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# $\epsilon$-Nash mean-field games for stochastic linear-quadratic systems with delay and applications

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**Abstract** In this paper, we focus on mean-field linear-quadratic games for stochastic large-population systems with time delays. The $\epsilon$-Nash equilibrium for decentralized strategies in linear-quadratic games is derived via the consistency condition. By means of variational analysis, the system of consistency conditions can be expressed by forward-backward stochastic differential equations. Numerical examples illustrate the sensitivity of solutions of advanced backward stochastic differential equations to time delays, the effect of the the population's collective behaviors, and the consistency of mean-field estimates.

**Keywords** Mean-field game, Linear-quadratic problem, Time delay, Large-population, $\epsilon$-Nash equilibrium

**2020 Mathematics Subject Classification** 91A15, 91A06, 93C43, 93C05, 93E20, 93E03

## 1. Introduction

The modelling and analysis of large-population game has attracted much attention in recent years. A major feature of large population system is that the role of individual agents is insignificant, but the role of the whole population is important to each agent. The motivation for studying the problem of large weakly coupled systems is that many complex phenomena arising in fields such as engineering, economy and other fields. Refer to Huang et al. [3, 15, 18, 20].

In a controlled large population system, centralized control is based on the full information of all agents. Therefore, it is more rational and effective to study the decentralized strategies that rely on their own individual states and off-line computation of mass effects. Huang, Caines and Malhame (2003, 2007) [14, 16] and Huang (2010) [13] provided the NCE methodology to address the difficulties caused by the highly complex interactions between agents. The main idea of this methoodology is to approximate the initial large population control problem with this limiting problem by means of a mean-field term. The theory of mean-field game consists in studying the

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limit behavior of the equilibrium to a stochastic differential game was first introduced independently by Lasry and Lions [21] and by Huang et al. [18]. Since then, it has been attracting increasing interest from both the mathematical and the engineering communities. Huge literature can be found in [5, 6, 10, 13]. For cooperative social optimization, Huang et al. [17] developed decentralized cooperative optimization using the mean-field structure in centralized optimal control problems. Feng et al. [11] applied a backward version of person-by-person optimality and constructed an auxiliary control problem for each agent based on decentralized information. For models with a major player, readers can refer to [22], [23] and [19]. Firoozi and Caines [12] studied the major agent with partial observations of its own state, as well as the partial observations of each minor agent of its own state and the major agent state. The existence of $\epsilon$-Nash equilibria and the individual agents' control laws that generates the equilibria were determined by the Separation Principle. Zhang and Li [31] focus on the linear-quadratic mean-field games with individual control domains as convex sets. The decentralized strategies and consistency condition are expressed by the coupled mean-field forward-backward stochastic differential equations with projection operators. Xu and Shi [26] construct a decentralized $\epsilon$-Nash equilibrium for large population linear-quadratic-Gaussian games driven by stochastic differential games with random jumps. Xu and Zhang [27] derived the asymptotic suboptimality of decentralized strategies for the linear-quadratic games for stochastic large population system. Amini et al. [1] considered a graphon game model with a continuum of players, where the dynamics of each participant involve mean-field interactions and individual jumps. Xu and Shen [28] studied decentralized strategy design for linear quadratic mean field games.

Time delays are a common feature in real-world systems to describe history-dependent behavior. In real financial markets, Arriojas et al. [2] used a stochastic differential delay equation to describe the dynamics of the stock price. Due to the wide rangeg of applications of stochastic systems with time-delay properties, their optimizations have been a popular topic. Refer to Chen and Wu [7, 9, 25, 32]. Compared to the dynamic programming principle approach, the maximum principle is a better method to study the optimal control problems for delay systems. However, the stochastic control systems with delay are further complicated by the infinite dimensional state space structure and the lack of an Itô's formula to deal with the delay part of the trajectory. Note that for the mean field stochastic delay systems, the drift coefficients and cost functions of the current state, delayed state and state distribution are coupled. This creates a greater technical difficulty in deriving an approximations between the limiting system and the corresponding closed-loop system. Therefore, it pays to develop the $\epsilon$-Nash equilibrium of the decentralized strategies of the linear quadratic games with delay. Very recently, Zhang [29] considered a stochastic optimal control problem for jump-diffusion system with moving-average and pointwise delays. Zhang [30] studied the stochastic maximum principle for system with mixed delay and gave the adjoint equation consisting of two coupled backward stochastic differential equations, neither of which requires a zero solution.

The innovations and contributions of this paper are as follows:

- Firstly, different from [26] and [27], we focus on a forward mean-field linear-quadratic Gaussian of large population systems in a more general framework where delays in states can be incorporated highly complex interactions between the agents. Therefore, the study in this paper can cover more stochastic phenomena and stochastic optimization problems.

- Secondly, our main goal is to establish a decentralized $\epsilon$-Nash equilibrium for the linear-quadratic Gaussian games with large populations. More precisely, inspired by [31], we first apply the stochastic maximum principle for the optimal decentralized response via Hamiltonian system on the convex control domain to obtain the corresponding explicitly optimal control. Meanwhile, a new mean-field anticipated backward stochastic differential equation is also employed. Then some new approximations are proposed to illustrate the relationship between the limiting optimal control system and the corresponding open-loop large population system.

- Finally, the perturbed open-loop system and perturbed limiting systems are involved to establish the related approximate Nash equilibrium property. It is also shown that the designed decentralization strategy is an $\epsilon$-Nash equilibrium of the original large-population game.

This paper is orgainzed as follows: Section 2 is devoted to the problem formulation and some preliminaries. Section 3 introduces mean-field anticipated backward stochastic differential equation, the limiting control system and the corresponding open-loop large population system. Section 4 verifies the $\epsilon$-Nash equilibrium for the decentralized strategies. In Section 5, we illustrate numerical examples of solving the adjoint equations with different time delays, as well as the effect of the collective behaviors and the consistency of the mean-field estimates. Finally, we conclude our work in Section 6 with some concluding remarks.

## 2. Preliminary results

Let $N$ be a sufficiently large natural number and denote by $\mathcal{N}$ the index set $\{1,2,...,N\}$. Let $T>0$ be a finite-time, $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geqslant 0},\mathbb{P})$ be a complete filtered probability space equipped with a natural filtration $N$-dimensional independent standard Brownian motion $\{W_i(t), 1\leqslant i\leqslant N\}_{0\leqslant t\leqslant T}$. We assume that $\mathbb{F}=\{\mathcal{F}_t,\}_{0\leqslant t\leqslant T}$, $\mathcal{F}=\mathcal{F}_T$ and $\mathcal{F}_t=\mathcal{F}_0$ for $t<0$, let $L^2_{\mathbb{F}}(0,T;\mathbb{R})$ denote the space of all $\mathcal{F}_t$-progressively measurable $\mathbb{R}$-valued processes satisfying $\mathbb{E}\int_0^T|\varphi(t)|^2\mathrm{d}t<\infty$. Let $\mathcal{S}^2_{\mathbb{F}}(0,T;\mathbb{R})$ be the class of $\mathbb{F}$-adapted continuous processes $\varphi(t):\Omega\times[0,T]\to\mathbb{R}$ such that $\mathbb{E}[\sup_{0\leqslant t\leqslant T}|\varphi(y)|^2]<\infty$. $\mathbb{F}^i=\{\mathcal{F}^i_t\}_{0\leqslant T}$ is the natural filtration generated by $\{W_i(t),0\leqslant t\leqslant T\}$ and augmented by all $\mathbb{P}$-null sets in $\mathcal{F}$. $\mathbb{F}=\{\mathcal{F}_t\}_{0\leqslant t\leqslant T}$ is the natural filtration generated by $\{W_i(t),1\leqslant i\leqslant N,0\leqslant t\leqslant T\}$ augmented by all $\mathbb{P}$-null sets in $\mathcal{F}$.

