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Semimartingale dynamics for a backward exchange rate process

Gregory Gagnon

Department of Economics, University of Toronto Mississauga, 3359 *Mississauga Road, Mississauga, Ontario, L5L 1C6,* Canada

Email: gregory.gagnon@utoronto.ca

Abstract Via a forward SDE solution $(k_t, t \geqslant 0)$ that captures money supply dynamics, a macroeconomic model known as the monetary model generates a backward exchange rate process $(y_t, t \geqslant 0)$. For any $t \geqslant 0$, $y_t = k_t + \alpha^{-1}\mu_t$ where $(\mu_t, t \geqslant 0)$ is a backward process and $\alpha > 0$ is a constant. Thus, $(y_t, t \geqslant 0)$ does not satisfy a conventional BSDE. Our paper proves $(y_t, t \geqslant 0)$ is a continuous semimartingale when restrictions on the SDE for $(k_t, t \geqslant 0)$ capture anti-inflationary initiatives. This new result in economic dynamics does not require the filtration to be the Brownian filtration.

Keywords Backward process, Semimartingale, Anti-inflationary SDE policy

2020 Mathematics Subject Classification 60G17, 91B55, 91B64, 91B70

1. Introduction

Macroeconomics presents a distinctive approach to exchange rates via the monetary model. In this model the exchange rate at any time $t \geqslant 0$ satisfies a backward equation involving money supply forecasts, c.f. Flood and Garber [5] as well as Froot and Obstfeld [6], rather than a forward stochastic differential equation (SDE) as in impulse control theory and mathematical finance c.f. Cadenillas and Zapatero [3, 4] as well as Shreve [14].

More precisely, let $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geqslant 0), P)$ be the complete filtered probability space where all action in the monetary model happens. Define k_t by $k_t = \log(M_t^d)$ for any $t \geqslant 0$ where $M_t^d > 0$ is a random variable giving the domestic money supply at t. Similarly define y_t by $y_t = \log(S_t)$ at any $t \geqslant 0$ where $S_t > 0$ is a random variable giving the exchange rate at t, meaning S_t is the domestic currency price of a unit of foreign currency at t. Economic variables are never distinguished from their natural logarithms in the exposition, thus we refer to k_t and y_t as the domestic money supply and exchange rate at $t \geqslant 0$ respectively.

In the model $(k_t, t \geqslant 0)$ solves a forward SDE and when $s \geqslant t$ the forecast at t of the money supply at s is $E(k_s|\mathcal{F}_t)$, the conditional expectation of k_s given $\mathcal{F}_t$. Let interest elasticity of money demand be $-\alpha$ where $\alpha > 0$ is a constant. Then Flood and Garber [5, p. 539-540] as well as Froot and Obstfeld [6, p.242] suggest that for any $t \geqslant 0$ a.s.

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$$y_t = \alpha^{-1} \int_t^{\infty} \exp\left(\frac{t-s}{\alpha}\right) E(k_s|\mathcal{F}_t)\mathrm{d}s. \tag{1.1}$$

More on the derivation of (1.1) under a general SDE can be found in Gagnon [8]. Flood and Garber [5] as well as Froot and Obstfeld [6] use a Brownian motion with drift for $(k_t, t \geqslant 0)$ but in our model the drift and diffusion coefficient depend on $t \geqslant 0$ and $x \in \mathbb{R}$, implying money creation responds to the current money supply. Via a specific version of $(E(k_s|\mathcal{F}_t), s \geqslant t)$ we actually use (1.1) to define $y_t(\omega)$ for all $\omega \in \Omega$ and $t \geqslant 0$ then view y_t as the exchange rate at t because of economic theory. Our paper proves $(y_t, t \geqslant 0)$ is a continuous semimartingale when drift and diffusion coefficient restrictions in the SDE for $(k_t, t \geqslant 0)$ capture anti-inflationary policy. Semimartingales are fundamental in control theory, c.f. Cadenillas and Zapatero [3, 4] as well as Yong and Zhou [17], so control problems for $(y_t, t \geqslant 0)$ under anti-inflationary policy may now be considered.

Let b be the drift in our SDE for $(k_t, t \geqslant 0)$. Our analysis of $(y_t, t \geqslant 0)$ is based on a forward-backward decomposition in which b is heavily involved. More precisely, for any $t \geqslant 0$ we use a specific version of $\left(E\left(\int_t^s b(u,k_u)\mathrm{d}u|\mathcal{F}_t\right), s \geqslant t\right)$ to show that for any $\omega \in \Omega$

$$y_t(\omega) = k_t(\omega) + \alpha^{-1} \int_t^{\infty} \exp\left(\frac{t-s}{\alpha}\right) E\left(\int_t^s b(u,k_u)\mathrm{d}u|\mathcal{F}_t\right)(\omega)\mathrm{d}s. \tag{1.2}$$

Pardoux and Peng [12] introduced the basic structure of a backward SDE (BSDE). Even though (1.1) ensures $(y_t, t \geqslant 0)$ is a backward process it does not satisfy a conventional BSDE because of the presence of k_t in (1.2). If (Y, Z) is a pair of processes solving a BSDE of the form in Pardoux and Peng [12] then Y is a semimartingale, but the failure of $(y_t, t \geqslant 0)$ to satisfy such a BSDE means other arguments must be invoked to prove it is a semimartingale. To this end, for each $t \geqslant 0$ define the random variable μ_t by

$$\mu_t = \int_t^{\infty} \exp\left(\frac{t-s}{\alpha}\right) E\left(\int_t^s b(u,k_u)\mathrm{d}u|\mathcal{F}_t\right)\mathrm{d}s. \tag{1.3}$$

The process $(\mu_t, t \geqslant 0)$ dominates our analysis. Section 3 proves it is a continuous adapted process so $(y_t, t \geqslant 0)$ has the same properties. If $(\mathcal{F}_t, t \geqslant 0)$ is the Brownian filtration standard techniques show $(\exp(\frac{-t}{\alpha})\mu_t, t \geqslant 0)$ satisfies a BSDE. Basic manipulations show $(\exp(\frac{-t}{\alpha})\mu_t, t \geqslant 0)$ is a semimartingale so integration by parts and (1.2) imply $(y_t, t \geqslant 0)$ is a semimartingale. However, our real focus is the case when $(\mathcal{F}_t, t \geqslant 0)$ is a general filtration so a different approach for our proof is needed.

We craft specific versions of conditional expectation for use in (1.1)–(1.3) and these drive our results. Via drift and diffusion coefficient restrictions our version of conditional expectation in (1.3) implies $(\mu_t, t \geqslant 0)$ is a continuous semimartingale. Once $(\mu_t, t \geqslant 0)$ is shown to be a semimartingale (1.2) implies that $(y_t, t \geqslant 0)$ is a semimartingale; (1.2) clearly links their decompositions. In the special case where b depends only on time μ_t is non-random for each $t \geqslant 0$. We also show $(\mu_t, t \geqslant 0)$ is a continuous function of bounded variation in this case, a fact which greatly simplifies the decomposition of $(y_t, t \geqslant 0)$.

The paper is organized as follows. Section 2 presents our anti-inflationary SDE monetary policy. Section 3 proves $(y_t, t \geqslant 0)$ is a semimartingale and section 4 concludes.

2. Anti-inflationary SDE policy dynamics

Consider a few mathematical preliminaries. Everything happens on complete probability space $(\Omega, \mathcal{F}, P)$ which is endowed with a right continuous filtration $(\mathcal{F}_t, t \geqslant 0)$ where $\mathcal{F}_0$ contains $\mathcal{N}$, the class of all P-null sets. Stochastic processes in our paper assume values in $\mathbb{R}$ and when we say a process is adapted or a martingale $(\mathcal{F}_t, t \geqslant 0)$ is the relevant filtration. Let $(M_t, t \geqslant 0)$ be a martingale then we say $(M_t, t \geqslant 0)$ is a L^2-martingale if for every $t \geqslant 0$ we have $EM_t^2 < \infty$. Additionally if $t \geqslant 0$ then $C([t,\infty), \mathbb{R})$ is the space of continuous real valued functions on $[t, \infty)$ and if $(X_s, s \geqslant t)$ is a process then for any $\omega \in \Omega$ its corresponding sample path is denoted by $(X_s(\omega), s \geqslant t)$. Finally, let $(W_t, t \geqslant 0)$ be a one dimensional standard Brownian motion relative to $(\mathcal{F}_t, t \geqslant 0)$.

Detailed discussions of the monetary model can be found in Flood and Garber [5] as well as Wang [16, ch. 8]. Logarithms of the foreign money supply and both GDPs at $t \geqslant 0$ also appear in k_t but attention in this model often gravitates toward domestic policy changes c.f. Flood and Garber [5], so when defining k_t we ignore these other variables.

When assumptions are introduced outside statements of theorems or lemmas they are active throughout the paper. A key assumption is that $(k_t, t \geqslant 0)$ solves an SDE, which is motivated by the fact that economic data often resembles an SDE sample path. Recall that $k_t = \log(M_t^d)$ so k_t may assume any sign, thus there is no reason to assume $(k_t, t \geqslant 0)$ solves a linear SDE with $k_0 > 0$. Other variables at t can be built into k_t but as long as $(k_t, t \geqslant 0)$ solves an SDE that satisfies certain conditions $(y_t, t \geqslant 0)$ is a semimartingale and the exact composition of k_t in terms of economic variables is unimportant.

Assumption 2.1 *Let $b, \sigma : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ be measurable relative to $\mathcal{B}([0,\infty)) \times \mathcal{B}(\mathbb{R})$ and $\mathcal{B}(\mathbb{R})$. In addition for some constant $K > 0$ the following conditions hold:*

(1) *For all $(t, x) \in [0, \infty) \times \mathbb{R}$,*

$$|b(t,x)| \leqslant K(1 + |x|),$$

$$|\sigma(t,x)| \leqslant K(1 + |x|);$$

(2) *For all $(t, (x, y)) \in [0, \infty) \times \mathbb{R}^2$,*

$$|b(t,x) - b(t,y)| \leqslant K|x - y|,$$

$$|\sigma(t,x) - \sigma(t,y)| \leqslant K|x - y|.$$

Then $(k_t, t \geqslant 0)$ satisfies the following SDE where k_0 is a real number:

$$k_t = k_0 + \int_0^t b(s, k_s)\mathrm{d}s + \int_0^t \sigma(s, k_s)\mathrm{d}W_s. \tag{2.1}$$

Flood and Garber [5], Froot and Obstfeld [6] as well as Smith [15] assume $(k_t, t \geqslant 0)$ is a Brownian motion with drift prior to the central bank pegging the exchange rate. Economic theory has long held that increasing the money supply is the driving force behind inflation. Jumps in inflation certainly cause real policymakers to reduce money growth. Since Brownian motion with drift policy does not allow reductions in money growth when the money supply is high it is not a meaningful way to model monetary policy.

