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Penalization schemes for BSDEs and reflected BSDEs with generalized driver

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Abstract The paper is directly motivated by the pricing of vulnerable European and American options in a general hazard process setup and a related study of the corresponding pre-default backward stochastic differential equations (BSDE) and pre-default reflected backward stochastic differential equations (RBSDE). The goal of this work is twofold. First, we aim to establish the well-posedness results and comparison theorems for a generalized BSDE and a reflected generalized BSDE with a continuous and nondecreasing driver A. Second, we study penalization schemes for a generalized BSDE and a reflected generalized BSDE in which we penalize against the driver in order to obtain in the limit either a constrained optimal stopping problem or a constrained Dynkin game in which the set of minimizer's admissible exercise times is constrained to the right support of the measure generated by A.

Keywords Generalized BSDEs, Reflected generalized BSDEs, Penalization scheme, Constrained optimal stopping, Constrained Dynkin game

2020 Mathematics Subject Classification 60H10, 60H30, 91G30, 91G40

1. Introduction

In this work, we consider the generalized backward stochastic differential equations and reflected generalized backward stochastic differential equations on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where the underlying filtration is fixed but otherwise arbitrary, except that it is assumed to satisfy the usual conditions. For a predetermined continuous, nondecreasing process A and a given terminal condition ξ_T, we are interested in a solution (Y, M) to the *generalized BSDE* (GBSDE) with data (A, ξ_T, g)

$$Y_t = \xi_T + \int_{]t,T]} g(s, Y_s)\, \mathrm{d}A_s - (M_T - M_t), \tag{1.1}$$

where Y is an $\mathbb{F}$-adapted, càdlàg process and M is a martingale with respect to the filtration $\mathbb{F}$

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(see Definition 3.2). We also study solutions (Y,M,K) to the *reflected generalized BSDE* (RGBSDE) with data (A,ζ_T,g,ζ), which is given by

$$Y_t = \zeta_T + \int_{]t,T]} g(s,Y_s)\,\mathrm{d}A_s - (M_T - M_t) + K_T - K_t, \tag{1.2}$$

where ζ is the lower obstacle, meaning that $Y \geqslant \zeta$, and a nondecreasing process K satisfies the appropriate Skorokhod conditions ensuring that inequality (see Definition 4.2). To avoid confusion, we shall refer to the process A as the *driver* of the GBSDE (1.1) (or the RGBSDE (1.2)) and the mapping $(\omega,s,y)\mapsto g(\omega,s,y)$ for $(\omega,s,y)\in\Omega\times[0,T]\times\mathbb{R}$ is called its *generator*. For brevity, in the remainder of this work we will only mention the dependence of the generator on ω when this is deemed necessary. For related works on generalized BSDEs and backward stochastic equations, the reader is referred to Cheridito and Nam [3] and Liang et al. [33] where the focus is on establishing the well-posedness of general classes of backward stochastic equations using fixed-point arguments, whereas our main goal is to examine various penalization schemes for generalized BSDEs and reflected generalized BSDEs and their relationship to constrained optimal stopping problems and constrained Dynkin games. In addition, the fact that time t is substituted with a nondecreasing process A is of crucial importance in certain applications of generalized BSDEs in financial mathematics.

We have deliberately chosen not to represent the GBSDE (1.1) in a more familiar form

$$Y_t = \xi_T + \int_{]t,T]} g(s,Y_s)\,\mathrm{d}A_s - \int_{]t,T]} Z_s\,\mathrm{d}M_s \tag{1.3}$$

for some predetermined d-dimensional $\mathbb{F}$-local martingale M and an unknown M-integrable process Z where a solution is a pair (Y,Z). Since the generator g does not depend explicitly on an unknown process Z, there is no need to study the GBSDE of the form (1.3) but all results for the GBSDE (1.1) obtained in this work can be easily reformulated to cover the GBSDE (1.3), provided that M has the (strong) predictable representation property for the filtration $\mathbb{F}$. Recall from He et al. [27] (see Definition 13.1 in [27]) that the (strong) predictable representation property is said to hold for a d-dimensional local martingale M with $M_0=0$ and a filtration $\mathbb{F}$ if $\mathcal{M}_{loc,0}=\mathcal{L}(M)$ where $\mathcal{M}_{loc,0}$ is the class of all $\mathbb{F}$-local martingales starting at zero and $\mathcal{L}(M) := \{H\bullet M : H\in\mathcal{P}(\mathbb{F}) \text{ and is M -integrable}\}$ where $\bullet$ denotes the Itô integral. On the other hand, it should also be acknowledged that most techniques used in the present work are not suitable to cover the case of a generator of the form $g(s,y,z)$ since several proofs rely on a thorough analysis of sample paths of the component Y under an additional monotonicity condition imposed on the generator $g(s,y)$.

A direct motivation to examine stochastic backward equations of the form (1.1) and (1.2) comes from a study of BSDEs and RBSDEs in the progressive enlargement $\mathbb{G}$ of $\mathbb{F}$ through observations of a random time, which is frequently interpreted as a default time of some credit-risky entity in the financial literature, and their subsequent applications to the pricing of vulnerable options of either a European or an American style. In particular, the form of the driver A in equations (1.1) and (1.2) is not arbitrary since it is directly related to the choice of a particular model for a default time, as was studied, e.g., in a recent paper by Jeanblanc and Li [28] and numerous previous works on credit risk modeling. A study of a BSDE in the progressively enlarged filtration $\mathbb{G}$ has been explored in two main directions. First, one can work directly in the enlarged filtration $\mathbb{G}$, as done in Blanchet-Scalliet et al. [2], Eyraud-Loisel and Royer-Carenzi [17], Grigorova et al. [24] and Dumitrescu et al. [7]. Second, in papers by Kharroubi and Lim [29] and Crépey and Song [4], the authors developed an alternative approach, which hinges on reducing an original BSDE in the filtration $\mathbb{G}$ to a reference filtration $\mathbb{F}$ and subsequently examining the $\mathbb{F}$-*reduced* BSDE, which is sometimes referred to as the *pre-default* BSDE in applications of a BSDE theory to credit risk models.

The present research draws inspiration from the above-mentioned papers on the $\mathbb{F}$-reduction approach and, to be more specific, from recent works by Aksamit et al. [1] and Li et al. [33] where the pre-default GBSDE, as well as the pre-default RGBSDE, were examined and applied to solve superhedging problems for vulnerable European and American options within a general setup. In that case, the driver process A in the pre-default GBSDE and RGBSDE satisfied by the pre-default lower and upper prices of a vulnerable European or American option corresponds to the hazard process associated with the default time while the generator g is used to model the uncertainty about the hazard process in an incomplete market model. We acknowledge that there is abundant literature on (generalized) BSDEs and RBSDEs, including papers by El Karoui et al. [15, 16], Pardoux and Zhang [37], Ren and El Otmani [41], Essaky and Hassani [9, 10], Essaky et al. [11, 12], Eddahbi et al. [8], Grigorova et al. [21, 22, 23], Hamadène et al. [25, 26], and Klimsiak et al. [30]. However, to the best of our knowledge, no single piece of work encompasses the general framework adopted in Aksamit et al. [1] and Li et al. [32] and hence we have found it justified to study the generalized BSDE within the framework consistent with market models examined in [32].

Our first goal is to obtain, under Assumptions 3.1 and 3.8 complemented by the postulate that the driver A is bounded, the well-posedness and comparison results for GBSDEs and RGBSDEs given by (1.1) and (1.2), respectively. In particular, we establish in Propositions 3.6 and 4.3 suitable comparison results, while the existence and uniqueness results are given in Propositions 3.12 and 4.7. Furthermore, we show in Proposition 3.14 that for a GBSDE the boundedness assumption on A can be relaxed and the existence and uniqueness result can be extended to the case where A is square-integrable under the additional assumption that the generator g is of class (P) (see Definition 3.13). We stress that this additional assumption is satisfied by generators arising in penalization schemes studied in the foregoing sections.

Our second goal is to examine various penalization schemes for (1.1) and (1.2) where we penalize against the driver A. Similar to the classical case where $A_t = t$, the limiting processes obtained from the proposed penalization schemes correspond to either the value process of an optimal stopping problem or a two-player Dynkin game. It is important to point out that, when compared to the existing results on penalization of BSDEs or RBSDEs, such as Hamadène et al. [25], the new feature here is that admissible stopping times are constrained to take values in the random set $\overline{S} := S^r \cup \{T\}$ where $S^r := \{s : \forall \epsilon > 0,\, A_{s+\epsilon} - A_s > 0\}$ is the right support of a nondecreasing process A. Let us briefly describe our main penalization results, which are given in Theorems 3.19, 4.11 and 4.13.

First, in Theorem 3.19, we show that if ξ_T is bounded and η is an $\mathbb{F}$-optional, bounded, nonnegative and right-continuous process, then the sequence (Y^n, M^n) of solutions to the GBSDE

$$Y_t^n = \xi_T + \int_{]t,T]} n(\eta_s - Y_s^n)^+ \,\mathrm{d}A_s - (M_T^n - M_t^n)$$

converges monotonically to the value process V of the *constrained optimal stopping* problem of the form, for every $t \in [0, T]$,

$$V_t = \operatorname*{ess\,sup}_{\tau \in \overline{\mathcal{T}}_{t,T}} \mathbb{E}\left[\xi_T \mathbb{1}_{\{\tau = T\}} + \eta_\tau \mathbb{1}_{\{\tau < T\}} \,|\, \mathcal{F}_t\right],$$

where $\overline{\mathcal{T}}_{t,T}$ is the set of all $\mathbb{F}$-stopping times with values in $[t, T] \cap \overline{S}$.

Next, in Theorem 4.11, we demonstrate that if the $\mathbb{F}$-optional and bounded process ζ (resp., the $\mathbb{F}$-optional and bounded process η) is right-upper-semicontinuous (resp., right-continuous), then the sequence (Y^n, M^n, K^n) of solutions to the reflected GBSDE

$$Y^n_\tau = \zeta_T + \int_{]\tau,T]} n(\eta_s - Y^n_s)^+ \,\mathrm{d}A_s - (M^n_T - M^n_\tau) + K^n_T - K^n_\tau,$$

where the $\mathbb{F}$-adapted, làdlàg, nondecreasing process K^n satisfies the Skorokhod conditions with the lower obstacle ζ, converges monotonically to the value process Y of the following variant of an optimal stopping problem, for every $t \in [0,T]$,

$$Y_t = \operatorname*{ess\,sup}_{\sigma\in\mathcal{T}_{t,T}} \mathbb{E}\left[\zeta_T \mathbb{1}_{\{T=\sigma\}} + (\zeta \vee \eta \mathbb{1}_{\overline{S}})_\sigma \mathbb{1}_{\{\sigma<T\}} \,|\, \mathcal{F}_t\right],$$

where $\mathcal{T}_{t,T}$ is the set of all $\mathbb{F}$-stopping times with values in $[t,T]$, as in the classical case, but the reward process depends on the right support of the nondecreasing process A.

Finally, it is proven in Theorem 4.13 that, under similar assumptions on ζ and η as in Theorem 4.11, if $\zeta_T = \eta_T$ then the sequence $(\widetilde{Y}^n, \widetilde{M}^n, \widetilde{K}^n)$ of solutions to the reflected GBSDE

$$\widetilde{Y}^n_\tau = \zeta_T - \int_{]\tau,T]} n(\widetilde{Y}^n_s - \eta_s)^+ \,\mathrm{d}A_s - (\widetilde{M}^n_T - \widetilde{M}^n_\tau) + \widetilde{K}^n_T - \widetilde{K}^n_\tau,$$

where $\widetilde{K}^n$ satisfies the Skorokhod conditions with the lower obstacle ζ, converges monotonically to the value process $\widetilde{Y}$ of the *constrained Dynkin game* given by, for every $t \in [0,T]$,

$$\widetilde{Y}_t = \operatorname*{ess\,inf}_{\tau\in\overline{\mathcal{T}}_{t,T}} \operatorname*{ess\,sup}_{\sigma\in\mathcal{T}_{t,T}} \mathbb{E}[\Theta(\sigma,\tau) \,|\, \mathcal{F}_t] = \operatorname*{ess\,sup}_{\sigma\in\mathcal{T}_{t,T}} \operatorname*{ess\,inf}_{\tau\in\overline{\mathcal{T}}_{t,T}} \mathbb{E}[\Theta(\sigma,\tau) \,|\, \mathcal{F}_t],$$

where $\Theta(\sigma,\tau) := \zeta_\sigma \mathbb{1}_{\{\tau>\sigma\}} + (\eta \vee \zeta)_\tau \mathbb{1}_{\{\tau\leqslant\sigma\}}$. As already mentioned, the penalization schemes for GBSDEs and RGBSDEs studied in Theorems 3.19, 4.11 and 4.13 underpin the superhedging approach to the arbitrage-free pricing of vulnerable European and American options in incomplete market models. The interested reader is referred to Li et al. [32] for applications of results from the present work to problems arising in financial mathematics.

Let us conclude the introduction by mentioning that the assumption of continuity of the driver A is not crucial and thus, at least in principle, it can be relaxed. For instance, it is possible to work with the case of a discontinuous driver by considering a purely discontinuous, left-continuous process A and hence a GBSDE with a làglàd process Y, as opposed to GBSDEs with a càdlàg process Y considered in the present work. In this modified setup, the set of admissible exercise times for the optimal stopping problem consists of all $\mathbb{F}$-stopping times taking values in the union of the graphs of the jump times of A and, in the context of financial applications, it formally corresponds to the case of an option of Bermudan style, that is, an American option with a predetermined discrete set of holder's allowable exercise times.

2. Preliminaries

We denote by $\mathcal{O}(\mathbb{F})$ and $\mathcal{P}(\mathbb{F})$ the class of all real-valued, $\mathbb{F}$-optional and $\mathbb{F}$-predictable processes, respectively. In order to simplify the notation, we denote by $X \bullet Y$ the Itô integral of X with respect to a classical (i.e., càdlàg) real-valued semimartingale Y, that is, $(X \bullet Y)_t := \int_{]0,t]} X_s \,\mathrm{d}Y_s$ for every $t \in [0,T]$. Let $\mathcal{T}$ denote the class of all $\mathbb{F}$-stopping times with values in $[0,T]$. We will use the property that the space $\mathcal{S}^2 = \{X \in \mathcal{O}(\mathbb{F}) : \|X\|^2_{\mathcal{S}^2} < \infty\}$ with the norm given by

$$\|X\|_{\mathcal{S}^2} := \left[\mathbb{E}\left(\operatorname*{ess\,sup}_{\tau\in\mathcal{T}} |X_\tau|^2\right)\right]^{1/2}$$

is a Banach space (see Proposition 2.1 in Grigorova et al. [21]).

We denote by $\mathcal{K}$ (resp., $\overline{\mathcal{K}}$) the class of all càdlàg, nondecreasing, $\mathbb{F}$-predictable (resp., làdlàg, nondecreasing, $\mathbb{F}$-predictable) processes. Recall that a stochastic process X with sample paths possessing right-hand limits is said to be $\mathbb{F}$-*strongly predictable* if X is $\mathbb{F}$-predictable and the process X_+ is $\mathbb{F}$-optional (see Definition 1.1 in Gal'čuk [19]). In particular, all processes from the class $\overline{\mathcal{K}}$ are $\mathbb{F}$-strongly predictable.

Furthermore, any process $K \in \mathcal{K}$ with $K_0 = 0$ has a unique decomposition $K = K^c + K^d$ where $K_0^c = K_0^d = 0$, K^c is an $\mathbb{F}$-adapted, continuous, nondecreasing process and K^d is an $\mathbb{F}$-predictable, càdlàg, purely discontinuous, nondecreasing process. More generally, if K belongs to $\overline{\mathcal{K}}$ and $K_0 = 0$ then the decomposition becomes $K = K^c + K^d + K^g$ where K^g with $K_0^g = 0$ is an $\mathbb{F}$-adapted, càglàd, purely discontinuous, nondecreasing process. If X and Y are arbitrary $\mathbb{F}$-optional processes, then the inequality $Y \geqslant X$ means that $Y_\tau \geqslant X_\tau$ for every $\tau \in \mathcal{T}$.

For future reference, we recall in Theorem 2.5 the classical version of the Doob–Meyer–Mertens decomposition theorem (see Mertens [34] and El Karoui [13]) and the basic properties of the Snell envelope (see Definition 2.3). We stress that we work under the assumption that the filtration $\mathbb{F}$ satisfies the usual conditions of Assumption 3.1 (i). The interested reader is referred to Gal'čuk [19] for an extended version of the Doob–Meyer–Mertens decomposition where these assumptions about the filtration $\mathbb{F}$ are relaxed.

Let us first recall the notion of a *strong supermartingale,* which is known to coincide with the classical concept of a *supermartingale* if a process is assumed to be càdlàg. We will also use later the concept of a *strong* $\mathcal{E}^g$*-supermartingale* where $\mathcal{E}^g$ denotes the nonlinear evaluation generated by solutions to a (generalized) BSDE with generator g (see Peng [38, 39]).

Definition 2.1 *A process X is called a strong supermartingale if it is $\mathbb{F}$-optional and for all $\tau, \nu \in \mathcal{T}$ such that $\tau \leqslant \nu$ we have that $X_\tau \geqslant \mathbb{E}[X_\nu \,|\, \mathcal{F}_\tau]$.*

Remark 2.2 *It is well known that any strong supermartingale is a làdlàg process, that is, it has almost all sample paths with right-hand and left-hand limits so that the processes X_- and X_+ are well-defined. Moreover, any strong $\mathbb{F}$-supermartingale is a right-upper-semicontinuous process so that $X \geqslant X_+$.*

Definition 2.3 *We say that X is the Snell envelope of an $\mathbb{F}$-optional process ξ* if:

(i) *X is a strong supermartingale such that $X \geqslant \xi$;*
(ii) *for any strong supermartingale such that $Y \geqslant \xi$ the inequality $Y \geqslant X$ holds.*

Remark 2.4 *The following simple observations will be useful:*

(i) *if X is a strong supermartingale, then the Snell envelope of X is equal to X;*
(ii) *the $\mathbb{F}$-Snell envelope is monotone, in the sense that if $\eta \geqslant \xi$, then the Snell envelope of η dominates the Snell envelope of ξ.*

Theorem 2.5 *Any strong supermartingale Y of class (D) has the unique Doob–Meyer–Mertens decomposition $Y = Y_0 + M - B^c - B^d - B^g$ where M is a uniformly integrable martingale, B^c is an $\mathbb{F}$-adapted, nondecreasing, continuous process, B^d is an $\mathbb{F}$-predictable, nondecreasing, purely discontinuous process, and B^g is an $\mathbb{F}$-adapted, càglàd, nondecreasing, purely discontinuous process. Furthermore, $M_0 = B_0^c = B_0^d = B_0^g = 0$.*

3. Generalized BSDEs and constrained optimal stopping

The goal of this section is to analyze some classes of generalized BSDEs of the form (1.1) and their applications to the constrained optimal stopping problem. With a slight abuse of notation, we denote by $\mathcal{O}(\mathbb{F})$ the σ-field of optional sets in $\Omega \times [0,T]$. We henceforth work under the following standing assumption, which formally specifies the inputs to the GBSDE (1.1): (i) the *terminal condition* ξ_T, (ii) the *driver* A, and (iii) the *generator* g.

Assumption 3.1 *We assume that we are given the following objects defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, which is endowed with a filtration* $\mathbb{F}$ *satisfying the usual conditions of right-continuity and* $\mathbb{P}$*-completeness:*

(i) *an* $\mathcal{F}_T$*-measurable random variable* ξ_T*;*
(ii) *an* $\mathbb{F}$*-adapted, nondecreasing, continuous process* A *with* $A_0 = 0$*;*
(iii) *an* $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$*-measurable mapping* $g : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$*.*

Let $\mathcal{M}_0$ denote the class of all real-valued $\mathbb{F}$-martingales with null initial value. The following definitions are natural variants of standard definitions from the theory of BSDEs.