The centralized admissible control $u^i\in\mathcal{U}^c_{ad}$, where the centralized admissible control set $\mathcal{U}^c_{ad}$ is defined as $\mathcal{U}^c_{ad}:=\{u^i(\cdot)|u^i(\cdot)\in L^2_{\mathbb{F}}(0,T;U),\ \ 1\leqslant i\leqslant N\}$. Here $U\subset\mathbb{R}$ is a closed convex set and $L^2_{\mathbb{F}}(0,T;U)$ denotes the space of all $\mathcal{F}_t$-progressively measurable $U$-valued processes satisfying $\mathbb{E}\int_0^T|\varphi(t)|^2\mathrm{d}t<\infty$. The "centralized" means that $\mathbb{F}$ is the centralized information produced by all Brownian motion components. In fact, $\mathbb{F}^i$ is the individual decentralized information of $i$th Brownian motion while $\mathbb{F}$ is the centralized information driven by all Brownian motion components. Moreover, we also define decentralized control as $u_i\in\mathcal{U}^{d,i}_{ad}$, where the decentralized admissible control set $\mathcal{U}^{d,i}_{ad}$ is defined as $\mathcal{U}^{d,i}_{ad}:=\{u^i(\cdot)|u^i(\cdot)\in L^2_{\mathbb{F}^i}(0,T;U)\}$, $1\leqslant i\leqslant N$ and $\mathcal{U}^{d,i}_{ad}\subset\mathcal{U}^c_{ad}$. Let $L^2_{\mathbb{F}^i}(0,T;U)$ denote the space of all $\mathcal{F}^i_t$-progressively measurable $U$-valued processes satisfying $\mathbb{E}\int_0^T|\varphi(t)|^2\mathrm{d}t<\infty$.

We assume that the state $x_i(\cdot)$ of the agent $\mathcal{A}_i$ satisfies the following linear stochastic differential equation with time delay:

$$\begin{cases} \mathrm{d}x_i(t)=[Ax_i(t)+Bx_i(t-\delta)+Cv_i(t)+bx^{(N)}(t)+ex^{(N)}(t-\delta)]\mathrm{d}t+\sigma\mathrm{d}W_i(t),\\ x_i(t)=a_{i0},\quad t\in[-\delta,0], \end{cases}\tag{2.1}$$

where $A,B,C,b$ and $e$ are given constants, for $i=1,...,N$, and $\delta$ is a positive constant and less than $T$. Suppose that $|a_{i0}|<\alpha$ holds for each $i=1,...,N$ where $\alpha$ is a positive constant independent of $N$. $x^{(N)}(t) = \frac{1}{N}\Sigma_{i\in\mathcal{N}}x_i(t)$ and $x^{(N)}(t-\delta) = \frac{1}{N}\Sigma_{i\in\mathcal{N}}x_i(t-\delta)$ are the state average across the whole population standing for the global population effects in macroscale. By [8, 29], (2.1) admits a unique solution $x_i(\cdot) \in \mathcal{L}, i = 1, ..., N$ for any $v_i(\cdot) \in \mathcal{U}_{ad}^c$.

Let $v = (v_1, ..., v_i, ..., v_N)$ be the set of control strategies of all $N$ agents, $v_{-i} = (v_1, v_2, ..., v_{i-1}, v_{i+1}, ..., v_N)$ be the set of control strategies except the $i$th agent $\mathcal{A}_i$. Here are the cost functional for agent $\mathcal{A}_i$:

$$J_i(v_i, v_{-i}) = \frac{1}{2}\mathbb{E}\left\{\int_0^T \left[Q(x_i(t) - \beta x^{(N)}(t))^2 + Rv_i^2(t)\right]\mathrm{d}t + Gx_i^2(T)\right\}, \quad 1 \leqslant i \leqslant N,$$

where $Q \geqslant 0, \beta \in \mathbb{R}, R > 0$ and $G \geqslant 0$ are given constants. Assume that each agent wants to minimize its own cost function by choosing approximately acceptable controls. So our game problem is as follows.

**Problem (LP)** The objective of agent $\mathcal{A}_i$ is to find an admissible control $u_i(\cdot) \in \mathcal{U}_{ad}^c$ such that

$$J_i(u_i, u_{-i}) = \inf_{v_i \in \mathcal{U}_{ad}^c} J_i(v_i, u_{-i}), \tag{2.2}$$

where $u_{-i}$ represents $(u_1, ..., u_{i-1}, u_{i+1}, ..., u_N)$, the strategies of all agents except $\mathcal{A}_i$. Then $u = (u_1, ..., u_N)$ is a Nash-equilibrium point of **Problem (LP).**

The **Problem (LP)** is computationally intensive due to the complex coupling structure between these agents. We construct a number of auxiliary control problems using frozen-state averaged limits and focus on the relationship between the limit system and the corresponding open-loop large population.

## 2.1 The optimal control of limiting system

Let's start with the limiting system. For any $v_i \in \mathcal{U}_{ad}^{d,i}$, the limit state $y_i(\cdot)$ of the agent $\mathcal{A}_i$ $(1 \leqslant i \leqslant N)$ satisfies the following linear stochastic control system:

$$\begin{cases} \mathrm{d}y_i(t) = [Ay_i(t) + By_i(t-\delta) + Cv_i(t) + bx^{(0)}(t) + ex^{(0)}(t-\delta)]\mathrm{d}t + \sigma \mathrm{d}W_i(t), \\ y_i(t) = a_{i0}, \quad t \in [-\delta, 0], \end{cases}$$

where $x^{(0)}(\cdot)$ is a deterministic continuous function and we will give it later. The limiting cost functional is

$$\tilde{J}_i(v_i) = \frac{1}{2}\mathbb{E}\left\{\int_0^T \left[Q(y_i(t) - \beta x^{(0)}(t))^2 + Rv_i^2(t)\right]\mathrm{d}t + Gy_i^2(T)\right\}, \quad 1 \leqslant i \leqslant N.$$

**Problem (LLP)** Our goal is to find an optimal control strategy $\bar{v}_i \in \mathcal{U}_{ad}^{d,i}(1 \leqslant i \leqslant N)$ such that

$$\tilde{J}_i(\bar{v}_i) = \inf_{v_i \in \mathcal{U}_{ad}^{d,i}} \tilde{J}_i(v_i).$$

Then $\bar{v}_i$ is called a decentralized optimal control for **Problem (LLP)**.