Building anti-inflationary policy into the model requires constraints on b and σ because they affect the growth of $(k_t, t \geqslant 0)$. Restrictions on b are important because if $\sigma = 0$ then $(k_t, t \geqslant 0)$ solves an ordinary differential equation (ODE). Formally $k_t = \log(M_t^d)$ so $\sigma = 0$ implies $(M_t^d, t \geqslant 0)$ must be a non-random function. Provided $(M_t^d, t \geqslant 0)$ is differentiable $b(t, k_t)$ gives the percentage change in the money supply at any $t \geqslant 0$ so we call b the money growth rate. Even when $\sigma \neq 0$ we call b the money growth rate. We allow b to assume any sign but it is easier to think of b as a nonnegative function.

Assumption 2.2 *The following conditions hold:*

(1) $|b(t,x) - b(t,y)| \leqslant \exp(-t)|x-y|$ *for all* $t \geqslant 0$ *and* $x, y \in \mathbb{R}$.
(2) $b, \sigma \in C^*([0,\infty) \times \mathbb{R})$, *the space of bounded continuous real valued functions on* $[0,\infty) \times \mathbb{R}$.

Policymakers consider a two percent inflation rate quite low but even this ensures that prices are unbounded across time. Restrictions on b and σ must therefore reflect low money growth instead of ensuring price stability and all the conditions in assumption 2.2 are crafted with this in mind. One motivation for condition (1) is that it can force the money growth rate to decay quickly over time. To see this consider the special case where $b(t,0) = 0$ for all $t \geqslant 0$ then by condition (1) for any $x \in \mathbb{R}$ and $t \geqslant 0$ we have

$$|b(t,x)| \leqslant \exp(-t)|x|. \tag{2.2}$$

Condition (2) reflects anti-inflationary policy if $\sup_{(t,x)\in[0,\infty)\times\mathbb{R}} |b(t,x)|$ is small. Random influences in policy are also mildly constrained via condition (2) because it forces σ to be bounded. Several proofs in section 3 make use of the fact that σ is bounded. Decay of the money growth rate over time, some restraint on random influences in monetary policy and a low money growth rate at all times can be expected with actual anti-inflationary policy, so assumption 2.2 moves the model in a more realistic direction.

Condition (1) is easy to fulfill as the following example shows. Let $C_c^1(\mathbb{R},\mathbb{R})$ be the space of once continuously differentiable compactly supported real valued functions on $\mathbb{R}$. Define $b : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ so that for all $(t,x) \in [0,\infty) \times \mathbb{R}$

$$b(t,x) = \exp(-t)h(x), \tag{2.3}$$

where $h \in C_c^1(\mathbb{R},\mathbb{R})$ and $|\frac{\mathrm{d}h}{\mathrm{d}x}| \leqslant 1$. Clearly, b satisfies the conditions in assumption 2.1 for a suitable constant K and the conditions of assumption 2.2. Since $h \in C_c^1(\mathbb{R},\mathbb{R})$ it follows that b cannot be a nonzero constant function, which is in marked contrast to the case of Brownian motion with drift monetary policy.

Many results in section 3 make use of the bound on b so we specify it here. For some constant $M > 0$ given any $(t,x) \in [0,\infty) \times \mathbb{R}$ we have

$$|b(t,x)| \leqslant M. \tag{2.4}$$

In the rest of the paper M always refers to this constant. One technical assumption is necessary to prove some theorems in section 3. These results involve K, the Lipschitz condition constant in assumption 2.1, and henceforth K always denotes this constant.

Assumption 2.3 $\frac{1}{\alpha} > 4(K^2+1)$ *where* α *is the parameter from (1.1).*

In the model $-\alpha$ is the interest elasticity of money demand in both the domestic and foreign economies, meaning $-\alpha$ captures the sensitivity of money demand to interest rate changes in each economy. Via assumption 2.3 a parameter linked to interest rate sensitivity of money holdings, namely α^{-1}, dominates a key parameter for monetary policy.

3. Semimartingale structure of the exchange rate

Small variations exist in the definition of a continuous semimartingale. For example, compare the definition in Karatzas and Shreve [11, p. 149] with the one in Kallianpur [10, p. 78]. Stating $(y_t, t \geqslant 0)$ is a continuous semimartingale here means $(y_t, t \geqslant 0)$ is a continuous adapted process such that for every $t \geqslant 0$ a.s.

$$y_t = \xi + M_t + B_t, \tag{3.1}$$

where ξ is a $\mathcal{F}_0$-measurable random variable, $(M_t, t \geqslant 0)$ is a continuous L^2-martingale with $M_0 = 0$ a.s. and $(B_t, t \geqslant 0)$ is a continuous adapted process of bounded variation with $B_0 = 0$ a. s. Note that if $t \geqslant 0$ and we say a process $(v_s, s \geqslant t)$ is continuous then for a.a. $\omega \in \Omega$ the sample path $s \to v_s(\omega)$ is continuous. Of course this is the usual definition of a continuous process c.f. Revuz and Yor [13, p. 16]. However, some processes in our paper have sample paths that are continuous for every $\omega \in \Omega$. Distinctions between sample path continuity for all $\omega \in \Omega$ versus a.a. ω are made throughout the paper.

Conditional expectation of a random variable is not unique although any two versions are equal a.s. Both (1.1) and (1.3) integrate conditional expectation sample paths across $[t, \infty)$ so altering the version of conditional expectation over a subinterval of $[t, \infty)$ in either (1.1) or (1.3) means a different value for the respective integral could emerge on a set of positive probability because an intersection of uncountably many measurable subsets is not necessarily measurable. Specifying the versions of conditional expectation entering (1.1) and (1.3) is therefore crucial to ensuring each equation turns out a precise value. Finding suitable versions of conditional expectation to use in (1.1) and (1.3) is a major endeavour.

Lemma 3.1 *Given $a, b \in \mathbb{R}$ with $a < b$ let $(f_t, t \in [a, b])$ be a bounded continuous process and $\mathcal{G}$ a sub-σ field of $\mathcal{F}$ such that $(E(f_t|\mathcal{G}), t \in [a, b])$ has a continuous modification. Then $E\left(\int_a^b f_t \, \mathrm{d}t|\mathcal{G}\right) = \int_a^b E(f_t|\mathcal{G})\mathrm{d}t$ a.s. where for a.a. $\omega \in \Omega$ the integrand in $\int_a^b E(f_t|\mathcal{G})\mathrm{d}t$ is the sample path of the continuous version of $(E(f_t|\mathcal{G}), t \in [a, b])$.*

Proof The result follows from the dominated convergence theorem for conditional expectation, c.f. Ash [1, Theorem 6.5.5 p. 257], as well as the fact that $\int_a^b f_t \mathrm{d}t$ and $\int_a^b E(f_t|\mathcal{G})\mathrm{d}t$ are Riemann integrals a.s. □

Each conditional expectation process constructed for use in (1.1), (1.2) and (1.3) has sample paths which are continuous for all $\omega \in \Omega$. Several of the following theorems construct these conditional expectation processes and explore their properties.

Theorem 3.2 *The following hold:*

(1) *Given any $t \geqslant 0$ there exists a version of $(E(b(u, k_u)|\mathcal{F}_t), u \geqslant t)$ such that*

$$(E(b(u, k_u)|\mathcal{F}_t)(\omega), u \geqslant t) \in C([t, \infty), \mathbb{R}) \quad \text{for all} \quad \omega \in \Omega;$$

(2) *Given any $s, t \geqslant 0$ with $t \leqslant s$ there exists a version of $\left(E\left(\int_s^u b(v, k_v)\mathrm{d}v|\mathcal{F}_t\right), u \geqslant s\right)$ such that*

$$\left(E\left(\int_s^u b(v, k_v)\mathrm{d}v|\mathcal{F}_t\right)(\omega), u \geqslant s\right) \in C([s, \infty), \mathbb{R}) \quad \text{for all} \quad \omega \in \Omega.$$

Proof Fix $t \geqslant 0$ then define $\mathcal{F}_{t,u}^W$ for $u \geqslant t$ by

$$\mathcal{F}_{t,u}^W = \sigma(W_v - W_j, t \leqslant j, v \leqslant u) \bigvee \mathcal{N}.$$

When $x \in \mathbb{R}$ consider the SDE solution $(\xi_{t,x}(u), u \geqslant t)$ which, given any $u \geqslant t$, satisfies

$$\xi_{t,x}(u) = x + \int_t^u b(v, \xi_{t,x}(v))\mathrm{d}v + \int_t^u \sigma(v, \xi_{t,x}(v))\mathrm{d}W_v \tag{3.2}$$

and where $\xi_{t,\cdot}(u) : \mathbb{R} \times \Omega \to \mathbb{R}$ is $\mathcal{B}(\mathbb{R}) \times \mathcal{F}_{t,u}^W$-measurable for any $u \geqslant t$ c.f. Kallianpur [11, ch. 5 lemma 5.4.3]. Recall that $k_v = \xi_{t,k_t}(v)$ a.s. when $v \geqslant t$ c.f. Kallianpur [11, ch. 5 lemma 5.4.6]. Moreover, if $v \geqslant t$ then $b(v, \xi_{t,\cdot}(v))$ is $\mathcal{B}(\mathbb{R}) \times \mathcal{F}_{t,v}^W$-measurable so, invoking Kallianpur [11, ch. 5 lemma 5.4.1], for almost all $\omega \in \Omega$ we have

$$E(b(v,k_v)|\mathcal{F}_t)(\omega) = E(b(v,\xi_{t,k_t}(v))|\mathcal{F}_t)(\omega) = \int_\Omega b(v,\xi_{t,k_t(\omega)}(v)(z))\mathrm{d}P(z). \tag{3.3}$$

Given $v \geqslant t$ define the random variable Z_v^t so that for any $\omega \in \Omega$

$$Z_v^t(\omega) \equiv \int_\Omega b(v,\xi_{t,k_t(\omega)}(v)(z))\mathrm{d}P(z). \tag{3.4}$$

Continuity of b together with the dominated convergence theorem imply that for any $\omega \in \Omega$

$$(Z_v^t(\omega), v \geqslant t) \in C([t,\infty),\mathbb{R}).$$

To see this observe that for any $x \in \mathbb{R}$, $(\xi_{t,x}(u)(z), u \geqslant t) \in C([t,\infty),\mathbb{R})$ for $z \in A_{t,x}$ where $P(A_{t,x}) = 1$. Let $\{u_n\}_{n=1}^\infty$ be a sequence such that $u_n \to u$, $u_n \geqslant t$ for all $n \geqslant 1$ where $u \geqslant t$. By continuity of b we have

$$b(u_n,\xi_{t,x}(u_n)) \to b(u,\xi_{t,x}(u)) \quad \text{as} \quad n\to\infty \quad \text{a.s.}$$

Recalling that b is bounded, the dominated convergence theorem ensures that

$$Eb(u_n,\xi_{t,x}(u_n)) \to Eb(u,\xi_{t,x}(u)) \quad \text{as} \quad n\to\infty,$$

which implies $(Eb(u,\xi_{t,x}(u)), u \geqslant t) \in C([t,\infty),\mathbb{R})$ for any $x \in \mathbb{R}$, so it follows that $(Z_v^t(\omega), v \geqslant t) \in C([t,\infty),\mathbb{R})$ for any $\omega \in \Omega$.

