respect to a filtration $\mathfrak{H}$.

For any random time θ and càdlàg process X, $\Delta_\theta X$ represents the jump of X at θ. We use the convention that $X_{0-} = X_0$ (hence $\Delta_0 X = 0$) and we write X^θ and $X^{\theta-}$ for the processes X stopped at θ and before θ, i.e.

$$X^\theta = X\mathbb{1}_{[0,\theta)} + X_\theta \mathbb{1}_{[\theta,+\infty)}, \quad X^{\theta-} = X\mathbb{1}_{[0,\theta)} + X_{\theta-}\mathbb{1}_{[\theta,+\infty)}. \tag{1.1}$$

The process X is said to be stopped at θ, respectively before θ, if $X = X^\theta$, respectively $X = X^{\theta-}$. We call compensator of a stopping time θ the compensator of $\mathbb{1}_{[\theta,\infty)}$. We say that θ has an intensity γ if θ is (strictly) positive and that its compensator is given as $\gamma \centerdot \boldsymbol{\lambda}$, for some predictable process γ (vanishing beyond time θ). For any event A, we denote by θ_A the stopping time $\mathbb{1}_A\theta + \mathbb{1}_{A^c}\infty$.

2. Setup

We work on a space Ω equipped with a σ field $\mathcal{A}$, a probability measure $\mathbb{Q}$ on $\mathcal{A}$, and a filtration $\mathfrak{G} = (\mathfrak{G}_t)_{t\in\mathbb{R}_+}$ of sub-σ fields of $\mathcal{A}$ satisfying the usual conditions. We are given a positive (nonnecessarily finite) $\mathfrak{G}$ stopping time τ and a subfiltration $\mathfrak{F} = (\mathfrak{F}_t)_{t\in\mathbb{R}_+}$ of $\mathfrak{G}$ satisfying the usual conditions, with $\mathfrak{F}$ optional and predictable projections denoted by ${}^{o}\cdot$ and ${}^{p}\cdot$. We consider the progressive enlargement of filtrations setup[1] defined by the condition that

$$\forall t \geqslant 0 \ \text{ and } \ B \in \mathfrak{G}_t, \quad \exists B' \in \mathfrak{F}_t \ \text{ such that } \ B \cap \{t < \tau\} = B' \cap \{t < \tau\}. \tag{2.1}$$

Example 2.1 *This holds in particular (but not only, see Section A) in the classical progressive enlargement of filtration setup*

$$\mathfrak{G}_t = \mathfrak{F}_t \vee \sigma(\tau \wedge t) \vee \sigma(\{\tau > t\}), \quad t \in \mathbb{R}_+,$$

i.e. when $\mathfrak{G}$ is the smallest filtration larger than $\mathfrak{F}$ making τ a stopping time.

Equivalently to (2.1)[2], any $\mathfrak{G}$ predictable (resp. optional) process L admits an $\mathfrak{F}$ predictable (resp. optional) process L' [3], dubbed $\mathfrak{F}$ predictable (resp. optional) of L, such that $\mathbb{1}_{(0,\tau]}L = \mathbb{1}_{(0,\tau]}L'$ (resp. $\mathbb{1}_{[0,\tau)}L = \mathbb{1}_{[0,\tau)}L'$). In particular, for any $\mathfrak{G}$ stopping time θ, there exists an $\mathfrak{F}$ stopping time θ', dubbed $\mathfrak{F}$ of θ, such that $\{\theta < \tau\} = \{\theta' < \tau\} \subseteq \{\theta = \theta'\}$.

Given a positive constant T, we work henceforth under the following condition, introduced with its first consequences in [11, Sections 4–6], and which is explored systematically in this work.

Condition (C) τ has a $(\mathfrak{G},\mathbb{Q})$ intensity, the Azéma supermartingale $\mathsf{S} = {}^{o}(\mathbb{1}_{[0,\tau)})$ of τ satisfies $\mathsf{S}_T > 0$ almost surely, and

[1] recalling from [15, Section 2.1] that the proofs of the classical progressive enlargement of filtration results in [28] or Chapitre 20 in [17], although only stated and proved in the specific setup of Example 2.1, all work in the extended setup (2.1).

[2] see [15, Eqn. (2.1)].

[3] the notation $\cdot'$ common to the predictable and optional projections is typically not an issue in practice (whenever useful we write explicitly "predictable" or "optional").

$$\begin{aligned}&\text{there exists a probability measure } \mathbb{P} \text{ equivalent to } \mathbb{Q} \text{ on } \mathfrak{F}_T, \text{ called invariance}\\&\text{probability measure, such that, for any } (\mathfrak{F},\mathbb{P}) \text{ local martingale } P,\\&P^{\tau-} \text{ is a } (\mathfrak{G},\mathbb{Q}) \text{ local martingale on } [0,T].\end{aligned} \tag{2.2}$$

Unless explicitly stated, the reference probability measure is the original measure $\mathbb{Q}$.

The conjunction of (2.1) and (2.2) corresponds to the notion of invariance time τ. Hence all the results of [15] are applicable in this work, sometimes in a stronger version due to the additional assumptions embodied in the first line of the condition (C). In particular:

Lemma 2.1 *Under the condition (C),*

(i) $\{\mathsf{S}_- > 0\} = \{{}^p\mathsf{S} > 0\} = \{\mathsf{S} > 0\} \supseteq [0,T]$;

(ii) *two* $\mathfrak{F}$ *optional processes that coincide before* τ *coincide on* $[0,T]$*; in particular, predictable and optional reductions are uniquely defined on* $[0,T]$*;*

(iii) *invariance probability measures* $\mathbb{P}$ *are uniquely determined on* $\mathfrak{F}_T$*, with* $(\mathfrak{F},\mathbb{Q})$ *density process*

$$\mathcal{Q} := \mathcal{E}\left(\frac{1}{{}^p\mathsf{S}} \boldsymbol{\cdot} \mathcal{M}\right) \text{ on } [0,T], \tag{2.3}$$

a positive $\mathfrak{F}$ *martingale on* $[0,T]$.

Proof (i) By Theorem 3.7 in [15], in the case of an invariance time τ endowed with a $(\mathfrak{G},\mathbb{Q})$ intensity, $\{\mathsf{S}_- > 0\} = \{{}^p\mathsf{S} > 0\} = \{\mathsf{S} > 0\}$. Under the additional assumption "$\mathsf{S}_T > 0$ a.s." that is postulated in the condition (C), we can add "$\supseteq [0,T]$".

(ii) The first part is Lemma 2.3 in [15], which readily implies the second part.

(iii) By Theorem 3.2 in [15] and (i). □

As $\mathbb{P}$ is only used for computations in $\mathfrak{F}$ on $[0,T]$, it only matters on $\mathfrak{F}_T$. Hence, in view of Lemma 2.1(iii), we can talk of "the invariance probability measure $\mathbb{P}$" in our setup. Moreover, by reduction, we may and do assume that the $(\mathfrak{G},\mathbb{Q})$ intensity of τ is of the form $\gamma \mathbb{1}_{(0,\tau]}$, for an $\mathfrak{F}$ predictable process γ uniquely defined on $[0,T]$, by Lemma 2.1(ii). We write $\Gamma = \int_0^{\cdot} \gamma_s \mathrm{d}s$, so that Γ^τ is the $(\mathfrak{G},\mathbb{Q})$ compensator of τ.

3. Conditional expectation transfer formulas

The $(\mathfrak{G}_t,\mathbb{Q})$ and $(\mathfrak{F}_t,\mathbb{P})$ conditional expectations are denoted by $\mathbb{E}_t$ and $\mathbb{E}'_t$ and we drop the index t at time 0.

The following result, the unconditional version of which corresponds to Theorem 4.1 in [11], provides an extension of classical results (see e.g. [4, Chapter 3]) beyond the basic immersion setup where $(\mathfrak{F},\mathbb{P}=\mathbb{Q})$ local martingales are $(\mathfrak{G},\mathbb{Q})$ local martingales without jump at τ.

Theorem 3.1 *For any constant* $t \in [0,T]$*, any* $[t,T]$ *valued* $\mathfrak{F}$ *stopping time* σ*, any* $\mathfrak{F}_\sigma$ *measurable nonnegative random variable* χ*, any* $\mathfrak{F}$ *predictable nonnegative process* K*, and any* $\mathfrak{F}$ *optional nondecreasing process* A *starting from 0, we have, on* $\{t < \tau\}$*,*

$$\mathbb{E}_t[\chi \mathbb{1}_{\{\sigma<\tau\}}] = \mathbb{E}'_t[\chi e^{-(\Gamma_\sigma - \Gamma_t)}], \tag{3.1}$$

$$\mathbb{E}_t[K_\tau \mathbb{1}_{\{\tau \leqslant T\}}] = \mathbb{E}'_t\left[\int_t^T K_s e^{-(\Gamma_s - \Gamma_t)} \gamma_s \mathrm{d}s\right], \tag{3.2}$$

$$\mathbb{E}_t[A_T^{\tau-} - A_t^{\tau-}] = \mathbb{1}_{\{t<\tau\}}\mathbb{E}_t^{'}\left[\int_t^T e^{-(\Gamma_s-\Gamma_t)}\mathrm{d}A_s\right]. \tag{3.3}$$

Proof Consider the $\mathfrak{F}$ canonical Doob-Meyer decomposition $\mathsf{S} = \mathsf{Q} - \mathsf{D}$ of S, where Q (with $\mathsf{Q}_0 = \mathsf{S}_0 = 1$) and D (with $\mathsf{D}_0 = 0$) are the $\mathfrak{F}$ local martingale component and the $\mathfrak{F}$ drift of S. By Lemma 2.2 5) in [15] and Lemma 2.1(i), in the present setup where τ is positive, so that $\mathsf{S}_0 = 1$, and Lemma 2.1(i) is satisfied, S admits the multiplicative decomposition

$$\mathsf{S} = \mathcal{Q}\mathcal{D} \text{ on } [0,T], \tag{3.4}$$

where $\mathcal{Q}$ is the $\mathfrak{F}$ martingale (2.3) on $[0,T]$ and $\mathcal{D} = \mathcal{E}(-\frac{1}{\mathsf{S}_-} \boldsymbol{\cdot} \mathsf{D})$ is an $\mathfrak{F}$ predictable nonincreasing process on $[0,T]$.