Let $(\bar{y}_i(\cdot), \bar{v}_i(\cdot))$ be an optimal pair of the above limiting problem. We introduce the following adjoint process:

$$\begin{cases} -\mathrm{d}p_i(t) = \{Ap_i(t) + B\mathbb{E}^{\mathcal{F}_t}[p_i(t+\delta)]\chi_{[0,T-\delta]} - Q(\bar{y}_i(t) - \beta\mathbb{E}[\bar{y}_i(t)])\}\mathrm{d}t - z_i(t)\mathrm{d}W_i(t), \\ p_i(T) = -G\bar{y}_i(T). \end{cases} \tag{2.3}$$

Here and in what follows, $\chi$ denotes the indicator function.

**Remark 2.1** *We subdivide $[0,T]$ into a finite number of subintervals. By using Burkholder-Davis-Gundy's inequality and contraction mapping principle on every subintervals, (2.3) admits a unique adjoint solution $(p_i(t),z_i(t))\in\mathcal{S}^2_{\mathbb{F}}(0,T;\mathbb{R})\times L^2_{\mathbb{F}}(0,T;\mathbb{R})$. For details, the readers can refer to Theorem 2.1 in [24].*

By Theorem 4.1 in Cadenillas [4], using the similar method, we obtain that the optimal control $\bar{v}_i(\cdot)$ satisfies $\bar{v}_i(t)=-R^{-1}Cp_i(t)$.

Here we write $x^{(0)}(t):=\lim_{N\to+\infty}\sum_{i=1}^N y_i(t)=\mathbb{E}[y_i(t)]$. Then the related Hamiltonian system becomes

$$\begin{cases}\mathrm{d}\bar{y}_i(t)=[A\bar{y}_i(t)+B\bar{y}_i(t-\delta)+C(-R^{-1}Cp_i(t))+b\mathbb{E}[\bar{y}_i(t)]+e\mathbb{E}[\bar{y}_i(t-\delta)]\mathrm{d}t+\sigma\mathrm{d}W_i(t),\\ -\mathrm{d}p_i(t)=\{Ap_i(t)+B\mathbb{E}^{\mathcal{F}_t}[p_i(t+\delta)]\chi_{[0,T-\delta]}-Q(\bar{y}_i(t)-\beta\mathbb{E}[\bar{y}_i(t)])\}\mathrm{d}t-z_i(t)\mathrm{d}W_i(t),\\ p_i(T)=-G\bar{y}_i(T),\\ \bar{y}_i(t)=a_{i0},\quad t\in[-\delta,0].\end{cases}\tag{2.4}$$

## 2.2 Approximation between the limiting system and the open-loop system

Define $\bar{x}^{(N)}(t)=\frac{1}{N}\sum_{i=1}^N\bar{x}_i(t)$. The state for the $i$th agent $\mathcal{A}_i$ satisfies

$$\begin{cases}\mathrm{d}\bar{x}_i(t)=[A\bar{x}_i(t)+B\bar{x}_i(t-\delta)+C(-R^{-1}Cp_i(t))+b\bar{x}^{(N)}(t)+e\bar{x}^{(N)}(t-\delta)]\mathrm{d}t+\sigma\mathrm{d}W_i(t),\\ \bar{x}_i(t)=a_{i0},\quad t\in[-\delta,0],\end{cases}\tag{2.5}$$

where the decentralized control law $u_i(t)=R^{-1}Cp_i(t)$, $1\leqslant i\leqslant N$. Here $p_i(t)$ satisfies

$$\begin{cases}-\mathrm{d}p_i(t)=\{Ap_i(t)+B\mathbb{E}^{\mathcal{F}_t}[p_i(t+\delta)]\chi_{[0,T-\delta]}-Q(\bar{x}_i(t)-\beta\mathbb{E}[\bar{x}_i(t)])\}\mathrm{d}t-z_i(t)\mathrm{d}W_i(t),\\ p_i(T)=-G\bar{x}_i(T).\end{cases}\tag{2.6}$$

The given deterministic continuous function $x^{(0)}(t)$ satisfies

$$\begin{cases}\mathrm{d}x^{(0)}(t)=(A+b)x^{(0)}(t)+(B+e)x^{(0)}(t-\delta)-CR^{-1}C\mathbb{E}[p_i(t)]\mathrm{d}t,\\ x^{(0)}(t)=\dfrac{1}{N}\sum_{i=1}^N a_{i0},\quad t\in[-\delta,0].\end{cases}\tag{2.7}$$

In view of the boundedness of $a_{i0}$, we have $\sup_{0\leqslant t\leqslant T}|x^{(0)}(t)|\leqslant H_0$. We use $H_0$ to denote positive constant, which is independent of $N$ and may vary from row to row.

**Lemma 2.2** *The following estimates hold:*

$$\sup_{0\leqslant t\leqslant T}\mathbb{E}|\bar{x}^{(N)}(t)-x^{(0)}(t)|^2=O\Big(\frac{1}{N}\Big),\tag{2.8}$$

$$\sup_{0\leqslant t\leqslant T}\mathbb{E}\Big||\bar{x}^{(N)}(t)|^2-|x^{(0)}(t)|^2\Big|=O\Big(\frac{1}{\sqrt{N}}\Big),\tag{2.9}$$

$$\sup_{0\leqslant t\leqslant T}\mathbb{E}|\bar{x}_i(t)-\bar{y}_i(t)|^2=O\Big(\frac{1}{N}\Big),\tag{2.10}$$

$$\sup_{0\leqslant t\leqslant T}\mathbb{E}\Big||\bar{x}_i(t)|^2-|\bar{y}_i(t)|^2\Big|=O\Big(\frac{1}{\sqrt{N}}\Big).\tag{2.11}$$

**Proof** We define

$$\Delta(t) := \bar{x}^{(N)}(t) - x^{(0)}(t).$$

By definition, $\bar{x}^{(N)}(t)$ satisfies

$$\begin{cases} \mathrm{d}\bar{x}^{(N)}(t) = [A\bar{x}^{(N)}(t) + B\bar{x}^{(N)}(t-\delta) - \dfrac{1}{N}C^2R^{-1}\sum_{i=1}^N p_i(t) \\ \qquad\qquad +b\bar{x}^{(N)}(t) + e\bar{x}^{(N)}(t-\delta)]\mathrm{d}t + \dfrac{1}{N}\sum_{i=1}^N \sigma \mathrm{d}W_i(t), \\ \bar{x}^{(N)}(t) = \dfrac{1}{N}\sum_{i=1}^N a_{i0}, \quad t \in [-\delta, 0]. \end{cases}$$