Since $x \to \int_\Omega b(v,\xi_{t,x}(v)(z))\mathrm{d}P(z)$ defines a Borel measurable function on $\mathbb{R}$ by Fubini's theorem, Z_v^t is $\mathcal{F}_t$-measurable for any $v \geqslant t$, meaning for any $\omega \in \Omega$ we may take

$$Z_v^t(\omega) = E(b(v,k_v)|\mathcal{F}_t)(\omega). \tag{3.5}$$

Thus, $(\int_\Omega b(v,\xi_{t,k_t}(v)(z))\mathrm{d}P(z), v \geqslant t)$ is the desired version of $(E(b(v,k_v)|\mathcal{F}_t), v \geqslant t)$.

Invoking (2.4), $|Z_v^t(\omega)| \leqslant M$ for all $\omega \in \Omega$ and any $t, v \geqslant 0$ with $t \leqslant v$. When $t \leqslant s < u$ it follows that $\int_s^u Z_v^t(\omega)\mathrm{d}v$ exists for all $\omega \in \Omega$. Moreover, $\int_s^u Z_v^t\mathrm{d}v$ is $\mathcal{F}_t$-measurable because $(Z_v^t(\omega), v \geqslant t)$ is continuous for all $\omega \in \Omega$. By Lemma 3.1,

$$E\left(\int_s^u b(v,k_v)\mathrm{d}v|\mathcal{F}_t\right) = \int_s^u Z_v^t\mathrm{d}v \quad \text{a.s.},$$

so for any $\omega \in \Omega$ when $t \leqslant s \leqslant u$ we may set

$$E\left(\int_s^u b(v,k_v)\mathrm{d}v|\mathcal{F}_t\right)(\omega) = \int_s^u Z_v^t(\omega)\mathrm{d}v. \tag{3.6}$$

For all $\omega \in \Omega$ we have

$$\left(\int_s^u Z_v^t(\omega)\mathrm{d}v, u \geqslant s\right) \in C([s,\infty),\mathbb{R}).$$

Thus, $(\int_s^u Z_v^t\mathrm{d}v, u \geqslant s)$ is the desired version of $(E\left(\int_s^u b(v,k_v)\mathrm{d}v|\mathcal{F}_t\right), u \geqslant s)$. □

Given any $t \geqslant 0$ and $\omega \in \Omega$, $(\int_t^s Z_v^t(\omega)\mathrm{d}v, s \geqslant t)$ is measurable relative to $\mathcal{B}([t,\infty))$ and $\mathcal{B}(\mathbb{R})$ because $(\int_t^s Z_v^t(\omega)\mathrm{d}v, s \geqslant t) \in C([t,\infty),\mathbb{R})$, so $\mu_t(\omega)$ is meaningful to consider for any $\omega \in \Omega$ when (3.6) specifies the version of $\left(E\left(\int_t^s b(v,k_v)\mathrm{d}v|\mathcal{F}_t\right), s \geqslant t\right)$. Assumption 3.3 formalizes some of our choices for versions of conditional expectation used in the model.

Assumption 3.3 *Given any* $t \geqslant 0$ *then*

(1) *When* $v \geqslant t$ *we use* Z_v^t *defined in (3.4) as the version of* $E(b(v,k_v)|\mathcal{F}_t)$.
(2) *When* $t \leqslant s \leqslant u$ *we use* $\int_s^u Z_v^t\mathrm{d}v$ *as the version of* $E\left(\int_s^u b(v,k_v)\mathrm{d}v|\mathcal{F}_t\right)$.

(3) *We use* $(\int_t^s Z_v^t \mathrm{d}v, s \geqslant t)$ *when defining* μ_t.

We often use the standard notation for conditional expectation so Z_v^t is denoted by $E(b(v, k_v)|\mathcal{F}_t)$ and $\int_s^u Z_v^t \mathrm{d}v$ by $E\left(\int_s^u b(v, k_v)\mathrm{d}v|\mathcal{F}_t\right)$.

Convenient properties are associated with our choices of conditional expectation. Theorem 3.2 used the fact that for any $t, v \geqslant 0$ with $t \leqslant v$ and any $\omega \in \Omega$ we have $|Z_v^t(\omega)| \leqslant M$. Lemma 3.4 presents an important property associated with our choice of $E\left(\int_s^u b(v, k_v)\mathrm{d}v|\mathcal{F}_t\right)$ where $t \leqslant s \leqslant u$. It follows immediately from the definitions and (2.4) so is stated without proof but is used in many results that follow.

Lemma 3.4 *Given any* $t, s, u \geqslant 0$ *with* $t \leqslant s \leqslant u$ *and any* $\omega \in \Omega$ *we have*

$$|E\left(\int_s^u b(v, k_v)\mathrm{d}v|\mathcal{F}_t\right)(\omega)| \leqslant M(u - s).$$

For every $t \geqslant 0$ it must be verified that the integral defining μ_t and the integral on the r.h.s. of (1.1) exist, exercises in which Lemma 3.4 plays a key role. Proving that the r.h.s. of (1.1) is well defined involves showing there exists a version of $(E(k_s|\mathcal{F}_t), s \geqslant t)$ such that $(E(k_s|\mathcal{F}_t)(\omega), s \geqslant t)$ is measurable relative to $\mathcal{B}([t, \infty))$ and $\mathcal{B}(\mathbb{R})$ for every $\omega \in \Omega$.

Lemma 3.5 *Given any* $t \geqslant 0$ *and* $\omega \in \Omega$, $\mu_t(\omega)$ *exists and is finite.*

Proof Recall $(\int_t^s Z_u^t(\omega)\mathrm{d}u, s \geqslant t) \in C([t, \infty), \mathbb{R})$ for all $\omega \in \Omega$ by Theorem 3.2 so the integrand in (1.3) is $\mathcal{B}([t, \infty))$-measurable. Using Lemma 3.4, for all $\omega \in \Omega$

$$\int_t^\infty \exp\left(\frac{t-s}{\alpha}\right) |E\left(\int_t^s b(u, k_u)\mathrm{d}u|\mathcal{F}_t\right)(\omega)|\mathrm{d}s \leqslant M \int_t^\infty \exp\left(\frac{t-s}{\alpha}\right)(s-t)\mathrm{d}s < \infty.$$

Hence $\mu_t(\omega)$ is finite for all $t \geqslant 0$ and $\omega \in \Omega$. □

Theorem 3.6 *Given any* $t \geqslant 0$ *there exists a version of* $(E(k_s|\mathcal{F}_t), s \geqslant t)$ *such that*

$$(E(k_s|\mathcal{F}_t)(\omega), s \geqslant t) \in C([t, \infty), \mathbb{R}), \quad \text{for all} \quad \omega \in \Omega.$$

Use this version of $(E(k_s|\mathcal{F}_t), s \geqslant t)$ *when constructing the integral on the r.h.s. of (1.1) then this integral exists and is finite for any* $t \geqslant 0$ *and* $\omega \in \Omega$. *Thus, we may define* $y_t(\omega)$ *by (1.1) for any* $t \geqslant 0$ *and* $\omega \in \Omega$. *Moreover,* y_t *satisfies (1.2) for all* $\omega \in \Omega$ *any* $t \geqslant 0$.

Proof By Lemma 3.1 for any $s, t \geqslant 0$ with $s \geqslant t$ we have

$$E(k_s|\mathcal{F}_t) = k_t + \int_t^s Z_u^t \mathrm{d}u \quad \text{a.s.}$$

Hence $k_t + \int_t^s Z_u^t \mathrm{d}u$ is a version of $E(k_s|\mathcal{F}_t)$ such that for all $\omega \in \Omega$

$$\left(k_t(\omega) + \int_t^s Z_u^t(\omega)\mathrm{d}u, s \geqslant t\right) \in C([t, \infty), \mathbb{R}).$$

Taking $E(k_s|\mathcal{F}_t) \equiv k_t + \int_t^s Z_u^t \mathrm{d}u$ it follows that $(E(k_s|\mathcal{F}_t)(\omega), s \geqslant t)$ is a measurable function on $[t, \infty)$ for every $\omega \in \Omega$. Lemma 3.4 then yields that the r.h.s. of (1.1) exists and is finite for all $\omega \in \Omega$ and all $t \geqslant 0$. Therefore, for all $t \geqslant 0$ and any $\omega \in \Omega$ we may set

$$y_t(\omega) = \alpha^{-1} \int_t^\infty \exp\left(\frac{t-s}{\alpha}\right)\left(k_t(\omega) + \int_t^s Z_u^t(\omega)\mathrm{d}u\right)\mathrm{d}s.$$

Direct calculation yields that (1.2) holds for all $t \geqslant 0$ and $\omega \in \Omega$. □

Assumption 3.7 *Our analysis treats* $(y_t, t \geqslant 0)$ *as a stochastic process generated by the r.h.s. of (1.1) when given any* $t \geqslant 0$ *the version of* $(E(k_s|\mathcal{F}_t), s \geqslant t)$ *used to define* y_t *is* $(k_t + \int_t^s Z_u^t \mathrm{d}u, s \geqslant t)$.

Economics states that the exchange rate at t satisfies (1.1) a.s. but since the r.h.s. of (1.1) is finite for every $\omega \in \Omega$ there is no loss of generality in defining $y_t(\omega)$ by (1.1) for every $\omega \in \Omega$ and $t \geqslant 0$ then thinking of y_t as the exchange rate at t. Two other facts are used in the theorems which follow. First, by lemma 3.4

$$\int_s^\infty \exp\left(\frac{t-u}{\alpha}\right)\left(E\left(\int_s^u b(v,k_v)\mathrm{d}v|\mathcal{F}_s\right)(\omega) - E\left(\int_s^u b(v,k_v)\mathrm{d}v|\mathcal{F}_t\right)(\omega)\right)\mathrm{d}u$$

is finite for all $\omega \in \Omega$ when $s, t \geqslant 0$ with $t < s$. Second, for all $s, t \geqslant 0$ with $t < s$

$$\int_s^\infty \exp\left(\frac{t-u}{\alpha}\right)\int_s^u \exp(4(K^2+1)(v-s))\mathrm{d}v\mathrm{d}u < \infty$$

by Assumption 2.3, a fact revealed when proving Theorem 3.10. Establishing one more fundamental result is necessary before turning to our major theorems.