Definition 3.2 *A pair* (Y, M) *is a solution to the GBSDE (1.1) with data* (A, ξ_T, g) *if* Y *is an* $\mathbb{F}$*-adapted, càdlàg process,* M *belongs to* $\mathcal{M}_0$*, and the following equality holds*

$$\mathbb{P}\bigg(Y_t = \xi_T + \int_{]t,T]} g(s, Y_s)\,\mathrm{d}A_s - (M_T - M_t),\ \forall\, t \in [0,T] \bigg) = 1. \tag{3.1}$$

Definition 3.3 *We say that the uniqueness of a solution to the GBSDE (1.1) holds if for any two solutions* (Y', M') *and* (Y'', M'') *to the GBSDE (1.1) the processes* (Y', M') *and* (Y'', M'') *are indistinguishable.*

It is important to observe that we consider GBSDEs with generators that do not depend on the component M of a solution (Y, M). In particular, it follows from Definition 3.2 that for any solution (Y, M) to the GBSDE (1.1) the process Y satisfies, for every $t \in [0,T]$,

$$Y_t = \mathbb{E}\bigg[\xi_T + \int_{]0,T]} g(s, Y_s)\,\mathrm{d}A_s \,\Big|\, \mathcal{F}_t \bigg] - \int_{]0,t]} g(s, Y_s)\,\mathrm{d}A_s = Y_0 + M_t - \int_{]0,t]} g(s, Y_s)\,\mathrm{d}A_s, \tag{3.2}$$

where the martingale M equals

$$M_t := \mathbb{E}\bigg[\xi_T + \int_{]0,T]} g(s, Y_s)\,\mathrm{d}A_s \,\Big|\, \mathcal{F}_t \bigg] - \mathbb{E}\bigg[\xi_T + \int_{]0,T]} g(s, Y_s)\,\mathrm{d}A_s \,\Big|\, \mathcal{F}_0 \bigg]. \tag{3.3}$$

Equations (3.2) and (3.3) lead to some important observations. First, it is clear from (3.2) that if (Y, M) is a solution to the GBSDE (1.1), then the uniqueness of M, in the sense of Definition 3.3, is an immediate consequence of the uniqueness of Y. Second, in view of (3.2) and (3.3), in order to establish the existence of a solution (Y, M) to the GBSDE (1.1) it suffices to search for an $\mathbb{F}$-adapted, càdlàg process Y satisfying (3.2) and then obtain the existence of the component M from (3.3). Therefore, we will mainly focus on the existence and uniqueness of the component Y in Definition 3.2 of a solution (Y, M) to the GBSDE (1.1).

Remark 3.4 *Definition 3.2 can be slightly extended by postulating that the process* M *in the GBSDE (1.1) is an* $\mathbb{F}$*-local martingale, rather than an* $\mathbb{F}$*-martingale. Then the problem can be reformulated as follows: for a given filtration* $\mathbb{F}$ *and a predetermined* $\mathcal{F}_T$*-measurable random variable* ξ_T*, we search for a special semimartingale* Y *with* $Y_T = \xi_T$ *and such that the* $\mathbb{F}$*-predictable finite variation process* B *in the canonical decomposition* $Y = Y_0 + N + B$ *of* Y

satisfies $B_t = -\int_{]0,t]} g(s, Y_s)\,\mathrm{d}A_s$. *Suppose that the existence and uniqueness of a special semimartingale* Y *with these properties can be established. Then the existence and uniqueness of an* $\mathbb{F}$*-local martingale* $N \in \mathcal{M}_{loc,0}$ *follows from the uniqueness of the canonical decomposition of a special semimartingale and, from the equalities* $Y = Y_0 + N + B$ *and* $Y_T = \xi_T$*, we deduce that the pair* $(Y, M = N)$ *is a unique solution to the GBSDE (1.1). The reader is referred to Liang et al. [33] for analogous arguments when the generator* g *in the GBSDE (1.3) may depend on* Z *and* $A_t = t$ *for all* $t \in [0, T]$ *where the latter postulate is not important.*

We will frequently follow the common practice in the existing literature of searching for solutions to BSDEs in specific spaces of stochastic processes. In particular, we examine the problem of the existence and uniqueness of a solution (Y, M) to the GBSDE (1.1) in the product space $\mathcal{S}^2 \times \mathcal{M}_0^2$ where $\mathcal{M}_0^2$ denotes the class of all real-valued, square-integrable $\mathbb{F}$-martingales with null initial value. Then Definition 3.2 can be reformulated as follows.

Definition 3.5 *We say that the uniqueness in* $\mathcal{S}^2 \times \mathcal{M}_0^2$ *of solutions to the GBSDE (1.1) holds if for any two solutions* (Y', M') *and* (Y'', M'') *belonging to* $\mathcal{S}^2 \times \mathcal{M}_0^2$ *we have that* $||Y' - Y''||_{\mathcal{S}^2} = 0$ *and* $||M' - M''||_{\mathcal{M}_0^2} = 0$.

It is worth noting that the uniqueness of the càdlàg component Y in the space $\mathcal{S}^2$ entails its uniqueness in the sense of indistinguishability of stochastic processes and thus also, due to (3.2), the uniqueness of M in the space $\mathcal{M}_0^2$.

Let us summarize further developments in this section. We start by establishing in Proposition 3.6 the comparison property for general solutions of the GBSDE (1.1). Next, we obtain in Proposition 3.9 some *a priori* estimates for solutions in $\mathcal{S}^2 \times \mathcal{M}_0^2$ when the driver A is bounded by a positive constant, that is, we have that $A_T \leqslant c_A$ for some constant c_A. These preliminary results allow us to study in Sections 3.3 and 3.4 the well-posedness results for the GBSDE (1.1) in the space $\mathcal{S}^2 \times \mathcal{M}_0^2$ when the generator is Lipschitz continuous and the driver A is either bounded (see Proposition 3.12) or A is a square-integrable process but the generator is Lipschitz continuous and belongs to the class (P) introduced in Definition 3.13 (see Proposition 3.14). Finally, in Section 3.5, for a given $\mathbb{F}$-optional and bounded process η, we examine the sequence $(Y^n, M^n)_{n\in\mathbb{N}}$ of solutions to the penalized GBSDEs of the form

$$Y_\tau^n = \xi_T + \int_{]\tau,T]} n(Y_s^n - \eta_s)^+\,\mathrm{d}A_s - (M_T^n - M_t^n)$$

and we show in Theorem 3.19 that the limit $Y := \lim_{n\to\infty} Y^n$ can be interpreted as the value process of a particular constrained optimal stopping problem related to the right support of the driver A.

3.1 Comparison theorem for generalized BSDEs

Our first goal is to establish the comparison property of solutions to a GBSDE and thus we work under a temporary assumption that a solution to a GBSDE exists but we do not postulate that it is unique and we do not make any specific assumptions about the driver and terminal condition such as, e.g., their boundedness or square-integrability. For brevity, we denote

$$G_t(Y) := \int_{]0,t]} g(s, Y_s)\,\mathrm{d}A_s, \quad H_t(\widetilde{Y}) := \int_{]0,t]} h(s, \widetilde{Y}_s)\,\mathrm{d}A_s,$$

where the mappings $g, h : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}_+$ are $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$-measurable.

Proposition 3.6 *Let the generators* $g,h:\Omega\times[0,T]\times\mathbb{R}\to\mathbb{R}$ *be such that the GBSDEs*

$$Y_t=\xi_T+\int_{]t,T]}g(s,Y_s)\,\mathrm{d}A_s-(M_T-M_t)=\xi_T+G_T(Y)-G_t(Y)-(M_T-M_t)$$

and

$$\widetilde{Y}_t=\widetilde{\xi}_T+\int_{]t,T]}h(s,\widetilde{Y}_s)\,\mathrm{d}A_s-(\widetilde{M}_T-\widetilde{M}_t)=\widetilde{\xi}_T+H_T(\widetilde{Y})-H_t(\widetilde{Y})-(\widetilde{M}_T-\widetilde{M}_t)$$

have solutions (Y,M) *and* $(\widetilde{Y},\widetilde{M})$*. Assume that* $\xi_T\geqslant\widetilde{\xi}_T$ *and the mappings* g *and* h *satisfy the following conditions:*

(i) $g(\omega,t,y)\geqslant h(\omega,t,y)$ *for every* $(\omega,t,y)\in\Omega\times[0,T]\times\mathbb{R}$,
(ii) $g(\omega,t,\cdot)$ *is a nonincreasing function for every* $(\omega,t)\in\Omega\times[0,T]$.

Then the inequality $Y\geqslant\widetilde{Y}$ *is valid, that is,* $\mathbb{P}(Y_t\geqslant\widetilde{Y}_t,\,\forall\,t\in[0,T])=1$.

Proof We adapt the proof of Lemma 8.3 in Peng [39]. It is important to recall that, by Definition 3.2, the processes Y and $\widetilde{Y}$ are càdlàg. For a fixed $\varepsilon>0$, we define the $\mathbb{F}$-stopping time $\tau^\varepsilon:=\inf\{t\geqslant 0: Y_t\leqslant\widetilde{Y}_t-\varepsilon\}$ where, by convention, $\inf\emptyset=T$. Note that if for all $\varepsilon>0$ we have $\mathbb{P}(\tau^\varepsilon=T)=1$, then $Y\geqslant\widetilde{Y}$. Indeed, for $\varepsilon_n=\frac{1}{n}$, there exists D_n such that $\mathbb{P}(D_n)=0$ and $\tau^{\varepsilon_n}(\omega)=T$ for $\omega\notin D_n$. Thus for $\omega\notin D_n$, from the definition of τ^ε and the equality $\tau^{\varepsilon_n}(\omega)=T$, we obtain $Y_t\geqslant\widetilde{Y}_t-\varepsilon_n$ for every $t\in[0,T[$. Then for $D:=\cup_{n=1}^\infty D_n$, we have $\mathbb{P}(D)=0$ and for $\omega\notin D$, we have $Y_t\geqslant\widetilde{Y}_t$ for every $t\in[0,T[$. Since $Y_T=\xi_T\geqslant\widetilde{\xi}_T=\widetilde{Y}_T$ we conclude that $Y\geqslant\widetilde{Y}$.

We now argue by contradiction. If the inequality $Y\geqslant\widetilde{Y}$ does not hold then, by the first step, there exists $\varepsilon>0$ such that $\mathbb{P}(E)>0$ where $E:=\{\tau^\varepsilon<T\}\in\mathcal{F}_{\tau^\varepsilon}$. We fix ε and we define $\tau:=\tau^\varepsilon\mathbb{1}_E+T\mathbb{1}_{E^c}$ and $\nu:=\inf\{t\geqslant\tau: Y_t\geqslant\widetilde{Y}_t\}$ so that $\nu\leqslant T$ since, by assumption, $Y_T\geqslant\widetilde{Y}_T$. Since Y and $\widetilde{Y}$ are càdlàg processes it is clear that $Y_\tau<\widetilde{Y}_\tau$ on E, the interval $[\![\tau,\nu[\![$ is nonempty on E, and the inequality $Y_\nu\geqslant\widetilde{Y}_\nu$ is valid.

For brevity, let us write $U:=G(Y)$ and $\widetilde{U}:=H(\widetilde{Y})$ so that the $\mathbb{F}$-adapted, continuous process $\overline{U}:=U-\widetilde{U}$ satisfies

$$\begin{aligned}\overline{U}_t&=G_t(Y)-H_t(\widetilde{Y})=(G_t(Y)-G_t(\widetilde{Y}))+(G_t(\widetilde{Y})-H_t(\widetilde{Y}))\\&=\int_{]0,t]}(g(s,Y_s)-g(s,\widetilde{Y}_s))\,\mathrm{d}A_s+\int_{]0,t]}(g(s,\widetilde{Y}_s)-h(s,\widetilde{Y}_s))\,\mathrm{d}A_s.\end{aligned}$$

We deduce that $\mathbb{1}_E\overline{U}$ is a continuous and nondecreasing process on $[\![\tau,\nu]\!]$ since $g(\omega,t,y)\geqslant h(\omega,t,y)$ for all $(\omega,t,y)\in\Omega\times[0,T]\times\mathbb{R}$, the inequality $Y<\widetilde{Y}$ holds on $[\![\tau,\nu[\![$, and for every $(\omega,t)\in\Omega\times[0,T]$ the function $g(\omega,t,\cdot)$ is nonincreasing. We observe that on E

$$Y_t-\widetilde{Y}_t=Y_\nu-\widetilde{Y}_\nu+\overline{U}_\nu-\overline{U}_t-(\widetilde{M}_\nu-\widetilde{M}_t).$$

Therefore, the process $Y-\widetilde{Y}$ is a supermartingale (hence also strong supermartingale) on $[\![\tau,\nu]\!]$ and $Y_\nu-\widetilde{Y}_\nu\geqslant 0$. Consequently, $Y_\tau-\widetilde{Y}_\tau\geqslant 0$ and thus $Y_\tau\geqslant\widetilde{Y}_\tau$ on E, which leads to a contradiction since $Y_\tau<\widetilde{Y}_\tau$ on $E\in\mathcal{F}_\tau$ and $\mathbb{P}(E)>0$. □

When using the penalization method to show the existence of a solution to the reflected GBSDE with a lower obstacle η, one may employ the mapping $f:\Omega\times[0,T]\times\mathbb{R}\to\mathbb{R}_+$, which is given by $f(t,y)=(\eta_t-y)^+$ where η is a predetermined $\mathbb{F}$-optional process. The following corollary to Proposition 3.6 will be useful in the proof of penalization result in Section 3.5.

Corollary 3.7 *Let the mapping $f:\Omega\times[0,T]\times\mathbb{R}\to\mathbb{R}_+$ be $\mathcal{O}(\mathbb{F})\otimes\mathcal{B}(\mathbb{R})$-measurable, nonnegative and such that $f(\omega,t,\cdot):\mathbb{R}\to\mathbb{R}_+$ is a nonincreasing function for every $(\omega,t)\in\Omega\times[0,T]$. Then the following assertions are valid:*

(i) *the uniqueness of a solution (Y,M) to the GBSDE (1.1) with generator $g=f$ holds;*

(ii) *if for every $n\in\mathbb{N}$, the pair (Y^n,M^n) is a unique solution to the GBSDE*

$$Y_t^n=\xi_T+\int_{]t,T]} nf(s,Y_s^n)\,\mathrm{d}A_s-(M_T^n-M_t^n),$$

then $Y^{n+1}\geqslant Y^n$ for every $n\in\mathbb{N}$, that is, $\mathbb{P}(Y_t^{n+1}\geqslant Y_t^n,\,\forall\,t\in[0,T])=1$.

Proof (i) For the first statement, it suffices to apply Proposition 3.6 with $\xi_T=\widetilde{\xi}_T$ and $g=f$ to obtain the equality $Y=\widetilde{Y}$ for any two solutions (Y,M) and $(\widetilde{Y},\widetilde{M})$. Then the uniqueness of M in a solution (Y,M) to the GBSDE (1.1) holds as well since the martingale M given by (3.3) is unique.

(ii) The inequality $Y^{n+1}\geqslant Y^n$ follows from Proposition 3.6 applied to the GBSDEs with $\xi_T=\widetilde{\xi}_T$ and generators $g(t,y)=(n+1)f(t,y)$ and $h(t,y)=nf(t,y)$. □

3.2 A priori estimates for solutions to generalized BSDEs

In addition to Assumption 3.1, we will frequently impose additional conditions on the terminal condition ξ_T and the generator g of the GBSDE (1.1).

Assumption 3.8 *Let the terminal value ξ_T be square-integrable and the generator $g:\Omega\times[0,T]\times\mathbb{R}\to\mathbb{R}$ satisfy the following conditions:*

(i) *the uniform Lipschitz condition: there exists a constant $L>0$ such that the inequality $|g(t,y)-g(t,y')|\leqslant L|y-y'|$ holds for every $t\in[0,T]$ and $y,y'\in\mathbb{R}$;*

(ii) *the inequality*

$$\mathbb{E}\bigg[\int_{]0,T]}|g(t,0)|^2\,\mathrm{d}A_t\bigg]<\infty.$$

Our next goal is to establish a useful *a priori* estimate for a postulated solution in $\mathcal{S}^2\times\mathcal{M}_0^2$ to the GBSDE (1.1). Notice that in most results in the remainder of this work, the driver A will be assumed to be either bounded so that $A\leqslant c_A$ for some constant $c_A>0$ or, more generally, square-integrable so that $\mathbb{E}[A_T^2]<\infty$. For the sake of clarity, this will be always explicitly stated in assumptions of each result.

Proposition 3.9 *Let Assumptions 3.1 and 3.8 hold and the driver A be bounded. If $(Y,M)\in\mathcal{S}^2\times\mathcal{M}_0^2$ is a solution to the GBSDE (1.1), then there exists a constant $c>0$ such that for every $\alpha,\beta>0$ such that $\beta>2L+\alpha^{-1}$ we have*

$$\mathbb{E}\bigg[\sup_{t\in[0,T]}\big[e^{\beta A_t}|Y_t|^2\big]+\int_{]0,T]}e^{\beta A_s}\,\mathrm{d}[M]_s\bigg]\leqslant c\,\mathbb{E}\bigg[e^{\beta A_T}|\xi_T|^2+\alpha\int_{]0,T]}e^{\beta A_s}|g(s,0)|^2\,\mathrm{d}A_s\bigg].$$

Proof For simplicity, we set $e_t:=e^{\beta A_t}$ and note that $de_t:=\beta e_t\,\mathrm{d}A_t$ and $1\leqslant e_t\leqslant e^{\beta c_A}$ for every $t\in[0,T]$ since $\beta>0$ and thus e is a bounded process. We first establish the *a priori* estimates for a postulated solution $(Y,M)\in\mathcal{S}^2\times\mathcal{M}_0^2$ to the GBSDE (1.1).