Notice (2.7), then $\Delta(t)$ satisfies

$$\begin{cases} \mathrm{d}\Delta(t) = (A+b)\Delta(t) + (B+e)\Delta(t-\delta) - (\dfrac{1}{N}R^{-1}C^2\sum_{i=1}^N p_i(t) \\ \qquad\qquad -R^{-1}C^2\mathbb{E}[p_i(t)])\mathrm{d}t + \dfrac{1}{N}\sum_{i=1}^N \mathrm{d}W_i(t), \\ \Delta(t) = 0, \quad t \in [-\delta, 0]. \end{cases}$$

Applying Itô's formula to $\Delta^2(t)$ and using the inequality $(x+y)^2 \leqslant 2x^2 + 2y^2$, we derive

$$\begin{aligned} \mathbb{E}[\sup_{0\leqslant s\leqslant t} |\Delta(s)|^2] \leqslant 2\mathbb{E} \sup_{0\leqslant s\leqslant t} \bigg| \int_0^s (A+b)\Delta(r) + (B+e)\Delta(r-\delta) \\ + \frac{1}{N}R^{-1}C^2 \sum_{i=1}^N p_i(t) - R^{-1}C^2\mathbb{E}[p_i(t)]\mathrm{d}r \bigg|^2 + 2\mathbb{E} \sup_{0\leqslant s\leqslant t} \bigg| \frac{1}{N}\int_0^s \sum_{i=1}^N \sigma \mathrm{d}W_i(t) \bigg|^2. \end{aligned}$$

By the well-known Cauchy-Schwarz inequality and the B-D-G inequality, we can conclude that there exists a constant $C_0$ independent of $N$ (which may vary line by line) such that

$$\begin{aligned} \mathbb{E}[\sup_{0\leqslant s\leqslant t} |\Delta(s)|^2] \leqslant C_0\mathbb{E} \sup_{0\leqslant s\leqslant t} \bigg[ \int_0^s |\Delta(r)|^2 + |\Delta(r-\delta)|^2 \\ + C_0 \bigg| \frac{1}{N}\sum_{i=1}^N p_i(r) - \mathbb{E}[p_i(r)] \bigg|^2 \mathrm{d}r + \frac{1}{N}C_0 \int_0^s \sigma^2 \mathrm{d}r \bigg]. \end{aligned} \tag{2.12}$$

Therefore, we obtain

$$\begin{aligned} \mathbb{E}[\sup_{0\leqslant s\leqslant t} |\Delta(s)|^2] \leqslant C_0\mathbb{E} \sup_{0\leqslant s\leqslant t} \bigg[ 2\int_0^s |\Delta(r)|^2 \mathrm{d}r \\ + R^{-1}C^2 \bigg| \frac{1}{N}\sum_{i=1}^N p_i(r) - \mathbb{E}[p_i(r)] \bigg|^2 \mathrm{d}r + \frac{1}{N}C_0 \int_0^s \sigma^2 \mathrm{d}r \bigg]. \end{aligned}$$

Moreover, since $\{W_i\}_{i=1}^N$ is $N$-dimensional Brownian motion whose components are independent and identically distributed, we have $(p_i(t), z_i(t))$, $1 \leqslant i \leqslant N$ are independent and identically distributed. We denote $\mu(s) = \mathbb{E}[p_i(s)]$, then we have

$$\mathbb{E}\left|\frac{1}{N}\sum_{i=1}^{N}p_i(t)-\mathbb{E}[p_i(t)]\right|^2=\frac{1}{N^2}\mathbb{E}\sum_{i=1}^{N}|p_i(t)-\mu(t)|^2 + \frac{2}{N^2}\mathbb{E}\left[\sum_{i=1,j=1,i\neq j}<p_i(t)-\mu(t),p_j(t)-\mu(t)>\right] \tag{2.13}$$

and

$$\begin{aligned}&\mathbb{E}\left[\sum_{i=1,j=1,i\neq j}(p_i(t)-\mu(t))(p_j(t)-\mu(t))\right]\\ &=\sum_{i=1,j=1,i\neq j}(\mathbb{E}[p_i(t)]-\mu(t))(\mathbb{E}[p_j(t)]-\mu(t))\\ &=0.\end{aligned} \tag{2.14}$$

Substituting (2.14) into (2.13) and integrating from 0 to $t$, we derive

$$\begin{aligned}\mathbb{E}\int_0^t\left|\frac{1}{N}\sum_{i=1}^{N}p_i(r)-\mathbb{E}[p_i(r)]\right|^2\mathrm{d}r&\leqslant\frac{1}{N^2}\mathbb{E}\int_0^t\sum_{i=1}^{N}|p_i(r)-\mu(r)|^2\mathrm{d}r\\ &\leqslant\frac{1}{N}\int_0^t\mathbb{E}|p_i(r)-\mu(r)|^2\mathrm{d}r\\ &=O\Big(\frac{1}{N}\Big).\end{aligned} \tag{2.15}$$

Substituting (2.15) into (2.12) and noticing that

$$\int_0^s|\Delta(r-\delta)|^2\mathrm{d}r=\int_{-\delta}^{s-\delta}|\Delta(r)|^2\mathrm{d}r\leqslant\int_0^s|\Delta(r)|^2\mathrm{d}r,$$

we give

$$\mathbb{E}\left[\sup_{0\leqslant s\leqslant t}|\Delta(s)|^2\right]\leqslant C_0\mathbb{E}\int_0^t|\Delta(s)|^2\mathrm{d}s+O\Big(\frac{1}{N}\Big).$$

By virtue of Gronwall's inequality, we get

$$\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}|\Delta(t)|^2\right]=O\Big(\frac{1}{N}\Big).$$

Therefore, we obtain the first estimate (2.8).

Next, since

$$\mathbb{E}\left||\bar{x}^{(N)}(t)|^2-|x^{(0)}(t)|^2\right|\leqslant\mathbb{E}|\bar{x}^{(N)}(t)-x^{(0)}(t)|^2+2|x^{(0)}(t)|(\mathbb{E}|\bar{x}^{(N)}(t)-x^{(0)}(t)|^2)^{\frac{1}{2}}.$$

From (2.8) and the boundedness of $x^{(0)}(t)$ we can get the result (2.9). By the first equation of (2.4) and (2.5), we have

$$\begin{cases}\mathrm{d}(\bar{x}_i(t)-\bar{y}_i(t))=[A(\bar{x}_i(t)-\bar{y}_i(t))+B(\bar{x}_i(t-\delta)-\bar{y}_i(t-\delta))\\ \qquad +b(\bar{x}^{(N)}(t)-\mathbb{E}[\bar{y}_i(t)])+e(\bar{x}^{(N)}(t-\delta)-\mathbb{E}[\bar{y}_i(t-\delta)])]\mathrm{d}t,\\ \bar{x}_i(t)-\bar{y}_i(t)=0.\end{cases}$$