Lemma 3.8 *Given any* $T > 0$ *take any* $s, t \geqslant 0$ *with* $t < s \leqslant T$ *then there exists a constant* $L = L(T) > 0$ *such that for all* $u \geqslant s$ *and any* $x, y \in \mathbb{R}$

$$\left(E|\xi_{t,x}(u) - \xi_{s,y}(u)|^2\right)^{\frac{1}{2}} \leqslant L\exp\left(4(K^2+1)(u-s)\right)\left(|x-y|^2 + (s-t)\right)^{\frac{1}{2}}, \tag{3.7}$$

where $(\xi_{s,y}(u), u \geqslant s)$ *as well as* $(\xi_{t,x}(u), u \geqslant t)$ *satisfy SDEs of the form in (3.2).*

Proof Let $T > 0$ and take $s, t \geqslant 0$ with $t < s \leqslant T$ then for $u \geqslant s$,

$$\begin{aligned}\xi_{s,y}(u) - \xi_{t,x}(u) = {}& y - x + \int_s^u b(v,\xi_{s,y}(v)) - b(v,\xi_{t,x}(v))\mathrm{d}v \\ &+ \int_s^u \sigma(v,\xi_{s,y}(v)) - \sigma(v,\xi_{t,x}(v))\mathrm{d}W_v - \int_t^s b(v,\xi_{t,x}(v))\mathrm{d}v - \int_t^s \sigma(v,\xi_{t,x}(v))\mathrm{d}W_v.\end{aligned} \tag{3.8}$$

Thus, part (1) of Assumption 2.2 and (2.4) yield

$$\begin{aligned}E|\xi_{s,y}(u) - \xi_{t,x}(u)|^2 \leqslant {}& 2|x-y|^2 + 4E\left(\int_s^u |b(v,\xi_{s,y}(v)) - b(v,\xi_{t,x}(v))|\mathrm{d}v\right)^2 \\ &+ 8E\left(\int_s^u \sigma(v,\xi_{s,y}(v)) - \sigma(v,\xi_{t,x}(v))\mathrm{d}W_v\right)^2 \\ &+ 16E\left(\int_t^s |b(v,\xi_{t,x}(v))|\mathrm{d}v\right)^2 \\ &+ 16E\left(\int_t^s \sigma(v,\xi_{t,x}(v))\mathrm{d}W_v\right)^2 \leqslant 2|x-y|^2 \\ &+ 8E\left(\int_s^u \exp(-v)|\xi_{s,y}(v) - \xi_{t,x}(v)|\mathrm{d}v\right)^2 \\ &+ 8K^2\int_s^u E|\xi_{s,y}(v) - \xi_{t,x}(v)|^2\mathrm{d}v \\ &+ 16M^2(s-t)^2 + 16E\int_t^s |\sigma(v,\xi_{t,x}(v))|^2\mathrm{d}v.\end{aligned} \tag{3.9}$$

Applying Hölder's inequality and Fubini's theorem yields

$$E\left(\int_s^u \exp(-v)|\xi_{s,y}(v)-\xi_{t,x}(v)|\mathrm{d}v\right)^2 \leqslant \left(\int_s^u \exp(-2v)\mathrm{d}v\right)\left(\int_s^u E|\xi_{s,y}(v)-\xi_{t,x}(v)|^2\mathrm{d}v\right).$$

Assumption 2.2 implies that $|\sigma(t,x)|^2 \leqslant D$ for all $(t,x)\in[0,\infty)\times\mathbb{R}$ where $D>0$ is a constant. Thus,

$$E\int_t^s |\sigma(v,\xi_{t,x}(v))|^2\mathrm{d}v \leqslant D(s-t). \tag{3.10}$$

Recalling that $t<s\leqslant T$, it follows that there exists a constant $J=J(T)>0$ such that

$$E|\xi_{s,y}(u)-\xi_{t,x}(u)|^2 \leqslant 2|x-y|^2+8(K^2+1)\int_s^u E|\xi_{s,y}(v)-\xi_{t,x}(v)|^2\mathrm{d}v+J(s-t).$$

By Gronwall's inequality, for any $u\geqslant s$

$$E|\xi_{t,x}(u)-\xi_{s,y}(u)|^2 \leqslant \exp(8(K^2+1)(u-s))(2|x-y|^2+J(s-t)). \tag{3.11}$$

Hence, we may find $L=L(T)$ that satisfies (3.7). □

Now we turn to proving key inequalities that illuminate properties of $(\mu_t, t\geqslant 0)$.

Theorem 3.9 *Given any $T>0$ take any $s,t\geqslant 0$ with $t<s\leqslant T$ and choose any $\omega\in\Omega$ then for the constant $L=L(T)$ of Lemma 3.8 we have*

$$\left|\int_s^\infty \exp\left(\frac{t-u}{\alpha}\right)\left(E\left(\int_s^u b(v,k_v)\mathrm{d}v|\mathcal{F}_s\right)(\omega)-E\left(\int_s^u b(v,k_v)\mathrm{d}v|\mathcal{F}_t\right)(\omega)\right)\mathrm{d}u\right|$$
$$\leqslant (|k_t(\omega)-k_s(\omega)|^2+(s-t))^{\frac{1}{2}}\int_s^\infty \exp\left(\frac{t-u}{\alpha}\right)\int_s^u KL\exp(4(K^2+1)(v-s))\mathrm{d}v\mathrm{d}u.$$

Proof Let $t<s\leqslant T$ then using the Lipschitz condition for b and Lemma 3.8, given any $v\geqslant s$ and any $\omega\in\Omega$ for the constant $L=L(T)$ of Lemma 3.8 we have

$$\begin{aligned}|Z_v^s(\omega)-Z_v^t(\omega)| &= |\int_\Omega b(v,\xi_{s,k_s(\omega)}(v)(z))\mathrm{d}P(z)-\int_\Omega b(v,\xi_{t,k_t(\omega)}(v)(z))\mathrm{d}P(z)|\\ &\leqslant \int_\Omega |b(v,\xi_{s,k_s(\omega)}(v)(z))-b(v,\xi_{t,k_t(\omega)}(v)(z))|\mathrm{d}P(z)\\ &\leqslant K(\int_\Omega |\xi_{s,k_s(\omega)}(v)(z)-\xi_{t,k_t(\omega)}(v)(z)|^2\mathrm{d}P(z))^{\frac{1}{2}}\\ &\leqslant KL\exp(4(K^2+1)(v-s))(|k_t(\omega)-k_s(\omega)|^2+(s-t))^{\frac{1}{2}}.\end{aligned} \tag{3.12}$$

By the inequality in (3.12), for all $\omega\in\Omega$ we have

$$\left|\int_s^\infty \exp\left(\frac{t-u}{\alpha}\right)\left(E\left(\int_s^u b(v,k_v)\mathrm{d}v|\mathcal{F}_s\right)(\omega)-E\left(\int_s^u b(v,k_v)\mathrm{d}v|\mathcal{F}_t\right)(\omega)\right)\mathrm{d}u\right|$$
$$=\left|\int_s^\infty \exp\left(\frac{t-u}{\alpha}\right)\int_s^u Z_v^s(\omega)-Z_v^t(\omega)\mathrm{d}v\mathrm{d}u\right| \leqslant \int_s^\infty \exp\left(\frac{t-u}{\alpha}\right)\int_s^u |Z_v^s(\omega)-Z_v^t(\omega)|\mathrm{d}v\mathrm{d}u$$
$$\leqslant (|k_t(\omega)-k_s(\omega)|^2+(s-t))^{\frac{1}{2}}\int_s^\infty \exp\left(\frac{t-u}{\alpha}\right)\int_s^u KL\exp(4(K^2+1)(v-s))\mathrm{d}v\mathrm{d}u.$$

□

Theorem 3.10 *Given any $T>0$ there exists a constant $C=C(T)>0$ such that for all $\omega\in\Omega$ and all $s,t\in[0,T]$ with $t<s$ we have*

$$|\mu_t(\omega)-\mu_s(\omega)|\leqslant C((s-t)+((s-t)+|k_t(\omega)-k_s(\omega)|^2)^{\frac{1}{2}}).$$

Proof Let $T>0$, take any s,t with $0\leqslant t<s\leqslant T$ and take any $\omega\in\Omega$. Consider

$$\begin{aligned}\mu_s(\omega)-\mu_t(\omega) &= \int_s^\infty \exp\left(\frac{s-u}{\alpha}\right)E\left(\int_s^u b(v,k_v)\mathrm{d}v|\mathcal{F}_s\right)(\omega)\mathrm{d}u\\ &\quad -\int_t^\infty \exp\left(\frac{t-u}{\alpha}\right)E\left(\int_t^u b(v,k_v)\mathrm{d}v|\mathcal{F}_t\right)(\omega)\mathrm{d}u\\ &= \int_s^\infty \exp\left(\frac{s-u}{\alpha}\right)E\left(\int_s^u b(v,k_v)\mathrm{d}v|\mathcal{F}_s\right)(\omega)\\ &\quad -\exp\left(\frac{t-u}{\alpha}\right)E\left(\int_t^u b(v,k_v)\mathrm{d}v|\mathcal{F}_t\right)(\omega)\mathrm{d}u\\ &\quad -\int_t^s \exp\left(\frac{t-u}{\alpha}\right)E\left(\int_t^u b(v,k_v)\mathrm{d}v|\mathcal{F}_t\right)(\omega)\mathrm{d}u. \end{aligned}\tag{3.13}$$

Let $I_1(s,t,\omega)$ and $I_2(s,t,\omega)$ be the following integrals from (3.13):

$$\begin{aligned}I_1(s,t,\omega) &= \int_s^\infty \exp\left(\frac{s-u}{\alpha}\right)E\left(\int_s^u b(v,k_v)\mathrm{d}v|\mathcal{F}_s\right)(\omega)\\ &\quad -\exp\left(\frac{t-u}{\alpha}\right)E\left(\int_t^u b(v,k_v)\mathrm{d}v|\mathcal{F}_t\right)(\omega)\mathrm{d}u,\\ I_2(s,t,\omega) &= \int_t^s \exp\left(\frac{t-u}{\alpha}\right)E\left(\int_t^u b(v,k_v)\mathrm{d}v|\mathcal{F}_t\right)(\omega)\mathrm{d}u.\end{aligned}$$

Given any $T>0$ we show that when $s,t\in[0,T]$ with $t<s$ there exist constants $J_1(T)$ and $J_2(T)$ such that for all $\omega\in\Omega$ and $i=1,2$,

$$|I_i(s,t,\omega)|\leqslant J_i(T)((s-t)+((s-t)+|k_t(\omega)-k_s(\omega)|^2)^{\frac{1}{2}}).\tag{3.14}$$