For any $B \in \mathfrak{G}_t$ and B' associated with B as in (2.1), we then have by definition of $\mathsf{S}_s = \mathbb{Q}(\tau > s\,|\,\mathfrak{F}_s), s \geqslant 0$, and $\mathfrak{F}_\sigma$ measurability of χ (using also the tower rule and recalling the assumption $\mathsf{S}_T > 0$ in the condition (C)):

$$\mathbb{E}\left[\mathbb{1}_{\{t<\tau\}}\mathbb{E}\big(\chi\mathsf{S}_\sigma/\mathsf{S}_t\big|\mathfrak{F}_t\big)\mathbb{1}_B\right] = \mathbb{E}\left[\mathsf{S}_t\mathbb{E}\big(\mathbb{1}_{B'}\chi\mathsf{S}_\sigma/\mathsf{S}_t|\mathfrak{F}_t\big)\right] = \mathbb{E}\left[\chi\mathsf{S}_\sigma\mathbb{1}_{B'}\right] = \mathbb{E}\left[\chi\mathbb{1}_{\{\sigma<\tau\}}\mathbb{1}_B\right].$$

Hence

$$\mathbb{1}_{\{t<\tau\}}\mathbb{E}\big(\chi\mathsf{S}_\sigma/\mathsf{S}_t\big|\mathfrak{F}_t\big) = \mathbb{E}\big(\mathbb{1}_{\{\sigma<\tau\}}\chi|\mathfrak{G}_t\big). \tag{3.5}$$

Then (3.4), under the assumption $\mathsf{S}_T > 0$ a.s., yields

$$\mathbb{E}\big(\chi\mathsf{S}_\sigma/\mathsf{S}_t\big|\mathfrak{F}_t\big) = \mathbb{E}\Big(\chi\mathcal{Q}_\sigma\mathcal{D}_\sigma/\big(\mathcal{Q}_t\mathcal{D}_t\big)\Big|\mathfrak{F}_t\Big) = \mathbb{E}'\left[\chi\mathcal{D}_\sigma/\mathcal{D}_t\Big|\mathfrak{F}_t\right], \tag{3.6}$$

by Lemma 2.1(iii) and the conditional Bayes formula corresponding to the $\mathbb{Q}$-to-$\mathbb{P}$ density process $\mathcal{Q}$ on $[0,T]$. Moreover, by [15, Lemma A.1 and Lemma 2.1(i)], D is continuous and

$$\mathcal{D}^{\pm 1} = \mathcal{E}(\pm\frac{1}{\mathsf{S}_-}\boldsymbol{\cdot}\mathsf{D}) = e^{\pm\frac{1}{\mathsf{S}_-}\boldsymbol{\cdot}\mathsf{D}}, \quad \frac{1}{\mathsf{S}_-}\boldsymbol{\cdot}\mathsf{D} = \gamma'\boldsymbol{\cdot}\boldsymbol{\lambda} \tag{3.7}$$

hold on $[0,T]$. Recalling that $\Gamma = \gamma\boldsymbol{\cdot}\boldsymbol{\lambda}$, (3.5)–(3.7) yield (3.1).

For (3.2), we compute, on $\{t < \tau\}$,

$$\begin{aligned}\mathbb{E}_t\left[K_\tau\mathbb{1}_{\{\tau\leqslant T\}}\right] &= \mathbb{E}_t\left[\int_t^T K_s\mathbb{1}_{\{s\leqslant\tau\}}\gamma_s\mathrm{d}s\right] = \int_t^T \mathbb{E}_t[K_s\mathbb{1}_{\{s<\tau\}}\gamma_s]\mathrm{d}s\\ &= \int_t^T \mathbb{E}_t^{'}[K_s e^{-(\Gamma_s-\Gamma_t)}\gamma_s]\mathrm{d}s = \mathbb{E}_t^{'}\left[\int_t^T K_s e^{-(\Gamma_s-\Gamma_t)}\gamma_s\mathrm{d}s\right],\end{aligned}$$

where (3.1) was used for passing to the second line.

Regarding (3.3), an application of (3.2) yields (still on $\{t < \tau\}$)

$$\begin{aligned}\mathbb{E}_t\left[(A_{\tau-} - A_t)\mathbb{1}_{\{\tau\leqslant T\}}\right] &= \mathbb{E}_t^{'}\left[\int_t^T (A_s - A_t)e^{-(\Gamma_s-\Gamma_t)}\gamma_s\mathrm{d}s\right]\\ &= -\,\mathbb{E}_t^{'}[(A_T - A_t)e^{-(\Gamma_T-\Gamma_t)}] + \mathbb{E}_t^{'}\left[\int_t^T e^{-(\Gamma_s-\Gamma_t)}\,\mathrm{d}A_s\right].\end{aligned}$$

Using (3.1), we deduce

$$\begin{aligned}&\mathbb{E}_t[(A_T^{\tau-}-A_t)]=\mathbb{E}_t[(A_T-A_t)\mathbb{1}_{\{T<\tau\}}]+\mathbb{E}_t\left[(A_{\tau-}-A_t)\mathbb{1}_{\{\tau\leqslant T\}}\right]\\&=\mathbb{E}_t^{'}[(A_T-A_t)e^{-(\Gamma_T-\Gamma_t)}]-\mathbb{E}_t^{'}[(A_T-A_t)e^{-(\Gamma_T-\Gamma_t)}]+\mathbb{E}_t^{'}\left[\int_t^T e^{-(\Gamma_s-\Gamma_t)}\,\mathrm{d}A_s\right]\\&=\mathbb{E}_t^{'}\left[\int_t^T e^{-(\Gamma_s-\Gamma_t)}\,\mathrm{d}A_s\right].\end{aligned}$$

□

See Section A for a discussion of two alternatives to the formula (3.2) that are known from the mathematical finance literature.

4. Martingale transfer formulas

We denote by:

- $\mathcal{M}_T(\mathfrak{F},\mathbb{P})$, the set of $(\mathfrak{F},\mathbb{P})$ local martingales stopped at T;
- $\mathcal{M}_{\tau-\wedge T}(\mathfrak{G},\mathbb{Q})$, the set of $(\mathfrak{G},\mathbb{Q})$ local martingales stopped at $\tau-\wedge T$, i.e. before τ and at T;
- $\mathcal{M}^c_T(\mathfrak{F},\mathbb{P})$ and $\mathcal{M}^d_T(\mathfrak{F},\mathbb{P})$, respectively $\mathcal{M}^c_{\tau\wedge T}(\mathfrak{G},\mathbb{Q})$, and $\mathcal{M}^d_{\tau-\wedge T}(\mathfrak{G},\mathbb{Q})$, their respective subsets of continuous local martingales and purely discontinuous local martingales.

Lemma 4.1 *For any $M,N\in\mathcal{M}_{\tau-\wedge T}(\mathfrak{G},\mathbb{Q})$, we have*

$$[M,N]'=[M',N'] \text{ on } [0,T], \tag{4.1}$$

where the quadratic variations $[M,N]$ (with $\mathfrak{F}$ optional reduction $[M,N]'$) and $[M',N']$ are respectively meant in $(\mathfrak{G},\mathbb{Q})$ and $(\mathfrak{F},\mathbb{P})$.

Proof As M and N are stopped before τ,

$$[M,N]=[M,N]^{\tau-}=[M',N']^{\tau-}, \tag{4.2}$$

where the quadratic variation $[M',N']$ is meant at this stage in $(\mathfrak{G},\mathbb{Q})$. But since $(M')^T$ and $(N')^T$ are $\mathfrak{F}$ adapted and that $\mathbb{P}$ and $\mathbb{Q}$ are equivalent on $\mathfrak{F}_T$, on $[0,T]$ this quadratic variation $[M',N']$ in $(\mathfrak{G},\mathbb{Q})$ is the same as the quadratic variation $[M',N']$ meant in $(\mathfrak{F},\mathbb{P})$. Hence on $[0,T]$ this quadratic variation and $[M,N]'$ coincide before τ, so that they coincide on $[0,T]$, by Lemma 2.1(ii). □

Theorem 4.1 *The following bijections hold:*

$$\begin{aligned}\mathcal{M}_T(\mathfrak{F},\mathbb{P})&\underset{(\cdot')^T}{\overset{\cdot^{\tau-}}{\rightleftarrows}}\mathcal{M}_{\tau-\wedge T}(\mathfrak{G},\mathbb{Q}),\\ \mathcal{M}^c_T(\mathfrak{F},\mathbb{P})&\underset{(\cdot')^T}{\overset{\cdot^{\tau}}{\rightleftarrows}}\mathcal{M}^c_{\tau\wedge T}(\mathfrak{G},\mathbb{Q}),\\ \mathcal{M}^d_T(\mathfrak{F},\mathbb{P})&\underset{(\cdot')^T}{\overset{\cdot^{\tau-}}{\rightleftarrows}}\mathcal{M}^d_{\tau-\wedge T}(\mathfrak{G},\mathbb{Q}),\end{aligned} \tag{4.3}$$

where $(\cdot')^T$ denotes the $\mathfrak{F}$ optional reduction operator composed with stopping at T.

Proof By the converse part in [15, Lemma 2.2 4] combined with Lemma 2.1(i)–(ii), for any $M\in\mathcal{M}_{\tau-\wedge T}(\mathfrak{G},\mathbb{Q})$, the process $\mathsf{S}_{-}\centerdot M'+[\mathsf{S},M']$ is an $(\mathfrak{F},\mathbb{Q})$ local martingale on $[0,T]$. This in turn implies that $(M')^T\in\mathcal{M}_T(\mathfrak{F},\mathbb{P})$, by Theorem 3.7 in [15] and Lemma 2.1(i). Hence, on $\mathcal{M}_{\tau-\wedge T}(\mathfrak{G},\mathbb{Q})$, the operator $(\cdot')^T$ takes its values in the space $\mathcal{M}_T(\mathfrak{F},\mathbb{P})$. Conversely, for any

$P \in \mathcal{M}_T(\mathfrak{F}, \mathbb{P})$, the condition (C) yields $P^{\tau-} \in \mathcal{M}_{\tau-\wedge T}(\mathfrak{G}, \mathbb{Q})$, i.e. on $\mathcal{M}_T(\mathfrak{F}, \mathbb{P})$ the map $\cdot^{\tau-}$ takes its values in the space $\mathcal{M}_{\tau-\wedge T}(\mathfrak{G}, \mathbb{Q})$.

To establish the first bijection in (4.3) it remains to show that $((M')^T)^{\tau-} = M$ and $((P^{\tau-})')^T = P$ in the above. As M is stopped before τ and at T, the first identity is trivially true. Regarding the second one, $(P^{\tau-})' = P$ holds before τ, hence on $[0, T]$, by Lemma 2.1(ii). Hence $((P^{\tau-})')^T = P$ holds on $\mathbb{R}_+$ (as P is stopped at T).

The second bijection in (4.3) follows by the same steps, noting that the reduction of a continuous process X is continuous on $[0, T]$, by Lemma 2.1(ii) applied to the jump process of X.

To prove the third bijection, following [23, Theorem 7.34], assuming $M \in \mathcal{M}_{\tau-\wedge T}(\mathfrak{G}, \mathbb{Q})$, we take a $(\mathfrak{G}, \mathbb{Q})$ continuous local martingale X and we consider the bracket $[M, X]$. By Lemma 4.1, on $[0, T]$, $[M', X']$ is the $\mathfrak{F}$ optional reduction of $[M, X]$. Consequently, using also Lemma 2.1(ii), $[M, X]^{\tau-} = 0$ on $[0, T]$ if and only if $[M', X'] = 0$ on $[0, T]$. The lemma then follows from the first and second bijections in (4.3). $\square$

5. Transfer of stochastic integrals in the sense of local martingales

Lemma 5.1 *Let $(\theta_n)_{n\geqslant 0}$ be a nondecreasing sequence of $\mathfrak{G}$ stopping times tending to infinity. There exists a nondecreasing sequence $(\sigma_n)_{n\geqslant 0}$ of $\mathfrak{F}$ stopping times such that σ_n tends to infinity and*

$$\theta_n \wedge T \wedge \tau = \sigma_n \wedge T \wedge \tau.$$

Proof We compute, using (3.1) at $t = 0$ for passing to the second line,

$$\begin{aligned}\mathbb{E}'[\mathbb{1}_{\{\theta'_n<T\}} e^{-\Gamma_T}] &\leqslant \mathbb{E}'[\mathbb{1}_{\{\theta'_n<T\}} e^{-\Gamma_{\theta'_n}}] \\ &= \mathbb{E}[\mathbb{1}_{\{\theta'_n<T\}} \mathbb{1}_{\{\theta'_n<\tau\}}] = \mathbb{E}[\mathbb{1}_{\{\theta_n<T\}} \mathbb{1}_{\{\theta_n<\tau\}}] \to 0 \text{ as } n \to \infty.\end{aligned}$$