The classical estimate for the stochastic differential equation yields that

$$\mathbb{E}\sup_{0\leqslant t\leqslant T}|\bar{x}_i(t)-\bar{y}_i(t)|^2\leqslant C_0\mathbb{E}\int_0^T\left|\bar{x}^{(N)}(t)-\mathbb{E}[\bar{y}_i(t)]\right|^2\mathrm{d}t.$$

By (2.8), we have

$$\sup_{0\leqslant t\leqslant T}\mathbb{E}|\bar{x}_i(t)-\bar{y}_i(t)|^2=O\Big(\frac{1}{N}\Big).$$

The last conclusion is proved similarly. □

**Proposition 2.3** *For* $1\leqslant i\leqslant N$, *we have*

$$|J_i(u_i,u_{-i})-\tilde{J}(\bar{v}_i)|=O\Big(\frac{1}{\sqrt{N}}\Big).$$

**Proof** We have

$$\begin{aligned}|J_i(u_i,u_{-i})-\tilde{J}(\bar{v}_i)|=\frac{1}{2}\mathbb{E}\Bigg[&\int_0^T Q_t<\bar{x}_i(t)-\bar{x}^{(N)}(t),\bar{x}_i(t)-\bar{x}^{(N)}(t)>\\&-Q_t<\bar{y}_i(t)-x^{(0)}(t),\bar{y}_i(t)-x^{(0)}(t)>\mathrm{d}t+<G(\bar{x}_i(T)-\bar{x}^{(N)}(T)),\bar{x}_i(T)\\&-\bar{x}^{(N)}(T)>-<G(\bar{y}_i(T)-x^{(0)}(T)),\bar{y}_i(T)-x^{(0)}(T)>\Bigg].\end{aligned}\tag{2.16}$$

From Lemma 2.2 and the boundedness of $x^{(0)}(t)$, for some constant $C_0$ independent of $N$ which may vary line by line in the following, we obtain

$$\begin{aligned}&\left|\mathbb{E}\int_0^T\Big[Q_t(\bar{x}_i(t)-\bar{x}^{(N)}(t))(\bar{x}_i(t)-\bar{x}^{(N)}(t))-Q_t(\bar{y}_i(t)-x^{(0)}(t))(\bar{y}_i(t)-x^{(0)}(t))\Big]\mathrm{d}t\right|\\&=\left|\mathbb{E}\int_0^T\Big[Q_t(\bar{x}_i(t)-\bar{x}^{(N)}(t))^2-Q_t(\bar{y}_i(t)-x^{(0)}(t))^2\Big]\mathrm{d}t\right|\\&=\left|\mathbb{E}\int_0^T\Big[Q_t(\bar{x}_i(t)-\bar{x}^{(N)}(t)-\bar{y}_i(t)+x^{(0)}(t))(\bar{x}_i(t)-\bar{x}^{(N)}(t)+\bar{y}_i(t)-x^{(0)}(t))\Big]\mathrm{d}t\right|\\&=\left|\mathbb{E}\int_0^T\Big[Q_t\Big((\bar{x}_i(t)-\bar{y}_i(t))-(\bar{x}^{(N)}(t)-x^{(0)}(t))\Big)\Big(\bar{x}_i(t)-\bar{x}^{(N)}(t)+\bar{y}_i(t)-x^{(0)}(t)\Big)\Big]\mathrm{d}t\right|\\&\leqslant C_0\left(\int_0^T\mathbb{E}|\bar{x}_i(t)-\bar{y}_i(t)|^2\mathrm{d}t\right)^{\frac{1}{2}}+C_0\left(\int_0^T\mathbb{E}|\bar{x}^{(N)}(t)-x^{(0)}(t)|^2\mathrm{d}t\right)^{\frac{1}{2}}\\&=O\left(\frac{1}{\sqrt{N}}\right).\end{aligned}\tag{2.17}$$

With the similar argument, one can show that

$$\begin{aligned}&\mathbb{E}\Big[(G(\bar{x}_i(T)-\bar{x}^{(N)}(T)))(\bar{x}_i(T)-\bar{x}^{(N)}(T))-(G(\bar{y}_i(T)-x^{(0)}(T))(\bar{y}_i(T)-x^{(0)}(T))\Big]\\&=O\Big(\frac{1}{\sqrt{N}}\Big).\end{aligned}\tag{2.18}$$

Substituting (2.17) and (2.18) into (2.16), for all $1\leqslant i\leqslant N$, we have $|J_i(u_i,u_{-i})-\tilde{J}(\bar{v}_i)|=O\Big(\frac{1}{\sqrt{N}}\Big)$. □

## 3. $\epsilon$-Nash equilibrium analysis

We will introduce two systems-a perturbed open-loop system and a perturbed limiting system. For any fixed $i$, $1 \leqslant i \leqslant N$, we consider an admissible alternative control $v_i \in \mathcal{U}_{ad}^c$ of the agent $\mathcal{A}_i$, and introduce the dynamics

$$\begin{cases} \mathrm{d}\tilde{x}_i(t) = [A\tilde{x}_i(t) + B\tilde{x}_i(t-\delta) + Cv_i(t) + b\tilde{x}^{(N)}(t) + e\tilde{x}^{(N)}(t-\delta)]\mathrm{d}t + \sigma \mathrm{d}W_i(t), \\ \tilde{x}_i(0) = a_{i0}, \end{cases}$$

whereas other agents keep the control $u_j(\cdot)$, $1 \leqslant j \leqslant N, j \neq i$, i.e.

$$\begin{cases} \mathrm{d}\tilde{x}_j(t) = [A\tilde{x}_j(t) + B\tilde{x}_j(t-\delta) + C(-CR^{-1}p_j(t)) + b\tilde{x}^{(N)}(t) + e\tilde{x}^{(N)}(t-\delta)]\mathrm{d}t + \sigma \mathrm{d}W_j(t), \\ \tilde{x}_j(0) = a_{i0}, \end{cases} \tag{3.1}$$

where $\tilde{x}^{(N)}(t) = \frac{1}{N}\sum_{i=1}^N \tilde{x}_i(t)$.