By applying first the Itô formula to $e_t|Y_t|^2$ and then the Young inequality with $\alpha>0$, we obtain

$$\begin{aligned}
&e_t|Y_t|^2 + \int_{]t,T]} e_s \,\mathrm{d}[M]_s \\
&\leqslant e_T|\xi_T|^2 - \beta \int_{]t,T]} e_s|Y_s|^2 \,\mathrm{d}A_s + 2\int_{]t,T]} e_s Y_s \left(|g(s,0)| + L|Y_s|\right) \mathrm{d}A_s - 2\int_{]t,T]} e_s Y_{s-} \,\mathrm{d}M_s \\
&\leqslant e_T|\xi_T|^2 + \alpha \int_{]t,T]} e_s|g(s,0)|^2 \,\mathrm{d}A_s + \left(2L + \alpha^{-1} - \beta\right) \int_{]t,T]} e_s|Y_s|^2 \,\mathrm{d}A_s - 2\int_{]t,T]} e_s Y_{s-} \,\mathrm{d}M_s
\end{aligned}$$

and thus the inequality

$$e_t|Y_t|^2 + \int_{]t,T]} e_s \,\mathrm{d}[M]_s \leqslant e_T|\xi_T|^2 + \alpha \int_{]t,T]} e_s|g(s,0)|^2 \,\mathrm{d}A_s - 2\int_{]t,T]} e_s Y_{s-} \,\mathrm{d}M_s \tag{3.4}$$

holds for any $\alpha, \beta > 0$ such that $\beta > 2L + \alpha^{-1}$. By taking the supremum and expectation in equation (3.4), we obtain

$$\mathbb{E}\Big[\sup_{t\in[0,T]} e_t|Y_t|^2\Big] \leqslant \mathbb{E}\Big[e_T|\xi_T|^2 + \alpha \int_{]0,T]} e_s|g(s,0)|^2 \,\mathrm{d}A_s\Big] + 2\mathbb{E}\Big[\operatorname*{ess\,sup}_{t\in[0,T]} \Big|\int_{]0,t]} e_s Y_{s-} \,\mathrm{d}M_s\Big|\Big]. \tag{3.5}$$

Since the driver A is assumed to be a bounded process, an application of the Burkholder–Davis–Gundy inequality with $p = 1$ gives

$$\begin{aligned}
&2\mathbb{E}\Big[\sup_{t\in[0,T]} \Big|\int_{]0,t]} e_s Y_{s-} \,\mathrm{d}M_s\Big|\Big] \leqslant 2c_1 \,\mathbb{E}\Big[\Big(\int_{]0,T]} e_s^2 |Y_{s-}|^2 \,\mathrm{d}[M]_s\Big)^{1/2}\Big] \\
&\leqslant \mathbb{E}\left[\Big(\frac{1}{2}\sup_{t\in[0,T]} e_t|Y_t|^2\Big)^{1/2} \Big(8c_1^2 \int_{]0,T]} e_s \,\mathrm{d}[M]_s\Big)^{1/2}\right] \\
&\leqslant \frac{1}{4}\,\mathbb{E}\Big[\sup_{t\in[0,T]} e_t|Y_t|^2\Big] + 4c_1^2 \,\mathbb{E}\Big[\int_{]0,T]} e_s \,\mathrm{d}[M]_s\Big],
\end{aligned} \tag{3.6}$$

where the constant c_1 is independent of α, β, L and the last inequality holds since $2ab \leqslant a^2 + b^2$ for all real numbers a, b. Furthermore, by taking the expectation in (3.4) and using the, just proven, martingale property of the integral with respect to M, we obtain

$$\mathbb{E}\Big[\int_{]0,T]} e_s \,\mathrm{d}[M]_s\Big] \leqslant \mathbb{E}\Big[e_T|\xi_T|^2 + \alpha \int_{]0,T]} e_s|g(s,0)|^2 \,\mathrm{d}A_s\Big] < \infty, \tag{3.7}$$

where the second inequality follows from Assumption 3.8 and the boundedness of A. By combining (3.5), (3.6) and (3.7), we obtain

$$\frac{3}{4}\,\mathbb{E}\Big[\sup_{t\in[0,T]} e_t|Y_t|^2\Big] \leqslant (1 + 4c_1^2)\,\mathbb{E}\Big[e_T|\xi_T|^2 + \alpha \int_{]0,T]} e_s|g(s,0)|^2 \,\mathrm{d}A_s\Big]$$

and thus

$$\mathbb{E}\Big[\sup_{t\in[0,T]} e_t|Y_t|^2 + \int_{]0,T]} e_s \,\mathrm{d}[M]_s\Big] \leqslant c\,\mathbb{E}\Big[e_T|\xi_T|^2 + \alpha \int_{]0,T]} e_s|g(s,0)|^2 \,\mathrm{d}A_s\Big],$$

where the constant $c := \frac{7+16c_1^2}{3} > 0$ is independent of α, β and L. □

The next result deals with the stability of solutions to a GBSDE with respect to the terminal condition ξ_T and generator g. Let us denote $\widehat{Y} = Y^1 - Y^2$, $\widehat{M} = M^1 - M^2$ and $\widehat{g}_t = g^1(t, Y_t^2) - g^2(t, Y_t^2)$.

Proposition 3.10 *Let Assumptions 3.1 and 3.8 hold for* g^i, $i=1,2$ *and the driver* A *be bounded. If* $(Y^i,M^i)\in\mathcal{S}^2\times\mathcal{M}_0^2$ *is a solution to the GBSDE with data* (A,ξ_T^i,g^i) *for* $i=1,2$, *then for every* $\alpha,\beta>0$ *such that* $\beta>2L+\alpha^{-1}$, *we have, for every* $t\in[0,T]$,

$$e^{\beta A_t}|\widehat{Y}_t|^2\leqslant\mathbb{E}\bigg[e^{\beta A_T}|\widehat{\xi}_T|^2+\alpha\int_{]t,T]}e^{\beta A_s}|\widehat{g}_s|^2\,\mathrm{d}A_s\,\bigg|\,\mathcal{F}_t\bigg].$$

Proof As in Proposition 3.9, we denote $e_t:=e^{\beta A_t}$. An application of the Itô formula to $e_t|\widehat{Y}_t|^2$ and the Young inequality with $\alpha>0$ gives

$$\begin{aligned}&e_t|\widehat{Y}_t|^2+\int_{]t,T]}e_s\,\mathrm{d}[\widehat{M}]_s\\&\leqslant e_T|\widehat{\xi}_T|^2-\beta\int_{]t,T]}e_s|\widehat{Y}_s|^2\,\mathrm{d}A_s+2\int_{]t,T]}e_s\widehat{Y}_s\left(L|\widehat{Y}_s|+|\widehat{g}_s|\right)\mathrm{d}A_s-2\int_{]t,T]}e_s\widehat{Y}_{s-}\,\mathrm{d}\widehat{M}_s\\&\leqslant e_T|\widehat{\xi}_T|^2+\alpha\int_{]t,T]}e_s|\widehat{g}_s|^2\,\mathrm{d}A_s+\left(2L+\alpha^{-1}-\beta\right)\int_{]t,T]}e_s|\widehat{Y}_s|^2\,\mathrm{d}A_s-2\int_{]t,T]}e_s\widehat{Y}_{s-}\,\mathrm{d}\widehat{M}_s\end{aligned}$$

and thus for any $\alpha,\beta>0$ such that $\beta>2L+\alpha^{-1}$

$$e_t|\widehat{Y}_t|^2\leqslant e_T|\widehat{\xi}_T|^2+\alpha\int_{]t,T]}e_s|\widehat{g}_s|^2\,\mathrm{d}A_s-2\int_{]t,T]}e_s\widehat{Y}_{s-}\,\mathrm{d}\widehat{M}_s.$$

To complete the proof it suffices to take the conditional expectation with respect to $\mathcal{F}_t$ and argue in a similar way as in (3.6). □

3.3 Well-posedness of GBSDEs with bounded driver

The next two propositions deal with the existence and uniqueness of a solution in the space $\mathcal{S}^2\times\mathcal{M}_0^2$ to the GBSDE (1.1) with a Lipschitz continuous generator. We first present the existence result under an additional postulate that the driver A is bounded. It will be subsequently extended in Proposition 3.14 to the case of square-integrable drivers.

Before proceeding, we point out that the well-posedness result presented in this subsection, under the assumption that the driver A is bounded, can be deduced from El Karoui and Huang [14] and Papapantoleon et al. [36]. However, the setup of [14] and [36] is rather general and hence, for the reader's convenience, we decided to present the detailed computations under our assumptions. The following lemma will be used in the proof of Proposition 3.12

Lemma 3.11 *Let Assumptions 3.1 and 3.8 hold and the driver* A *be bounded. Then for every* $w\in\mathcal{S}^2$ *the GBSDE*

$$Y_t^w=\xi_T+\int_{]t,T]}g(s,w_s)\,\mathrm{d}A_s-(M_T^w-M_t^w)\tag{3.8}$$

with a fixed generator has a unique solution (Y^w,M^w) *in the space* $\mathcal{S}^2\times\mathcal{M}_0^2$.

Proof The GBSDE (3.8) is linear and necessarily

$$Y_t^w:=\mathbb{E}\bigg[\xi_T+\int_{]t,T]}g(s,w_s)\,\mathrm{d}A_s\,\bigg|\,\mathcal{F}_t\bigg].$$

Next, we observe that the process Θ, which is given by

$$\Theta_t:=Y_t^w+\int_{]0,t]}g(s,w_s)\,\mathrm{d}A_s\tag{3.9}$$

is a square-integrable martingale since $\xi_T\in L^2(\mathcal{F}_T)$ and, by Assumption 3.1 and repeated application of the Jensen inequality, we have that, for every $t \in [0,T]$,

$$|(g \bullet A)_t|^2 = \Big| \int_{]0,t]} g(s,w_s)\,\mathrm{d}A_s \Big|^2 \leqslant 2 \Big| \int_{]0,t]} L|w_s|\,\mathrm{d}A_s \Big|^2 + 2 \Big| \int_{]0,t]} |g(s,0)|\,\mathrm{d}A_s \Big|^2$$
$$\leqslant 2L^2 \sup_{s \leqslant t} |w_s|^2 A_t^2 + 2A_t \int_{]0,t]} |g(s,0)|^2\,\mathrm{d}A_s,$$

which entails that the process $g \bullet A$ belongs to $\mathcal{S}^2$. The process $Y = Y^w$ satisfies

$$Y_t := \mathbb{E}\Big[\xi_T + \int_{]0,T]} g(s,w_s)\,\mathrm{d}A_s \,\Big|\, \mathcal{F}_t\Big] - \int_{]0,t]} g(s,w_s)\,\mathrm{d}A_s = Y_0 + M_t - (g \bullet A)_t,$$

where M is a square-integrable martingale with $M_0 = 0$. Hence the Burkholder–Davis–Gundy inequality with $p=2$ applied to the square-integrable martingale M combined with the property that $g \bullet A$ is in $\mathcal{S}^2$ show that Y is a càdlàg process belonging to the space $\mathcal{S}^2$. □

Proposition 3.12 *Let Assumptions 3.1 and 3.8 hold and the driver A be bounded. Then the GBSDE (1.1) has a unique solution (Y,M) in the space $\mathcal{S}^2 \times \mathcal{M}_0^2$.*

Proof To prove the existence and uniqueness of a solution to the GBSDE (1.1), we use the standard method based on the Banach fixed point theorem. For any $Y \in \mathcal{S}^2$ and $\beta > 0$, we define $||Y||^2_{\mathcal{S}^2_\beta} := \mathbb{E}\left[\sup_{t \in [0,T]} e^{\beta A_t} |Y_t|^2\right]$ and we observe that the norms $||\cdot||_{\mathcal{S}^2}$ and $||\cdot||_{\mathcal{S}^2_\beta}$ are equivalent on $\mathcal{S}^2$. We denote by $\mathcal{S}^2_\beta$ the space $\mathcal{S}^2$ endowed with the norm $||\cdot||_{\mathcal{S}^2_\beta}$. Let the mapping $\Phi : \mathcal{S}^2_\beta \to \mathcal{S}^2_\beta$ be defined as follows: for any given $w \in \mathcal{S}^2_\beta$ we set $\Phi(w) := Y^w$ where the pair $(Y^w, M^w) \in \mathcal{S}^2 \times \mathcal{M}^2_0$ is a unique solution to the GBSDE (3.8). Our first goal is to demonstrate that there exists a unique process $\widehat{w} \in \mathcal{S}^2_\beta$ such that $\Phi(\widehat{w}) = \widehat{w}$. Then the corresponding process $\widehat{m} \in \mathcal{M}^2_0$ can be found from equality (3.9), that is, from the equality

$$\widehat{w}_t + \int_{]0,t]} g(s,\widehat{w}_s)\,\mathrm{d}A_s = \widehat{w}_0 + \widehat{m}_t.$$

It is clear that it suffices to show that the mapping $\Phi : \mathcal{S}^2_\beta \to \mathcal{S}^2_\beta$ is a contraction for a sufficiently large β. To this end, we take $w', w'' \in \mathcal{S}^2_\beta$ and denote $Y^{w'} = \Phi(w')$ and $Y^{w''} = \Phi(w'')$. For the simplicity of notation, we write $y := Y^{w'} - Y^{w''} = \Phi(w') - \Phi(w'')$, $m := M^{w'} - M^{w''}$ and $w := w' - w''$. It is clear from (3.8) that y satisfies the GBSDE

$$y_t = \int_{]t,T]} (g(s,w'_s) - g(s,w''_s))\,\mathrm{d}A_s - (m_T - m_t),$$

where $|g(s,w'_s) - g(s,w''_s)| \leqslant L|w_s|$ since $g(s,\cdot)$ is a Lipschitz continuous function with constant L.

The foregoing computations are similar to those used in the proof of Proposition 3.9, so we only sketch some significant steps. By applying the Itô formula to $e_t|y_t|^2$, where $e_t = e^{\beta A_t}$, and subsequently the Young inequality with $\alpha > 0$, we obtain

$$e_t|y_t|^2 + \int_{]t,T]} e_s\,\mathrm{d}[m]_s \leqslant -\beta \int_{]t,T]} e_s|y_s|^2\,\mathrm{d}A_s + 2L \int_{]t,T]} e_s y_s\,|w_s| \mathrm{d}A_s - 2 \int_{]t,T]} e_s y_{s-}\,\mathrm{d}m_s$$
$$\leqslant -\beta \int_{]t,T]} e_s|y_s|^2\,\mathrm{d}A_s + \frac{1}{\alpha} \int_{]t,T]} e_s|y_s|^2\,\mathrm{d}A_s + \alpha L^2 \int_{]t,T]} e_s|w_s|^2\,\mathrm{d}A_s - 2 \int_{]t,T]} e_s y_{s-}\,\mathrm{d}m_s$$
$$\leqslant \alpha L^2 \int_{]t,T]} e_s|w_s|^2\,\mathrm{d}A_s - 2 \int_{]t,T]} e_s y_{s-}\,\mathrm{d}m_s, \tag{3.10}$$

where we have assumed that $\beta > \alpha^{-1}$. For brevity, we denote

$$J(w) := \alpha L^2 \int_{]0,T]} e_s |w_s|^2 \,\mathrm{d}A_s.$$

By setting $t = 0$ in (3.10) and taking the expectation, we obtain

$$\mathbb{E}\left[\int_{]0,T]} e_s \,\mathrm{d}[m]_s\right] \leqslant \mathbb{E}[J(w)]. \tag{3.11}$$

Furthermore, it follows from (3.8) that, for every $t \in [0,T]$,

$$e_t|y_t|^2 \leqslant J(w) - 2\int_{]t,T]} e_s y_{s-} \,\mathrm{d}m_s \leqslant J(w) + 2\left|\int_{]t,T]} e_s y_{s-} \,\mathrm{d}m_s\right|$$

and thus, by taking the supremum over $t \in [0,T]$ and the expectation on both sides, we get

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]} e_t|y_t|^2\right] &\leqslant \mathbb{E}[J(w)] + 2\,\mathbb{E}\left[\operatorname*{ess\,sup}_{t\in[0,T]}\left|\int_{]t,T]} e_s y_{s-}\,\mathrm{d}m_s\right|\right]\\ &\leqslant \mathbb{E}[J(w)] + \frac{1}{4}\,\mathbb{E}\left[\sup_{t\in[0,T]} e_t|y_t|^2\right] + 4c_1^2\,\mathbb{E}\left[\int_{]0,T]} e_s\,\mathrm{d}[m]_s\right],\end{aligned} \tag{3.12}$$

where we have used the Burkholder–Davis–Gundy inequality, similarly as in (3.6), and thus the constant c_1 is independent of L, α and β. By combining (3.11) and (3.12), we obtain

$$\frac{3}{4}\,\mathbb{E}\left[\sup_{t\in[0,T]} e_t|y_t|^2\right] \leqslant (1+4c_1^2)\,\mathbb{E}[J(w)],$$

which in turn implies that

$$\begin{aligned}\frac{3}{4}\,\mathbb{E}\left[\sup_{t\in[0,T]} e_t|y_t|^2\right] &\leqslant \alpha L^2(1+4c_1^2)\,\mathbb{E}\left[\int_{]0,T]} e_s|w_s|^2\mathrm{d}A_s\right]\\ \leqslant \alpha L^2(1+4c_1^2)\,\mathbb{E}\left[A_T\sup_{t\in[0,T]} e_t|w_t|^2\right] &\leqslant \alpha c_A L^2(1+4c_1^2)\,\mathbb{E}\left[\sup_{t\in[0,T]} e_t|w_t|^2\right],\end{aligned}$$

since $A_T \leqslant c_A$. Consequently, for all $w', w'' \in \mathcal{S}^2_\beta$,

$$||\Phi(w') - \Phi(w'')||^2_{\mathcal{S}^2_\beta} \leqslant \frac{4}{3}\alpha c_A L^2(1+4c_1^2)\,||w'-w''||^2_{\mathcal{S}^2_\beta} = \gamma||w'-w''||^2_{\mathcal{S}^2_\beta}.$$

We conclude that Φ is a contraction when $\beta > \alpha^{-1}$ and $\alpha > 0$ is such that $\gamma < 1$, that is, when $\alpha < \frac{3}{4}c_A^{-1}L^{-2}(1+4c_1^2)^{-1}$ and $\beta > \alpha^{-1}$. Then, from the Banach fixed point theorem, there exists a unique fixed point, denoted as Y, of the mapping $\Phi : \mathcal{S}^2_\beta \to \mathcal{S}^2_\beta$ and we conclude using Lemma 3.8 that the GBSDE (1.1) has a unique solution (Y, M) in the space $\mathcal{S}^2 \times \mathcal{M}^2_0$. □

3.4 Well-posedness of GBSDEs with square-integrable driver

In the next proposition, we relax the assumption that the driver A is bounded and we work instead under Assumption 3.1 (iii) that A is a square-integrable process. However, we need to make additional assumptions about the generator g in order to establish the well-posedness result for the GBSDE (1.1) with a square-integrable driver.

Definition 3.13 *A generator g is said to be of class (P) if $g(\omega,t,\cdot)$ is nonnegative, nonincreasing for every $(\omega,t) \in \Omega \times [0,T]$ and there exists a process $\theta \in \mathcal{S}^2$ such that $xg(\omega,t,x) \leqslant \theta_t(\omega)g(\omega,t,x)$ for every $(\omega,t,x) \in \Omega\times[0,T]\times\mathbb{R}$.*

Proposition 3.14 *Let Assumptions 3.1 and 3.8 hold and the driver A be square-integrable. If the generator g is of class (P), then the GBSDE (1.1) has a unique solution (Y, M) and it belongs to $\mathcal{S}^2 \times \mathcal{M}^2_0$.*

Proof Define the sequence $A^n := A \wedge n$ of continuous, nondecreasing, bounded processes so that $c_{A^n} = n$ for every $n \in \mathbb{N}$. It is already known from Proposition 3.12 that for every $n \in \mathbb{N}$ the GBSDE

$$Y_t^n = \xi_T + \int_{]t,T]} g(s, Y_s^n)\,\mathrm{d}A_s^n - (M_T^n - M_t^n) \tag{3.13}$$

has a unique solution $(Y^n, M^n) \in \mathcal{S}^2 \times \mathcal{M}_0^2$. We will use the comparison property for solutions to the GBSDE in order to obtain a solution (Y, M) to (1.1) as a limit of the sequence $(Y^n, M^n)_{n\in\mathbb{N}}$. Since the generator $g(\omega, t, \cdot)$ is nonnegative and nonincreasing, the uniqueness of a solution to (1.1) is a consequence of Corollary 3.7. Therefore, it suffices to show that the limit $Y := \lim_{n\to\infty} Y^n$ is well defined in $\mathcal{S}^2$ and hence the pair (Y, M) is a unique solution to the GBSDE (1.1).

The uniqueness of a solution in $\mathcal{S}^2 \times \mathcal{M}_0^2$ to (1.1) can also be shown directly, that is, without using Corollary 3.7 (i). The computations are analogous to those in the proof of Propositions 3.9 and 3.10, therefore we will not present all details below. To this end, suppose (Y^1, M^1) and (Y^2, M^2) are two solutions in $\mathcal{S}^2 \times \mathcal{M}_0^2$ to (1.1). Then by applying the Itô formula to $|\widehat{Y}^2|$, where $\widehat{Y} := Y^1 - Y^2$ and $\widehat{M} = M^1 - M^2$, we obtain from the fact that A is continuous and $g(t, \cdot)$ is nonincreasing

$$\begin{aligned}|\widehat{Y}_t|^2 + \int_{]t,T]} \mathrm{d}[\widehat{M}]_s &\leqslant 2\int_{]t,T]} \widehat{Y}_s\left(g(s, Y_s^1) - g(s, Y_s^2)\right)\mathrm{d}A_s - 2\int_{]t,T]} \widehat{Y}_{s-}\,\mathrm{d}\widehat{M}_s \\ &\leqslant 2\int_{]t,T]} \widehat{Y}_{s-}\,\mathrm{d}\widehat{M}_s.\end{aligned}$$

From similar computations as in (3.6), we deduce that $2\int_{]0,t]} \widehat{Y}_{s-}\,\mathrm{d}\widehat{M}_s$ is a uniformly integrable martingale and hence, by taking the expectation and setting $t = 0$ in the above inequality, we find that $||\widehat{M}||_{\mathcal{M}_0^2} = 0$. Finally, from computations akin to those leading to (3.5), we deduce that $||\widehat{Y}||_{\mathcal{S}^2} = 0$. To complete the proof, we establish the following assertions:

Step 1 the sequence $(Y^n)_{n\in\mathbb{N}}$ is increasing, in the sense that $Y^{n+1} \geqslant Y^n$ for every $n \in \mathbb{N}$;

Step 2 there exists a constant c independent of n such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t^n|^2 + \int_{]0,T]} \mathrm{d}[M^n]_s\right] \leqslant c\,\mathbb{E}\left[|\xi_T|^2 + 1\right];$$

Step 3 there exists a pair $(Y, M) \in \mathcal{S}^2 \times \mathcal{M}_0^2$, which is a unique solution to the GBSDE (1.1).