This implies that $\mathbb{P}[\theta'_n < T] \to 0$. Hence $\mathbb{Q}[\theta'_n < T] \to 0$, as $\mathbb{P}$ is equivalent to $\mathbb{Q}$ on $\mathfrak{F}_T$. The sequence $\sigma_n = (\theta'_n)_{\{\theta'_n<T\}}, n \geqslant 0$, satisfies all the desired properties. $\square$

Lemma 5.2 *Let A be a $\mathfrak{G}$ adapted nondecreasing càdlàg process. The process $A^{\tau-}$ is $(\mathfrak{G}, \mathbb{Q})$ locally integrable on $[0, T]$ if and only if A' is $(\mathfrak{F}, \mathbb{P})$ locally integrable on $[0, T]$.*

Proof First note from [36, Lemma 6.10] (with $\mathsf{S}_T > 0$ a.s. under the condition (C)) that A' is a nondecreasing process on $[0, T]$. Let $(\theta_n)_{n\geqslant 0}$ be a nondecreasing sequence of $\mathfrak{G}$ stopping times tending to infinity. Let $(\sigma_n)_{n\geqslant 0}$ be associated with $(\theta_n)_{n\geqslant 0}$ as in Lemma 5.1. We compute

$$\begin{aligned}\mathbb{E}[\int_0^{\theta_n\wedge T} \mathbb{1}_{\{s<\tau\}} e^{\Gamma_s} \mathrm{d}A_s\ ] &= \mathbb{E}[\int_0^{\theta_n\wedge T\wedge\tau} \mathbb{1}_{\{s<\tau\}} e^{\Gamma_s} \mathrm{d}A_s\ ] \\ = \mathbb{E}[\int_0^{\sigma_n\wedge T\wedge\tau} \mathbb{1}_{\{s<\tau\}} e^{\Gamma_s} \mathrm{d}A'_s\ ] &= \mathbb{E}[\int_0^{\sigma_n\wedge T} \mathbb{1}_{\{s<\tau\}} e^{\Gamma_s} \mathrm{d}A'_s\ ] = \mathbb{E}'[A'_{\sigma_n\wedge T}],\end{aligned}$$

by (3.3) (used at $t = 0$). As Γ is continuous, the factor e^{Γ_s} can be handled by another layer of localization. This implies the result. $\square$

Theorem 5.1 *Let W be a $(\mathfrak{G}, \mathbb{Q})$ local martingale stopped beforeτ and L be a $\mathfrak{G}$ predictable process. The process L is W integrable on $[0, T]$ in the sense of $(\mathfrak{G}, \mathbb{Q})$ local martingales if and only if L' is W' integrable on $[0, T]$ in the sense of $(\mathfrak{F}, \mathbb{P})$ local martingales. If so, then (with "$\centerdot$ in $(\mathfrak{F}, \mathbb{P})$", resp. "$\centerdot$ in $(\mathfrak{G}, \mathbb{Q})$", in reference to the stochastic integrals in $(\mathfrak{F}, \mathbb{P})$ and $(\mathfrak{G}, \mathbb{Q})$)*

$$(L' \centerdot W' \text{ in } (\mathfrak{F}, \mathbb{P}))^{\tau-} = (L \centerdot W \text{ in } (\mathfrak{G}, \mathbb{Q})) \text{ holds on } [0, T].$$

Proof In view of [23, Definition 9.1], we only need to check the local integrability of the processes $\sqrt{\int_0^t L_s^2 \mathrm{d}[W,W]_s}$ and $\sqrt{\int_0^t (L')_s^2 \mathrm{d}[W',W']_s}$ under respectively $(\mathfrak{G},\mathbb{Q})$ and $(\mathfrak{F},\mathbb{P})$. But, on $[0,T]$, $[W',W']$ is the $\mathfrak{F}$ optional reduction of $[W,W]$, by Lemma 4.1. Hence the local integrabilities above are equivalent, because of Lemma 5.2.

To prove the identity between the stochastic integrals when they exist, we first note that the identity holds for any L in the class of $\mathfrak{G}$ predictable bounded step processes. By monotone class theorem, this is then extended to the class of $\mathfrak{G}$ predictable bounded processes L. By stochastic dominated convergence, i.e. Theorem 9.30 in [23], this is extended further to all $\mathfrak{G}$ predictable processes L which are W integrable under $(\mathfrak{G},\mathbb{Q})$. □

6. Transfer of random measures stochastic integrals

Given a Polish space E endowed with its Borel σ algebra $\mathcal{B}(E)$, we recall from[23, Theorem 11.13] that, for any (optional) integer valued random measure π, there exists an E valued optional process β and an optional thin set, of the form $\cup_{n\in\mathbb{N}}[\theta_n]$ for some sequence of stopping times $(\theta_n)_{n\geqslant 0}$, such that

$$\pi = \sum_s \boldsymbol{\delta}_{(s,\beta_s)} \mathbb{1}_{\{s\in\cup_{n\in\mathbb{N}}[\theta_n]\}} \tag{6.1}$$

(where $\boldsymbol{\delta}_{(s,\beta_s)}$ is the Dirac measure at (s,β_s)). By definition, for any nonnegative $\mathcal{A}\times\mathcal{B}(\mathbb{R}_+)\times\mathcal{B}(E)$ measurable function Ψ,

$$\Psi * \pi = \sum_{s<\cdot} \Psi_s(\beta_s) \mathbb{1}_{\{s\in\cup_{n\in\mathbb{N}}[\theta_n]\}} = \sum_{\theta_n<\cdot} \Psi(\theta_n,\beta_{\theta_n}) \mathbb{1}_{\{\theta_n<\infty\}}. \tag{6.2}$$

Lemma 6.1 *The $\mathfrak{G}$ optional integer valued random measure π on $\mathbb{R}_+\times E$ admits an $\mathfrak{F}$ optional reduction, i.e. an $\mathfrak{F}$ optional integer valued random measure π' on $\mathbb{R}_+\times E$ such that $\mathbb{1}_{[0,\tau)}.\pi = \mathbb{1}_{[0,\tau)}\cdot\pi'$.*

Proof We have, for any nonnegative $\mathcal{A}\times\mathcal{B}(\mathbb{R}_+)\times\mathcal{B}(E)$ measurable function Ψ,

$$\begin{aligned}\Psi * (\mathbb{1}_{[0,\tau)}.\pi) &= \sum_{s<\cdot} \mathbb{1}_{\{s<\tau\}} \Psi_s(\beta'_s) \mathbb{1}_{\{s\in\cup_{n\in\mathbb{N}}[\theta_n]\}} \\ &= \sum_{s<\cdot} \mathbb{1}_{\{s<\tau\}} \Psi_s(\beta'_s) \mathbb{1}_{\{s\in\cup_n[\theta'_n]\}} = \Psi * (\mathbb{1}_{[0,\tau)}\cdot\pi'),\end{aligned}$$

where $\pi' = \sum_{s<\cdot} \boldsymbol{\delta}_{(s,\beta'_s)} \mathbb{1}_{\{s\in\cup_n[\theta'_n]\}}$ defines an $\mathfrak{F}$ optional integer valued random measure, by [23, Theorem 11.13]. □

In the remainder of the paper, we fix the space E and a $\mathfrak{G}$ optional integer valued random measure π, with the related notation in the above. We introduce the spaces of random functions $\widehat{\mathcal{P}}(\mathfrak{F}) = \mathcal{P}(\mathfrak{F})\times\mathcal{B}(E)$ and $\widehat{\mathcal{P}}(\mathfrak{G}) = \mathcal{P}(\mathfrak{G})\times\mathcal{B}(E)$. We denote the $(\mathfrak{F},\mathbb{P})$ compensator of $\mu = \pi'$ by ν.

Lemma 6.2 *The $(\mathfrak{G},\mathbb{Q})$ compensator of $\mathbb{1}_{[0,\tau)}\cdot\mu$ is $\mathbb{1}_{[0,\tau]}\cdot\nu$ on $[0,T]$.*

Proof By Lemma 5.2, for any $\Psi\in\widehat{\mathcal{P}}(\mathfrak{G})$ such that the process $|\Psi| * \pi$ is $(\mathfrak{G},\mathbb{Q})$ integrable on $[0,T]$, the processes $|\Psi'| * \mu$ and $|\Psi'| * \nu$ are $(\mathfrak{F},\mathbb{P})$ locally integrable on $[0,T]$. It follows that the process

$$P = \Psi' * \mu - \Psi' * \nu$$

is an $(\mathfrak{F},\mathbb{P})$ local martingale on $[0,T]$ (cf. [23, p. 301]). By the condition (C), the stopped process

$$P^{\tau-} = \mathbb{1}_{[0,\tau)}\Psi' * \mu - \mathbb{1}_{[0,\tau)}\Psi' * \nu = \mathbb{1}_{[0,\tau)}\Psi * \mu - \mathbb{1}_{[0,\tau]}\Psi * \nu$$

is a $(\mathfrak{G}, \mathbb{Q})$ local martingale on $[0, T]$, where $\mathbb{1}_{[0,\tau)}\Psi * \nu = \mathbb{1}_{[0,\tau]}\Psi * \nu$ because τ avoids the predictable stopping times. As $\mathbb{1}_{[0,\tau]}.\nu$ is a $\mathfrak{G}$ predictable random measure, this proves the lemma. □

Theorem 6.1 *For any $\Psi \in \widehat{\mathcal{P}}(\mathfrak{G})$, Ψ is $(\mathbb{1}_{[0,\tau)} \centerdot \mu - \mathbb{1}_{[0,\tau]} \centerdot \nu)$ stochastically integrable in $(\mathfrak{G}, \mathbb{Q})$ on $[0, T]$ if and only if Ψ' is $(\mu - \nu)$ stochastically integrable in $(\mathfrak{F}, \mathbb{P})$ on $[0, T]$. If so, then*

$$\left(\Psi' * (\mu - \nu) \ \text{ in } (\mathfrak{F}, \mathbb{P})\right)^{\tau-} = \left(\Psi * (\mathbb{1}_{[0,\tau)}.\mu - \mathbb{1}_{[0,\tau]}.\nu) \text{ in } (\mathfrak{G}, \mathbb{Q})\right) \text{ holds on } [0, T].$$

Proof In view of [23, Definition 11.16], the integrability relationship between Ψ and Ψ' is the consequence of Lemma 5.2. To prove the identity between the corresponding integrals when they exist, we note that

$$(\Psi' * (\mu - \nu))^{\tau-} \quad \text{and} \quad \Psi * (\mathbb{1}_{[0,\tau)}.\mu - \mathbb{1}_{[0,\tau]}.\nu)$$

are $(\mathfrak{G}, \mathbb{Q})$ purely discontinous local martingales. By virtue of [23, Theorem 7.42 and Definition 11. 16], they are then equal because they have the same jumps, namely

$$\begin{aligned}
\Delta_t(\Psi' * (\mu - \nu))^{\tau-} &= \left(\Psi'_t(\beta'_t)\mathbb{1}_{\{t \in \cup_{n\in\mathbb{N}}[\theta'_n]\}} - \int_{\{t\}\times E} \Psi'_s(e)\nu(\mathrm{d}s, \mathrm{d}e)\right)\mathbb{1}_{\{t<\tau\}} \\
&= \left(\Psi_t(\beta_t)\mathbb{1}_{\{t \in \cup_{n\in\mathbb{N}}[\theta_n]\}} - \int_{\{t\}\times E} \Psi_s(e)\nu(\mathrm{d}s, \mathrm{d}e)\right)\mathbb{1}_{\{t<\tau\}} \\
&= \Delta_t\left(\Psi * (\mathbb{1}_{[0,\tau)}.\mu - \mathbb{1}_{[0,\tau]}.\nu)\right),
\end{aligned}$$

as $\mathbb{1}_{[0,\tau)}.\nu = \mathbb{1}_{[0,\tau]}.\nu$ (because τ avoids the predictable stopping times). □

7. Transfer of martingale representation properties

We consider martingale representations with respect to martingales and compensated jump measures as in [24], which corresponds to the notion of weak representation in [23]. As in [23], when no jump measure is involved, we talk of strong representation.