The corresponding cost functional for agent $\mathcal{A}_i$ is given as follows:

$$\begin{aligned} J_i(v_i, u_{-i}) = \frac{1}{2}\mathbb{E}\Big[\int_0^T [Q_t(\tilde{x}_i(t) - \tilde{x}^{(N)}(t))(\tilde{x}_i(t) - \tilde{x}^{(N)}(t)) + Rv_i^2(t)]\mathrm{d}t \\ + G(\tilde{x}_i(T) - \tilde{x}^{(N)}(T))(\tilde{x}_i(T) - \tilde{x}^{(N)}(T)))\Big]. \end{aligned}$$

Then we introduce the perturbed limiting system

$$\begin{cases} \mathrm{d}\tilde{y}_i(t) = (A\tilde{y}_i(t) + B\tilde{y}_i(t-\delta) + Cv_i(t) + bx^{(0)}(t) + ex^{(0)}(t-\delta))\mathrm{d}t + \sigma \mathrm{d}W_i(t), \\ \mathrm{d}\tilde{y}_j(t) = (A\tilde{y}_j(t) + B\tilde{y}_j(t-\delta) + C(-CR^{-1}p_j(t)) + bx^{(0)}(t) + ex^{(0)}(t-\delta))\mathrm{d}t + \sigma \mathrm{d}W_j(t), \\ \tilde{y}_i(t) = a_{i0}, \quad \tilde{y}_j(t) = a_{j0}, \quad t \in [-\delta, 0]. \end{cases} \tag{3.2}$$

The corresponding cost functional for agent $\mathcal{A}_i$ is

$$\begin{aligned} \tilde{J}_i(v_i) = \frac{1}{2}\mathbb{E}\Big[\int_0^T [Q_t(\tilde{y}_i(t) - x^{(0)}(t))(\tilde{y}_i(t) - x^{(0)}(t)) + Rv_i^2(t)]\mathrm{d}t \\ + G(\tilde{y}_i(T) - x^{(0)}(T))(\tilde{y}_i(T) - x^{(0)}(T))\Big]. \end{aligned}$$

Then we will give the approximation between these two perturbed systems.

**Proposition 3.1** *The following estimates hold:*

$$\sup_{0 \leqslant t \leqslant T} \mathbb{E}|\tilde{x}^{(N)}(t) - x^{(0)}(t)|^2 = O\Big(\frac{1}{N}\Big), \tag{3.3}$$

$$\sup_{0 \leqslant t \leqslant T} \mathbb{E}|\tilde{x}_i(t) - \tilde{y}_i(t)|^2 = O\Big(\frac{1}{N}\Big). \tag{3.4}$$

**Proof** Note that $\tilde{x}^{(N)}(t) = \frac{1}{N}\sum_{i=1}^N \tilde{x}_i(t)$, we have

$$\begin{cases} \mathrm{d}\tilde{x}^{(N)}(t) = \Big[(A+b)\tilde{x}^{(N)}(t) + (B+e)\tilde{x}^{(N)}(t-\delta) + \frac{1}{N}Cv_i(t) \\ \qquad +\frac{1}{N}C\sum_{j=1,j\neq i}(-CR^{-1}p_j(t))\Big]\mathrm{d}t + \frac{1}{N}\sum_{j=1}^{N}\sigma\mathrm{d}W_j(t), \\ \tilde{x}^{(N)}(0) = \frac{1}{N}\sum_{j=1}^{N}a_{j0}. \end{cases}$$

By (2.7) and let us denote $\Pi(t) := \tilde{x}^{(N)}(t) - x^{(0)}(t)$, we have

$$\begin{cases} \mathrm{d}\Pi(t) = \Big((A+b)\Pi(t) + (B+e)\Pi(t-\delta) + \frac{1}{N}Cv_i(t) + \frac{1}{N}C\sum_{j=1,j\neq i}(-CR^{-1}p_j(t)) \\ \qquad +C^2R^{-1}\mathbb{E}[p_i(t)]\Big)\mathrm{d}t + \frac{1}{N}\sum_{j=1}^{N}\sigma\mathrm{d}W_j(t), \\ \Pi(0) = 0. \end{cases}$$

By the Cauchy-Schwarz inequality and the B-D-G inequality, we can conclude that there exists a constant $C_0$ independent of $N$, which may vary line by line such that, for any $t \in [0,T]$

$$\begin{aligned} \mathbb{E}\sup_{0\leqslant s\leqslant t}|\Pi(s)|^2 \leqslant{}& C_0\mathbb{E}\int_0^t \left[|\Pi(s)|^2 + |\Pi(s-\delta)|^2 + \frac{1}{N^2}|v_i(s)|^2\right]\mathrm{d}s \\ &+ C_0\mathbb{E}\int_0^t\left[\left|\frac{1}{N}\sum_{j=1,j\neq i}^{N}p_j(s) - \mathbb{E}[p_i(s)]\right|^2\right]\mathrm{d}s + \frac{C_0}{N^2}\mathbb{E}\sum_{j=1}^{N}\int_0^t\sigma^2\mathrm{d}s. \end{aligned}$$

Note that $p_i(t),\ 1\leqslant i\leqslant N$ are independent and identically distributed and let $\mu(t) = \mathbb{E}[p_i(t)]$, then

$$\begin{aligned} E\left|\frac{1}{N}\sum_{j=1,j\neq i}^{N}p_j(s) - \mu(s)]\right|^2 &\leqslant 2\mathbb{E}\left|\frac{1}{N}\sum_{j=1,j\neq i}^{N}p_j(s) - \frac{N-1}{N}\mu(s)\right|^2 + 2\mathbb{E}\left|\frac{1}{N}\mu(s)\right|^2 \\ &= 2\frac{(N-1)^2}{N^2}\mathbb{E}\left|\frac{1}{N-1}\sum_{j=1,j\neq i}^{N}p_j(s) - \mu(s)\right|^2 + \frac{2}{N^2}\mathbb{E}|\mu(s)|^2. \end{aligned}$$

We conclude that

$$\begin{aligned} &\mathbb{E}\int_0^t\left[\left|\frac{1}{N}\sum_{j=1,j\neq i}^{N}p_j(s) - \mathbb{E}[p_i(s)]\right|^2\right]\mathrm{d}s \\ &\leqslant C_0\frac{(N-1)^2}{N^2}\int_0^t\mathbb{E}\left|\frac{1}{N-1}\sum_{j=1,j\neq i}^{N}p_j(s) - \mu(s)\right|^2\mathrm{d}s + \frac{C_0}{N^2}\int_0^t\mathbb{E}|\mu(s)|^2\mathrm{d}s \\ &= C_0\frac{(N-1)^2}{N^2}\int_0^t\mathbb{E}\left|\frac{1}{N-1}\Big(\sum_{j=1,j\neq i}^{N}p_j(s) - (N-1)\mu(s)\Big)\right|^2\mathrm{d}s + \frac{C_0}{N^2}\int_0^t\mathbb{E}|\mu(s)|^2\mathrm{d}s \\ &= \frac{C_0}{N^2}\int_0^t\mathbb{E}\Big|p_1(s)-\mu(s)+p_2(s)-\mu(s)+\ldots+p_N(s)-\mu(s)\Big|^2\mathrm{d}s + \frac{C_0}{N^2}\int_0^t\mathbb{E}|\mu(s)|^2\mathrm{d}s \\ &= \frac{C_0(N-1)}{N^2}\int_0^t\mathbb{E}|p_j(s)-\mu(s)|^2\mathrm{d}s + \frac{C_0}{N^2}\int_0^t\mathbb{E}|\mu(s)|^2\mathrm{d}s \\ &= O\Big(\frac{1}{N}\Big). \end{aligned}$$