Step 1 To establish the claimed comparison property $Y^{n+1} \geqslant Y^n$, we fix n and we apply the method from the proof of Proposition 3.6 to the pair Y and $\widetilde{Y}$ where $Y := Y^{n+1}$ and $\widetilde{Y} := Y^n$. We set $\xi = \eta = \xi_T$ and $g = h$ and we define

$$U_t := G_t(Y) = \int_{]0,t]} g(s, Y_s)\,\mathrm{d}A_s^{n+1}, \quad \widetilde{U}_t := H_t(\widetilde{Y}) = \int_{]0,t]} g(s, \widetilde{Y}_s)\,\mathrm{d}A_s^n.$$

Then we have that

$$\begin{aligned}V_t := U_t - \widetilde{U}_t = G_t(Y) - H_t(\widetilde{Y}) &= \left(G_t(Y) - G_t(\widetilde{Y})\right) + \left(G_t(\widetilde{Y}) - H_t(\widetilde{Y})\right) \\ &= \int_{]0,t]} \left(g(s, Y_s) - g(s, \widetilde{Y}_s)\right)\mathrm{d}A_s^{n+1} + \int_{]0,t]} g(s, \widetilde{Y}_s)\,\mathrm{d}\left(A_s^{n+1} - A_s^n\right).\end{aligned}$$

Since the generator $g(\omega, t, y)$ is assumed to be nonnegative and nonincreasing in y and the process $A^{n+1} - A^n$ is nondecreasing, the continuous and $\mathbb{F}$-adapted process $\mathbb{1}_E V$ is nondecreasing on $[\![\tau, \nu]\!]$ where the event E and the stopping times τ and ν are defined as in the proof of Proposition 3.6. To complete the proof of the assertion from Step 1, it now suffices to check that all other arguments from the proof of Proposition 3.6 are still valid.

Step 2 From Step 1, we note that $(Y^n)_{n\in\mathbb{N}}$ converges increasingly to an optional process Y which, for the moment, might not be càdlàg. We define

$$K_t^n = \int_{]0,t]} g(s, Y_s^n)\,\mathrm{d}A_s^n.$$

Then (3.13) is equivalent to

$$Y_t^n = \xi_T + K_T^n - K_t^n - (M_T^n - M_t^n).$$

By applying the Itô formula to $|Y_t^n|^2$ and using the fact that $Y_t^n g(t, Y_t^n) \leqslant \theta_t g(t, Y_t^n)$ for every $n \in \mathbb{N}$, we obtain

$$\begin{aligned} |Y_t^n|^2 + M_T^n - M_t^n &\leqslant |\xi_T|^2 + 2\int_{]t,T]} \theta_s\,\mathrm{d}K_s^n - 2\int_{]t,T]} Y_{s-}\,\mathrm{d}M_s^n \\ &\leqslant |\xi_T|^2 + \alpha(K_T^n - K_t^n)^2 + \alpha^{-1} \sup_{s\in]t,T]} \theta_s^2 - 2\int_{]t,T]} Y_{s-}\,\mathrm{d}M_s^n, \end{aligned} \tag{3.14}$$

where the last inequality follows from the Hölder inequality and the Young inequality with $\alpha > 0$. Next, from the equality

$$K_T^n - K_t^n = Y_t^n - \xi_T + M_T^n - M_t^n,$$

it follows that

$$\mathbb{E}\left[(K_T^n - K_t^n)^2 \,\middle|\, \mathcal{F}_t\right] \leqslant c_1 \mathbb{E}\left[|Y_t^n|^2 + |\xi_T|^2 + ([M^n]_T - [M^n]_t) \,\middle|\, \mathcal{F}_t\right].$$

By choosing $\alpha = (3c_1)^{-1}$ in (3.14), we have

$$\mathbb{E}\left[\frac{2}{3}|Y_t^n|^2 + \frac{2}{3}([M^n]_T - [M^n]_t) \,\middle|\, \mathcal{F}_t\right] \leqslant \mathbb{E}\left[\frac{4}{3}|\xi_T|^2 + 3c_1 \sup_{s\in]t,T]} \theta_s^2 \,\middle|\, \mathcal{F}_t\right],$$

where c_1 is independent of n. Finally, by using Burkholder–Davis–Gundy inequality and similar arguments as in (3.5) and (3.6), we obtain

$$\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t^n|^2 + ([M^n]_T - [M^n]_0)\right] \leqslant c\,\mathbb{E}\left[|\xi_T|^2 + 1\right],$$

where c depends only on α and c_1.

Step 3 We will demonstrate the existence of a pair $(Y, M) \in \mathcal{S}^2 \times \mathcal{M}_0^2$ which satisfies (1.1). To do this, we deduce from Step 1 that the sequence of optional processes $(Y^n)_{n\in\mathbb{N}}$, viewed as random variables on $([0,T] \times \Omega, \mathcal{O}(\mathbb{F}))$, converges increasingly to a unique optional process, which we denote as Y. We observe that $|Y_t|^2 \leqslant \liminf_{n\to\infty} \sup_{t\leqslant T} |Y_t^n|^2$ which by Fatou's lemma implies that

$$\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t|^2\right] \leqslant \liminf_{n\to\infty} \mathbb{E}\left[\sup_{t\in[0,T]} |Y_t^n|^2\right] < \infty,$$

which implies that $Y \in \mathcal{S}^2$. Note that it is not yet known whether Y is a càdlàg process but this property will follow from representation (3.15). Since the generator g does not depend on M^n, it follows that

$$Y_t^n = \mathbb{E}\left[\xi_T + \int_{]t,T]} g(s, Y_s^n)\,\mathrm{d}A_s^n \,\middle|\, \mathcal{F}_t\right]$$

and the existence of a unique $M \in \mathcal{M}_0^2$ will follow once we show that

$$\lim_{n\to\infty} \mathbb{E}\left[\xi_T + \int_{]t,T]} g(s, Y_s^n)\, \mathrm{d}A_s^n \,\Big|\, \mathcal{F}_t\right] = \mathbb{E}\left[\xi_T + \int_{]t,T]} g(s, Y_s)\, \mathrm{d}A_s \,\Big|\, \mathcal{F}_t\right]$$

and $K_t := \int_{]0,t]} g(s, Y_s)\, \mathrm{d}A_s \in \mathcal{S}^2$. To proceed, we first show that K_T has finite expectation. To this end, we note that by Assumption 3.8 (i)

$$\begin{aligned}\int_{]0,T]} g(s, Y_s)\, \mathrm{d}A_s &\leqslant \int_{]0,T]} |g(s,0)|\, \mathrm{d}A_s + L \int_{]0,T]} |Y_s|\, \mathrm{d}A_s \\ &\leqslant \int_{]0,T]} 1 + |g(s,0)|^2\, \mathrm{d}A_s + L \int_{]0,T]} |Y_s|\, \mathrm{d}A_s.\end{aligned}$$

The expected value of the first term in the right hand side above is finite in view of Assumption 3.8 (ii). On the other hand, by applications of the Hölder inequality, the expected value of the second term can be upper estimated as follows

$$\mathbb{E}\left[\int_{]0,T]} |Y_s|\, \mathrm{d}A_s\right] \leqslant \mathbb{E}\left[\sqrt{\sup_{s\in[0,T]} |Y_s|^2} A_T\right] \leqslant \|Y\|_{\mathcal{S}^2} \mathbb{E}[A_T^2] < \infty.$$

Note that similar conclusions hold when Y or A is replaced by Y^n or A^n in the integral K. Next, to prove that K^n converges to K as n tends to ∞, we observe that

$$\begin{aligned}&\int_{]0,t]} g(s, Y_s)\, \mathrm{d}A_s - \int_{]0,t]} g(s, Y_s^n)\, \mathrm{d}A_s^n \\ &= \int_{]0,t]} g(s, Y_s)\, \mathrm{d}\,(A_s - A_s^n) + \int_{]0,t]} (g(s, Y_s) - g(s, Y_s^n))\, \mathrm{d}A_s^n \\ &= \int_{]0,t]} g(s, Y_s) \mathbb{1}_{\{A_s > n\}}\, \mathrm{d}A_s + \int_{]0,t]} (g(s, Y_s) - g(s, Y_s^n))\, \mathrm{d}\,(A_s^n - A_s) \\ &\quad + \int_{]0,t]} (g(s, Y_s) - g(s, Y_s^n))\, \mathrm{d}A_s := I_1^n + I_2^n + I_3^n.\end{aligned}$$

Both I_1^n and $-I_3^n$ has positive integrands which decreases to zero as n approaches infinity. Therefore by the monotone convergence theorem, both I_1^n and $-I_3^n$ decrease to zero as n tends to ∞. For I_2^n, we observe that

$$\begin{aligned}0 \leqslant I_2^n &= \int_{]0,t]} (g(s, Y_s) - g(s, Y_s^n))\, \mathrm{d}\,(A_s^n - A_s) = \int_{]0,t]} (g(s, Y_s^n) - g(s, Y_s))\, \mathbb{1}_{\{A_s > n\}} \mathrm{d}A_s \\ &\leqslant \int_{]0,t]} (g(s, Y_s^n) - g(s, Y_s))\, \mathrm{d}A_s = -I_3^n.\end{aligned}$$

To see that K belongs to $\mathcal{S}^2$, we note that $K_T^n \geqslant 0$ and by the Fatou lemma we have

$$\mathbb{E}[K_T^2] \leqslant \liminf_{n\to\infty} \mathbb{E}[(K_T^n)^2] \leqslant c\mathbb{E}[\xi_T^2 + 1].$$

Finally, by the monotone convergence theorem, we obtain

$$\begin{aligned}Y_t &= \mathbb{E}\left[\xi_T + \int_{]t,T]} g(s, Y_s)\, \mathrm{d}A_s \,\Big|\, \mathcal{F}_t\right] = \mathbb{E}\left[\xi_T + \int_{]0,T]} g(s, Y_s)\, \mathrm{d}A_s \,\Big|\, \mathcal{F}_t\right] - \int_{]0,t]} g(s, Y_s)\, \mathrm{d}A_s \\ &= Y_0 + M_t - \int_{]0,t]} g(s, Y_s)\, \mathrm{d}A_s, \qquad (3.15)\end{aligned}$$

where M is given by (3.3) and thus (Y, M) a unique solution to the GBSDE (1.1). It also follows from (3.15) that the process Y has a càdlàg version.

To show that Y^n converges to Y in $\mathcal{S}^2$ as n tends to ∞, we first note that, since K^n and K are both continuous and the predictable projection of a uniformly integrable martingale M is equal to M_- (see Remark 5.3 on page 137 of He et al. [27]), we have ${}^p(Y^n) = Y^n_-$ and ${}^pY = Y_-$. By the monotone convergence theorem, we have $\lim_{n\to\infty} Y^n_{t-} = \lim_{n\to\infty} {}^p(Y^n_t) = {}^pY_t = Y_{t-}$ for every $t \in [0,T]$. Therefore, by the càdlàg version of Dini's theorem (see page 202 of Dellacherie and Meyer [6]), for almost every $\omega \in \Omega$, the sequence of càdlàg processes $(Y - Y^n)^2(\omega)$ converges uniformly to zero on $[0,T]$ as $n \to \infty$. From inequalities $Y^1 \leqslant Y^n \leqslant Y$ we obtain

$$\sup_{t\leqslant T} |Y_t - Y^n_t|^2 \leqslant \sup_{t\leqslant T} |Y_t - Y^1_t|^2,$$

where the right-hand side is integrable because we have previously shown that Y and Y^1 are in $\mathcal{S}^2$. Hence from the dominated convergence theorem we deduce that $\lim_{n\to\infty} ||Y^n - Y||_{\mathcal{S}^2} = 0$, as was required to show. □

If the generator satisfies the assumptions of Proposition 3.14 and is bounded so that $|g| \leqslant c_g$ for some constant c_g, then the sequence $(K^n)_{n\in\mathbb{N}}$ of nondecreasing processes given by

$$K^n_t := \int_{]0,t]} g(s, Y^n_s)\, \mathrm{d}A^n_s$$

converges to the nondecreasing process K, which equals

$$K_t := \int_{]0,t]} g(s, Y_s)\, \mathrm{d}A_s$$

and the convergence is uniform in t, for almost all ω. Indeed, we have that

$$\begin{aligned} \left|K_t - K^n_t\right| &= \left| \int_{]0,t]} g(s, Y_s)\, \mathrm{d}A_s - \int_{]0,t]} g(s, Y^n_s)\, \mathrm{d}A^n_s \right| \\ &\leqslant \left| \int_{]0,t]} g(s, Y_s)\, \mathrm{d}A_s - \int_{]0,t]} g(s, Y^n_s)\, \mathrm{d}A_s \right| + \left| \int_{]0,t]} g(s, Y^n_s)\, \mathrm{d}A_s - \int_{]0,t]} g(s, Y^n_s)\, \mathrm{d}A^n_s \right| \\ &\leqslant L \int_{]0,t]} \left|Y_s - Y^n_s\right| \mathrm{d}A_s + c_g L\,(A_T - A^n_T) \end{aligned}$$

and thus

$$\begin{aligned} \sup_{t\in[0,T]} \left|K_t - K^n_t\right| &\leqslant L \int_{]0,T]} \left|Y_s - Y^n_s\right| \mathrm{d}A_s + c_g L\,(A_T - A^n_T) \\ &\leqslant L A_T \sup_{t\in[0,T]} \left|Y_t - Y^n_t\right| + c_g L\,(A_T - A^n_T)\,, \end{aligned}$$

which entails that $\sup_{t\in[0,T]} |K_t - K^n_t|$ converges to 0 almost surely when n tends to ∞.

Corollary 3.15 *Let Assumption 3.1 hold and ξ_T be square-integrable. If η is an $\mathbb{F}$-optional, bounded process, then for every $n \in \mathbb{N}$ the GBSDE*

$$Y^n_t = \xi_T + \int_{]t,T]} n(\eta_s - Y^n_s)^+\, \mathrm{d}A_s - (M^n_T - M^n_t) \tag{3.16}$$

has a unique solution (Y^n, M^n), which belongs to $\mathcal{S}^2 \times \mathcal{M}^2_0$, and the inequality $Y^{n+1} \geqslant Y^n$ is satisfied for every $n \in \mathbb{N}$.

Proof The first assertion is an immediate consequence of Proposition 3.14 since the generator $g(t,y) = n(\eta_t - y)^+$ is of class (P). To show that $Y^{n+1} \geqslant Y^n$, it suffices to take $g(t,y) := (n+1)(\eta_t - y)^+$ and $h(t,y) := n(\eta_t - y)^+$ in Proposition 3.6 (see also Corollary 3.7). □

3.5 Constrained optimal stopping and penalization scheme for GBSDEs

To examine a constrained optimal stopping problem and the associated penalization scheme for GBSDEs, we define $\overline{S} := S^r \cup \{T\}$ where $S^r = S^r(A)$ is the right support of the driver A. Recall that the *right support* of a nondecreasing process A is given by

$$S^r(A) := \{(\omega, t) \in \Omega \times [0,T[\, : \forall\, \varepsilon > 0 \ A_{t+\varepsilon}(\omega) - A_t(\omega) > 0\}.$$

Furthermore, we denote by $\overline{\mathcal{T}}$ the class of all $\mathbb{F}$-stopping times τ with values in $[0,T]$ and such that $\mathbb{P}(\tau \in \overline{S}) = 1$. Similarly, $\overline{\mathcal{T}}_{t,T}$ is the set of $\mathbb{F}$-stopping times from $\overline{\mathcal{T}}$ such that $\mathbb{P}(\tau \in \overline{S} \cap [t,T]) = 1$. We consider the following constrained optimal stopping problem

$$V_0 = \sup_{\tau \in \overline{\mathcal{T}}} \mathbb{E}\left[\xi_T \mathbb{1}_{\{\tau = T\}} + \eta_\tau \mathbb{1}_{\{\tau < T\}}\right],$$

which leads to the following definition of the value process.

Definition 3.16 *The process V is the value process of the constrained optimal stopping with data $(\eta, \xi_T, \overline{\mathcal{T}})$ if the following equality holds, for every $t \in [0,T]$,*

$$V_t = \operatorname*{ess\,sup}_{\tau \in \overline{\mathcal{T}}_{t,T}} \mathbb{E}\left[\xi_T \mathbb{1}_{\{\tau = T\}} + \eta_\tau \mathbb{1}_{\{\tau < T\}} \,|\, \mathcal{F}_t\right]. \tag{3.17}$$

Remark 3.17 *Notice that to formulate the optimal stopping problem of Definition 3.16 it is enough to specify the inputs $(\eta, \xi_T, \overline{S}, \mathbb{F})$, that is, there is no need to explicitly select a process A such that the equality $\overline{S} = S^r(A) \cup \{T\}$ holds. In particular, if B is another $\mathbb{F}$-adapted, nondecreasing, square-integrable, and continuous process such that $S^r(B) = S^r(A)$, then either A or B can be used in the penalisation scheme of Theorem 3.19.*

It should be made clear that form (3.17) of a stopping problem is by no means arbitrary. In fact, the constrained optimal stopping introduced in Definition 3.16 arises naturally in a study of the arbitrage-free pricing of a vulnerable European option in an incomplete market model with default event modeled by an optional hazard process (see Section 3 in [32]). Before proceeding to an analysis of the penalization scheme for the value process V, we present the following auxiliary result, which is an extension of Lemma 6.1 in [15].

Lemma 3.18 *Let η be an $\mathbb{F}$-optional, bounded and right-continuous process. Then for any stopping time $\nu \in \overline{\mathcal{T}}$ we have*

$$\lim_{n\to\infty} \left[\xi_T \mathcal{E}_{\nu,T}(-A^n) + (\mathbb{1}_{]\!]\nu,T]\!]} \eta \mathcal{E}_{\nu,\cdot}(-A^n) \bullet A^n)_T\right] = \xi_T \mathbb{1}_{\{\nu = T\}} + \eta_\nu \mathbb{1}_{\{\nu < T\}},$$

where $A^n := nA$.

Proof Since $A^n_\nu - A^n_T = 0$ on the event $\{\nu = T\}$ we have

$$\lim_{n\to\infty} \mathcal{E}_{\nu,T}(-A^n) = \lim_{n\to\infty} e^{A^n_\nu - A^n_T} = \mathbb{1}_{\{\nu = T\}}.$$

Since ν takes values in the right support we have that $A^n_T - A^n_\nu > 0$ on the event $\{\nu < T\}$ and thus $\lim_{n\to\infty}(A^n_\nu - A^n_T) = -\infty$. We claim that, on the event $\{\nu < T\}$,

$$\lim_{n\to\infty} \int_{]\nu,T]} \eta_s e^{A^n_\nu - A^n_s}\, \mathrm{d}A^n_s = \eta_\nu \mathbb{1}_{\{\nu < T\}} \tag{3.18}$$

since the sequence of bounded and positive measures $\mu_n := \mathbb{1}_{]\nu,T]} e^{A^n_\nu - A^n_s}\, \mathrm{d}A^n_s$, $n \in \mathbb{N}$ converges to the Dirac measure at ν on the event $\{\nu < T\}$, that is, to the measure $\mu := \mathbb{1}_{\{\nu < T\}}\delta_\nu$. Equality (3.18) can be formally established using the time change on $[0,T]$ generated by A. To this end, we define the nondecreasing, right-continuous process C by $C_s = \inf\{t \in \mathbb{R}_+ : A_t > s\}$.