Let W be a d variate $(\mathfrak{G}, \mathbb{Q})$ local martingale stopped before τ and at T. We assume the random measure π of Section 6 stopped before τ and at T, in the sense that $\cup_{n\in\mathbb{N}}[\theta_n] \subseteq (0, \tau) \cap (0, T]$. We write $B = (W')^T$, $\mu = (\pi')^T$. Let ρ and ν denote the $(\mathfrak{G}, \mathbb{Q})$ compensator of π and the $(\mathfrak{F}, \mathbb{P})$ compensator of μ, so that $\rho = \mathbb{1}_{[0,\tau]}\cdot\nu$, by Lemma 6.2.

Lemma 7.1 *Given $(\mathcal{P}(\mathfrak{G}))^{\times d}$ and $\widehat{\mathcal{P}}(\mathfrak{G})$ measurable integrands L and Ψ, if*

$$M = L \centerdot W + \Psi * (\pi - \rho) \tag{7.1}$$

holds in $(\mathfrak{G}, \mathbb{Q})$, then $(M')^T = L' \centerdot B + \Psi' * (\mu - \nu)$ holds in $(\mathfrak{F}, \mathbb{P})$.

Conversely, given $(\mathcal{P}(\mathfrak{F}))^{\times d}$ and $\widehat{\mathcal{P}}(\mathfrak{F})$ measurable integrands K and Φ, if

$$P = K \centerdot B + \Phi * (\mu - \nu) \tag{7.2}$$

holds in $(\mathfrak{F}, \mathbb{P})$, then $P^{\tau-} = K \centerdot B^{\tau-} + \Phi * (\mathbb{1}_{[0,\tau)}\cdot\mu - \mathbb{1}_{[0,\tau]}\cdot\nu)$ holds in $(\mathfrak{G}, \mathbb{Q})$ on $[0, T]$.

Proof This is the consequence of Theorems 5.1 and 6.1. □

Remark 7.1 *In the representation (7.1), the integrands L and Ψ corresponding to a given process M are unique modulo $\mathrm{d}[W, W]$ (with $\mathrm{d}[W, W]_s$-a.e. in the multivariate sense of [25]) and*

ρ negligible sets, respectively. Likewise, in the representation (7.2), the integrands K and Φ corresponding to a given process P are unique modulo $\mathrm{d}[B,B]$ (with $\mathrm{d}[B,B]_s$-a.e. in the multivariate sense of [25]) and ν negligible sets.

As an immediate consequence of Lemma 7.1:

Theorem 7.1 *The space $\mathcal{M}_{\tau-\wedge T}(\mathfrak{G},\mathbb{Q})$ admits a weak representation by W and π if and only if the space $\mathcal{M}_T(\mathfrak{F},\mathbb{P})$ admits a weak representation by $B=W'$ and $\mu=\pi'$.*

Applying Theorem 7.1 with $\mu\equiv 0$, one obtains the strong martingale representation transfer property.

We refer the reader to [21, 22] for other transfers of martingale representation properties, in respective Brownian and marked point process enlargement of filtration setups (progressive but also initial as already before in [20]) satisfying Jacod's equivalence hypothesis, i.e. the existence of a positive $\mathfrak{F}$ conditional density for τ, as opposed to a semimartingale progressive enlargement of filtration setup under the condition (C) in this work. See also [27] or (until τ) [8] and (also after τ) [7] for rather general transfers of martingale representation properties in a progressive enlargement of filtration setup. From a technical viewpoint, our setup stopped before τ is elementary once the underlying Theorems 5.1 and 6.1 are in place. Of course one cannot say anything beyond τ in our setup, but our motivating application of Example 10.1 never requires to go beyond τ.

8. Semimartingale characteristic triplets transfer formula

Let there be given a semimartingale X stopped before τ in some filtration $\mathfrak{H}$ under a probability measure $\mathbb{M}$, with jump measure π^X. The characteristic triplet of X is composed of:

$b^{X,\mathfrak{H},\mathbb{M}}$, the drift part of the truncated semimartingale $X-(x\mathbb{1}_{\{|x|>1\}})_*\pi^X$;

$a^{X,\mathfrak{H},\mathbb{M}}=\langle X^c,X^c\rangle$, the angle bracket of the continuous martingale part of X (i.e. the diffusion part of X);

$c^{X,\mathfrak{H},\mathbb{M}}=(\pi^X)^{p,\mathfrak{H},\mathbb{M}}$, the predictable dual projection of π^X, called in [23] the Lévy system of X (i.e. the extension to a semimartingale setup of the notion of a Lévy measure).

The following results show that the $(\mathfrak{F},\mathbb{P})$ characteristic triplet of the optional reduction X' of a $(\mathfrak{G},\mathbb{Q})$ semimartingale stopped before τ, X, is the predictable reduction of the $(\mathfrak{G},\mathbb{Q})$ characteristic triplet of X. Moreover, if the $(\mathfrak{G},\mathbb{Q})$ semimartingale X is special, then so is X' and the $(\mathfrak{F},\mathbb{P})$ drift of X' is the predictable reduction of the $(\mathfrak{G},\mathbb{Q})$ drift of X.

Theorem 8.1 *Let $X=X^{\tau-}$ be a $(\mathfrak{G},\mathbb{Q})$ semimartingale stopped before τ and at T. We have*

$$\big(b^{X,\mathfrak{G},\mathbb{Q}},a^{X,\mathfrak{G},\mathbb{Q}},c^{X,\mathfrak{G},\mathbb{Q}}\big)=\Big((b^{(X')^T,\mathfrak{F},\mathbb{P}})^{\tau},(a^{(X')^T,\mathfrak{F},\mathbb{P}})^{\tau},\mathbb{1}_{[0,\tau]}.c^{(X')^T,\mathfrak{F},\mathbb{P}}\Big). \tag{8.1}$$

Proof First note that $(X')^T$ is an $(\mathfrak{F},\mathbb{Q})$ semimartingale, by [36, Lemma 6.5] (with $\mathsf{S}_T>0$ a.s. under the condition (C)), hence an $(\mathfrak{F},\mathbb{P})$ semimartingale as well, which justifies the writings in the right hand side of (8.1). Denoting by $\pi^{(X')^T}$ the $(\mathfrak{F},\mathbb{P})$ jump measure of $(X')^T$, we have $\mathbb{1}_{[0,\tau)}.\pi^X=\mathbb{1}_{[0,\tau)}.\pi^{(X')^T}$. So

$$X-(x\mathbb{1}_{\{|x|>1\}})_*\pi^X=\Big((X')^T-(x\mathbb{1}_{\{|x|>1\}})_*\pi^{(X')^T}\Big)^{\tau-}=(P^c)^{\tau-}+(P^d)^{\tau-}+(b^{(X')^T,\mathfrak{F},\mathbb{P}})^{\tau-}, \tag{8.2}$$

where P is the $(\mathfrak{F},\mathbb{P})$ canonical Doob–Meyer local martingale component of the $(\mathfrak{F},\mathbb{P})$ special semimartingale $(X')^T - (x\mathbb{1}_{\{|x|>1\}})_*\pi^{(X')^T}$, with continuous and purely discontinuous parts P^c and P^d. By the condition (C), $P^{\tau-}$ is a $(\mathfrak{G},\mathbb{Q})$ local martingale. Therefore, we conclude from (8.2) that

$$b^{X,\mathfrak{G},\mathbb{Q}} = (b^{(X')^T,\mathfrak{F},\mathbb{P}})^{\tau-} = (b^{(X')^T,\mathfrak{F},\mathbb{P}})^{\tau}$$

(as $\Delta_\tau b^{(X')^T,\mathfrak{F},\mathbb{P}} = 0$, because τ is totally inaccessible.) Now, applying Lemma 6.2 with $E=\mathbb{R}$, we also conclude

$$c^{X,\mathfrak{G},\mathbb{Q}} = (\pi^X)^{p,\mathfrak{G},\mathbb{Q}} = (\mathbb{1}_{[0,\tau)}.\pi^X)^{p,\mathfrak{G},\mathbb{Q}} = \mathbb{1}_{[0,\tau]}.(\pi^{(X')^T})^{p,\mathfrak{F},\mathbb{P}} = \mathbb{1}_{[0,\tau]}.c^{(X')^T,\mathfrak{F},\mathbb{P}}.$$

Finally, according to the second and third bijections in (4.3), we have

$$(P^c)^{\tau-} \in \mathcal{M}^c_{\tau\wedge T}(\mathfrak{G},\mathbb{Q}), \quad (P^d)^{\tau-} \in \mathcal{M}^d_{\tau-\wedge T}(\mathfrak{G},\mathbb{Q}).$$

Hence we conclude from (8.2) that $X^c = (P^c)^{\tau-}$ is the continuous local martingale part of X in $(\mathfrak{G},\mathbb{Q})$ and therefore

$$a^{X,\mathfrak{G},\mathbb{Q}} = [X^c,X^c] = [(P^c)^{\tau-},(P^c)^{\tau-}] = [P^c,P^c]^\tau = (a^{(X')^T,\mathfrak{F},\mathbb{P}})^\tau. \qquad \square$$

Corollary 8.1 *Suppose that a $(\mathfrak{G},\mathbb{Q})$ semimartingale $X = X^{\tau-}$ is special on $[0,T]$. Then $(X')^T$ is an $(\mathfrak{F},\mathbb{P})$ special semimartingale. Denoting by $\beta^{X,\mathfrak{G},\mathbb{Q}}$ and $\beta^{(X')^T,\mathfrak{F},\mathbb{P}}$ the $(\mathfrak{G},\mathbb{Q})$ drift of X and the $(\mathfrak{F},\mathbb{P})$ drift of $(X')^T$, we have*

$$\beta^{X,\mathfrak{G},\mathbb{Q}} = (\beta^{(X')^T,\mathfrak{F},\mathbb{P}})^\tau. \tag{8.3}$$

Proof As $(X')^T$ is already known to be an $(\mathfrak{F},\mathbb{P})$ semimartingale and because a special semimartingale means one with locally integrable jumps, the special feature of $(X')^T$ follows from Lemma 5.2. Note that, by [23, Lemma 7.16 and Theorem 11.24], the function $|x|\mathbb{1}_{\{|x|>1\}}$ is $c^{(X')^T,\mathfrak{F},\mathbb{P}}$ integrable on $[0,T]$. Consequently

$$\beta^{(X')^T,\mathfrak{F},\mathbb{P}} = b^{(X')^T,\mathfrak{F},\mathbb{P}} + (x\mathbb{1}_{\{|x|>1\}})_* c^{(X')^T,\mathfrak{F},\mathbb{P}}.$$

The analogous $(\mathfrak{G},\mathbb{Q})$ relationship holds for X. Hence (8.3) follows from (8.1). $\square$

9. Markov transfer formulas

In this section we study the transfer of Markov properties between $(\mathfrak{G},\mathbb{Q})$ and $(\mathfrak{F},\mathbb{P})$. The reader is referred to [35, proof of Proposition (60.2)] regarding the definition of the semigroup generated by a Markov family.