Achieving this involves showing there are three integrals, $I_1^l(s,t,\omega)$, $l=1,2,3$ such that $I_1(s,t,\omega)=\sum_{l=1}^3 I_1^l(s,t,\omega)$. For all $\omega\in\Omega$ we show

$$|I_1^l(s,t,\omega)|\leqslant J_1^l((s-t)+((s-t)+|k_t(\omega)-k_s(\omega)|^2)^{\frac{1}{2}})\tag{3.15}$$

for $l=1,2,3$ where $J_1^l=J_1^l(T)>0$ are constants. Letting $J_1=\sum_{l=1}^3 J_1^l$, for all $\omega\in\Omega$ when $0\leqslant t<s\leqslant T$ we will then have

$$|\mu_t(\omega)-\mu_s(\omega)|\leqslant C((s-t)+((s-t)+|k_t(\omega)-k_s(\omega)|^2)^{\frac{1}{2}}),\tag{3.16}$$

where $C=C(T)=J_1(T)+J_2(T)$. To begin, observe that by Lemma 3.4 $I_2(s,t,\omega)$ satisfies

$$\left|\int_t^s \exp\left(\frac{t-u}{\alpha}\right)E\left(\int_t^u b(v,k_v)\mathrm{d}v|\mathcal{F}_t\right)(\omega)\mathrm{d}u\right|\leqslant M\int_t^s \exp\left(\frac{t-u}{\alpha}\right)(u-t)\mathrm{d}u\leqslant MT(s-t).\tag{3.17}$$

Thus, we can focus on $I_1(s,t,\omega)$ which can be written as

$$\begin{aligned}&\int_s^\infty \left(\exp\left(\frac{s-u}{\alpha}\right)-\exp\left(\frac{t-u}{\alpha}\right)\right)E\left(\int_s^u b(v,k_v)\mathrm{d}v|\mathcal{F}_s\right)(\omega)\mathrm{d}u\\ &+\int_s^\infty \exp\left(\frac{t-u}{\alpha}\right)(E\left(\int_s^u b(v,k_v)\mathrm{d}v|\mathcal{F}_s\right)(\omega)-E\left(\int_t^u b(v,k_v)\mathrm{d}v|\mathcal{F}_t\right)(\omega))\mathrm{d}u.\end{aligned}\tag{3.18}$$

We prove the bound holds for

$$\left|\int_s^\infty \left(\exp\left(\frac{s-u}{\alpha}\right)-\exp\left(\frac{t-u}{\alpha}\right)\right)E\left(\int_s^u b(v,k_v)\mathrm{d}v|\mathcal{F}_s\right)(\omega)\mathrm{d}u\right|\tag{3.19}$$

and then that it holds for

$$\left|\int_s^\infty \exp\left(\frac{t-u}{\alpha}\right)(E\left(\int_s^u b(v,k_v)\mathrm{d}v|\mathcal{F}_s\right)(\omega) - E\left(\int_t^u b(v,k_v)\mathrm{d}v|\mathcal{F}_t\right)(\omega))\mathrm{d}u\right|. \tag{3.20}$$

Let $u \geqslant s$ then by the mean value theorem

$$\exp\left(\frac{s-u}{\alpha}\right) - \exp\left(\frac{t-u}{\alpha}\right) = \alpha^{-1}\exp\left(\frac{t-u}{\alpha} + \theta(s,t,u)\left(\frac{s-t}{\alpha}\right)\right)(s-t) \tag{3.21}$$

for some $\theta(s,t,u) \in (0,1)$ so invoking Lemma 3.4 we have

$$\begin{aligned}
&\left|\int_s^\infty \left(\exp\left(\frac{s-u}{\alpha}\right) - \exp\left(\frac{t-u}{\alpha}\right)\right) E\left(\int_s^u b(v,k_v)\mathrm{d}v|\mathcal{F}_s\right)(\omega)\mathrm{d}u\right| \\
&\leqslant \alpha^{-1}M\exp\left(\frac{s-t}{\alpha}\right)(s-t)\int_s^\infty \exp\left(\frac{t-u}{\alpha}\right)(u-s)\mathrm{d}u \\
&\leqslant \alpha^{-1}M\exp\left(\frac{s-t}{\alpha}\right)(s-t)\int_s^\infty \exp\left(\frac{s-u}{\alpha}\right)(u-s)\mathrm{d}u.
\end{aligned} \tag{3.22}$$

Since $\alpha^{-1}\int_s^\infty \exp\left(\frac{s-u}{\alpha}\right)(u-s)\mathrm{d}u = \alpha$ and $0 \leqslant t < s \leqslant T$ we have $\exp\left(\frac{s-t}{\alpha}\right) \leqslant \exp\left(\frac{T}{\alpha}\right)$ so (3.19) satisfies the desired bound.

When $t < s < u$, for every $\omega \in \Omega$

$$\begin{aligned}
E\left(\int_t^u b(v,k_v)\mathrm{d}v|\mathcal{F}_t\right)(\omega) &= \int_t^u Z_v^t(\omega)\mathrm{d}v = \int_t^s Z_v^t(\omega)\mathrm{d}v + \int_s^u Z_v^t(\omega)\mathrm{d}v \\
&= E\left(\int_t^s b(v,k_v)\mathrm{d}v|\mathcal{F}_t\right)(\omega) + E\left(\int_s^u b(v,k_v)\mathrm{d}v|\mathcal{F}_t\right)(\omega),
\end{aligned} \tag{3.23}$$

so the integral in (3.20) can be written as

$$\begin{aligned}
&\int_s^\infty \exp\left(\frac{t-u}{\alpha}\right)(E\left(\int_s^u b(v,k_v)\mathrm{d}v|\mathcal{F}_s\right)(\omega) - E\left(\int_t^u b(v,k_v)\mathrm{d}v|\mathcal{F}_t\right)(\omega))\mathrm{d}u \\
= &\int_s^\infty \exp\left(\frac{t-u}{\alpha}\right)\left(E\left(\int_s^u b(v,k_v)\mathrm{d}v|\mathcal{F}_s\right)(\omega) - E\left(\int_s^u b(v,k_v)\mathrm{d}v|\mathcal{F}_t\right)(\omega)\right)\mathrm{d}u \\
&- \int_s^\infty \exp\left(\frac{t-u}{\alpha}\right)E\left(\int_t^s b(v,k_v)\mathrm{d}v|\mathcal{F}_t\right)(\omega)\mathrm{d}u.
\end{aligned} \tag{3.24}$$

By Lemma 3.4

$$\left|\int_s^\infty \exp\left(\frac{t-u}{\alpha}\right)E\left(\int_t^s b(v,k_v)\mathrm{d}v|\mathcal{F}_t\right)(\omega)\mathrm{d}u\right| \leqslant \alpha M(s-t). \tag{3.25}$$

Thus, we can focus on

$$\int_s^\infty \exp\left(\frac{t-u}{\alpha}\right)\left(E\left(\int_s^u b(v,k_v)\mathrm{d}v|\mathcal{F}_s\right)(\omega) - E\left(\int_s^u b(v,k_v)\mathrm{d}v|\mathcal{F}_t\right)(\omega)\right)\mathrm{d}u. \tag{3.26}$$

By Theorem 3.9 there is a constant $L = L(T) > 0$ such that for any $\omega \in \Omega$

$$\begin{aligned}
&\left|\int_s^\infty \exp\left(\frac{t-u}{\alpha}\right)\left(E\left(\int_s^u b(v,k_v)\mathrm{d}v|\mathcal{F}_s\right)(\omega) - E\left(\int_s^u b(v,k_v)\mathrm{d}v|\mathcal{F}_t\right)(\omega)\right)\mathrm{d}u\right| \\
&\leqslant (|k_t(\omega) - k_s(\omega)|^2 + (s-t))^{\frac{1}{2}}\int_s^\infty \exp\left(\frac{t-u}{\alpha}\right)\int_s^u KL\exp(4(K^2+1)(v-s))\mathrm{d}v\mathrm{d}u \\
&\leqslant KL(4(K^2+1))^{-1}(|k_t(\omega) - k_s(\omega)|^2 + (s-t))^{\frac{1}{2}}\int_s^\infty \exp\left(\frac{t-u}{\alpha}\right)\exp\left(4(K^2+1)(u-s)\right)\mathrm{d}u.
\end{aligned}$$

By Assumption 2.3 we have $\frac{1}{\alpha} > 4(K^2+1)$ so that

$$\begin{aligned} &\int_s^\infty \exp\left(\frac{t-u}{\alpha}\right) \exp\left(4(K^2+1)(u-s)\right) \mathrm{d}u \\ &= \exp\left(\frac{t}{\alpha} - 4(K^2+1)s\right) \int_s^\infty \exp\left(-\left(\frac{1}{\alpha} - 4(K^2+1)\right) u\right) \mathrm{d}u \\ &= \left(\frac{1}{\alpha} - 4(K^2+1)\right)^{-1} \exp\left(\frac{t}{\alpha} - 4(K^2+1)s\right) \exp\left(-\left(\frac{1}{\alpha} - 4(K^2+1)\right) s\right) \\ &= \exp\left(\frac{t-s}{\alpha}\right) \left(\frac{1}{\alpha} - 4(K^2+1)\right)^{-1} \leqslant \exp\left(\frac{T}{\alpha}\right) \left(\frac{1}{\alpha} - 4(K^2+1)\right)^{-1}. \end{aligned} \tag{3.27}$$

Combining inequalities establishes the bound for the integral in (3.26) since

$$\begin{aligned} &\left| \int_s^\infty \exp\left(\frac{t-u}{\alpha}\right) \left(E\left(\int_s^u b(v,k_v)\mathrm{d}v|\mathcal{F}_s\right)(\omega) - E\left(\int_s^u b(v,k_v)\mathrm{d}v|\mathcal{F}_t\right)(\omega)\right) \mathrm{d}u \right| \\ &\leqslant \exp\left(\frac{T}{\alpha}\right) \left(\frac{1}{\alpha} - 4(K^2+1)\right)^{-1} KL(4(K^2+1))^{-1}(|k_t(\omega) - k_s(\omega)|^2 + (s-t))^{\frac{1}{2}}. \end{aligned}$$

□

Theorem 3.11 *The process $(\mu_t, t \geqslant 0)$ is adapted and continuous.*

Proof First it is established that $(\mu_t, t \geqslant 0)$ is adapted then that its sample paths are continuous. Recall that for any $t \geqslant 0$ we have $\mu_t \equiv \int_t^\infty \exp\left(\frac{t-s}{\alpha}\right) \int_t^s Z_u^t \mathrm{d}u\mathrm{d}s$.