Then an application of the time change formula (see, e.g., Chapter 0 in Revuz and Yor [42]) gives

$$\int_{]\nu,T]} \eta_s e^{A^n_\nu - A^n_s}\,\mathrm{d}A^n_s = \int_{]\nu,T]} \eta_s n e^{n(A_\nu - A_s)}\,\mathrm{d}A_s = \int_0^\infty \mathbb{1}_{\{\nu < C_s \leqslant T\}} \eta_{C_s} n e^{n(A_\nu - s)}\,\mathrm{d}s,$$

where to obtain the second equality, we have used the equality $A_{C_s} = s$, which holds since A is a continuous process. From the fact that $\{\nu < C_s\} \subseteq \{A_\nu \leqslant s\}$ and the change of variable $u = s - A_\nu$ we obtain

$$\int_0^\infty \mathbb{1}_{\{\nu < C_s \leqslant T\}} \eta_{C_s} n e^{n(A_\nu - s)}\,\mathrm{d}s = \int_0^\infty \mathbb{1}_{\{\nu < C_{A_\nu + u} \leqslant T\}} \eta_{C_{A_\nu + u}} n e^{-nu}\,\mathrm{d}u = \mathbb{E}_X \left[\mathbb{1}_{\{\nu < C_{A_{\nu + X/n}} \leqslant T\}} \eta_{C_{A_{\nu + X/n}}} \right],$$

where in the last equality we have used the observation that ne^{-nu}, $u > 0$ is the density of $\frac{1}{n}X$ where $X \sim \exp(1)$ and is independent of the σ-field $\mathcal{F}_\infty$.

Since on the event $\{\nu < T\}$, the stopping time ν takes values in the right support of A and $\nu < \nu + X/n$, we deduce that $A_{\nu + X/n} \geqslant A_{\nu + X/(n+1)} > A_\nu$ for a sufficiently large $n \in \mathbb{N}$. This observation, together with the right-continuity of the processes η, C and A, allows us to conclude that

$$\lim_{n\to\infty} \mathbb{E}_X \left[\mathbb{1}_{\{\nu < C_{A_\nu + X/n} \leqslant T\}} \eta_{C_{A_\nu + X/n}} \right] = \mathbb{1}_{\{\nu \leqslant C_{A_\nu} < T\}} \eta_{C_{A_\nu}}.$$

Finally, since on $\{\nu < T\}$ the stopping time ν takes values in the right support of the process A we have that $C_{A_\nu} = \inf\{s : A_s > A_\nu\} = \nu$, which in turn allows us to conclude that the equality $\mathbb{1}_{\{\nu \leqslant C_{A_\nu} < T\}} \eta_{C_{A_\nu}} = \mathbb{1}_{\{\nu < T\}} \eta_\nu$ is valid. □

We are ready to establish the main result of this section where we show that a sequence of processes Y^n converges to the value process V of a constrained optimal stopping problem (3.17) in which stopping is allowed at times belonging to the right support of the measure generated by the process A.

Theorem 3.19 *Let Assumptions 3.1 and 3.8 hold and the driver A be square-integrable. For a bounded terminal condition ξ_T and an $\mathbb{F}$-optional, bounded, nonnegative, and right-continuous process η, consider the sequence of unique solutions $(Y^n, M^n) \in \mathcal{S}^2 \times \mathcal{M}^2_0$ to the GBSDE* (3.16)*. Then the sequence Y^n is increasing and converges almost surely to the value process V of the constrained optimal stopping problem given by equation* (3.17).

Proof We know from Proposition 3.14 that the GBSDE (3.16) has a unique solution (Y^n, M^n) in $\mathcal{S}^2 \times \mathcal{M}^2_0$ and it follows from Corollary 3.15 that the sequence Y^n of processes is monotonically increasing as n tends to ∞ and the limit $Y = \lim_{n\to\infty} Y^n$ is well defined.

Step 1 Our first goal is to show that the càdlàg, $\mathbb{F}$-adapted process Y^n satisfies, for every $n \in \mathbb{N}$ and $t \in [0, T]$,

$$Y^n_t = \operatorname*{ess\,sup}_{\tau \in \overline{\mathcal{T}}_{t,T}} \mathbb{E} \left[\xi_T \mathbb{1}_{\{\tau = T\}} + (\eta_\tau \wedge Y^n_\tau) \mathbb{1}_{\{\tau < T\}} \,|\, \mathcal{F}_t \right]. \tag{3.19}$$

For a fixed $n \in \mathbb{N}$, we set $\gamma^n_t := \xi_T \mathbb{1}_{\{t=T\}} + (\eta_t \wedge Y^n_t) \mathbb{1}_{\{t<T\}}$ and we observe that

$$\gamma^n_t = (\xi_T \wedge Y^n_T) \mathbb{1}_{\{t=T\}} + (\eta_t \wedge Y^n_t) \mathbb{1}_{\{t<T\}} \leqslant Y^n_t.$$

Furthermore, the GBSDE (3.16) can be represented as

$$Y^n_t = \xi_T + K^n_T - K^n_t - (M^n_T - M^n_t), \tag{3.20}$$

where the $\mathbb{F}$-adapted, continuous, nondecreasing process K^n is given by

$$K^n_t := \int_{]0,t]} n(\eta_s - Y^n_s)^+ \, \mathrm{d}A_s.$$

Recall that we assumed that η (and hence also γ^n) is a right-continuous process. We claim that (3.19) is valid, that is, for every $t \in [0,T]$,

$$Y^n_t = \operatorname*{ess\,sup}_{\tau \in \overline{\mathcal{T}}_{t,T}} \mathbb{E}\left[\gamma^n_\tau \,|\, \mathcal{F}_t\right]. \tag{3.21}$$

Equality (3.21) is obvious for $t = T$ so it suffices to consider any $t < T$. We have, for any $\tau \in \overline{\mathcal{T}}_{t,T}$,

$$Y^n_t = \mathbb{E}\left[Y^n_\tau + K^n_\tau - K^n_t \,|\, \mathcal{F}_t\right] \geqslant \mathbb{E}\left[Y^n_\tau \,|\, \mathcal{F}_t\right] \geqslant \mathbb{E}\left[\gamma^n_\tau \,|\, \mathcal{F}_t\right],$$

which implies that $Y^n_t \geqslant \operatorname{ess\,sup}_{\tau \in \overline{\mathcal{T}}_{t,T}} \mathbb{E}\left[\gamma^n_\tau \,|\, \mathcal{F}_t\right]$.

For the converse inequality, we define the stopping time $\tau_t := \inf\{s \in [t,T] \,|\, K^n_s - K^n_t > 0\}$, which belongs to $\overline{\mathcal{T}}_{t,T}$, and we observe that $K^n_{\tau_t} - K^n_t = 0$ due to the continuity of K^n. Furthermore, on the event $\{\tau_t < T\}$ we have $K^n_s > K^n_{\tau_t}$ on $]\!]\, \tau_t, T]\!]$, which entails the inequality $\limsup_{u \downarrow \tau_t} (\eta_u - Y^n_u) \geqslant 0$ and thus, since the processes η and Y^n are right-continuous, we conclude that $\eta_{\tau_t} - Y^n_{\tau_t} \geqslant 0$, which in turn implies that $Y^n_{\tau_t} = \gamma^n_{\tau_t}$. On the event $\{\tau_t = T\}$ we have that $K^n_{\tau_t} - K^n_t = K^n_T - K^n_t = 0$ and $Y^n_{\tau_t} = \gamma^n_{\tau_t}$. From (3.20), we now get

$$Y^n_t = Y^n_{\tau_t} - (M^n_{\tau_t} - M^n_t) = \gamma^n_{\tau_t} - (M^n_{\tau_t} - M^n_t) = \mathbb{E}\left[\gamma^n_{\tau_t} \,|\, \mathcal{F}_t\right],$$

which leads to the inequality $Y^n_t \leqslant \operatorname{ess\,sup}_{\tau \in \overline{\mathcal{T}}_{t,T}} \mathbb{E}\left[\gamma^n_\tau \,|\, \mathcal{F}_t\right]$ since $\tau_t \in \overline{\mathcal{T}}_{t,T}$. We conclude that equality (3.21) (or, equivalently, (3.19)) holds.

Step 2 We set $\gamma_t := \xi_T \mathbb{1}_{\{t=T\}} + \eta_t \mathbb{1}_{\{t<T\}}$ and we show that $\lim_{n\to\infty} Y^n_t = V_t$ where

$$V_t := \operatorname*{ess\,sup}_{\nu \in \overline{\mathcal{T}}_{t,T}} \mathbb{E}\left[\gamma_\nu \,|\, \mathcal{F}_t\right]. \tag{3.22}$$

It follows from (3.19) and (3.22) that $Y^n \leqslant V$ and $Y^n \uparrow Y \leqslant V$ where the $\mathbb{F}$-optional process Y is given by $Y = \lim_{n\to\infty} Y^n$. Furthermore, it is clear that the process Y is nonnegative and belongs to $\mathcal{S}^2$ since it is dominated by the process V belonging to $\mathcal{S}^2$.

It is clear that $Y_T = \xi_T = \gamma_T$ and thus it suffices to show that $Y_\nu \mathbb{1}_{\{\nu<T\}} \geqslant \eta_\nu \mathbb{1}_{\{\nu<T\}}$ for any $\mathbb{F}$-stopping time ν taking values in $\overline{S}$. From the monotone convergence theorem and the comparison property established in Proposition 3.6 we obtain, for every $0 \leqslant \tau \leqslant \nu \leqslant T$,

$$\mathbb{E}\left[Y_\nu \,|\, \mathcal{F}_\tau\right] = \lim_{n\to\infty} \mathbb{E}\left[Y^n_\nu \,|\, \mathcal{F}_\tau\right] \leqslant \lim_{n\to\infty} Y^n_\tau = Y_\tau.$$

Using the fact that V is bounded, we deduce that Y is a bounded strong supermartingale and, as a consequence of Theorem 2 in Mertens [34] (or, more specifically, the Lemma on page 51 of [34]), we have $Y \geqslant Y_+$. Next, from the form of Y^n we have, for every $\tau \in \mathcal{T}$,

$$\frac{1}{n}\mathbb{E}[Y^n_\tau] = \frac{1}{n}\mathbb{E}[\xi_T] + \mathbb{E}\left[\int_{]\!]\,\tau,T]\!]} (\eta_s - Y^n_s)^+ \, \mathrm{d}A_s\right].$$

By letting n go to ∞ and applying the dominated convergence theorem, we obtain the equality

$$\int_{]\!]\,\tau,T]\!]} (\eta_s - Y_s)^+ \, \mathrm{d}A_s = 0.$$

We claim that the last equality implies that $Y_\nu \mathbb{1}_{\{\nu<T\}} \geqslant \eta_\nu \mathbb{1}_{\{\nu<T\}}$ for all $\nu \in \overline{\mathcal{T}}$. Suppose, on

the contrary, that there exists a stopping time $\nu \in \overline{\mathcal{T}}$ such that the event $E = \{\eta_\nu > Y_\nu \geqslant Y_{\nu+}\} \cap \{\nu < T\}$ has a positive probability. Then we deduce from the right-continuity of η that there exists ε, which may depend on $\omega \in E$, such that $\eta_t - Y_t > 0$ for all $t \in [\![\nu, \nu+\varepsilon]\!] \cap [\![\nu, T]\!]$. However, since $\nu \in \overline{\mathcal{T}}$, we deduce that $A_{\nu+\varepsilon} - A_\nu > 0$ for almost every $\omega \in E$ and this contradicts the equality $\int_{]\!]\nu,T]\!]} (\eta_s - Y_s)^+ \, \mathrm{d}A_s = 0$. We this conclude that Y dominates γ on $\overline{S}$.

Finally, by using the fact that the essential supremum and conditional expectation can be interchanged, we can conclude that V is the smallest strong supermartingale dominating γ on $\overline{S}(A)$, which in turn implies that, for every $t \in [0, T]$,

$$Y_t \geqslant \operatorname*{ess\,sup}_{\nu \in \overline{\mathcal{T}}_{t,T}} \mathbb{E}\left[\gamma_\nu \,|\, \mathcal{F}_t\right] = V_t,$$

as was required to show. □

4. Reflected GBSDEs and constrained Dynkin games

In this section, we study reflected generalized BSDEs (reflected GBSDEs or, briefly, RGBSDEs) with a lower obstacle given by a predetermined process ζ. Hence Assumption 3.1 is substituted with Assumption 4.1, which specifies the basic conditions for the *lower obstacle* ζ, the driver A, and the generator g.

Assumption 4.1 *We are given the following objects defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$, *which is endowed with a filtration* $\mathbb{F}$ *satisfying the usual conditions of right-continuity and* $\mathbb{P}$-*completeness:*

(i) *an* $\mathbb{F}$-*optional process* ζ;
(ii) *an* $\mathbb{F}$-*adapted, nondecreasing, continuous process* A *with* $A_0 = 0$;
(iii) *an* $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$-*measurable mapping* $g : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$.

If ζ is an $\mathbb{F}$-optional process, then the process $\bar{\zeta}$ given by $\bar{\zeta}_t := \limsup_{s\uparrow t,\, s<t} \zeta_t$ for all $t \in]0, T]$ is known to be $\mathbb{F}$-predictable on $]0, T]$ and left-upper-semicontinuous (see, e.g., [23] or Theorem 90 on page 225 in [6]). The process $\bar{\zeta}$ is called the *left-upper-semicontinuous envelope* of ζ.

Recall that by $\mathcal{K}$ (resp., $\overline{\mathcal{K}}$) we denote the class of all càdlàg, nondecreasing, $\mathbb{F}$-predictable (resp., làdlàg, nondecreasing, $\mathbb{F}$-predictable) processes and $\mathcal{T}$ (resp., $\mathcal{T}^p$) stands for the class of all $\mathbb{F}$-stopping times (resp., $\mathbb{F}$-predictable stopping times) taking values in $[0, T]$.

The following definition is consistent with the classical case where $\mathbb{F}$ is a Brownian filtration and the driver $A_t = t$ for every $t \in [0, T]$. It is important to notice that the component Y of a solution (Y, M, K) is now postulated to be a làdlàg process, as opposed to a càdlàg component Y in a solution (Y, M) in Definition 3.1. Recall also that the conditions for the processes K^c, K^d and K^g specified in Definition 4.1 are called the *Skorokhod conditions.*

Definition 4.2 *A triplet* (Y, M, K) *is a solution to the reflected GBSDE (1.2) with data* (A, ζ_T, g, ζ) *if* Y *is an* $\mathbb{F}$-*adapted, làdlàg process,* M *is an* $\mathbb{F}$-*martingale,* K *is a nondecreasing,* $\mathbb{F}$-*predictable, làdlàg process, and the following conditions are met*

$$\begin{cases} Y_\tau = \zeta_T + \displaystyle\int_{]\tau,T]} g(s, Y_s) \, \mathrm{d}A_s - (M_T - M_\tau) + K_T - K_\tau, \quad \forall\, \tau \in \mathcal{T}, \\ Y_t \geqslant \zeta_t, \quad \forall\, t \in [0, T], \quad \left(\mathbb{1}_{\{Y_- > \bar{\zeta}\}} \bullet K^c\right)_T = 0, \\ (Y_{\tau-} - \bar{\zeta}_\tau)\Delta K^d_\tau = 0, \quad \forall\, \tau \in \mathcal{T}^p, \quad (Y_\tau - \zeta_\tau)\Delta^+ K^g_\tau = 0, \quad \forall\, \tau \in \mathcal{T}, \end{cases} \tag{4.1}$$

where the integral $\int_{]0,t]} g(s, Y_s)\,\mathrm{d}A_s$ *is an* $\mathbb{F}$*-adapted, continuous process of finite variation.*

Let us summarize the content of this section. We first establish in Section 4.1 a variant of the comparison theorem for a reflected GBSDE under the assumption that the filtration $\mathbb{F}$ is quasi-left-continuous. Then we obtain in Section 4.2 some useful *a priori* estimates and we study in Section 4.3 the existence and uniqueness of a solution (Y, M, K) to the RGBSDE (1.2) in the space $\mathcal{S}^2 \times \mathcal{M}_0^2 \times \overline{\mathcal{K}}$. Another version of the comparison theorem for reflected GBSDEs is proven in Section 4.4 under the assumption that the driver A is bounded. Finally, for given $\mathbb{F}$-optional and bounded processes ζ and η, we consider in Sections 4.5 and 4.6 the sequences of penalized RGBSDEs with the lower obstacle ζ of the form

$$Y_\tau^n = \zeta_T + \int_{]\tau,T]} n(Y_s^n - \eta_s)^+\,\mathrm{d}A_s - (M_T^n - M_\tau^n) + K_T^n - K_\tau^n$$

and

$$\widetilde{Y}_\tau^n = \zeta_T - \int_{]\tau,T]} n(\widetilde{Y}_s^n - \eta_s)^+\,\mathrm{d}A_s - (\widetilde{M}_T^n - \widetilde{M}_\tau^n) + \widetilde{K}_T^n - \widetilde{K}_\tau^n$$

and we examine the limits $Y := \lim_{n\to\infty} Y^n$ and $\widetilde{Y} := \lim_{n\to\infty} \widetilde{Y}^n$. It will be shown that the process Y (resp., the process $\widetilde{Y}$) has a useful interpretation as the value process of a particular optimal stopping problem (resp., a constrained Dynkin game) where the right support of the driver A plays an important role in the specification of the reward process and/or the class of allowable stopping times.

4.1 Comparison theorem for reflected GBSDEs with quasi-left-continuous filtration

We aim to show that the comparison property of solutions to a GBSDE established in Proposition 3.6 can be extended to the case of a reflected GBSDE. The first variant of the comparison theorem for a reflected GBSDE is established under an additional postulate that the filtration $\mathbb{F}$ is quasi-left-continuous so that any $\mathbb{F}$-martingale does not jump at any $\mathbb{F}$-predictable stopping time.

Proposition 4.3 *Suppose that the filtration* $\mathbb{F}$ *is quasi-left-continuous. Let the mappings* $g, h : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ *be such that the reflected GBSDEs*

$$Y_\tau = \zeta_T + \int_{]\tau,T]} g(s, Y_s)\,\mathrm{d}A_s - (M_T - M_\tau) + K_T - K_\tau$$

and

$$\widetilde{Y}_\tau = \widetilde{\zeta}_T + \int_{]\tau,T]} h(s, \widetilde{Y}_s)\,\mathrm{d}A_s - (\widetilde{M}_T - \widetilde{M}_\tau) + \widetilde{K}_T - \widetilde{K}_\tau$$

with the lower obstacles ζ *and* $\widetilde{\zeta}$*, respectively, have solutions* (Y, M, K) *and* $(\widetilde{Y}, \widetilde{M}, \widetilde{K})$*, respectively. If* $\zeta \geqslant \widetilde{\zeta}$ *and the functions* g *and* h *satisfy the following conditions:*

(i) $g(\omega, t, y) \geqslant h(\omega, t, y)$ *for every* $(\omega, t, y) \in \Omega \times [0,T] \times \mathbb{R}$,
(ii) $g(\omega, t, \cdot)$ *is a nonincreasing function for every* $(\omega, t) \in \Omega \times [0,T]$,

then the inequality $Y \geqslant \widetilde{Y}$ *is valid.*

Proof We extend the proof of Proposition 3.6. Notice, however, that the processes Y and $\widetilde{Y}$ in the statement of Proposition 4.3 are làdlàg, whereas in Proposition 3.6 the processes denoted by Y and $\widetilde{Y}$ were càdlàg. For a fixed $\varepsilon > 0$, we define the $\mathbb{F}$-stopping time $\tau^\varepsilon :=$

$\inf\{t \geqslant 0 : Y_t \leqslant \widetilde{Y}_t - \varepsilon\}$ where, by convention, $\inf\emptyset = T$. As in the proof of Proposition 3.6, we argue that if for all $\varepsilon > 0$ we have $\mathbb{P}(\tau^\varepsilon = T) = 1$, then the asserted inequality $Y \geqslant \widetilde{Y}$ holds.