We suppose that the filtration $\mathfrak{G}$ is generated by a $(\mathfrak{G},\mathbb{Q})$ quasi-left continuous strong Markov semimartingale X with state space $\mathbb{R}^d$. We denote by an index $\cdot^{(t)}$ everything related to the Markov process X translated by time t. We assume that τ is a terminal time of X, i.e. (see e.g. [5, (3.7) Remark p.108][4]) $\tau = \tau^{(t)} + t$ if $\tau > t$[5]. We assume further that the $(\mathfrak{G},\mathbb{Q})$ intensity process of τ takes the form $\gamma(X_.)\mathbb{1}_{(0,\tau]}$, for some $\mathcal{B}(\mathbb{R}^d)$ measurable function $\gamma \geqslant 0$. Let

$$\mathsf{M}_s = e^{\Gamma_s}\mathbb{1}_{\{s<\tau\}} = e^{\int_0^s \gamma(X_u)\mathrm{d}u}\mathbb{1}_{\{s<\tau\}}.$$

Lemma 9.1 M *is a multiplicative functional of X, i.e. $\mathsf{M}_t = \mathsf{M}_s\mathsf{M}^{(s)}_{t-s}$ holds for any $t \geqslant s \geqslant 0$, and a $(\mathfrak{G},\mathbb{Q})$ local martingale. The multiplicative functional M defines a probability transition function $(\mathcal{T}_t)_{t\in\mathbb{R}_+}$.*

[4] or Definition 1.3 page 98 in the original 1968 edition of their book.

[5] or “ $\tau = \tau \circ \theta_t + t$ ” in the classical notation where θ is the translation operator associated with the Markov process X (but we already use θ to denote a stopping time in this paper).

Proof The first part can be checked by definition of M. For the second part, we check by the Doléans-Dade exponential formula that

$$\mathsf{M} = \mathcal{E}(-\mathbb{1}_{[\tau,\infty)} + \Gamma^\tau).$$

The last part follows from [35, (65.3), proof of Proposition (56.5), Hypothesis (62.9) and Theorem (62.19)]. □

Theorem 9.1 *The reduction X' of X is an $(\mathfrak{F},\mathbb{P})$ strong Markov process on $[0,T]$, with the transition semigroup $(\mathcal{T}_t)_{t\in[0,T]}$.*

Proof For $A \in \mathfrak{F}_s, h$ Borel bounded and $0 < s < s+t \leqslant T$, we have

$$\begin{aligned}
&\mathbb{E}\left[\mathbb{1}_A h(X_{t+s}) e^{\int_0^{t+s}\gamma(X_u)\mathrm{d}u}\mathbb{1}_{\{t+s<\tau\}}\right]\\
&= \mathbb{E}\left[\mathbb{1}_A e^{\int_0^{s}\gamma(X_u)\mathrm{d}u}\mathbb{1}_{\{s<\tau\}} h(X_t^{(s)}) e^{\int_0^{t}\gamma(X_u^{(s)})\mathrm{d}u}\mathbb{1}_{\{t<\tau^{(s)}\}}\right]\\
&= \mathbb{E}\left[\mathbb{1}_A e^{\int_0^{s}\gamma(X_u)\mathrm{d}u}\mathbb{1}_{\{s<\tau\}} \mathcal{T}_t h(X_s)\right],
\end{aligned}$$

which is rewritten in terms of X' through the first expectation transfer formula (3.1) as

$$E^\tau[\mathbb{1}_A h(X'_{t+s})] = E^\tau[\mathbb{1}_A(\mathcal{T}_t h)(X'_s)].$$

This proves that X' is an $(\mathfrak{F},\mathbb{P})$ Markov process with the transition semigroup $(\mathcal{T}_t)_{t\in[0,T]}$. If we rewrite the above computation for s replaced by an $\mathfrak{F}$ stopping time σ, we prove that X' is an $(\mathfrak{F},\mathbb{P})$ strong Markov process on $[0,T]$. □

The next question is how to determine the generator of the semigroup $(\mathcal{T}_t)_{t\in[0,T]}$. We suppose that the Markov process X is of the form $X = (Y,Z)$, for some process Y stopped before τ and some process Z constant $(= 0$, say) before τ, the role of which is to store some information, encoded into the jump of Z at τ, about what happens at τ;

Example 9.1 *We may consider for X the following dynamic copula models of portfolio credit risk, with any of the portfolio default times in the role of τ in this paper:*

(i) *the dynamic Marshall-Olkin copula (DMO) model, shown in [13, Theorem 9.2] to satisfy the condition (C) for $\mathbb{P} = \mathbb{Q}$ there (case of a strict pseudo-stopping time in the terminology of [26, Definition 2.1]);*

(ii) *the dynamic Gaussian copula (DGC) model, shown in [14] to satisfy the condition (C) with $\mathbb{P} \neq \mathbb{Q}$, provided the correlation coefficient $\varrho > 0$ in the model is small enough. In particular, the condition (C) holds in the univariate DGC model (there is then no correlation ϱ involved) of Section A below.*

Suppose further that X solves the following $(\mathfrak{G},\mathbb{Q})$ martingale problem:

$$v(X_t) - \int_0^t \mathcal{L}v(X_s)\mathrm{d}s \ \text{ is a } \ (\mathfrak{G},\mathbb{Q}) \ \text{ local martingale for all } v \in \mathcal{D}(\mathcal{L}),$$

where $\mathcal{L}$ is the generator of X, with domain $\mathcal{D}(\mathcal{L}) \subseteq$ the set of the $\mathcal{B}(\mathbb{R}^d)$ measurable bounded functions. For $u \equiv u(y)$ we define $\widehat{u} \equiv \widehat{u}(y,z)$ by $\widehat{u}(y,z) = u(y)$. Let $\mathcal{D}' = \{u \equiv u(y); \widehat{u} \in \mathcal{D}(\mathcal{L})\}$ and let $\mathcal{L}'$ be the operator on $\mathcal{D}'$ defined by

$$(\mathcal{L}'u)(y) = (\mathcal{L}\widehat{u})(y,0), \quad u \in \mathcal{D}'. \tag{9.1}$$

Theorem 9.2 *We suppose that $(\mathcal{D}', \mathcal{L}')$ satisfies the conditions of [19, Theorem 4.1 of Chapter 4, p.182]. Then $X' = (Y', 0)$, Y' is an $(\mathfrak{F}, \mathbb{P})$ strong Markov process on $[0, T]$, and the generator of Y' is an extension of $(\mathcal{D}', \mathcal{L}')$.*

Proof Clearly, $X' = (Y', 0)$. Hence, Theorem 9.1 implies that Y' is an $(\mathfrak{F}, \mathbb{P})$ strong Markov process on $[0, T]$. For $u \in \mathcal{D}'$,

$$\widehat{u}(X_t) - \int_0^t \mathcal{L}\widehat{u}(X_s)\mathrm{d}s = u(Y_t) - \int_0^t \mathcal{L}\widehat{u}(Y_s, 0)\mathrm{d}s$$

is a $(\mathfrak{G}, \mathbb{Q})$ local martingale. As Y is stopped before τ,

$$u(Y_t)^{\tau-} - \int_0^{t\wedge\tau} \mathcal{L}\widehat{u}(Y_s, 0)\mathrm{d}s \text{ is a } (\mathfrak{G}, \mathbb{Q}) \text{ local martingale stopped before } \tau.$$

The first bijection in (4.3) then implies that

$$u((Y')_t^T) - \int_0^t \mathcal{L}' u((Y')_s^T)\mathrm{d}s \ \text{ is an } \ (\mathfrak{F}, \mathbb{P}) \text{ local martingale on } \ [0, T].$$

Therefore, Y' is the solution of the $(\mathfrak{F}, \mathbb{P})$ martingale problem associated with $\mathcal{L}'$ on $[0, T]$. The result then follows from [19, Theorem 4.1 of Chapter 4, p.182]. □

10. BSDE transfer properties

In this section, τ satisfying the condition (C) on $[0, T]$ as before, we reduce a $(\mathfrak{G}, \mathbb{Q})$ backward stochastic differential equation (BSDE) stopped before τ and at T to a simpler $(\mathfrak{F}, \mathbb{P})$ BSDE stopped at T.

We suppose E Euclidean and $(E, \mathcal{B}(E))$ endowed with a σ finite measure m integrating $(1 \wedge |e|^2)$ on E. We consider the space $\mathbb{L}_0$ of the $\mathcal{B}(E)$ measurable functions u endowed with the topology of convergence in measure induced by m.

Given a $\mathcal{P}(\mathfrak{G}) \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{L}_0)$ measurable function $g = g_t(z, l, \psi)$, we can define, by monotone class theorem, a $\mathcal{P}(\mathfrak{F}) \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{L}_0)$ reduction $g' = g'_t(z, l, \psi)$ of g such that $\mathbb{1}_{(0,\tau]} g = \mathbb{1}_{(0,\tau]} g'$. Let A be a $\mathfrak{G}$ finite variation (càdlàg) process.

Adopting the setup of Section 7, we consider the $(\mathfrak{G}, \mathbb{Q})$ BSDE with data (g, A) and solution sought for as a triplet (Z, L, Ψ), where Z is a $\mathfrak{G}$ adapted process, L is a $(\mathcal{P}(\mathfrak{G}))^{\times d}$ measurable process integrable against $B^{\tau-}$ in $(\mathfrak{G}, \mathbb{Q})$, and Ψ is a $\widehat{\mathcal{P}}(\mathfrak{G})$ measurable function stochastically integrable against $(\mathbb{1}_{[0,\tau)} \cdot \mu - \mathbb{1}_{[0,\tau]} \cdot \nu)$ in $(\mathfrak{G}, \mathbb{Q})$, satisfying in $(\mathfrak{G}, \mathbb{Q})$

$$\begin{cases} \int_0^{\tau\wedge T} |g_s(Z_{s-}, L_s, \Psi_s)|\mathrm{d}s < \infty \ \text{ and } \ \int_0^{\cdot} \mathbb{1}_{\{s<\tau\}} |\mathrm{d}A_s| \ \text{ is } \ (\mathfrak{G}, \mathbb{Q}) \text{ locally integrable on } \ [0, T], \\ Z_t^{\tau-\wedge T} + \int_0^{t\wedge\tau\wedge T} \left(g_s(Z_{s-}, L_s, \Psi_s)\mathrm{d}s + \mathrm{d}A_s^{\tau-}\right) = L \centerdot B_t^{\tau-} + \Psi * (\mathbb{1}_{[0,\tau)} \cdot \mu - \mathbb{1}_{[0,\tau]} \cdot \nu)_t, \quad t \in \mathbb{R}_+, \\ Z \ \text{ vanishes on } \ [\tau \wedge T, +\infty). \end{cases} \tag{10.1}$$

We also consider the $(\mathfrak{F}, \mathbb{P})$ BSDE with data (g', A') and solution sought for as a triplet (U, K, Φ), where U is an $\mathfrak{F}$ adapted process, K is a $(\mathcal{P}(\mathfrak{F}))^{\times d}$ measurable process integrable against B in $(\mathfrak{F}, \mathbb{P})$ on $[0, T]$, and Φ is a $\widehat{\mathcal{P}}(\mathfrak{F})$ measurable function stochastically integrable against $(\mu - \nu)$ in $(\mathfrak{F}, \mathbb{P})$ on $[0, T]$, satisfying in $(\mathfrak{F}, \mathbb{P})$

$$\begin{cases} \int_0^T |g'_s(U_{s-}, K_s, \Phi_s)|\mathrm{d}s < \infty \ \text{ and } \int_0^{\cdot} |\mathrm{d}A'_s| \ \text{ is } (\mathfrak{F}, \mathbb{P}) \text{ locally integrable on } [0,T], \\ U^T_t + \int_0^{t\wedge T} \big(g'_s(U_{s-}, K_s, \Phi_s)\mathrm{d}s + \mathrm{d}A'_s\big) = K \centerdot B_t + \Phi * (\mu - \nu)_t, \quad t \in \mathbb{R}_+, \\ U^T \text{ vanishes on } [T, +\infty) \ \text{ (i.e. } U_T = 0). \end{cases} \tag{10.2}$$

Note that the $(\mathfrak{G}, \mathbb{Q})$ BSDE (10.1) is stopped at $\tau - \wedge T$ (in particular, the terminal condition $Z_T = 0$ only holds on $\{T < \tau\}$), whereas the $(\mathfrak{F}, \mathbb{P})$ BSDE (10.2) is stopped at T.