Let $t \leqslant s$ then the proof of Theorem 3.2 established that $\int_t^s Z_u^t \mathrm{d}u$ is $\mathcal{F}_t$-measurable. It also showed that for all $\omega \in \Omega$ the sample path $(\int_t^s Z_u^t(\omega)\mathrm{d}u, s \geqslant t) \in C([t,\infty),\mathbb{R})$. Hence, given any $T > t$ it follows that $\mu_{t,T}$ is $\mathcal{F}_t$-measurable where

$$\mu_{t,T} \equiv \int_t^T \exp\left(\frac{t-s}{\alpha}\right) \int_t^s Z_u^t \mathrm{d}u\mathrm{d}s. \tag{3.28}$$

Let $\{T_n\}_{n=1}^\infty$ be a sequence such that $T_n \uparrow \infty$ as $n \to \infty$ and $T_n > t$ for all $n \geqslant 1$ then $[t, T_n) \uparrow [t,\infty)$ as $n \to \infty$ so for every $\omega \in \Omega$

$$\mu_{t,T_n}(\omega) \to \mu_t(\omega) \quad \text{as} \quad n \to \infty.$$

Hence, μ_t is $\mathcal{F}_t$-measurable for any $t \geqslant 0$, implying $(\mu_t, t \geqslant 0)$ is adapted.

To prove continuity let $A \in \mathcal{F}$ be the set where $\omega \in A$ implies

$$(k_t(\omega), t \geqslant 0) \in C([0,\infty),\mathbb{R}).$$

Choose any $T > 0$ and let $\omega \in A$ then by Theorem 3.10 the sample path $(\mu_t(\omega), t \in [0,T])$ is uniformly continuous since $(k_t(\omega), t \in [0,T])$ is uniformly continuous. Since T is arbitrary $(\mu_t(\omega), t \geqslant 0) \in C([0,\infty),\mathbb{R})$ for $\omega \in A$, implying $(\mu_t, t \geqslant 0)$ is a continuous process. □

Deriving that $(y_t, t \geqslant 0)$ is a semimartingale first requires showing $(\mu_t, t \geqslant 0)$ is a semimartingale, as mentioned in the introduction. Via (1.2) we then link their specific decompositions. Theorem 3.12 sets the stage for this key result of the paper.

Theorem 3.12 *The following hold:*

(1) *We can define a measurable function on Ω by*

$$\omega \to \int_0^\infty \exp\left(-\frac{s}{\alpha}\right) \int_0^s b(u, k_u(\omega))\mathrm{d}u\mathrm{d}s$$

and the random variable $\int_0^\infty \exp\left(-\frac{s}{\alpha}\right)\int_0^s b(u,k_u)\mathrm{d}u\mathrm{d}s \in L^1(\Omega,\mathcal{F},P)$.

(2) *Given any* $t \geqslant 0$ a.s.

$$E\left(\int_0^\infty \exp\left(-\frac{s}{\alpha}\right)\int_0^s b(u,k_u)\mathrm{d}u\mathrm{d}s|\mathcal{F}_t\right) = \int_0^t \exp\left(-\frac{s}{\alpha}\right)\int_0^s b(u,k_u)\mathrm{d}u\mathrm{d}s$$
$$+\int_t^\infty \exp\left(-\frac{s}{\alpha}\right)E\left(\int_t^s b(u,k_u)\mathrm{d}u|\mathcal{F}_t\right)\mathrm{d}s + \alpha\exp\left(-\frac{t}{\alpha}\right)\int_0^t b(u,k_u)\mathrm{d}u, \tag{3.29}$$

where all of the integrals on the r.h.s. of (3.29) are finite for all $\omega \in \Omega$.

(3) *The r.h.s. of (3.29) defines a version of* $\left(E\left(\int_0^\infty \exp\left(-\frac{s}{\alpha}\right)\int_0^s b(u,k_u)\mathrm{d}u\mathrm{d}s|\mathcal{F}_t\right), t \geqslant 0\right)$ *such that for a.a.* $\omega \in \Omega$

$$\left(E\left(\int_0^\infty \exp\left(-\frac{s}{\alpha}\right)\int_0^s b(u,k_u)\mathrm{d}u\mathrm{d}s|\mathcal{F}_t\right)(\omega), t \geqslant 0\right) \in C([0,\infty),\mathbb{R}).$$

Proof To establish (1) observe that since b is bounded and $(k_t, t \geqslant 0)$ measurable, $\int_0^t b(s,k_s(\omega))\mathrm{d}s$ exists and is finite given any $\omega \in \Omega$ and $t \geqslant 0$. Moreover, $(b(t,k_t), t \geqslant 0)$ is a continuous adapted process and $\mathcal{F}_t$ a complete σ-field for any $t \geqslant 0$. Thus, given any $t \geqslant 0$ it follows that $\int_0^t b(s,k_s)\mathrm{d}s$ is $\mathcal{F}_t$-measurable, a fact used throughout the proof.

Also for every $\omega \in \Omega$ we have

$$\left(\int_0^t b(s,k_s(\omega))\mathrm{d}s, t \geqslant 0\right) \in C([0,\infty),\mathbb{R}).$$

Consider the process $(\exp\left(-\frac{t}{\alpha}\right)\int_0^t b(s,k_s)\mathrm{d}s, t \geqslant 0)$ and observe that for all $\omega \in \Omega$

$$\left(\exp\left(-\frac{t}{\alpha}\right)\int_0^t b(s,k_s(\omega))\mathrm{d}s, t \geqslant 0\right) \in C([0,\infty),\mathbb{R}).$$

Hence, for every $\omega \in \Omega$ it follows that $(\exp\left(-\frac{t}{\alpha}\right)\int_0^t b(s,k_s(\omega))\mathrm{d}s, t \geqslant 0)$ is measurable relative to $\mathcal{B}([0,\infty))$ and $\mathcal{B}(\mathbb{R})$. Recalling that $|b| \leqslant M$, for all $\omega \in \Omega$ we have

$$\int_0^\infty \exp\left(-\frac{s}{\alpha}\right)\left|\int_0^s b(u,k_u(\omega))\mathrm{d}u\right|\mathrm{d}s \leqslant M\int_0^\infty \exp\left(-\frac{s}{\alpha}\right)s\mathrm{d}s < \infty. \tag{3.30}$$

Thus, $\omega \to \int_0^\infty \exp\left(-\frac{s}{\alpha}\right)\int_0^s b(u,k_u(\omega))\mathrm{d}u\mathrm{d}s$ is a well defined function on Ω.

Since $(\exp\left(-\frac{t}{\alpha}\right)\int_0^t b(s,k_s)\mathrm{d}s, t \geqslant 0)$ is a continuous adapted process, given any $t \geqslant 0$ it follows that $\int_0^t \exp\left(-\frac{s}{\alpha}\right)\int_0^s b(u,k_u)\mathrm{d}u\mathrm{d}s$ is $\mathcal{F}_t$-measurable, which is another fact used throughout the proof. Moreover, for every $\omega \in \Omega$

$$\int_0^t \exp\left(-\frac{s}{\alpha}\right)\int_0^s b(u,k_u(\omega))\mathrm{d}u\mathrm{d}s \to \int_0^\infty \exp\left(-\frac{s}{\alpha}\right)\int_0^s b(u,k_u(\omega))\mathrm{d}u\mathrm{d}s \quad \text{as} \quad t \to \infty,$$

so $\int_0^\infty \exp\left(-\frac{s}{\alpha}\right)\int_0^s b(u,k_u)\mathrm{d}u\mathrm{d}s$ is measurable relative to $\mathcal{F}$ and $\mathcal{B}(\mathbb{R})$. By (3.30) we have

$$\int_0^\infty \exp\left(-\frac{s}{\alpha}\right)\int_0^s b(u,k_u)\mathrm{d}u\mathrm{d}s \in L^1(\Omega,\mathcal{F},P).$$

To prove (2) we need to break $\int_0^\infty \exp\left(-\frac{s}{\alpha}\right)\int_0^s b(u,k_u)\mathrm{d}u\mathrm{d}s$ into a number of integrals. To this end take any $t \geqslant 0$ and $\omega \in \Omega$ then for any $T \geqslant t$ arguments which establish (1) imply that $\int_T^\infty \exp\left(-\frac{s}{\alpha}\right)\int_t^s b(u,k_u(\omega))\mathrm{d}u\mathrm{d}s$ exists and is finite. They also establish that

$$\int_T^\infty \exp\left(-\frac{s}{\alpha}\right)\int_t^s b(u,k_u)\mathrm{d}u\mathrm{d}s \in L^1(\Omega,\mathcal{F},P).$$

Likewise for $t < T$ and $\omega \in \Omega$ it follows that $\int_t^T \exp\left(-\frac{s}{\alpha}\right)\int_t^s b(u,k_u(\omega))\mathrm{d}u\mathrm{d}s$ exists and is finite.

Moreover, our arguments imply that $\int_t^T \exp\left(-\frac{s}{\alpha}\right) \int_t^s b(u,k_u)\mathrm{d}u\mathrm{d}s \in L^1(\Omega,\mathcal{F},P)$.

Observe that for any $t \geqslant 0$ and $\omega \in \Omega$

$$\int_0^\infty \exp\left(-\frac{s}{\alpha}\right) \int_0^s b(u,k_u(\omega))\mathrm{d}u\mathrm{d}s = \int_0^t \exp\left(-\frac{s}{\alpha}\right) \int_0^s b(u,k_u(\omega))\mathrm{d}u\mathrm{d}s$$
$$+\alpha \exp\left(-\frac{t}{\alpha}\right) \int_0^t b(u,k_u(\omega))\mathrm{d}u + \int_t^\infty \exp\left(-\frac{s}{\alpha}\right) \int_t^s b(u,k_u(\omega))\mathrm{d}u\mathrm{d}s. \tag{3.31}$$

The first two terms on the r.h.s. of (3.31) are $\mathcal{F}_t$-measurable and integrable so a.s.

$$E\left(\int_0^\infty \exp\left(-\frac{s}{\alpha}\right) \int_0^s b(u,k_u)\mathrm{d}u\mathrm{d}s|\mathcal{F}_t\right) = \int_0^t \exp\left(-\frac{s}{\alpha}\right) \int_0^s b(u,k_u)\mathrm{d}u\mathrm{d}s$$
$$+\alpha \exp\left(-\frac{t}{\alpha}\right) \int_0^t b(u,k_u)\mathrm{d}u + E\left(\int_t^\infty \exp\left(-\frac{s}{\alpha}\right) \int_t^s b(u,k_u)\mathrm{d}u\mathrm{d}s|\mathcal{F}_t\right). \tag{3.32}$$

Once we show that a.s.

$$E\left(\int_t^\infty \exp\left(-\frac{s}{\alpha}\right) \int_t^s b(u,k_u)\mathrm{d}u\mathrm{d}s|\mathcal{F}_t\right) = \int_t^\infty \exp\left(-\frac{s}{\alpha}\right) E\left(\int_t^s b(u,k_u)\mathrm{d}u|\mathcal{F}_t\right)\mathrm{d}s, \tag{3.33}$$

then (2) holds. To this end take any $T > t$ then using Lemma 3.1 and Theorem 3.2 a.s.