Let us now assume that the inequality $Y \geqslant \widetilde{Y}$ does not hold. Then there exists $\varepsilon > 0$ such that $\mathbb{P}(\tau^\varepsilon < T) > 0$. We fix ε and we denote $E := \{\tau^\varepsilon < T\} \in \mathcal{F}_{\tau^\varepsilon}$. Next, we define $\tau := \tau^\varepsilon \mathbb{1}_E + T\mathbb{1}_{E^c}$ so that $\{\tau < T\} = \{\tau^\varepsilon < T\} \in \mathcal{F}_\tau$.

Step 1 We start by showing that the inequality $Y_{\tau+} < \widetilde{Y}_{\tau+}$ holds on E. To this end, let us consider any event ω from E. We have that either (1.1) $Y_\tau \leqslant \widetilde{Y}_\tau - \varepsilon$ so that $Y_\tau < \widetilde{Y}_\tau$ or (1.2) $Y_\tau > \widetilde{Y}_\tau - \varepsilon$ but $Y_{\tau+} \leqslant \widetilde{Y}_{\tau+} - \varepsilon$ so that $Y_{\tau+} < \widetilde{Y}_{\tau+}$, which is the desired inequality.

Hence it remains to show that in case (1.1) we also have that $Y_{\tau+} < \widetilde{Y}_{\tau+}$. To this end, we first observe that $\widetilde{Y}_\tau > \widetilde{Y}_\tau - \varepsilon > Y_\tau \geqslant \zeta_\tau \geqslant \widetilde{\zeta}_\tau$ and thus the process $\widetilde{K}$ is right-continuous at τ (from the respective Skorokhod condition), which in turn implies that $\widetilde{Y}$ is right-continuous at τ. Recall that $Y_\tau = \zeta_\tau \vee Y_{\tau+}$. If $Y_\tau > \zeta_\tau$, then Y is also right-continuous at τ and thus $\widetilde{Y}_{\tau+} = \widetilde{Y}_\tau > Y_\tau = Y_{\tau+}$. Finally, if $Y_\tau = \zeta_\tau$, then $Y_{\tau+} \leqslant \zeta_\tau$ and thus $\widetilde{Y}_{\tau+} = \widetilde{Y}_\tau > \zeta_\tau \geqslant Y_{\tau+}$.

We have thus shown that the inequality $Y_{\tau+} > \widetilde{Y}_{\tau+}$ holds on E, for almost all ω. Then we define the $\mathbb{F}$-stopping time $\nu := \inf\{t \geqslant \tau : Y_t \geqslant \widetilde{Y}_t\}$ and we note that $\nu \leqslant T$ since, by assumption, $Y_T = \zeta_T \geqslant \widetilde{\zeta}_T = \widetilde{Y}_T$. Since $Y_{\tau+} > \widetilde{Y}_{\tau+}$ on E it is clear that the interval $]\!]\,\tau, \nu]\!]$ is nonempty on the event $E = \{\tau < T\} \in \mathcal{F}_\tau$. It is also worth noting that E belongs also to $\mathcal{F}_{\nu-}$ since $\tau < \nu$ on E (see Proposition 2.4 in Nikeghbali [35]).

Step 2 Our next goal is to show that the inequality $Y_\nu \geqslant \widetilde{Y}_\nu$ is satisfied. It manifestly holds on the event $\nu = T$ and thus it suffices to shown that it is valid on $\{\nu < T\}$ as well. If $Y_\nu \geqslant \widetilde{Y}_\nu$, then the desired inequality manifestly holds and thus it suffices to examine the event $\{Y_\nu < \widetilde{Y}_\nu,\, Y_{\nu+} \geqslant \widetilde{Y}_{\nu+}\}$. We will show by contradiction that the probability of that event is null. Since $Y_\nu = \zeta_\nu \vee Y_{\nu+}$ it suffices to consider two cases: (2.1) $Y_\nu > Y_{\nu+}$ and (2.2) $Y_\nu = Y_{\nu+}$.

In case (2.1), we have $Y_\nu = \zeta_\nu$ and thus $\widetilde{Y}_\nu > Y_\nu = \zeta_\nu \geqslant \widetilde{\zeta}_\nu$. This implies that $\widetilde{Y}$ is right-continuous at ν, which in turn yields $\widetilde{Y}_\nu = \widetilde{Y}_{\nu+} > Y_\nu > Y_{\nu+}$ and hence contradicts the assumption that $Y_\nu < \widetilde{Y}_\nu$.

In case (2.2), we have $Y_\nu = Y_{\nu+}$ and thus $\widetilde{Y}_\nu > Y_\nu = Y_{\nu+} \geqslant \widetilde{Y}_{\nu+}$, which implies $\widetilde{Y}_\nu = \widetilde{\zeta}_\nu > Y_\nu \geqslant \zeta_\nu$. Hence $\widetilde{\zeta}_\nu > \zeta_\nu$, which is a contradiction since, by assumption, the inequality $\widetilde{\zeta} \leqslant \zeta$ holds. We thus see that the inequality $Y_\nu \geqslant \widetilde{Y}_\nu$ is proven.

Step 3 We are now ready to show that if $\mathbb{P}(E) > 0$ then a contradiction arises. From the first step, we deduce that there exists a sufficiently small constant $\delta = \delta(\varepsilon) > 0$ such that $\mathbb{P}(C) := \mathbb{P}(\tau + \delta < \nu) > 0$ and $Y_{\tau+\delta} < \widetilde{Y}_{\tau+\delta}$. Hence we define the $\mathbb{F}$-stopping time $\sigma := (\tau + \delta)\mathbb{1}_E + T\mathbb{1}_{E^c}$ and we consider the interval $[\![\sigma, \nu]\!]$. We henceforth work on the event $E \in \mathcal{F}_\sigma$. As in Section 3.1, we denote

$$G_t(Y) := \int_{]0,t]} g(s, Y_s)\,\mathrm{d}A_s, \quad H_t(\widetilde{Y}) := \int_{]0,t]} h(s, \widetilde{Y}_s)\,\mathrm{d}A_s$$

and we also write $U := G(Y)$ and $\widetilde{U} := H(\widetilde{Y})$ so that the process $\overline{U} := U - \widetilde{U}$ satisfies

$$\overline{U}_t = G_t(Y) - H_t(\widetilde{Y}) = (G_t(Y) - G_t(\widetilde{Y})) + (G_t(\widetilde{Y}) - H_t(\widetilde{Y})).$$

Since $g(\omega, t, y) \geqslant h(\omega, t, y)$ for all $(\omega, t, y) \in \Omega \times [0, T] \times \mathbb{R}$, the inequality $Y < \widetilde{Y}$ holds on $[\![\sigma, \nu[\![$ and for every $(\omega, t) \in \Omega \times [0, T]$ the function $g(\omega, t, \cdot)$ is nonincreasing, it is easy to check that $\mathbb{1}_E \overline{U}$ is a continuous and nondecreasing process on $[\![\sigma, \nu]\!]$. We observe that on $[\![\sigma, \nu]\!]$

$$Y_t - \widetilde{Y}_t = Y_\nu - \widetilde{Y}_\nu - \int_{]t,\nu]} \mathrm{d}(M_s - \widetilde{M}_s) + (K_\nu - K_t) - (\widetilde{K}_\nu - \widetilde{K}_{\nu-}) + \overline{U}_\nu - \overline{U}_t,$$

where in fact the process $\widetilde{K}$ is constant on $[\![\sigma,\nu[\![$ since $\widetilde{Y} > Y \geqslant \zeta \geqslant \widetilde{\zeta}$ on $[\![\sigma,\nu[\![$ and thus $\widetilde{K}_t = \widetilde{K}_{\nu-}$ for $t \in [\![\sigma,\nu[\![$. Consequently,

$$Y_t - \widetilde{Y}_t = Y_\nu - \widetilde{Y}_\nu + \Delta K_\nu - \Delta\widetilde{K}_\nu - \int_{]t,\nu]} \mathrm{d}(M_s - \widetilde{M}_s) + (K_{\nu-} - K_t) + \overline{U}_\nu - \overline{U}_t.$$

By taking the conditional expectation with respect to $\mathcal{F}_\sigma$, we obtain

$$\begin{aligned} Y_\sigma - \widetilde{Y}_\sigma &\geqslant \mathbb{E}\left[Y_\nu - \widetilde{Y}_\nu + \Delta K_\nu - \Delta\widetilde{K}_\nu \,|\, \mathcal{F}_\sigma\right] \\ &= \mathbb{E}\left[(Y_\nu - \widetilde{Y}_\nu + \Delta K_\nu)\mathbb{1}_{\{\Delta\widetilde{K}_\nu = 0\}} \,|\, \mathcal{F}_\sigma\right] + \mathbb{E}\left[(Y_\nu - \widetilde{Y}_\nu + \Delta K_\nu - \Delta\widetilde{K}_\nu)\mathbb{1}_{\{\Delta\widetilde{K}_\nu > 0\}} \,|\, \mathcal{F}_\sigma\right] \\ &\geqslant \mathbb{E}\left[(Y_\nu - \widetilde{Y}_\nu + \Delta K_\nu - \Delta\widetilde{K}_\nu)\mathbb{1}_{\{\Delta\widetilde{K}_\nu > 0\}} \,|\, \mathcal{F}_\sigma\right], \end{aligned}$$

where in the last inequality we have used the facts that $Y_\nu - \widetilde{Y}_\nu \geqslant 0$ and $\Delta K_\nu \geqslant 0$. Next, we show that

$$\mathbb{E}\left[(Y_\nu - \widetilde{Y}_\nu + \Delta K_\nu - \Delta\widetilde{K}_\nu)\mathbb{1}_{\{\Delta\widetilde{K}_\nu > 0\}} \,|\, \mathcal{F}_\sigma\right] = 0.$$

We first notice that on the event $\{\Delta\widetilde{K}_\nu > 0\}$

$$(Y_\nu - \widetilde{Y}_\nu + \Delta K_\nu - \Delta\widetilde{K}_\nu)\mathbb{1}_{\{\Delta\widetilde{K}_\nu > 0\}} = \Delta(M_\nu - \widetilde{M}_\nu)\mathbb{1}_{\{\Delta\widetilde{K}_\nu > 0\}}$$

since it is easy to check that $Y_{\nu-} - \widetilde{Y}_{\nu-} = 0$ on $\{\Delta\widetilde{K}_\nu > 0\}$. The filtration $\mathbb{F}$ is assumed to be quasi-left-continuous and thus the equality $\Delta M_\tau = 0$ holds for any $\mathbb{F}$-predictable stopping time τ, as shown in Theorem 5.39 in [27]. The process $\widetilde{K}$ is strongly $\mathbb{F}$-predictable and thus, by Theorem 3. 33 in [27], the set $\{\Delta\widetilde{K} > 0\} = \{\Delta\widetilde{K}^d > 0\}$ is included in the union of the graphs of a family of $\mathbb{F}$-predictable stopping times. Hence we can conclude that $\Delta M \mathbb{1}_{\{\Delta\widetilde{K} > 0\}} = 0$ so that $Y_\sigma \geqslant \widetilde{Y}_\sigma$. However, $Y_\sigma - \widetilde{Y}_\sigma < 0$ on $E \in \mathcal{F}_\sigma$ and thus $Y_\sigma = \widetilde{Y}_\sigma$ on E, which clearly contradicts the definition of σ. □

4.2 A priori estimates for solutions to reflected GBSDEs

Similar to Section 3.2, we make additional assumptions about the lower obstacle ζ and the generator g of the RGBSDE (1.2).

Assumption 4.4 *Let the processes ζ and A be bounded and the mapping $g : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfy the following conditions:*

(i) *the uniform Lipschitz condition: there exists a constant $L > 0$ such that the inequality $|g(t,y) - g(t,y')| \leqslant L|y - y'|$ holds for every $t \in [0,T]$ and $y, y' \in \mathbb{R}$;*
(ii) *the inequality*

$$\mathbb{E}\left[\int_{]0,T]} |g(t,0)|^2 \,\mathrm{d}A_t\right] < \infty.$$

We now deal with *a priori* estimates for solutions to the reflected GBSDE.

Proposition 4.5 *For $i = 1,2$, let $(Y^i, M^i, K^i) \in \mathcal{S}^2 \times \mathcal{M}^2_0 \times \overline{\mathcal{K}}$ be a solution to the reflected GBSDE* (1.2) *with data (A, ζ_T, g^i, ζ) satisfying Assumptions 4.1 and 4.4. Then for every $\beta > 0$ there exists a constant $c > 0$ such that the processes $\widehat{Y} := Y^1 - Y^2$ and $\widehat{M} := M^1 - M^2$ satisfy*

$$\mathbb{E}\bigg[\sup_{t\in[0,T]} e^{\beta A_t}|\widehat{Y}_t|^2 + \int_{]0,T]} e^{\beta A_s}\,\mathrm{d}[\widehat{M}]_s\bigg] \leqslant c\,\mathbb{E}\bigg[\int_{]0,T]} e^{\beta A_s}|\bar{g}_s|^2\,\mathrm{d}A_s\bigg],$$

where $\bar{g}_s := g^1(s,Y_s^1) - g^2(s,Y_s^2)$. *In addition, if* $g^1(\omega,s,\cdot)$ *is Lipschitz continuous then there exists a constant* $C>0$ *such that*

$$\mathbb{E}\bigg[\sup_{t\in[0,T]} e^{\beta A_t}|\widehat{Y}_t|^2 + \int_{]0,T]} e^{\beta A_s}\,\mathrm{d}[\widehat{M}]_s\bigg] \leqslant C\,\mathbb{E}\bigg[\int_{]0,T]} e^{\beta A_s}|\widehat{g}_s|^2\,\mathrm{d}A_s\bigg],$$

where $\widehat{g}_s := g^1(s,Y_s^2) - g^2(s,Y_s^2)$.

Proof The proof is similar to the proof of Proposition 3.9. Again we set $e_t = e^{\beta A_t}$ and since $\beta>0$ we have that $de_t := \beta e_t\,\mathrm{d}A_t$ and $1 \leqslant e_t \leqslant e^{\beta c_A}$ for every $t\in[0,T]$. Denote $\widehat{K} = K^1 - K^2$ and recall that K has a unique decomposition $\widehat{K} = \widehat{K}^r + \widehat{K}^g$ where $\widehat{K}^r$ is an $\mathbb{F}$-adapted, càdlàg, nondecreasing process of finite variation and $\widehat{K}^g$ is an $\mathbb{F}$-adapted, càglàd, purely discontinuous, nondecreasing process of finite variation. Hence, by setting $\bar{g}_s := g^1(s,Y_s^1) - g^2(s,Y_s^2)$, we obtain

$$\widehat{Y}_\tau = \int_{]\tau,T]} \bar{g}_s\,\mathrm{d}A_s - (\widehat{M}_T - \widehat{M}_\tau) + \widehat{K}_T^r - \widehat{K}_\tau^r + \widehat{K}_T^g - \widehat{K}_\tau^g.$$

By applying the Gal'čuk-Lenglart formula (see, e.g., Theorem 8.2 in Gal'čuk [18]) to $e_t|\widehat{Y}_t|^2$ and the Young inequality with a constant $\alpha>0$, we obtain

$$\begin{aligned}
& e_t|\widehat{Y}_t|^2 + \int_{]t,T]} e_s\,\mathrm{d}[\widehat{M}]_s \\
&= -\beta\int_{]t,T]} e_s|\widehat{Y}_s|^2\,\mathrm{d}A_s + 2\int_{]t,T]} e_s\widehat{Y}_s\bar{g}_s\,\mathrm{d}A_s - 2\int_{]t,T]} e_s\widehat{Y}_{s-}\,\mathrm{d}\widehat{M}_s + 2\int_{]t,T]} e_s\widehat{Y}_{s-}\,\mathrm{d}K_s^r \\
&\quad + 2\int_{[t,T[} e_s\widehat{Y}_s\,\mathrm{d}\widehat{K}_{s+}^g - \sum_{t<s\leqslant T} e_s(\widehat{Y}_s - \widehat{Y}_{s-})^2 - \sum_{t\leqslant s<T} e_s(\widehat{Y}_{s+} - \widehat{Y}_s)^2 \\
&\leqslant (\alpha^{-1}-\beta)\int_{]t,T]} e_s|\widehat{Y}_s|^2\,\mathrm{d}A_s + \alpha\int_{]t,T]} e_s|\bar{g}_s|^2\,\mathrm{d}A_s - 2\int_{]t,T]} e_s\widehat{Y}_{s-}\,\mathrm{d}M_s + 2\int_{]t,T]} e_s\widehat{Y}_{s-}\,\mathrm{d}\widehat{K}_s^r \\
&\quad + 2\int_{[t,T[} e_s\widehat{Y}_s\,\mathrm{d}\widehat{K}_{s+}^g - \sum_{t<s\leqslant T} e_s(\widehat{Y}_s - \widehat{Y}_{s-})^2 - \sum_{t\leqslant s<T} e_s(\widehat{Y}_{s+} - \widehat{Y}_s)^2.
\end{aligned} \tag{4.2}$$

Next we show that $\int_{]t,T]} e_s\widehat{Y}_{s-}\,\mathrm{d}\widehat{K}_s^r \leqslant 0$ and $\int_{[t,T[} e_s\widehat{Y}_s\,\mathrm{d}\widehat{K}_{s+}^g \leqslant 0$ for every $t\in[0,T]$. We note that $\widehat{K}^r = (K^{1,c} - K^{2,c}) + (K^{1,d} - K^{2,d})$. From the Skorokhod conditions in (4.1) we obtain, for all $s\in[0,T]$,

$$\begin{aligned}
\widehat{Y}_s\,\mathrm{d}K_s^{1,c} &= (Y_s^1 - \zeta_s)\,\mathrm{d}K_s^{1,c} - (Y_s^2 - \zeta_s)\,\mathrm{d}K_s^{1,c} \\
&\leqslant (Y_s^1 - \bar{\zeta}_s)\,\mathrm{d}K_s^{1,c} - (Y_s^2 - \zeta_s)\,\mathrm{d}K_s^{1,c} = -(Y_s^2 - \zeta_s)\,\mathrm{d}K_s^{1,c} \leqslant 0,
\end{aligned}$$

where the last inequality holds since the process $K^{1,c}$ is nondecreasing. By symmetry, we obtain $\widehat{Y}_s\,\mathrm{d}K_s^{2,c} \geqslant 0$. Furthermore, for any $s\in[0,T]$,

$$\begin{aligned}
\widehat{Y}_{s-}\Delta K_s^{1,d} &= (Y_{s-}^1 - \zeta_s)\Delta K_s^{1,d} - (Y_{s-}^2 - \zeta_s)\Delta K_s^{1,d} \\
&\leqslant (\widehat{Y}_{s-}^1 - \bar{\zeta}_s)\Delta K_s^{1,d} - (Y_{s-}^2 - \zeta_s)\Delta K_s^{1,d} = -(Y_{s-}^2 - \zeta_s)\Delta K_s^{1,d} \leqslant 0
\end{aligned}$$

and $\widehat{Y}_{s-}\Delta K_s^{2,d} \geqslant 0$. Similarly, $\int_{[t,T[} e_s\widehat{Y}_s\,\mathrm{d}\widehat{K}_{s+}^g = \sum_{t\leqslant s<T} e_s\widehat{Y}_s\Delta^+\widehat{K}_s^g$ for all $s\in[0,T]$. Note that

$$\widehat{Y}_s\Delta^+\widehat{K}_s^g = (Y_s^1 - Y_s^2)\Delta^+K_s^{1,g} - (Y_s^1 - Y_s^2)\Delta^+K_s^{2,g}$$

and, for all $s\in[0,T]$,

$$(Y^1_s - Y^2_s)\Delta^+ K^{1,g}_s = (Y^1_s - \zeta_s)\Delta^+ K^{1,g}_s - (Y^2_s - \zeta_s)\Delta^+ K^{1,g}_s = -(Y^2_s - \zeta_s)\Delta^+ K^{1,g}_s \leqslant 0$$

and $(Y^1_s - Y^2_s)\Delta^+ K^{2,g}_s \geqslant 0$. Hence (4.2) gives, for any $\beta > \alpha^{-1}$ and $t \in [0,T]$,

$$e_t|\widehat{Y}_t|^2 + \int_{]t,T]} e_s \,\mathrm{d}[\widehat{M}]_s \leqslant \alpha \int_{]t,T]} e_s|\bar{g}_s|^2 \,\mathrm{d}A_s - 2\int_{]t,T]} e_s \widehat{Y}_{s-} \,\mathrm{d}\widehat{M}_s \tag{4.3}$$

and thus, by taking the expectation on both sides, we obtain

$$\mathbb{E}\Big[\int_{]t,T]} e_s \,\mathrm{d}[\widehat{M}]_s\Big] \leqslant \alpha\,\mathbb{E}\Big[\int_{]t,T]} e_s|\bar{g}_s|^2 \,\mathrm{d}A_s\Big]. \tag{4.4}$$