Example 10.1 *Given a bank with default time τ, a $\mathfrak{G}$ stopping time θ representing the default time of a client of the bank, and a nonnegative $\mathfrak{G}$ optional process G representing the liability of the client to the bank, then the process $A = \int_0^{\cdot} G_s \boldsymbol{\delta}_\theta(\mathrm{d}s)$ represents the counterparty credit exposure of the bank to its client. In this case*

$$|\mathrm{d}A_s| = G_s \boldsymbol{\delta}_\theta(\mathrm{d}s), \quad A^{\tau-} = \int_0^{\cdot} \mathbb{1}_{\{s<\tau\}} G_s \boldsymbol{\delta}_\theta(\mathrm{d}s), \quad A' = \int_0^{\cdot} G'_s \boldsymbol{\delta}_{\theta'}(\mathrm{d}s).$$

The coefficient g represents the risky funding costs of the bank entailed by its own credit riskiness. For the reason explained in the second paragraph of Section 1, all cash flows are stopped before the bank default time τ. This results in a BSDE of the form (10.1) for the valuation of counterparty risk (CVA) and of its funding implications to the bank (FVA). The cost of capital (KVA) also obeys an equation of the form (10.1): see [10, Eqns. (2.12), (2.13), and (2.17)].

10.1 Transfer of local martingale solutions

The result that follows states the equivalence between the $(\mathfrak{G}, \mathbb{Q})$ BSDE (10.1) and the $(\mathfrak{F}, \mathbb{P})$ BSDE (10.2) considered within the above-introduced spaces of solutions for the triplets (Z, L, Ψ) and (U, K, Φ), called local martingale solutions henceforth (in reference to the fact that the right-hand sides in the second lines of (10.1) and (10.2) are then respectively $(\mathfrak{G}, \mathbb{Q})$ and $(\mathfrak{F}, \mathbb{P})$ local martingales).

Theorem 10.1 *The $(\mathfrak{G}, \mathbb{Q})$ BSDE (10.1) and the $(\mathfrak{F}, \mathbb{P})$ BSDE (10.2) are equivalent in their respective spaces of local martingale solutions. Specifically, if (Z, L, Ψ) solves (10.1) in $(\mathfrak{G}, \mathbb{Q})$, then $(U, K, \Phi) = (Z, L, \Psi)'$ solves (10.2) in $(\mathfrak{F}, \mathbb{P})$. Conversely, if (U, K, Φ) solves (10.2) in $(\mathfrak{F}, \mathbb{P})$, then $(Z, L, \Psi) = (\mathbb{1}_{[0,\tau)}U, \mathbb{1}_{[0,\tau]}K, \mathbb{1}_{[0,\tau]}\Phi)$ solves (10.1) in $(\mathfrak{G}, \mathbb{Q})$.*

Proof Through the correspondence stated in the theorem between the involved processes:

- The equivalence between the Lebesgue integrability conditions (first lines) in (10.1) and (10.2) follows from Lemma 5.2;
- The equivalence between the martingale conditions (second lines) in (10.1) and (10.2) follows from Theorems 5.1 and 6.1;
- The terminal condition for U in (10.2) obviously implies the one for $Z = \mathbb{1}_{[0,\tau)}U$ in (10.1), whereas the terminal condition in (10.1) implies $Z_T \mathbb{1}_{\{T<\tau\}} = 0$, hence by taking the $\mathfrak{F}_T$ conditional expectation:

$$0 = \mathbb{E}[Z_T \mathbb{1}_{\{T<\tau\}} | \mathfrak{F}_T] = \mathbb{E}[Z'_T \mathbb{1}_{\{T<\tau\}} | \mathfrak{F}_T] = Z'_T \mathsf{S}_T,$$

yielding $U_T = Z'_T = 0$ (as S_T is positive under the condition (C)). □

10.2 Transfer of square integrable solutions

We now consider the $(\mathfrak{G}, \mathbb{Q})$ BSDE (10.1) and the $(\mathfrak{F}, \mathbb{P})$ BSDE (10.2) within suitable spaces of square integrable solutions.

We assume that the compensator ν of $\mu = \pi'$ is given as $\zeta_t(e)m(\mathrm{d}e)\mathrm{d}t$, where ζ is a nonnegative and bounded integrand in $\mathcal{P}(\mathfrak{F})$. We write, for any $t \geqslant 0$ and $\mathcal{B}(E)$ measurable function u,

$$|u|_t^2 = \int_E u(e)^2 \zeta_t(e) m(\mathrm{d}e).$$

Let also $Y_t^* = \sup_{s\in[0,t]} |Y_s|$, for any càdlàg process Y.

Lemma 10.1 *For any real valued càdlàg $\mathfrak{F}$ adapted process V, respectively nonnegative $\mathfrak{F}$ predictable process X, we have*

$$\mathbb{E}\Big[V_0^2 + \int_0^T e^{\int_0^s \gamma_u \mathrm{d}u} \mathbb{1}_{\{s<\tau\}} \mathrm{d}(V^*)_s^2\Big] = \mathbb{E}'[(V^*)_T^2]; \tag{10.3}$$

$$\mathbb{E}\left[\int_0^T e^{\int_0^s \gamma_u \mathrm{d}u} \mathbb{1}_{\{s<\tau\}}\, X_s \mathrm{d}s\right] = \mathbb{E}'\left[\int_0^T X_s \mathrm{d}s\right]. \tag{10.4}$$

Proof The formula (3.3) used at $t = 0$ yields:

- For $A = \int_0^{\cdot} e^{\int_0^s \gamma_u \mathrm{d}u} \mathrm{d}(V^*)_s^2$,

$$\mathbb{E}\left[\int_0^T e^{\int_0^s \gamma_u \mathrm{d}u} \mathbb{1}_{\{s<\tau\}} \mathrm{d}(V^*)_s^2\right] = \mathbb{E}'[(V^*)_T^2] - \mathbb{E}'[V_0^2];$$

- For $A = \int_0^{\cdot} e^{\int_0^s \gamma_u \mathrm{d}u} X_s \mathrm{d}s$,

$$\mathbb{E}\left[\int_0^T e^{\int_0^s \gamma_u \mathrm{d}u} \mathbb{1}_{\{s<\tau\}}\, X_s \mathrm{d}s\right] = \mathbb{E}'\left[\int_0^T X_s \mathrm{d}s\right].$$

□

Considering the $(\mathfrak{G}, \mathbb{Q})$ BSDE (10.1) for (Z, L, Ψ) and the reduced $(\mathfrak{F}, \mathbb{P})$ BSDE (10.2) for (U, K, Φ), with local martingale solutions (if any) such that

$$(U, K, \Phi) = (Z, L, \Psi)',\ (Z, L, \Psi) = (\mathbb{1}_{[0,\tau)} U, \mathbb{1}_{[0,\tau]} K, \mathbb{1}_{[0,\tau]} \Phi) \tag{10.5}$$

(cf. Theorem 10.1), we define

$$\|(Z, L, \Psi)\|_2^2 = \mathbb{E}\left[|Z_0|^2 + \int_0^T e^{\int_0^s \gamma_u \mathrm{d}u} \mathbb{1}_{\{s<\tau\}} \mathrm{d}(Z^*)_s^2\right] + \mathbb{E}\left[\int_0^T e^{\int_0^s \gamma_u \mathrm{d}u} \mathbb{1}_{\{s<\tau\}} \big(|L_s|^2 + |\Psi_s|_s^2\big)\mathrm{d}s\right],$$

$$(\|(U, K, \Phi)\|_2')^2 = \mathbb{E}'[(U^*)_T^2] + \mathbb{E}'\left[\int_0^T \big(|K_s|^2 + |\Phi_s|_s^2\big)\mathrm{d}s\right].$$

We consider the respective subspaces of square integrable solutions of the $(\mathfrak{G}, \mathbb{Q})$ BSDE (10.1) and of the $(\mathfrak{F}, \mathbb{P})$ BSDE (10.2) defined by $\|\cdot\|_2 < +\infty$ and $Z = 0$ on $[\tau \wedge T, +\infty)$, respectively $\|\cdot\|_2' < +\infty$ and $U^T = 0$ on $[T, +\infty)$ (i.e. $U_T = 0$), dubbed $\|\cdot\|_2$ and $\|\cdot\|_2'$ solutions hereafter.

Theorem 10.2 *Given local martingale solutions (Z, L, Ψ) to the $(\mathfrak{G}, \mathbb{Q})$ BSDE (10.1) and (U, K, Φ) to the reduced $(\mathfrak{F}, \mathbb{P})$ BSDE (10.2), we have*

$$\|(Z, L, \Psi)\|_2 = \|(U, K, \Phi)\|_2'. \tag{10.6}$$

The $(\mathfrak{G},\mathbb{Q})$ *BSDE* (10.1) *considered in terms of* $\|\cdot\|_2$ *solutions and the* $(\mathfrak{F},\mathbb{P})$ *BSDE* (10.2) *considered in terms of* $\|\cdot\|_2'$ *solutions are equivalent through the correspondence* (10.5).

Proof Given respective local martingale solutions (Z,L,Ψ) and (U,K,Φ) to (10.1) and (10.2), then related through (10.5) as seen in Theorem 10.1, Lemma 10.1 applied to $V=U$ and $X=|K_\cdot|^2+|\Phi_\cdot|^2_\cdot$ proves the transfer of norms formula (10.6). Given the equivalence of Theorem 10.1 between (10.1) and (10.2) in the sense of local martingale solutions, their equivalence in the sense of square integrable solutions follows from (10.6). □

10.3 Application

Assuming $\int_0^T|\mathrm{d}A'_s|$ integrable under $\mathbb{P}$ and a (weak) martingale representation of the form studied in Theorem 7.1, we define the processes R and P given as

$$R_t=\mathbb{E}'\left[\int_{t\wedge T}^T \mathrm{d}A'_s\,|\,\mathfrak{F}_t\right]\quad\text{and}\quad P_t=\mathbb{E}'\left[\int_0^T \mathrm{d}A'_s\,|\,\mathfrak{F}_t\right],\qquad t\in\mathbb{R}_+.$$

Let $f_s(v,k,\phi)=g'_s(R_{s-}+v,K^P_s+k,\Phi^P_s+\phi)$, where K^P and Φ^P are the integrands in the representation (7.2) of the $(\mathfrak{F},\mathbb{P})$ martingale P (cf. Remark 7.1).