Moreover,

$$\frac{C_0}{N^2}\mathbb{E}\int_0^t |v_i(s)|^2\mathrm{d}s+\frac{C_0}{N^2}\mathbb{E}\sum_{j=1}^N\int_0^t|\sigma|^2\mathrm{d}s=O\Big(\frac{1}{N}\Big)$$

and

$$\int_0^t|\Pi(s-\delta)|^2\mathrm{d}s\leqslant\int_0^t|\Pi(s)|^2\mathrm{d}s.$$

Therefore, from the above estimates, for any $t\in[0,T]$, we have

$$\mathbb{E}\sup_{0\leqslant s\leqslant t}|\Pi(s)|^2\leqslant C_0\int_0^t|\Pi(s)|^2\mathrm{d}s+O\Big(\frac{1}{N}\Big).$$

Using Gronwall's inequality, we obtain

$$\sup_{0\leqslant t\leqslant T}\mathbb{E}|\tilde{x}^{(N)}(t)-x^{(0)}(t)|^2=O\Big(\frac{1}{N}\Big).$$

Next, we will prove (3.4). From the above equalities, we know $\tilde{x}_i(t)-\tilde{y}_i(t)$ satisfying

$$\begin{cases}\mathrm{d}(\tilde{x}_i(t)-\tilde{y}_i(t))=\Big(A(\tilde{x}_i(t)-\tilde{y}_i(t))+B(\tilde{x}_i(t-\delta)-\tilde{y}_i(t-\delta))\\ \qquad\qquad +b(\tilde{x}^{(N)}(t)-x^{(0)}(t))+e(\tilde{x}^{(N)}(t-\delta)-x^{(0)}(t-\delta))\Big)\mathrm{d}t,\\ \tilde{x}_i(t)-\tilde{y}_i(t)=0,\quad t\in[-\delta,0].\end{cases}$$

Note that $\sup_{0\leqslant t\leqslant T}\mathbb{E}|\tilde{x}^{(N)}(t)-x^{(0)}(t)|^2=O\Big(\frac{1}{N}\Big)$, then with the help of classical estimates of stochastic differential equation and Gronwall's inequality, it is easy to obtain

$$\sup_{0\leqslant t\leqslant T}\mathbb{E}|\tilde{x}_i(t)-\tilde{y}_i(t)|^2=O\Big(\frac{1}{N}\Big).$$

The proof is completed. □

**Proposition 3.2** *For all $1\leqslant i\leqslant N$, for the perturbation control $v_i$, we have*

$$|J_i(v_i,u_{-i})-\tilde{J}_i(v_i)|=O\Big(\frac{1}{\sqrt{N}}\Big). \tag{3.5}$$

**Proof**

$$\begin{aligned}&|J_i(v_i,u_{-i})-\tilde{J}_i(v_i)|\\ &=\frac{1}{2}\mathbb{E}\bigg[\int_0^T\{Q_t(\tilde{x}_i(t)-\tilde{x}^{(N)}(t))(\tilde{x}_i(t)-\tilde{x}^{(N)}(t))-Q_t(\tilde{y}_i(t)-x^{(0)}(t))(\tilde{y}_i(t)-x^{(0)}(t))\}\mathrm{d}t\\ &\quad+G(\tilde{x}_i(T)-\tilde{x}^{(N)}(T))(\tilde{x}_i(T)-\tilde{x}^{(N)}(T))-G(\tilde{y}_i(T)-x^{(0)}(T))(\tilde{y}_i(T)-x^{(0)}(T))\bigg].\end{aligned} \tag{3.6}$$

Recall to (3.3), (3.4) and the boundedness of $x^{(0)}(t)$, we easily deduce (3.5). □

We now give the definition of $\epsilon$-Nash equilibrium.

**Definition 3.3** *For $\epsilon>0$, a set of controls $v_i\in\mathcal{U}_{ad}^c, 1\leqslant i\leqslant N$ for $N$ agents is called an $\epsilon$-Nash equilibrium with respect to the costs $J_i$, $1\leqslant i\leqslant N$, if for any fixed $1\leqslant i\leqslant N$, it holds that*

$$J_i(u_i, u_{-i}) \leqslant J_i(v_i, u_{-i}) + \epsilon,$$

*when any alternative control* $v_i \in \mathcal{U}_{ad}^c$ *is applied by* $\mathcal{A}_i$.

We now state the main result of this paper.

**Theorem 3.4** *Let* $\bar{x}_i(t)$ *satisfy* (3.1). *Then the set of strategies*

$$u_i(t) = -R^{-1}Cp_i(t), \quad 1 \leqslant i \leqslant N, \tag{3.7}$$

*is an* $\epsilon$*-Nash equilibrium of* ***Problem (LP)***, *with* $\epsilon = O\Big(\frac{1}{\sqrt{N}}\Big) \to 0$ as $N \to +\infty$.

**Proof** By Propositions 2.3 and 3.2 and the optimal of $\bar{v}_i$, we have

$$\begin{aligned} J_i(u_i, u_{-i}) &= \tilde{J}_i(\bar{v}_i) + O\Big(\frac{1}{\sqrt{N}}\Big) \\ &\leqslant \tilde{J}_i(v_i) + O\Big(\frac{1}{\sqrt{N}}\Big) \\ &\leqslant J_i(v_i, u_{-i}) + O\Big(\frac{1}{\sqrt{N}}\Big). \end{aligned}$$

The proof is then completed by taking $\epsilon = O\Big(\frac{1}{\sqrt{N}}\Big)$. □

## 4. Numerical examples

In this section, we will give a numerical example to illustrate sensitivity of the solution to advanced backward stochastic differential equation to time delays, the effect of the collective behavior of the population, and the consistency of the mean-field estimation. For $u_i \in \mathcal{U}_{ad}^c$, the dynamics of agent $\mathcal{A}_i$ is given by

$$\begin{cases} \mathrm{d}x_i(t) = [0.1x_i(t-\delta) - v_i(t) + x^{(200)}(t)]\mathrm{d}t + \mathrm{d}W_i(t), \\ x_i(t) = a_{i0}, \end{cases} \tag{4.1}$$

where $x^{(200)}(t) = \frac{1}{200}\sum_{i=1}^{200} x_i(t)$ and $W_i(t), i = 1, ..., 200$ denote 200 independent standard Brownian motions. Let the initial states of the agents $a_{i0}, i = 1, ..., 200$ be independent and identically distributed random variables.

Let $v = (v_1, v_2, ..., v_{200})$ denote the set of control strategies of all 200 agents and $v_{-i} = (v_1, v_2, ..., v_{i-1}, v_{i+1}, ..., v_{200})$ be the set of the control strategies except the $i$th agent $\mathcal{A}_i$.