In addition, taking the essential supremum and expectation in (4.3) gives

$$\mathbb{E}\Big[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}} e_\tau|\widehat{Y}_\tau|^2\Big] \leqslant \mathbb{E}\Big[\alpha \int_{]0,T]} e_s|\bar{g}_s|^2 \,\mathrm{d}A_s\Big] + 2\,\mathbb{E}\Big[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}}\Big|\int_{]0,\tau]} e_\tau \widehat{Y}_{\tau-}\,\mathrm{d}\widehat{M}_\tau\Big|\Big]. \tag{4.5}$$

An application of the Burkholder–Davis–Gundy inequality with $p=1$ similar to (3.16) yields

$$2\,\mathbb{E}\Big[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}}\Big|\int_{]0,\tau]} e_s \widehat{Y}_{s-}\,\mathrm{d}\widehat{M}_s\Big|\Big] \leqslant \frac{1}{4}\,\mathbb{E}\Big[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}} e_\tau|\widehat{Y}_\tau|^2\Big] + 4c_1^2\,\mathbb{E}\Big[\int_{]0,T]} e_s \,\mathrm{d}[\widehat{M}]_s\Big], \tag{4.6}$$

where the constant c_1 is independent of α,β and the last inequality holds since $2ab \leqslant a^2 + b^2$ for all real numbers a,b. By combining (4.4), (4.5) and (4.6), we obtain

$$\frac{3}{4}\,\mathbb{E}\Big[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}} e_\tau|\widehat{Y}_\tau|^2\Big] \leqslant (1+4c_1^2)\alpha\,\mathbb{E}\Big[\int_{]0,T]} e_s|\bar{g}_s|^2 \,\mathrm{d}A_s\Big],$$

and, finally,

$$\mathbb{E}\Big[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}} e_\tau|\widehat{Y}_\tau|^2 + \int_{]0,T]} e_s \,\mathrm{d}[\widehat{M}]_s\Big] \leqslant c\,\mathbb{E}\Big[\int_{]0,T]} e_s|\bar{g}_s|^2 \,\mathrm{d}A_s\Big],$$

where $c = \frac{7+16c_1^2}{3}\alpha$. Assume that the generator g^1 is Lipschitz continuous with a constant L. Then

$$\begin{aligned}|\bar{g}_s|^2 &= |g^1(s,Y^1_s) - g^1(s,Y^2_s) + g^1(s,Y^2_s) - g^2(s,Y^2_s)|^2\\ &\leqslant 2|g^1(s,Y^1_s) - g^1(s,Y^2_s)|^2 + 2|g^1(s,Y^2_s) - g^2(s,Y^2_s)|^2\\ &\leqslant 2L^2|Y^1_s - Y^2_s|^2 + 2|\widehat{g}_s|^2,\end{aligned}$$

where we denote $\widehat{g}_s = g^1(s,Y^2_s) - g^2(s,Y^2_s)$. Consequently, in (4.2) we get

$$(\alpha^{-1}-\beta)\int_{]t,T]} e_s|\widehat{Y}_s|^2 \,\mathrm{d}A_s + \alpha\int_{]t,T]} e_s|\bar{g}_s|^2 \,\mathrm{d}A_s \leqslant c_{\alpha,\beta,L}\int_{]t,T]} e_s|\widehat{Y}_s|^2 \,\mathrm{d}A_s + 2\alpha\int_{]t,T]} e_s|\widehat{g}_s|^2 \,\mathrm{d}A_s,$$

where $c_{\alpha,\beta,L} = (\alpha^{-1} + 2\alpha L^2 - \beta)$. Then, by arguing as in the proof of Proposition 4.3, we obtain the following modification of (4.3), which is valid for every $\beta > \alpha^{-1} + 2\alpha L^2$

$$e_t|\widehat{Y}_t|^2 + \int_{]t,T]} e_s \,\mathrm{d}[\widehat{M}]_s \leqslant \alpha \int_{]t,T]} e_s|\widehat{g}_s|^2 \,\mathrm{d}A_s - 2\int_{]t,T]} e_s \widehat{Y}_{s-} \,\mathrm{d}\widehat{M}_s.$$

By taking the conditional expectation of both sides with respect to $\mathcal{F}_t$ we get

$$e_t|\widehat{Y}_t|^2 + \mathbb{E}\Big[\int_{]0,T]} e_s \,\mathrm{d}[\widehat{M}]_s \,\Big|\, \mathcal{F}_t\Big] \leqslant \mathbb{E}\Big[\int_{]t,T]} e_s|\widehat{g}_s|^2 \,\mathrm{d}A_s \,\Big|\, \mathcal{F}_t\Big],$$

which holds for a sufficiently large β. □

We note that if, in addition, the nondecreasing process A is bounded, we obtain the existence of a constant c such that for all $t \in [0,T]$

$$|\widehat{Y}_t|^2 \leqslant c\,\mathbb{E}\left[\sup_{s\in[t,T]} |\widehat{g}_s|^2 \,\middle|\, \mathcal{F}_t\right].$$

In particular, if $|g^1(s,y) - g^2(s,y)| \leqslant \varepsilon$ for all (s,y), then $\sup_{t\in[0,T]} |Y_t^1 - Y_t^2| \leqslant c\varepsilon$ for every $t \in [0,T]$.

4.3 Existence and uniqueness of solutions to reflected GBSDEs

We now examine the existence and uniqueness of a solution (Y, M, K) to the reflected GBSDE (1.2) under Assumptions 4.1 and 4.4. As customary in the theory of BSDES, we first establish the existence and uniqueness of a solution to the reflected GBSDE (1.2) when the generator g does not depend on Y using the theory of the classical optimal stopping problem with irregular ($\mathbb{F}$-optional) payoff and a general filtration, as presented in Section 3 of Grigorova et al. [23]. The following lemma is based on Lemmas 3.2, 3.3, and 3.5 in [23] (see also Lemma 3.3 in [21]) but with slightly modified proof since we consider reflected GBSDE with a generalized driver. For completeness, we sketch here the main arguments and we refer the interested reader to [21] and [23] for a more detailed presentation, in particular, a thorough analysis of Skorokhod conditions in the second step of the proof of Lemma 4.6. Finally, we mention that similar well-posedness results can be found in the recent work of Possamaï and Rodrigues [40].

Lemma 4.6 *Let Assumptions 4.1 and 4.4 hold and $g(\omega,s,y) = g(\omega,s) := g_s(\omega)$ be a fixed process. Then the reflected GBSDE* (1.2) *with data (A, ζ_T, g, ζ) admits a unique solution (Y, M, K) and it belongs to $\mathcal{S}^2 \times \mathcal{M}_0^2 \times \overline{\mathcal{K}}$.*

Proof This is a direct application of Lemma 3.2 in [23]. □

We are in a position to prove the existence and uniqueness of the solution to reflected GBSDE (1.2) with a general generator g satisfying Assumption 4.4. For analogous results and a similar proof, we refer to Theorem 3.4 in Grigorova et al. [21] and Theorem 4.1 in Grigorova et al. [23].

Proposition 4.7 *Let Assumptions 4.1 and 4.4 hold. If the obstacle ζ is nonnegative, then the reflected GBSDE* (1.2) *with data (A, ζ_T, g, ζ) has a unique solution $(Y, M, K) \in \mathcal{S}^2 \times \mathcal{M}_0^2 \times \overline{\mathcal{K}}$.*

Proof We will extend the proof of Proposition 3.12. Note that $Y \in \mathcal{S}^2$ in (1.2) is not necessarily right-continuous and we equipped $\mathcal{S}^2$ with the norm $||Y||^2_{\mathcal{S}^2_\beta} = \mathbb{E}[\operatorname{ess\,sup}_{\tau\in\mathcal{T}} e^{\beta A_\tau} |Y_\tau|^2]$ under which $\mathcal{S}^2$ is still a Banach space. We observe that for $\beta > 0$ the norms $||\cdot||_{\mathcal{S}^2}$ and $||\cdot||_{\mathcal{S}^2_\beta}$ are equivalent on $\mathcal{S}^2$ and we denote by $\mathcal{S}^2_\beta$ the space $\mathcal{S}^2$ endowed with the norm $||\cdot||_{\mathcal{S}^2_\beta}$.

Let the mapping $\Phi : \mathcal{S}^2_\beta \to \mathcal{S}^2_\beta$ be defined as follows: for any given $w \in \mathcal{S}^2_\beta$ we set $\Phi(w) := Y^w$ where the triplet (Y^w, M^w, K^w) is a unique solution to the reflected GBSDE (see Lemma 4.6)

$$Y_t^w = \zeta_T + \int_{]t,T]} g(s, w_s)\,\mathrm{d}A_s - (M_T^w - M_t^w) + K_T^w - K_t^w, \tag{4.7}$$

where $Y^w \geqslant \zeta$ and the fixed generator is independent of M^w.

We wish to demonstrate that there exists a unique process $\widehat{w} \in \mathcal{S}^2_\beta$ such that $\Phi(\widehat{w}) = \widehat{w}$. Then the corresponding process $\widehat{m} \in \mathcal{M}_0^2$ and $\widehat{k} \in \overline{\mathcal{K}}$ can be determined by (4.7), that is, we would obtain

$$\widehat{w}_t = \zeta_T + \int_{]t,T]} g(s, \widehat{w}_t)\,\mathrm{d}A_s - (\widehat{m}_T - \widehat{m}_t) + \widehat{k}_T - \widehat{k}_t$$

with $\widehat{k}_0 = 0$ and the nondecreasing process $\widehat{k}$ satisfies the Skorokhod conditions in (4.1). It is suffice to show that the mapping $\Phi : \mathcal{S}^2_\beta \to \mathcal{S}^2_\beta$ is a contraction for a sufficiently large β.

We take $w', w'' \in \mathcal{S}^2_\beta$ and denote $Y^{w'} = \Phi(w')$ and $Y^{w''} = \Phi(w'')$. For the simplicity of notation, we write $y := Y^{w'} - Y^{w''} = \Phi(w') - \Phi(w'')$, $m := M^{w'} - M^{w''}$, $k := K' - K''$ and $w := w' - w''$. It is clear from (3.19) that y satisfies the reflected GBSDE

$$y_t = \int_{]t,T]} (g(s, w'_s) - g(s, w''_s))\, \mathrm{d}A_s - (m_T - m_t) + k_T - k_t,$$

where $|g(s, w'_s) - g(s, w''_s)| \leqslant L|w_s|$ since $g(s, \cdot)$ is Lipschitz continuous with a constant L. By applying Proposition 4.5, we obtain

$$\mathbb{E}\bigg[\operatorname*{ess\,sup}_{\tau \in \mathcal{T}} e^{\beta A_\tau} |w_\tau|^2 + \int_{]0,T]} e^{\beta A_s}\, \mathrm{d}[m]_s \bigg] \leqslant \alpha L^2 \frac{7 + 16c_1^2}{3}\, \mathbb{E}\bigg[\int_{]0,T]} e^{\beta A_s} |w_s|^2\, \mathrm{d}A_s \bigg],$$

where $\alpha > 0$, $\beta > \frac{1}{\alpha}$ and a constant $c_1 > 0$ is independent of α, β, L and thus

$$\mathbb{E}\bigg[\operatorname*{ess\,sup}_{\tau \in \mathcal{T}} e^{\beta A_\tau} |w_\tau|^2 + \int_{]0,T]} e^{\beta A_s}\, \mathrm{d}[m]_s \bigg] \leqslant \alpha c_A L^2 (T+1) \frac{7 + 16c_1^2}{3}\, \mathbb{E}\bigg[\operatorname*{ess\,sup}_{\tau \in \mathcal{T}} e^{\beta A_\tau} |w_\tau|^2 \bigg],$$

where $c_A > 0$ is such that $A_T \leqslant c_A$. Thus we conclude that Φ is a contraction when $0 < \alpha < c_A^{-1} L^{-2} \frac{3}{(7+16c_1^2)(T+1)}$ and $\beta > \alpha^{-1}$ and thus there exists a unique solution (Y, M, K) in $\mathcal{S}^2 \times \mathcal{M}^2_0 \times \overline{\mathcal{K}}$ to the reflected GBSDE (1.2). □

4.4 Comparison theorem for reflected GBSDEs with bounded driver

In the second comparison theorem we relax the assumption that the filtration $\mathbb{F}$ is quasi-left-continuous, which was made in Proposition 4.3. However, the method of the proof of Proposition 4.8 requires to postulate that the driver A is bounded. We denote the Doléans exponential of a semimartingale X by $\mathcal{E}(X)$ (see Theorem 5.1 in [20]) and we write $\mathcal{E}_{s,t}(X) = \mathcal{E}_t(X)/\mathcal{E}_s(X)$ for every $s \leqslant t$.

Proposition 4.8 *Let Assumptions 4.1 and 4.4 be satisfied by the mappings* $g, h : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ *so that the reflected GBSDEs with the lower obstacles* ζ *and* $\widetilde{\zeta}$

$$Y_\tau = \zeta_T + \int_{]\tau,T]} g(s, Y_s)\, \mathrm{d}A_s - (M_T - M_\tau) + K_T - K_\tau$$

and

$$\widetilde{Y}_\tau = \widetilde{\zeta}_T + \int_{]\tau,T]} h(s, \widetilde{Y}_s)\, \mathrm{d}A_s - (\widetilde{M}_T - \widetilde{M}_\tau) + \widetilde{K}_T - \widetilde{K}_\tau$$

have unique solutions (Y, M, K) *and* $(\widetilde{Y}, \widetilde{M}, \widetilde{K})$ *in* $\mathcal{S}^2 \times \mathcal{M}^2_0 \times \overline{\mathcal{K}}$. *Suppose, in addition, that the mappings* g *and* h *satisfy the following conditions:*

(i) $g(\omega, t, y) \geqslant h(\omega, t, y)$ *for every* $(\omega, t, y) \in \Omega \times [0,T] \times \mathbb{R}$,

(ii) $g(\omega, t, \cdot)$ *is a nonincreasing function for every* $(\omega, t) \in \Omega \times [0,T]$.

If the obstacles satisfy $\zeta \geqslant \widetilde{\zeta}$ *and are right-upper-semicontinuous, then the inequality* $Y \geqslant \widetilde{Y}$ *is valid.*

Proof Step 1 We first show that the process Y given by

$$Y_\tau = \zeta_T + \int_{]\tau,T]} g(s, Y_s)\, \mathrm{d}A_s - (M_T - M_\tau) + K_T - K_\tau$$

is a strong $\mathcal{E}^g$-supermartingale where the nonlinear evaluation $\mathcal{E}^g$ (see, e.g., Peng [38, 39]) is defined through the solution to the GBSDE

$$Y'_\tau = \zeta_T + \int_{]\tau,T]} g(s, Y'_s)\, \mathrm{d}A_s - (M'_T - M'_\tau). \tag{4.8}$$

We fix $\sigma \in \mathcal{T}$ and we denote $\overline{Y}_s := Y_s - \mathcal{E}^g_{s,\sigma}(Y_\sigma)$, $\bar{M} := M - M'$ and $\overline{g}_s := g(s, Y_s) - g(s, Y'_s)$. We define $\rho_s := (\overline{g}_s/\overline{Y}_s)\mathbb{1}_{\{\overline{Y}_s \neq 0\}}$ and we note that the process ρ is clearly bounded since the generator g is a Lipschitz continuous function in y. Therefore, the continuous process $X_t := \int_{]0,t]} \rho_s \, \mathrm{d}A_s$ is well defined. The integration by parts formula yields, for every $\mathbb{F}$-stopping times $\tau \leqslant \sigma$,

$$\begin{aligned}\mathcal{E}_{\tau,\tau}(X)\overline{Y}_\tau = & \int_{]\tau,\sigma]} \mathcal{E}_{\tau,s-}(X)\overline{g}_s \, \mathrm{d}A_s - \int_{]\tau,\sigma]} \mathcal{E}_{\tau,s}(X)\rho_s\overline{Y}_s \, \mathrm{d}A_s \\ & + \int_{]\tau,\sigma]} \mathcal{E}_{\tau,s}(X) \, \mathrm{d}K^r_s + \int_{[\tau,\sigma[} \mathcal{E}_{\tau,s}(X) \, \mathrm{d}K^g_{s+} - \int_{]\tau,\sigma]} \mathcal{E}_{\tau,s}(X) \, \mathrm{d}\bar{M}_s.\end{aligned}$$

Since the processes ρ and A are bounded the stochastic exponential $\mathcal{E}(X)$ belongs to $\mathcal{S}^2$ and thus, by the Kunita–Watanabe inequality, the stochastic integral with respect to M is a uniformly integrable martingale. By taking the $\mathcal{F}_\tau$-conditional expectation and using the assumption that $\overline{g}_\tau = \rho_\tau \overline{Y}_\tau$, we obtain

$$\overline{Y}_\tau = \mathcal{E}_{\tau,\tau}(X)\overline{Y}_\tau = \mathbb{E}\left[\int_{]\tau,T]} \mathcal{E}_{\tau,s-}(X) \, \mathrm{d}K^r_s + \int_{[\tau,T[} \mathcal{E}_{\tau,s-}(X) \, \mathrm{d}K^g_{s+} \,\Big|\, \mathcal{F}_\tau\right] \geqslant 0,$$

which shows that $\overline{Y}_\tau \geqslant 0$ and thus $Y_\tau \geqslant \mathcal{E}^g_{\tau,\sigma}(Y_\sigma)$. We have thus shown that Y is a strong $\mathcal{E}^g$-supermartingle.