Proposition 10.1 *Suppose that* $\int_0^T|\mathrm{d}A'_s|$ *is* $\mathbb{P}$ *square integrable and*

(i) *the functions* $\mathbb{R}\ni v\mapsto f_t(v,k,\phi)\in\mathbb{R}$ *are continuous, for each* $(k,\phi)\in\mathbb{R}^d\times\mathbb{L}_0$*. Moreover,* f *is monotonous with respect to* v*, i.e.*

$$(f_t(v_1,k,\phi)-f_t(v_2,k,\phi))(v_1-v_2)\leqslant C(v_1-v_2)^2;$$

(ii) $\mathbb{E}'\int_0^T\sup_{|v|\leqslant c}|f_t(v,0,0)-f_t(0,0,0)|\mathrm{d}t<\infty$ *holds for every positive* c*;*

(iii) f *is Lipschitz continuous with respect to* k *and* ϕ*, i.e.*

$$|f_t(v,k_1,\phi_1)-f_t(v,k_2,\phi_2)|\leqslant C(|k_1-k_2|+|\phi_1-\phi_2|_t);$$

(iv) $\mathbb{E}'\int_0^T|f_t(0,0,0)|^2\mathrm{d}t<+\infty$.

Then the $(\mathfrak{G},\mathbb{Q})$ *BSDE* (10.1) *and the* $(\mathfrak{F},\mathbb{P})$ *BSDE* (10.2) *have unique* $\|\cdot\|_2$ *and* $\|\cdot\|_2'$ *solutions, respectively, and these solutions are related through* (10.5).

Proof Note that $\int_0^T|\mathrm{d}A'_s|$ being $\mathbb{P}$ square integrable implies that $\mathbb{E}'\big[(R^*)^2_T\big]<\infty$. Through the correspondence

$$U=R+V,\quad K^U=K^P+K^V,\quad \Phi^U=\Phi^P+\Phi^V,$$

the $(\mathfrak{F},\mathbb{P})$ BSDE (10.2) for (U,K^U,Φ^U) is equivalent (in both senses of $(\mathfrak{F},\mathbb{P})$ local martingale solutions and $\|\cdot\|_2'$ solutions) to the following $(\mathfrak{F},\mathbb{P})$ BSDE for (V,K^V,Φ^V) :

$$\begin{cases}\int_0^T|f_s(V_{s-},K^V_s,\Phi^V_s)|\mathrm{d}s<\infty,\\ V^T_t+\int_0^{t\wedge T}f_s(V_{s-},K^V_s,\Phi^V_s)\mathrm{d}s=K^V\centerdot B_t+\Phi^V*(\mu-\nu)_t,\\ V_T=0.\end{cases}\tag{10.7}$$

Under the assumptions of the proposition, the $(\mathfrak{F},\mathbb{P})$ BSDE (10.7) for (V,K^V,Φ^V) satisfies the assumptions of [31, Theorem 1]. Hence it has a unique $\|\cdot\|_2'$ solution. So has in turn the BSDE (10.2). The result then follows by an application of Theorem 10.2. □

Remark 10.1 *[31, Theorem 1] is only derived in the case a Poisson measure π, but one can readily check that all their computations and results derived under square integrable assumptions are still valid in our more general integer valued random measure setup. Also, in view of Bouchard, et al. [6], Kruse and Popier [31]'s condition of a quasi-left continuous filtration is in fact not needed.*

The reader is referred to [10, Lemma B.1, Proposition B.1 and Theorem 6.1] and [11, Section 6] for variations on the above results, in the respective cases where f only depends on v (in the notation of Proposition 10.1 above) and no martingale representation property needs to be assumed, or where f is assumed to be Lipschitz (versus monotonous in Proposition 10.1) but also exhibits a dependence on a conditional expected shortfall of a future increment of the martingale part of the solution.

Remark 10.2 *Earlier occurrences of such results are [12, 13], with the difference that these earlier works were about BSDEs stopped at a random time. Our more recent papers, instead, with the motivation recalled in the second paragraph of Section 1, are about BSDEs stopped before a random time: compare e.g. stopping at ϑ in the second line of [12, Eqn. (2.1)] versus stopping before τ in the second part of (10.1).*

Analogous techniques could be used to simplify $(\mathfrak{G}, \mathbb{Q})$ optimal stopping or stochastic control problems into reduced $(\mathfrak{F}, \mathbb{P})$ reformulations: cf., in the case of BSDEs or control problems stopped at time τ (as opposed to stopped before τ in our setup, and without the flexibility induced by our measure change from $\mathbb{Q}$ to a possibly different $\mathbb{P}$), [30] and [29] (assuming that a driving $(\mathfrak{F}, \mathbb{Q})$ Brownian motion, stopped at τ, is a $(\mathfrak{G}, \mathbb{Q})$ martingale), Aksamit, Li, and Rutkowski [1] (who provide some comparative comments with our approach in their Remark 8.2), or [2].

11. Conclusion

The present paper provides new developments on the concept of invariance times τ introduced in [15]. More precisely, this paper provides more complete results under stronger (practical) assumptions, summarized as the condition (C), whereas [15] was mainly about the conditions (B) and (A) corresponding respectively to the Eqns. (2.1) and its combination with (2.2) in the above. The condition (C) was first introduced, along with its first consequences, in [11, Section 4]: it is explored systematically in this work. Whereas [15] was in quest of generality and "minimal conditions", this paper aims at identifying a "comfort zone", provided by the condition (C) as the paper demonstrates, where all the standard apparatus of semimartingale calculus is equivalently available in both a larger stochastic basis $(\mathfrak{G}, \mathbb{Q})$ and a smaller one (making τ a stopping time) $(\mathfrak{F}, \mathbb{P})$. But many problems reformulated under the reduced basis are simpler than their original formulation under the full stochastic basis, whence the benefit of the approach.

Specifically, assuming the condition (C) of an invariance time τ endowed with an intensity and a positive Azéma supermartingale as detailed in Section 2, the present paper establishes a dictionary of transfer properties between the semimartingale calculi in the original and changed stochastic bases $(\mathfrak{G}, \mathbb{Q})$ and $(\mathfrak{F}, \mathbb{P})$:

• Theorem 3.1 extends the classical reduced-form credit risk pricing formulas beyond the basic progressive enlargement of filtration setup where the Azéma supermartingale of τ has no martingale component;

• Theorem 4.1 establishes a bijection between the $(\mathfrak{G}, \mathbb{Q})$ (resp. $(\mathfrak{G}, \mathbb{Q})$ continuous/ $(\mathfrak{G}, \mathbb{Q})$ purely discontinuous) local martingales stopped before τ and the $(\mathfrak{F}, \mathbb{P})$ (resp. $(\mathfrak{F}, \mathbb{P})$ continuous/

$(\mathfrak{F}, \mathbb{P})$ purely discontinuous) local martingales;

• Theorem 5.1 establishes the connection between stochastic integrals in the sense of local martingales in $(\mathfrak{G}, \mathbb{Q})$ and in $(\mathfrak{F}, \mathbb{P})$;

• Theorem 6.1 establishes the connection between the $(\mathfrak{G}, \mathbb{Q})$ and $(\mathfrak{F}, \mathbb{P})$ random measures stochastic integrals;

• Theorem 7.1 establishes the correspondence between (weak or strong) $(\mathfrak{G}, \mathbb{Q})$ and $(\mathfrak{F}, \mathbb{P})$ martingale representation properties;

• Theorem 8.1 yields the relationship between the $(\mathfrak{G}, \mathbb{Q})$ local characteristics of a $\mathfrak{G}$ semimartingale X stopped before τ and the $(\mathfrak{F}, \mathbb{P})$ local characteristics of the $\mathfrak{F}$ semimartingale X', called reduction of X, that coincides with X before τ;

• Theorems 9.1 and 9.2 state conditions under which Markov properties, transition semigroups and infinitesimal generators can be transferred between $(\mathfrak{G}, \mathbb{Q})$ and $(\mathfrak{F}, \mathbb{P})$;

• Theorems 10.1 and 10.2 show the equivalences, within various spaces of solutions, between a nonstandard $(\mathfrak{G}, \mathbb{Q})$ backward SDE (BSDE) stopped before τ and a reduced $(\mathfrak{F}, \mathbb{P})$ BSDE with null terminal condition.

As illustrated in Section A, the notion of invariance time is also related to various approaches that were introduced in the mathematical finance literature for coping with defaultable cash flows based on default intensities. These different approaches could perhaps be related via the generalized Girsanov formulas of [32] and [37]. We leave this for further research.

Appendix

A. Intensity based pricing formulas, survival measure and invariance times

This section puts Theorem 3.1 (specifically, the formula (3.2)) in perspective with [18, Proposition 1] and [9, Theorem 1]. This is done in the setup of the Markov model corresponding to the univariate case in Example 9.1(ii), where the issues at stake can be understood based on Feynman-Kac representations. See [26] for other renewed views on the seminal formulas of [18].

A.1 The univariate dynamic Gaussian copula model

We consider the following single-name version of the dynamic Gaussian copula model of [14]. Let

$$\tau = \Psi\left(\int_0^{+\infty} \varsigma(s)\mathrm{d}B_s\right), \tag{A.1}$$

where Ψ is a continuously differentiable increasing function from $\mathbb{R}$ to $(0, +\infty), \varsigma$ is a Borel function on $\mathbb{R}_+$ such that $\int_0^{+\infty} \varsigma^2(u)\mathrm{d}u = 1$, and B is an $(\mathfrak{F}, \mathbb{Q})$ Brownian motion, with $\mathfrak{F}$ taken as the augmented natural filtration of B. The full model filtration $\mathfrak{G}$ is given as the augmented filtration of the progressive enlargement of $\mathfrak{F}$ by τ. Hence, the random time τ is an $\mathfrak{F}_\infty$ measurable $\mathfrak{G}$ stopping time.

By Theorem 2.2, Lemma 3.2, and Remark 4.1 in [14], the condition (C) holds in this setup, for some probability measure $\mathbb{P}$ distinct from $\mathbb{Q}$ but equivalent to it on $\mathfrak{F}_T$, on which $\mathbb{P}$ is uniquely determined through (2.3). Let

$$h_t = \mathbb{1}_{\{t \geqslant \tau\}}, \quad m_t = \int_0^t \varsigma(s)\mathrm{d}B_s, \quad k_t = (h_t, \tau \wedge t), \quad \nu^2(t) = \int_t^{+\infty} \varsigma^2(s)\mathrm{d}s, \tag{A.2}$$

and assume ν positive for all t. By application of results in [19], one can show that the process (m,k) is $(\mathfrak{G},\mathbb{Q})$ Markov.

Remark A.1 *The reason why we introduce $(\tau\wedge t)$ on top of the indicator h_t in k_t is because of a dependence of the post-τ behavior of the model on the value of τ itself. The state augmentation by $(\tau\wedge t)$ takes care of this path-dependence.*

By definition of S and (A.1) of τ, we have

$$\mathsf{S}_t=\mathbb{Q}(\tau>t\,|\,\mathfrak{F}_t)=\Phi\left(\frac{\Psi^{-1}(t)-m_t}{\nu(t)}\right),\quad t\in\mathbb{R}_+, \tag{A.3}$$

where Φ denotes the standard normal cdf. The process on the right hand side of (A.3) has infinite variation. This shows that the reference filtration $\mathfrak{F}$ is not immersed into the full model filtration $\mathfrak{G}$. This lack of immersion makes it more interesting from the point of view of the different approaches that we want to compare. This is our motivation for working in this particular model in this part.