The cost functional for $\mathcal{A}_i$ $(1 \leqslant i \leqslant 200)$ is defined by

$$J_i(v_i, v_{-i}) = \frac{1}{2}\mathbb{E}\left\{\int_0^T Rv_i^2(t)\mathrm{d}t + y_i(T)\right\}. \tag{4.2}$$

In this case, $p_i(t)$ satisfies

$$\begin{cases} -\mathrm{d}p_i(t) = \Big(\mathbb{E}[p_i(t)] + 0.1\mathbb{E}^{\mathcal{F}_t}[p_i(t+\delta)]\chi_{[0,T-\delta]}\Big)\mathrm{d}t - z_i(t)\mathrm{d}W_i(t), \\ p_i(T) = 1. \end{cases} \tag{4.3}$$

We can solve (4.3) by successive Itô's integrations over steps of length $\delta$, i.e.

$$p_i(t) = e^{T-t}, \quad z_i(t) = 0, \quad t \in [T-\delta, T];$$
$$p_i(t) = -\frac{1}{20}e^{T-\delta}e^{-t} + \left(\frac{1}{20}e^{-(T-\delta)} + e^{2\delta-T}\right)e^t, \quad z_i(t) = 0, \quad t \in [T-2\delta, T-\delta];$$
$$\cdots$$

Since $p_i(T)$ and the coefficients of $\mathbb{E}^{\mathcal{F}_t}[p(t+\delta)]$ are deterministic, we can obtain $z_i(t) \equiv 0$ $(t \in [0,T])$ fortunately.

Figure 1 plots the solution of advanced backward stochastic differential (4.3) with different time delays. The parameter values used in the calculations are $T=5, a=0.1, \delta_1=1, \delta_2=0.5, \delta_3=0.25$. We find that the solution of the advanced backward stochastic differential equation (4.3) is sensitive to the time delay $\delta$, i.e., the larger $\delta$ is, the smaller $p_i$ is.

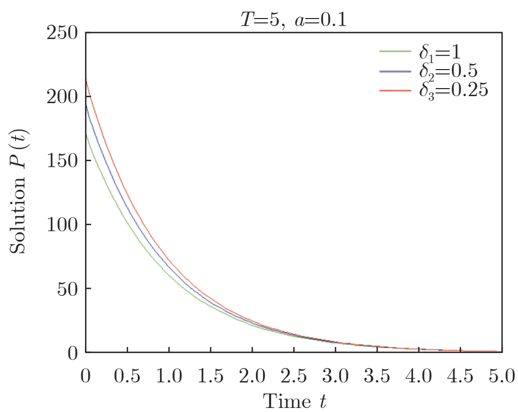


**Figure 1** The solution $p_i(t)$ with different time delays

By Theorem 3.4, we give the set of the following decentralized control strategies:

$$u_i(t) = -R^{-1}p_i(t), \quad 1 \leqslant i \leqslant 200,$$

which is an $\epsilon$-Nash equilibrium of the large population game (4.1) and (4.2). Here $\bar{x}_i(t)$ satisfies

$$\begin{cases} \mathrm{d}\bar{x}_i(t) = [0.1\bar{x}_i(t-\delta) - R^{-1}p_i(t) + \bar{x}^{(200)}(t)]\mathrm{d}t + \mathrm{d}W_i(t), \\ \bar{x}_i(t) = a_{i0}, \quad t \in [-\delta, 0] \end{cases}$$

and $\bar{x}^{(200)}(t)$ satisfies the following equation

$$\begin{cases} \mathrm{d}\bar{x}^{(200)}(t) = [\bar{x}^{(200)}(t) + 0.1\bar{x}^{(200)}(t-\delta) - \frac{1}{200}\sum_{i=1}^{200} R^{-1}p_i(t)]\mathrm{d}t + \frac{1}{200}\sum_{i=1}^{200}\mathrm{d}W_i(t), \\ \bar{x}^{(200)}(t) = \frac{1}{200}\sum_{i=1}^{200} a_{i0}, \quad t \in [-\delta, 0]. \end{cases}$$

The deterministic continuous function $x^{(0)}(t)$ satisfies

$$\begin{cases} \mathrm{d}x^{(0)}(t) = \{x^{(0)}(t) + 0.1x^{(0)}(t-\delta) - \frac{1}{200}\sum_{i=1}^{200}\mathbb{E}[R^{-1}p_i(t)]\}\mathrm{d}t, \\ x^{(0)}(t) = \frac{1}{200}\sum_{i=1}^{200} a_{i0}, \quad t \in [-\delta, 0]. \end{cases}$$

Figure 2 (a) illustrates the trajectories of the agent's state under $N=200, T=1$ and $R=1$ conditions. As shown in Figure 2(a), in addition to the influence of the agent itself (i.e. the initial state and Brownian motion), the trajectory of $\bar{x}_i(t)$ is also influenced by the collective behavior of the group.

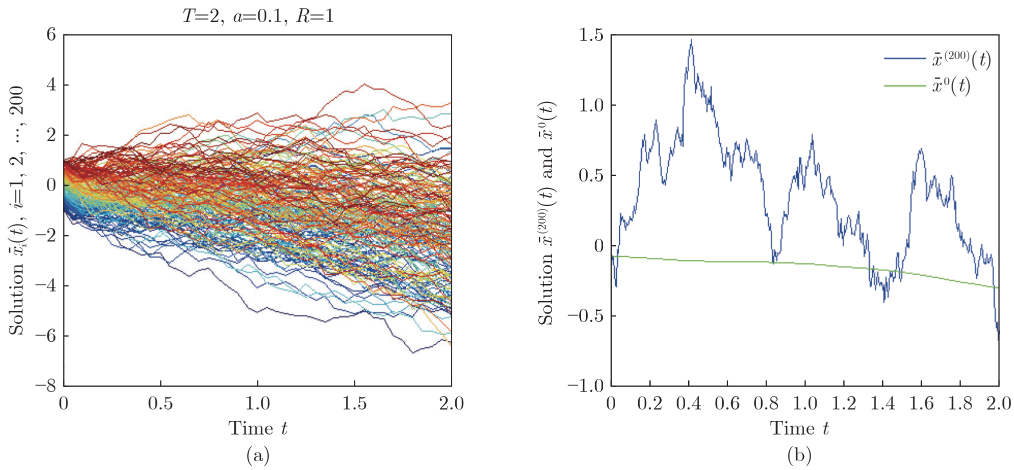


**Figure 2** (a) State trajectories of agents when $N = 200$; (b) The consistency of mean-field estimation

Figure 2 (b) shows the consistency of mean-field estimation. It can be seen that the trajectory of $\bar{x}^{(200)}(t)$ almost coincides with the trajectory of $x^{(0)}(t)$, which illustrates the consistency of the mean-field estimates.

## 5. Conclusions

In this paper, we study the mean-field linear-quadratic games with delay. Time delay generates some additional difficulties in obtaining the desired results. The resulting decentralized control turns out to be an $\epsilon$-Nash equilibrium. An interesting direction for future research is to extend the model to the mixed delays case. Compared with pointwise delays, it is more reasonable and practical to add moving-average delays, which shows that the current situation of the system depends on a past history in the form of an integral. Another direction is to consider the partially observed linear-quadratic Gaussian mean field games. We hope to discuss these topics in a forthcoming paper.


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