$$E\left(\int_t^\infty \exp\left(-\frac{s}{\alpha}\right) \int_t^s b(u,k_u)\mathrm{d}u\mathrm{d}s|\mathcal{F}_t\right)$$
$$= \int_t^T \exp\left(-\frac{s}{\alpha}\right) E\left(\int_t^s b(u,k_u)\mathrm{d}u|\mathcal{F}_t\right)\mathrm{d}s + E\left(\int_T^\infty \exp\left(-\frac{s}{\alpha}\right) \int_t^s b(u,k_u)\mathrm{d}u\mathrm{d}s|\mathcal{F}_t\right). \tag{3.34}$$

To apply lemma 3.1 when deriving (3.34) for every $\omega \in \Omega$ and $s \geqslant t$ we take

$$E(\exp\left(-\frac{s}{\alpha}\right) \int_t^s b(u,k_u)\mathrm{d}u|\mathcal{F}_t)(\omega) = \exp\left(-\frac{s}{\alpha}\right) \int_t^s Z_u^t(\omega)\mathrm{d}u.$$

Given any $\epsilon > 0$ we can choose some large $T > t$ such that for any $\omega \in \Omega$

$$\left|\int_T^\infty \exp\left(-\frac{s}{\alpha}\right) \int_t^s b(u,k_u(\omega))\mathrm{d}u\mathrm{d}s\right| \leqslant M \int_T^\infty \exp\left(-\frac{s}{\alpha}\right)(s-t)\mathrm{d}s < \epsilon. \tag{3.35}$$

Choose some $T > t$ so that (3.35) holds then a.s.

$$\left|E\left(\int_T^\infty \exp\left(-\frac{s}{\alpha}\right) \int_t^s b(u,k_u)\mathrm{d}u\mathrm{d}s|\mathcal{F}_t\right)\right| \leqslant \epsilon. \tag{3.36}$$

Now consider $\int_t^\infty \exp\left(-\frac{s}{\alpha}\right) E\left(\int_t^s b(u,k_u)\mathrm{d}u|\mathcal{F}_t\right)(\omega)\mathrm{d}s$, which exists and is finite for all $\omega \in \Omega$ by Lemma 3.4. Given any $T > t$ and $\omega \in \Omega$

$$\int_t^\infty \exp\left(-\frac{s}{\alpha}\right) E\left(\int_t^s b(u,k_u)\mathrm{d}u|\mathcal{F}_t\right)(\omega)\mathrm{d}s$$
$$= \int_t^T \exp\left(-\frac{s}{\alpha}\right) E\left(\int_t^s b(u,k_u)\mathrm{d}u|\mathcal{F}_t\right)(\omega)\mathrm{d}s + \int_T^\infty \exp\left(-\frac{s}{\alpha}\right) E\left(\int_t^s b(u,k_u)\mathrm{d}u|\mathcal{F}_t\right)(\omega)\mathrm{d}s. \tag{3.37}$$

Using Lemma 3.4 again, for any $\epsilon > 0$ and $\omega \in \Omega$ there is a large $T > t$ such that

$$\left|\int_T^\infty \exp\left(-\frac{s}{\alpha}\right) E\left(\int_t^s b(u,k_u)\mathrm{d}u|\mathcal{F}_t\right)(\omega)\mathrm{d}s\right| \leqslant M \int_T^\infty \exp\left(-\frac{s}{\alpha}\right)(s-t)\mathrm{d}s < \epsilon. \tag{3.38}$$

Using (3.34), (3.36), (3.37) and (3.38), for any $\epsilon > 0$ a.s.

$$\left|E\left(\int_t^\infty \exp\left(-\frac{s}{\alpha}\right)\int_t^s b(u,k_u)\mathrm{d}u\mathrm{d}s|\mathcal{F}_t\right)-\int_t^\infty \exp\left(-\frac{s}{\alpha}\right)E\left(\int_t^s b(u,k_u)\mathrm{d}u|\mathcal{F}_t\right)\mathrm{d}s\right|<\epsilon. \tag{3.39}$$

For any $n \geqslant 1$ let A_n be the subset with $P(A_n)=1$ where $\omega \in A_n$ implies

$$\left|E\left(\int_t^\infty \exp\left(-\frac{s}{\alpha}\right)\int_t^s b(u,k_u)\mathrm{d}u\mathrm{d}s|\mathcal{F}_t\right)(\omega)-\int_t^\infty \exp\left(-\frac{s}{\alpha}\right)E\left(\int_t^s b(u,k_u)\mathrm{d}u|\mathcal{F}_t\right)(\omega)\mathrm{d}s\right|<\frac{1}{n}.$$

Let $A=\bigcap_{n=1}^\infty A_n$ then $P(A)=1$ and for any $\omega \in A$

$$E\left(\int_t^\infty \exp\left(-\frac{s}{\alpha}\right)\int_t^s b(u,k_u)\mathrm{d}u\mathrm{d}s|\mathcal{F}_t\right)(\omega)=\int_t^\infty \exp\left(-\frac{s}{\alpha}\right)E\left(\int_t^s b(u,k_u)\mathrm{d}u|\mathcal{F}_t\right)(\omega)\mathrm{d}s.$$

By Theorem 3.11 all terms on the r.h.s. of (3.29) are $\mathcal{F}_t$-measurable so the r.h.s. of (3.29) is a version of $E\left(\int_0^\infty \exp(\frac{-s}{\alpha})\int_0^s b(u,k_u)\mathrm{d}u\mathrm{d}s|\mathcal{F}_t\right)$. Theorem 3.11 also implies that the process defined by the r.h.s. of (3.29) has sample paths that are continuous a.s. □

Lemma 3.13 *Define M_t for each $t \geqslant 0$ by*

$$M_t = E\left(\int_0^\infty \exp\left(-\frac{s}{\alpha}\right)\int_0^s b(u,k_u)\mathrm{d}u\mathrm{d}s|\mathcal{F}_t\right),$$

where for any $t \geqslant 0$ the version of $E\left(\int_0^\infty \exp\left(-\frac{s}{\alpha}\right)\int_0^s b(u,k_u)\mathrm{d}u\mathrm{d}s|\mathcal{F}_t\right)$ is the one in (3.29). Then $(M_t, t \geqslant 0)$ is a continuous L^2-martingale.

Proof Since $\int_0^\infty \exp\left(-\frac{s}{\alpha}\right)\int_0^s b(u,k_u)\mathrm{d}u\mathrm{d}s \in L^1(\Omega,\mathcal{F},P)$ the process $(M_t, t \geqslant 0)$ is a martingale. Theorem 3.12 established that the r.h.s. of (3.29) defines a continuous process. Via (3.30) we showed that $\int_0^\infty \exp\left(-\frac{s}{\alpha}\right)\int_0^s b(u,k_u)\mathrm{d}u\mathrm{d}s$ is bounded so $EM_t^2 < \infty$. □

Definition 3.14 *For each $t \geqslant 0$ define M_t^1 by*

$$M_t^1 = M_t - M_0,$$

then $(M_t^1, t \geqslant 0)$ is a continuous L^2-martingale with $M_0^1 = 0$.

Theorem 3.15 *The process $(\mu_t, t \geqslant 0)$ is a continuous semimartingale with the following decomposition: for all $t \geqslant 0$*

$$\mu_t = M_0 + \int_0^t \exp\left(\frac{s}{\alpha}\right)\,\mathrm{d}M_s^1 + \left(\exp\left(\frac{t}{\alpha}\right)M_0 - M_0\right) + \alpha^{-1}\int_0^t M_s^1 \exp\left(\frac{s}{\alpha}\right)\mathrm{d}s$$
$$-\int_0^t \exp\left(\frac{t-s}{\alpha}\right)\int_0^s b(u,k_u)\mathrm{d}u\mathrm{d}s - \alpha\int_0^t b(u,k_u)\mathrm{d}u. \tag{3.40}$$

Moreover, $(y_t, t \geqslant 0)$ is a continuous semimartingale with the following decomposition: for all $t \geqslant 0$

$$y_t = k_0 + \alpha^{-1}M_0 + \int_0^t \sigma(s,k_s)\mathrm{d}W_s + \alpha^{-1}\int_0^t \exp(\frac{s}{\alpha})\,\mathrm{d}M_s^1 + \alpha^{-1}(\exp(\frac{t}{\alpha})M_0 - M_0)$$
$$+\alpha^{-2}\int_0^t M_s^1\exp(\frac{s}{\alpha})\mathrm{d}s - \alpha^{-1}\int_0^t \exp\left(\frac{t-s}{\alpha}\right)\int_0^s b(u,k_u)\mathrm{d}u\mathrm{d}s. \tag{3.41}$$

Proof Observe that for any $t \geqslant 0$ we have

$$M_t^1 + M_0 = \int_0^t \exp\left(-\frac{s}{\alpha}\right)\int_0^s b(u,k_u)\mathrm{d}u\mathrm{d}s + \int_t^\infty \exp\left(-\frac{s}{\alpha}\right)E\left(\int_t^s b(u,k_u)\mathrm{d}u|\mathcal{F}_t\right)\mathrm{d}s$$
$$+\alpha\exp\left(-\frac{t}{\alpha}\right)\int_0^t b(u,k_u)\mathrm{d}u. \tag{3.42}$$

Moreover, for any $t \geqslant 0$ we have

$$\mu_t = \exp\left(\frac{t}{\alpha}\right)\int_t^\infty \exp\left(-\frac{s}{\alpha}\right) E\left(\int_t^s b(u,k_u)\mathrm{d}u|\mathcal{F}_t\right)\mathrm{d}s.$$

Thus, for any $t \geqslant 0$

$$\mu_t = \exp\left(\frac{t}{\alpha}\right)(M_t^1 + M_0 - \int_0^t \exp\left(-\frac{s}{\alpha}\right)\int_0^s b(u,k_u)\mathrm{d}u\mathrm{d}s) - \alpha\int_0^t b(u,k_u)\mathrm{d}u. \tag{3.43}$$

Via the integration by parts formula (3.43) yields (3.40). To complete the proof recall Theorem 3.6 showed (1.2) holds for all $t \geqslant 0$ and $\omega \in \Omega$, so for any $t \geqslant 0$

$$y_t = k_t + \alpha^{-1}\mu_t. \tag{3.44}$$

By Theorem 3.11, $(\mu_t, t \geqslant 0)$ is a continuous adapted process so Theorem 3.11 and (3.44) imply that $(y_t, t \geqslant 0)$ is a continuous adapted process. Using (3.40) and (3.44) we obtain (3.41).