Theorems 2.2 and 2.4 in [14] show the existence of processes of the form

$$\beta_t=\beta(t,m_t,k_t)\ \text{ and }\ \gamma_t=\gamma\,(t,m_t,k_t)=\gamma_t\mathbb{1}_{(0,\tau]},\quad t\in\mathbb{R}_+, \tag{A.4}$$

for continuous functions β and γ with linear growth in m, such that

$$\begin{array}{l}\mathrm{d}W_t=\mathrm{d}B_t-\beta_t\mathrm{d}t\ \text{ is a }(\mathfrak{G},\mathbb{Q})\text{ Brownian motion and}\\ \text{the process }\gamma\text{ is the }(\mathfrak{G},\mathbb{Q})\ \text{ intensity of }\tau.\end{array} \tag{A.5}$$

The proof of the following result is deferred to Section A.3.

Proposition A.1 *Let a process $m^\star$ satisfy*

$$\mathrm{d}m^\star_t=\varsigma(t)\big(\mathrm{d}W^\star_t+\beta(t,m^\star_t,(0,t))\mathrm{d}t\big),\quad 0\leqslant t\leqslant T, \tag{A.6}$$

starting from $m^\star_0=0$, for some Brownian motion $W^\star$ with respect to some stochastic basis $(\mathfrak{G}^\star,\mathbb{Q}^\star)$. Denoting the $\mathbb{Q}^\star$ expectation by $\mathbb{E}^\star$, we have, for any bounded Borel function $G(t,m)$,

$$\mathbb{E}\big[\mathbb{1}_{\{\tau\leqslant T\}}G(\tau,m_\tau)\big]=\mathbb{E}^\star\left[\int_0^T e^{-\int_0^t\gamma(s,m^\star_s,(0,s))\mathrm{d}s}\gamma(t,m^\star_t,(0,t))G(t,m^\star_t)\mathrm{d}t\right]. \tag{A.7}$$

A.2 Discussion

From (A.2) and (A.4)–(A.5), it holds that

$$\mathrm{d}m_t=\varsigma(t)\big(\mathrm{d}W_t+\beta(t,m_t,k_t)\mathrm{d}t\big),\quad t\in\mathbb{R}_+, \tag{A.8}$$

which, for $t\geqslant\tau$ (so that $k_t=(1,\tau)$), diverges from the specification (A.6). Hence the quadruplet $(m,W,\mathfrak{G},\mathbb{Q})$ is not an eligible choice for $(m^\star,W^\star,\mathfrak{G}^\star,\mathbb{Q}^\star)$ in Proposition A.1. As a consequence, we expect a contrario from Proposition A.1 that

$$\mathbb{E}\big[\mathbb{1}_{\{\tau<T\}}G(\tau,m_\tau)\big]\neq\mathbb{E}\left[\int_0^T e^{-\int_0^t\gamma(s,m_s,(0,s))\mathrm{d}s}\gamma(t,m_t,(0,t))G(t,m_t)\mathrm{d}t\right] \tag{A.9}$$

(except in special cases, including obviously $G=0$). In fact, let

$$V_t=\mathbb{E}\left[\int_t^T e^{-\int_t^s\gamma(u,m_u,(0,u))\mathrm{d}u}\gamma(s,m_s,(0,s))G(s,m_s)\mathrm{d}s\,\Big|\,\mathfrak{G}_t\right],\quad t\in\mathbb{R}_+,$$

so that V_0 is equal to the right hand side in (A.9). By an application of [18, Proposition 1] with

$X = r = 0$ and $h_\cdot = \gamma(\cdot, m_\cdot, (0, \cdot))$ on $[0, T]$ there (noting that any process coinciding with the $(\mathfrak{G}, \mathbb{Q})$ intensity of τ before τ can be used as a process h in their setup), we have

$$\mathbb{E}\big[\mathbb{1}_{\{\tau<T\}} G(\tau, m_\tau)\big] = V_0 - \mathbb{E}(V_\tau - V_{\tau-}). \tag{A.10}$$

In a basic immersed setup, $\mathbb{E}(V_\tau - V_{\tau-})$ vanishes and equality actually holds in (A.9) for any G: see the comments before Section 3 in [18], page 1379 in [9], or following (3.22), (H.3) and Proposition 6.1 in [3]). But beyond this immersion case, $\mathbb{E}(V_\tau - V_{\tau-})$ is typically nonnull and intractable, whence inequality in (A.9) (except in special cases such as $G = 0$).

Instead, an eligible choice in Proposition A.1 consists in using $m^* = m$, $W^* = W$ as per (A.5), $\mathbb{Q}^*$ = the so called survival measure $\mathbb{S}$ with $(\mathfrak{G}, \mathbb{Q})$ density process

$$e^{\int_0^{\cdot\wedge T} \gamma(u, m_u, (0,u))\mathrm{d}u} \mathbb{1}_{\{\tau > \cdot\wedge T\}}$$

(assuming $e^{\int_0^{\tau\wedge T} \gamma(u, m_u, (0,u))\mathrm{d}u}$ integrable under $\mathbb{Q}$), and $\mathfrak{G}^*$ = the $\mathbb{S}$ augmentation $\bar{\mathfrak{G}}$ of $\mathfrak{G}$, obtained by adding to each $\mathfrak{G}_t$ all the $\mathbb{S}$ null sets $A \in \mathcal{A}$ such that $A \subseteq \{\tau \leqslant T\}$. Indeed, as noted in [9, Lemma 1(i)], $\mathbb{1}_{[\tau,+\infty)} = 0$ holds $\mathbb{S}$ almost surely on $[0, T]$, hence (A.6) holds in this setup, while [9, Lemma 1(ii)] shows that $W^* = W$ is a $(\mathfrak{G}^\star, \mathbb{Q}^\star) = (\bar{\mathfrak{G}}, \mathbb{S})$ Brownian motion. The corresponding specification of the formula (A.7) corresponds to [9, Theorem 1]. This "survival measure" idea and terminology were first introduced in [33, 34]. One can thus fix the discrepancy in (A.9) (in a progressive enlargement of filtration setup without immersion) by singularly changing the probability measure $\mathbb{Q}$ to $\mathbb{Q}^* = \mathbb{S}$, while sticking to the original model filtration $\mathfrak{G}$ (or, more precisely, resorting to its $\mathbb{S}$ augmentation $\bar{\mathfrak{G}}$).

Another eligible choice for $(m^\star, W^\star, \mathfrak{G}^\star, \mathbb{Q}^\star)$ in Proposition A.1 is

$$m^* = m, \quad \mathrm{d}W^*_t = \mathrm{d}B_t - \beta(t, m_t, (0,t))\mathrm{d}t, \quad \mathfrak{G}^* = \mathfrak{F}, \quad \mathbb{Q}^* = \mathbb{P}.$$

Indeed, as it follows from Lemma 3.5 and Section 4.4 in [14], this process W^* is an $(\mathfrak{F}, \mathbb{P})$ Brownian motion. Hence, just like $(m, B - \beta(\cdot, m_\cdot, k_\cdot) \centerdot \boldsymbol{\lambda}, \bar{\mathfrak{G}}, \mathbb{S})$ in the previous specification, $(m, B - \beta(\cdot, m_\cdot, (0, \cdot)) \centerdot \boldsymbol{\lambda}, \mathfrak{F}, \mathbb{P})$ satisfies all the conditions required on $(m^*, W^*, \mathfrak{G}^*, \mathbb{Q}^*)$ in Proposition A.1. The formula (A.7) corresponding to this second legit specification for $(m^*, W^*, \mathfrak{G}^*, \mathbb{Q}^*)$ is none other than our formula (3.2) (for $t = 0$).

The approach $(m^*, W^*, \mathfrak{G}^*, \mathbb{Q}^*) = (m, B - \beta(\cdot, m_\cdot, (0, \cdot)) \centerdot \boldsymbol{\lambda}, \mathfrak{F}, \mathbb{P})$ of this paper thus fixes the discrepancy in (A.9) (in a non-immersed setup) by reducing the filtration from $\mathfrak{G}$ to a smaller $\mathfrak{F}$, while changing the probability measure "as little as possible", i.e. equivalently on $\mathfrak{F}_T$. Collin-Dufresne, Goldstein, and Hugonnier [9]'s choice $(m^*, W^*, \mathfrak{G}^*, \mathbb{Q}^*) = (m, B - \beta(\cdot, m_\cdot, k_\cdot) \centerdot \boldsymbol{\lambda}, \bar{\mathfrak{G}}, \mathbb{S})$ does the opposite, touching the filtration as little as possible but changing the measure singularly (in a basic immersive setup, an invariance time approach would not change $\mathbb{Q}$ at all, whereas Collin-Dufresne, Goldstein, and Hugonnier [9]'s measure change would still be singular). But Collin-Dufresne, Goldstein, and Hugonnier [9] only provide a transfer of conditional expectation formulas. The present paper demonstrates how the invariance times approach, instead, results in a transfer of semimartingale calculus as a whole. One concrete motivation for this work is the solution of BSDEs stopped before their terminal time. As Section 10 illustrates, in order to deal with these, conditional expectation formulas are not enough: the entire semimartingale calculus of this paper is required.

A.3 Proof of proposition A.1

By the $(\mathfrak{G}, \mathbb{Q})$ Markov property of the process (m, k), noting that τ is the hitting time of 1 by the h component of the process k, we have

$$\begin{aligned}&\mathbb{E}\big[\mathbb{1}_{\{\tau\leqslant T\}}G(\tau,m_\tau)\,|\,\mathfrak{G}_t\big]=\mathbb{E}\big[\mathbb{1}_{\{\tau<T\}}G(\tau,m_\tau)\,|\,(m_t,k_t)\big]\\&=v(t,m_t,k_t)=u_{h_t}(t,m_t),\ t\in[0,\tau\wedge T],\end{aligned}\tag{A.11}$$

for suitable Borel bounded functions $v(t,m,k)$ and $u_h(t,m)=v(t,m,k=(h,t))$. As a $(\mathfrak{G},\mathbb{Q})$ martingale, the process $u_{h_t}(t,m_t)$, $t\in[0,\tau\wedge T]$, has a vanishing $(\mathfrak{G},\mathbb{Q})$ drift. Hence, by an application of the Itô formula to this process, using (A.5), the pair function $u=(u_0(t,m),u_1(t,m))$ formally solves $u_1=G, u_0(T,\cdot)=0$ and

$$\begin{aligned}&\partial_t u_0(t,m)+\varsigma(t)\beta(t,m,(0,t))\partial_m u_0(t,m)+\frac{\varsigma(t)^2}{2}\partial^2_{m^2}u_0(t,m)\\&+\gamma(t,m,0)\big[G(t,m)-u_0(t,m)\big]=0,\quad t<T,\quad m\in\mathbb{R}.\end{aligned}\tag{A.12}$$

Remark A.2 *At least, the above holds assuming u regular enough for applicability of the Itô formula. Given the discussion nature of this appendix, we content ourselves with the above formal argument, without introducing weak solutions of* (A.12).

Putting together (A.11) *and the Feynman-Kac representation of the solution u_0 of* (A.12) *at the origin yields*

$$\begin{aligned}&\mathbb{E}\big[\mathbb{1}_{\{\tau\leqslant T\}}G(\tau,m_\tau)\big]=u_0(0,0)\\&=\mathbb{E}^\star\left[\int_0^T e^{-\int_0^t\gamma(s,m_s^\star,(0,s))\mathrm{d}s}\gamma(t,m_t^\star,(0,t))G(t,m_t^\star)\mathrm{d}t\right],\end{aligned}$$

for any process $m^\star$ as stated in the proposition, which is therefore proven.

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