We now show (3.40) and (3.41) are valid semimartingale decompositions for $(\mu_t, t \geqslant 0)$ and $(y_t, t \geqslant 0)$ respectively. Since σ is bounded $(\int_0^t \sigma(s,k_s)\mathrm{d}W_s, t \geqslant 0)$ is a continuous L^2-martingale. Clearly, $(\int_0^t \exp(\frac{s}{\alpha})\, dM_s^1, t \geqslant 0)$ is a continuous L^2-martingale while both $(\exp(\frac{t}{\alpha})M_0 - M_0$, $t \geqslant 0)$ and $(\int_0^t M_s^1 \exp(\frac{s}{\alpha})\mathrm{d}s, t \geqslant 0)$ are continuous adapted processes of bounded variation. Finally, $(\int_0^t \exp\left(\frac{t-s}{\alpha}\right)\int_0^s b(u,k_u)\mathrm{d}u\mathrm{d}s, t \geqslant 0)$ is also a continuous adapted process of bounded variation since for every $t \geqslant 0$

$$\int_0^t \exp\left(\frac{t-s}{\alpha}\right)\int_0^s b(u,k_u)\mathrm{d}u\mathrm{d}s = \exp\left(\frac{t}{\alpha}\right)\int_0^t \exp\left(\frac{-s}{\alpha}\right)\int_0^s b(u,k_u)\mathrm{d}u\mathrm{d}s. \tag{3.45}$$

Being as $(\int_0^t b(s,k_s(\omega))\mathrm{d}s, t \geqslant 0) \in C([0,\infty),\mathbb{R})$ for all $\omega \in \Omega$, the r.h.s. of (3.45) defines a process whose every sample path resides in $C^1([0,\infty),\mathbb{R})$, the space of once continuously differentiable real valued functions on $[0,\infty)$. Thus, every sample path of the process $(\int_0^t \exp\left(\frac{t-s}{\alpha}\right)\int_0^s b(u,k_u)\mathrm{d}u\mathrm{d}s, t \geqslant 0)$ is of bounded variation. □

A semimartingale is a fundamental kind of stochastic process. Proving that $(y_t, t \geqslant 0)$ is a semimartingale places greater structure on a process arising naturally in macroeconomics as investors forecast monetary policy. However, the decomposition in (3.41) is very involved. When b depends on time exclusively the semimartingale property of $(y_t, t \geqslant 0)$ emerges very easily because $(\mu_t, t \geqslant 0)$ is a function of bounded variation. A more tractable decomposition of $(y_t, t \geqslant 0)$ than the one in (3.41) also emerges.

Theorem 3.16 *Let $b(t,x) = b(t)$ for all $(t,x) \in [0,\infty) \times \mathbb{R}$ where $b \in C([0,\infty),\mathbb{R})$ is bounded then μ_t is non-random for each $t \geqslant 0$ and $(\mu_t, t \geqslant 0)$ is a continuous function of bounded variation. Moreover, $(y_t, t \geqslant 0)$ is a continuous semimartingale and possesses the following decomposition: for all $t \geqslant 0$*

$$y_t = k_0 + \alpha^{-1}\mu_0 + \int_0^t b(s)\mathrm{d}s + \int_0^t \sigma(s,k_s)\mathrm{d}W_s + \alpha^{-1}(\mu_t - \mu_0).$$

Proof In this special case b still satisfies the conditions required of it in Assumptions 2.1–2.2. Thus, for any $\omega \in \Omega$ and $s, t \geqslant 0$ with $s \geqslant t$ we may set

$$E\left(\int_t^s b(u)\mathrm{d}u|\mathcal{F}_t\right)(\omega) = \int_t^s b(u)\mathrm{d}u, \tag{3.46}$$

since the r.h.s. of (3.46) is the relevant version of conditional expectation introduced in Theorem 3.2 in this case. Thus, μ_t is non-random for each $t \geqslant 0$. To reveal properties of $(\mu_t, t \geqslant 0)$ we simplify (1.3). First, observe $|b| \leqslant M$ for some constant $M > 0$ so

$$\exp\left(\frac{t-s}{\alpha}\right)\int_t^s |b_u|\mathrm{d}u \leqslant M\exp\left(\frac{t-s}{\alpha}\right)(s-t). \tag{3.47}$$

Now since $\exp\left(\frac{t-s}{\alpha}\right)(s-t) \to 0$ as $s \to \infty$ it follows that

$$\exp\left(\frac{t-s}{\alpha}\right)\int_t^s b_u\mathrm{d}u \to 0 \quad \text{as} \quad s \to \infty.$$

Then for all $t \geqslant 0$ integration by parts implies

$$\mu_t = \int_t^\infty \exp\left(\frac{t-s}{\alpha}\right)\int_t^s b(u)\mathrm{d}u\mathrm{d}s = \alpha\int_t^\infty \exp\left(\frac{t-s}{\alpha}\right)b(s)\mathrm{d}s. \tag{3.48}$$

Consider $v : [0,\infty) \to \mathbb{R}$ defined for each $t \geqslant 0$ by

$$v_t = \int_t^\infty \exp\left(\frac{-s}{\alpha}\right)b(s)\mathrm{d}s. \tag{3.49}$$

By continuity of b we have $v \in C^1([0,\infty),\mathbb{R})$, which implies $(\mu_t, t \geqslant 0)$ is a continuous function of bounded variation.

Turn attention to establishing that $(y_t, t \geqslant 0)$ is a semimartingale. Since $(k_t, t \geqslant 0)$ is a continuous adapted process and $(\mu_t, t \geqslant 0)$ is a continuous function of bounded variation, $(y_t, t \geqslant 0)$ is a continuous adapted process. Using (3.44) again implies that for all $t \geqslant 0$

$$y_t = k_0 + \alpha^{-1}\mu_0 + \int_0^t b(s)\mathrm{d}s + \int_0^t \sigma(s,k_s)\mathrm{d}W_s + \alpha^{-1}(\mu_t - \mu_0). \tag{3.50}$$

Both $(\alpha^{-1}(\mu_t - \mu_0), t \geqslant 0)$ and $(\int_0^t b(s)\mathrm{d}s, t \geqslant 0)$ are continuous functions of bounded variation while $(\int_0^t \sigma(s,k_s)\mathrm{d}W_s, t \geqslant 0)$ is a continuous L^2-martingale. □

We allow $(\mathcal{F}_t, t \geqslant 0)$ to be a general filtration but suppose instead it is the Brownian filtration. Then we can apply the martingale representation theorem (MRT) to the martingale $(M_t, t \geqslant 0)$ defined in Lemma 3.13. Manipulations like those in Theorem 3.15 imply $(y_t, t \geqslant 0)$ is a semimartingale, but very little in the structure of the decomposition would change. Researchers often assume the filtration is the Brownian filtration but there is no innate reason it should be part of a macroeconomic or financial model. Proving $(y_t, t \geqslant 0)$ is a semimartingale when $(\mathcal{F}_t, t \geqslant 0)$ is not the Brownian filtration is quite natural.

Our paper not only fills a void in mathematical economics but opens the door to a range of possibilities. The monetary model exchange rate is not part of general macroeconomic or financial models, rather it has been largely confined to regime switching research of the type mentioned in section 2. Including the monetary model exchange rate process in a wide range of models is tractable now that standard techniques from Ito's calculus can be applied to it. For example, portfolio optimization problems such as the one in Benth and Karlsen [2] ignore exchange rates, meaning investment in foreign stock markets is implicitly excluded. Deriving an optimal portfolio when foreign stock market investment is allowed and the exchange rate comes from the monetary model is one interesting problem.

Jeanblanc-Piqué [9] uses conditions on an elliptic operator to analyze the value function of an impulse control problem for $(y_t, t \geqslant 0)$ when $(k_t, t \geqslant 0)$ is a Brownian motion with drift. A central bank may influence its exchange rate via occasional interventions that depart from its basic anti-inflationary stance in an effort to raise exports, implying impulse control problems for the exchange rate naturally arise under anti-inflationary policy. Equation (3.41) suggests that conditions on elliptic operators will not be enough to characterize the value function of an impulse control problem for $(y_t, t \geqslant 0)$ under our anti-inflationary policy. Despite the challenges progress on such a control problem is possible. For instance, if b depends on time exclusively and σ depends on $(t,x) \in [0,\infty) \times \mathbb{R}$, Gagnon [7] uses constraints on integrals to analyze the value function of a control problem for $(y_t, t \geqslant 0)$.

4. Conclusion

Methods for modeling anti-inflationary monetary policy in SDE based models appear to be absent from the mathematical finance and economics literatures, thus our efforts in this direction and results for the properties of $(y_t, t \geqslant 0)$ are distinctive. Proving $(y_t, t \geqslant 0)$ is a semimartingale when policy is anti-inflationary is of practical interest because central banks implemented anti-inflationary policies for decades and thirty-six countries have freely floating exchange rates c.f. Wang [16, p. 19]. Recent econometric studies prove that connections between money supplies and exchange rates can be strong c.f. Wang [16, p.180-183], which further underscores the importance of analyzing $(y_t, t \geqslant 0)$.

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References

[1] Ash, R., Real Analysis and Probability, Academic Press, San Diego, 1972.

[2] Benth, F. E. and Karlsen, K. H., A note on Merton's portfolio selection problem for the Schwartz mean-reversion model, Stochastic Analysis and Applications, 2005, 23: 687–704.

[3] Cadenillas, A. and Zapatero, F., Optimal central bank intervention in the foreign exchange market, Journal of Economic Theory, 1999, 87: 218–242.

[4] Cadenillas, A. and Zapatero, F., Classical and stochastic impulse control of the exchange rate using interest rates and reserves, Mathematical Finance, 2000, 10: 141–156.

[5] Flood, R. and Garber, P., A model of stochastic process switching, Econometrica, 1983, 51: 537–551.

[6] Froot, K. and Obstfeld, M., Stochastic process switching: Some simple solutions, Econometrica, 1991, 59: 241–250.

[7] Gagnon, G., Vanishing central bank intervention in stochastic impulse control, Annals of Finance, 2019, 15: 125–153.

[8] Gagnon, G., Diffusion processes and a random ODE arising in macroeconomics, Working paper, 2023.

[9] Jeanblanc-Piqué, M., Impulse control method and exchange rate, Mathematical Finance, 1993, 8: 161–177.

[10] Kallianpur, G., Stochastic Filtering Theory, Applications of Mathematics, Vol. 13, Springer-Verlag, New York, 1980.

[11] Karatzas, I. and Shreve, S., Brownian Motion and Stochastic Calculus, 2nd ed., Graduate Texts in Mathematics, Vol. 113, Springer-Verlag, New York, 1991.

[12] Pardoux, E. and Peng, S. G., Adapted solution of a backward stochastic differential equation, Systems & Control Letters, 1990, 14: 55–61.

[13] Revuz, D. and Yor, M., Continuous Martingales and Brownian Motion, 2nd ed., A Series of Comprehensive Studies in Mathematics, Vol. 243, Springer-Verlag, Berlin, 1994.

[14] Shreve, S., Stochastic Calculus for Finance Ⅱ: Continuous Time Models, Springer, New York, 2004.

[15] Smith, G., Solution to a problem of stochastic process switching, Econometrica, 1991, 59: 237–239.

[16] Wang, P., The Economics of Foreign Exchange and Global Finance, 2nd ed., Springer-Verlag, New York, 2009.

[17] Yong, J. and Zhou, X. Y., Stochastic Controls, Applications of Mathematics, Vol. 43, Springer-Verlag, New York, 1999.