Step 2 Our next goal is to show Y can be characterized as the value process for a nonlinear optimal stopping problem associated with $\mathcal{E}^g$ and ζ. Since Y is a strong $\mathcal{E}^g$-supermartingale and the nonlinear evaluation $\mathcal{E}^g$ has the monotonicity property we have that $Y_\tau \geqslant \mathcal{E}^g_{\tau,\sigma}(Y_\sigma) \geqslant \mathcal{E}^g_{\tau,\sigma}(\zeta_\sigma)$ for all $\mathbb{F}$-stopping times $\tau \leqslant \sigma$, which in turn implies that

$$Y_\tau \geqslant \sup_\sigma \mathcal{E}^g_{\tau,\sigma}(\zeta_\sigma).$$

To show the reverse inequality, we fix $\tau \in \mathcal{T}$ and we define the $\mathbb{F}$-stopping time $\sigma^\varepsilon_t := \inf\{s \geqslant \tau : Y_s \leqslant \zeta_s + \varepsilon\}$. Since the obstacle ζ is upper-semicontinuous, using similar techniques to those in Section 4.5, one can show that $Y_{\sigma^\varepsilon_\tau} \leqslant \zeta_{\sigma^\varepsilon_\tau} + \varepsilon$ and Y is an $\mathcal{E}^g$-martingale on $[\![\tau, \sigma^\varepsilon_\tau]\!]$, that is, Y is the solution to (4.8) on $[\![\tau, \sigma^\varepsilon_\tau]\!]$ with the terminal condition $Y_{\sigma^\varepsilon_\tau}$. Using first the monotonicity property of $\mathcal{E}^g$ stemming from Proposition 3.6 and then Proposition 3.10 with $g^1 = g^2 = g$, we deduce that, for arbitrary $\varepsilon > 0$,

$$Y_\tau = \mathcal{E}^g_{\tau,\sigma^\varepsilon_\tau}(Y_{\sigma^\varepsilon_\tau}) \leqslant \mathcal{E}^g_{\tau,\sigma^\varepsilon_\tau}(\zeta_{\sigma^\varepsilon_t} + \varepsilon) \leqslant \mathcal{E}^g_{\tau,\sigma^\varepsilon_\tau}(\zeta_{\sigma^\varepsilon_t}) + C\varepsilon \leqslant \sup_{\sigma \in \mathcal{T}_{\tau,T}} \mathcal{E}^g_{\tau,\sigma}(\zeta_\sigma) + C\varepsilon.$$

We have thus shown that for every $\tau, \sigma \in \mathcal{T}$

$$Y_\tau = \sup_{\sigma \in \mathcal{T}_{\tau,T}} \mathcal{E}^g_{\tau,\sigma}(\zeta_\sigma),$$

which means that Y is the value process for a nonlinear optimal stopping problem associated with the nonlinear evaluation $\mathcal{E}^g$ and the reward process ζ.

Step 3 Suppose that $g \geqslant h$ and the mapping g is nonincreasing in y. Then from the comparison property of solutions to a GBSDE established in Proposition 3.6, we deduce that, for every $\tau \leqslant \sigma$,

$$\mathcal{E}^h_{\tau,\sigma}(\widetilde{\zeta}_\sigma) \leqslant \mathcal{E}^g_{\tau,\sigma}(\zeta_\sigma)$$

and the asserted inequality $Y \geqslant \widetilde{Y}$ now follows by taking the supremum over all stopping times from $\mathcal{T}$. □

The following corollary holds under the assumptions of either Proposition 4.3 or Proposition 4.8. In particular, it is either assumed that the filtration $\mathbb{F}$ is quasi-left-continuous and the driver A is square-integrable or the filtration $\mathbb{F}$ is arbitrary and A is bounded.

Corollary 4.9 *Let the assumptions of either Proposition 4.3 or Proposition 4.8 be valid. Let the mapping $f:\Omega\times[0,T]\times\mathbb{R}\to\mathbb{R}_+$ be nonnegative (resp., nonpositive) and such that $f(\omega,t,\cdot)$ is a nonincreasing function for every $(\omega,t)\in\Omega\times[0,T]$. For every $n\in\mathbb{N}$, let the triplet (Y^n,M^n,K^n) be a unique solution to the reflected GBSDE*

$$Y^n_\tau=\zeta_T+\int_{]\tau,T]} nf(s,Y^n_s)\,\mathrm{d}A_s-(M^n_T-M^n_\tau)+K^n_T-K^n_\tau$$

with the lower obstacle ζ. Then the inequality $Y^{n+1}\geqslant Y^n$ (resp., $Y^{n+1}\leqslant Y^n$) holds for every $n\in\mathbb{N}$.

Proof It suffices to fix $n\in\mathbb{N}$ and apply Proposition 4.3 (or Proposition 4.8) to $g(t,y)=(n+1)f(t,y)$ and $h(t,y)=nf(t,y)$ where f is a nonnegative (resp., nonpositive) mapping. It is obvious that $g(\omega,t,\cdot)$ is a nonincreasing function for every $(\omega,t)\in\Omega\times[0,T]$ and $g(\omega,t,y)\geqslant h(\omega,t,y)$ (resp., $g(\omega,t,y)\leqslant h(\omega,t,y)$) since f is a nonnegative (resp., nonpositive) function. Hence Proposition 4.3 (or Proposition 4.8) implies that $Y^{n+1}\geqslant Y^n$ (resp., $Y^{n+1}\leqslant Y^n$) for every $n\in\mathbb{N}$. □

We will later apply Corollary 4.9 to two variants of penalization theorem for the reflected GBSDEs with the mappings $f,\tilde f:\Omega\times[0,T]\times\mathbb{R}\to\mathbb{R}_+$ given by $f(\omega,t,y)=(\eta_t(\omega)-y)^+$ and $\tilde f(\omega,t,y)=-(y-\eta_t(\omega))^+$, respectively, where η is a predetermined $\mathbb{F}$-optional process. Then the functions $g(\omega,t,\cdot):=nf(\omega,t,\cdot)$ and $h(\omega,t,\cdot):=(n+1)f(\omega,t,\cdot)$ nonnegative, nonincreasing and such that $g\geqslant h$. Similarly, the functions $\tilde g(\omega,t,\cdot):=n\tilde f(\omega,t,\cdot)$ and $\tilde h(\omega,t,\cdot):=(n+1)\tilde f(\omega,t,\cdot)$ are nonpositive, nonincreasing and such that $g\leqslant h$.

4.5 Optimal stopping problem and penalization scheme for RGBSDE

To formulate penalization schemes for the reflected GBSDE, we recall that $\overline{S}=S^r\cup\{T\}$ where $S^r=S^r(A)$ is the right support of the process A (see, however, Remark 3.17). We assume that we are given two processes, denoted as ζ and η, and we define the reward process γ by

$$\gamma_t:=\zeta_T\mathbb{1}_{\{t=T\}}+(\zeta\vee\eta\mathbb{1}_{\overline{S}})_t\mathbb{1}_{\{t<T\}}. \tag{4.9}$$

The optimal stopping problem introduced in Definition 4.10 is classical, in the sense that the set of all available stopping times is unrestricted, but it is also unusual since the reward process explicitly depends on the right support of the measure generated by the nondecreasing process A. It is worth noting that the formulation of the optimal stopping problem in Definition 4.10 is motivated by an important application to the problem of finding the *reduced upper price* (i.e., the pre-default seller's price) of a vulnerable American option in an incomplete market model. The interested reader is referred to Theorem 5.5 in [32].

Definition 4.10 *The process Y is the value process of the optimal stopping problem with data $(\zeta,\eta,\overline{S},\mathcal{T})$ if the following equality holds, for every $t\in[0,T]$,*

$$Y_t=\operatorname*{ess\,sup}_{\sigma\in\mathcal{T}_{t,T}}\mathbb{E}\left[\gamma_\sigma\,|\,\mathcal{F}_t\right], \tag{4.10}$$

where the reward process γ is given by equation (4.9).

The first penalization scheme is analogous to the case of the penalization scheme for the GBSDEs examined in Section 3.5. Similar to Theorem 3.19, we show in Theorem 4.11 that the process $Y=\lim_{n\to\infty} Y^n$ can be interpreted as the value process of the classical optimal stopping problem with the reward process γ. We henceforth postulate that Assumptions 4.1 and 4.4 are satisfied with a bounded process A. Then we have the following result on a relationship between the optimal stopping problem with data $(\zeta,\eta,\overline{S},\mathcal{T})$ and penalization scheme (4.11) for reflected GBSDEs.

Theorem 4.11 *Let Assumptions 4.1 and 4.4 be satisfied and the $\mathbb{F}$-optional and bounded process ζ (resp., the $\mathbb{F}$-optional and bounded process η) be right-upper-semicontinuous (resp., right-continuous). Consider the sequence of unique solutions (Y^n,M^n,K^n) to the reflected GBSDEs*

$$Y^n_\tau=\zeta_T+\int_{]\tau,T]} n(\eta_s-Y^n_s)^+\,\mathrm{d}A_s-(M^n_T-M^n_t)+K^n_T-K^n_\tau, \tag{4.11}$$

where an $\mathbb{F}$-adapted, làdlàg, nondecreasing process K^n satisfies the Skorokhod conditions with the lower obstacle ζ. Then the sequence Y^n converges monotonically to the process Y given by (4.10). *In addition, the triplet $(Y,M,K)=\lim_{n\to\infty}(Y^n,M^n,K^n)$ is a unique solution to the reflected BSDE*

$$Y_t=\zeta_T-(M_T-M_t)+K_T-K_t, \tag{4.12}$$

where $Y\geqslant\gamma$ and an $\mathbb{F}$-adapted, làdlàg, nondecreasing process K satisfies the Skorokhod conditions with the lower obstacle γ.

Proof We start by noticing that the existence of a unique solution $(Y^n,M^n,K^n)\in\mathcal{S}^2\times\mathcal{M}^2_0\times\overline{\mathcal{K}}$ to the reflected GBSDE (4.11) follows from Proposition 4.7. Furthermore, the sequence Y^n of processes is monotonically increasing as n tends to ∞ (see Corollary 4.9) and the limit $Y=\lim_{n\to\infty}Y^n$ is well defined.

Step 1 Our first goal is to show that, for every $n\in\mathbb{N}$,

$$Y^n_t=\operatorname*{ess\,sup}_{\sigma\in\mathcal{T}_{t,T}}\mathbb{E}\left[Y^n_\sigma\wedge\gamma_\sigma\,|\,\mathcal{F}_t\right]. \tag{4.13}$$

To prove (4.13), we fix n and we observe that Y^n is a supermartingale and thus, for every $\sigma\in\mathcal{T}_{t,T}$,

$$Y^n_t\geqslant\mathbb{E}\left[Y^n_\sigma\,|\,\mathcal{F}_t\right]\geqslant\mathbb{E}\left[Y^n_\sigma\wedge\gamma_\sigma\,|\,\mathcal{F}_t\right], \tag{4.14}$$

where the second inequality is obvious. To show the reverse inequality, we fix $t\in[0,T[$ and we define $\nu=\sigma^n_t\wedge\tau^n_t\in\mathcal{T}_{t,T}$ where for any fixed $\delta>0$ we set (as usual, $\inf\emptyset=T$)

$$\sigma^n_t:=\inf\{s\in[t,T]:Y^n_s\leqslant\zeta_s+\varepsilon\},\quad \tau^n_t:=\inf\{s\in[t,T]:\widetilde{K}^n_s-\widetilde{K}^n_t>0\},$$

where $\varepsilon:=0.5(Y^n_t-\zeta_t)\delta$ and the continuous, nondecreasing process $\widetilde{K}^n$ is given by $\widetilde{K}^n_t:=\int_{]0,t]}n(\eta_s-Y^n_s)^+\,\mathrm{d}A_s$. We will check that $Y^n_\nu=Y^n_\nu\wedge((\zeta+\varepsilon)\vee\eta\mathbb{1}_S)_\nu$ on the event $\{\nu<T\}=\{\sigma^n_t\leqslant\tau^n_t<T\}\cup\{\tau^n_t<\sigma^n_t\}=E_1\cup E_2$. It is obvious that $Y^n_\nu=\zeta_T$ on the event $E_3:=\{\nu=T\}$.

On the event $E_1=\{t<\sigma^n_t\leqslant\tau^n_t<T\}$, we have $Y^n_{\sigma^n_t-}-\bar{\zeta}_{\sigma^n_t}\geqslant\varepsilon$ and thus $\Delta K^{n,d}_{\sigma^n_t}=0$, which implies that Y^n is a martingale on $[\![t,\sigma^n_t]\!]$. Furthermore, if $\Delta^+K^{n,g}_{\sigma^n_t}>0$, then the Skorokhod condition gives $Y^n_{\sigma^n_t}=\zeta_{\sigma^n_t}$ and if $\Delta^+K^{n,g}_{\sigma^n_t}=0$, then Y^n is continuous at σ^n_t and $\zeta_{\sigma^n_t}\leqslant Y^n_{\sigma^n_t}\leqslant\zeta_{\sigma^n_t}+\varepsilon$ since ζ is assumed to be right-upper-semicontinuous. We conclude that on E_1 we have $Y^n_\nu=Y^n_\nu\wedge(\zeta_\nu+\varepsilon)=Y^n_\nu\wedge((\zeta+\varepsilon)\vee\eta\mathbb{1}_S)_\nu$ where the second equality is a trivial consequence of the first one.

On the event $E_2 = \{\tau_t^n < \sigma_t^n\}$, the process Y^n is right-continuous at τ_t^n and hence from the definition of τ_t^n we obtain $Y^n_{\tau_t^n} = Y^n_{\tau_t^n+} \leqslant \eta_{\tau_t^n+} = \eta_{\tau_t^n}$ where the last inequality follows from the right-continuity of η. We note also that the $\mathbb{F}$-stopping time τ_t^n has values in S so that $\eta_{\tau_t^n} = (\eta \mathbb{1}_S)_{\tau_t^n}$ and thus we have $Y^n_\nu = Y^n_\nu \wedge (\eta\mathbb{1}_S)_\nu = Y^n_\nu \wedge ((\zeta+\varepsilon)\vee \eta\mathbb{1}_S)_\nu$ on E_2 where the second equality is obvious. It is also clear that Y^n is a martingale on $[\![t,\tau_t^n]\!]$ since the continuous, nondecreasing process $\widetilde{K}^n$ and the nondecreasing process K^n are constant on that interval.

Recall that $\varepsilon = 0.5(Y^n_t - \zeta_t)\delta$ and the processes Y^n and ζ are bounded so that $\varepsilon \leqslant c\delta$ for some constant c. Let us denote $\gamma^\varepsilon_t := ((\zeta+\varepsilon)\vee\eta\mathbb{1}_S)_t \mathbb{1}_{\{t<T\}} + \zeta_T \mathbb{1}_{\{t=T\}}$. Since Y^n is a martingale on $[\![t,\nu]\!]$ we have

$$Y^n_t = \mathbb{E}\,[Y^n_\nu \,|\, \mathcal{F}_t] = \mathbb{E}\,[Y^n_\nu \wedge \gamma^\varepsilon_\nu \,|\, \mathcal{F}_t] \leqslant \mathbb{E}\,[Y^n_\nu \wedge \gamma_\nu \,|\, \mathcal{F}_t] + c\delta \leqslant \mathbb{E}\,[Y^n_\nu \wedge \gamma_\nu \,|\, \mathcal{F}_t]\,, \tag{4.15}$$

where the last inequality holds since δ is any positive number. By combining (4.14) with (4.15) we conclude that (4.13) is satisfied for every $n \in \mathbb{N}$.

Step 2 We are now ready to show that (4.10) is valid. For any $\tau \in \overline{\mathcal{T}}_{t,T}$ equation (4.11) gives

$$Y^n_\tau = \zeta_T + \int_{]\tau,T]} n(\eta_s - Y^n_s)^+ \,\mathrm{d}A_s - (M^n_T - M^n_t) + K_T - K_\tau$$

and, by applying the comparison theorem for reflected GBSDEs (see Proposition 4.8), we obtain the inequality $Y^n \geqslant \widehat{Y}^n$ where $(\widehat{Y}^n, \widehat{M}^n, \widehat{K}^n)$ solves the linear reflected GBSDE

$$\begin{aligned}
\widehat{Y}^n_\tau &= \zeta_T + \int_{]\tau,T]} n(\eta_s - \widehat{Y}^n_s)\,\mathrm{d}A_s - (\widehat{M}^n_T - \widehat{M}^n_\tau) + \widehat{K}^n_T - \widehat{K}^n_\tau \\
&= \zeta_T + \int_{]\tau,T]} n(\eta_s - \widehat{Y}^n_s)\,\mathrm{d}A_s - (\widehat{M}^n_T - \widehat{M}^n_\tau) + \widehat{K}^n_T - \widehat{K}^n_\tau \\
&\geqslant \zeta_T + \int_{]\tau,T]} n(\eta_s - \overline{Y}^n_s)\,\mathrm{d}A_s - (\bar{M}^n_T - \bar{M}^n_\tau) = \overline{Y}^n_\tau,
\end{aligned}$$

where the inequality holds since the generator $g^n(t,y) = n(\eta_t - y)$ is linear and the process $\widehat{K}$ is nondecreasing on $[\![\tau,T]\!]$. Furthermore, by solving the linear GBSDE

$$\overline{Y}^n_\tau = \zeta_T + \int_{]\tau,T]} n(\eta_s - \overline{Y}^n_s)\,\mathrm{d}A_s - (\bar{M}^n_T - \bar{M}^n_\tau),$$

we obtain $Y^n_\tau \geqslant \widehat{Y}^n_\tau \geqslant \overline{Y}^n_\tau$ where

$$\overline{Y}^n_\tau = \mathbb{E}\left[\zeta_T \mathcal{E}_{\tau,T}(-A^n) + (\mathbb{1}_{]\!]\tau,T]\!]}\eta\mathcal{E}_{\tau,\cdot}(-A^n)\bullet A^n)_T \,|\mathcal{F}_\tau\right],$$

where the sequence of random variables $\mathcal{E}_{\tau,T}(-A^n)$ converges to $\mathbb{1}_{\{\tau=T\}}$ as n tends to ∞ and thus, by Lemma 3.18 and the right-continuity of η, we obtain

$$Y_\tau = \lim_{n\to\infty} Y^n_\tau \geqslant \lim_{n\to\infty} \overline{Y}^n_\tau = \zeta_\tau \mathbb{1}_{\{\tau=\sigma\}} + \eta_\tau \mathbb{1}_{\{\tau<\sigma\}}. \tag{4.16}$$

Using the fact that $Y \geqslant \overline{Y}^n \geqslant 0$ and $Y \geqslant \zeta$, we deduce that for any stopping time $\sigma \in \mathcal{T}_{t,T}$, on the event $\{\sigma < T\}$, we have from (4.16)

$$Y_\sigma \geqslant \zeta_\sigma \vee (\eta\mathbb{1}_{\overline{S}})_\sigma$$

and $Y_T = \zeta_T$ on the event $\{\sigma = T\}$. Consequently,

$$Y_\sigma \geqslant \zeta_T \mathbb{1}_{\{\sigma=T\}} + \zeta_\sigma \vee (\eta\mathbb{1}_{\overline{S}})_\sigma \mathbb{1}_{\{\sigma<T\}} = \gamma_\sigma.$$

Since we clearly have $Y^n_t \leqslant \operatorname{ess\,sup}_{\sigma\in\mathcal{T}_{t,T}} \mathbb{E}\,[\gamma_\sigma \,|\, \mathcal{F}_t]$, it suffices to observe that the inequality $Y_t \geqslant \operatorname{ess\,sup}_{\sigma\in\mathcal{T}_{t,T}} \mathbb{E}\,[\gamma_\sigma \,|\, \mathcal{F}_t]$ holds since, by the monotone convergence theorem, the process Y is a supermartingale dominating the reward process γ and the minimality property of the Snell envelope. We thus conclude that (4.10) holds.

Finally, the representation of Y as the solution of the reflected BSDE (4.12) follows by noticing that the process γ is upper-semicontinuous and applying the classical Doob–Meyer–Mertens decomposition to (4.10). □

4.6 Constrained Dynkin game and penalization scheme for RGBSDE

Assume now that we are given two reward processes, denoted as ζ and η, and we consider a particular zero-sum stochastic stopping game, which is associated with the *reduced lower price* (i.e., the pre-default holder's price) of a vulnerable American option under market incompleteness (see Theorem 5.6 in [32]). We have the following definition of the constrained Dynkin game where, obviously, it is not yet known whether the value process given by (4.17) is well defined. As before, $\overline{S} = S^r(A) \cup \{T\}$ where $S^r(A)$ is the right support of the measure generated by the nondecreasing process A. As in Definitions 3.16 and 4.10, the process A enters the specification of the reward Θ, and hence also the value process $\widetilde{Y}$, only through the set $S^r(A)$ (see Remark 3.17).

Definition 4.12 *We say that the process $\widetilde{Y}$ is the value process of the constrained Dynkin game with data $(\zeta, \mathcal{T} \,|\, \eta, \overline{\mathcal{T}})$ if the following holds, for every $t \in [0,T]$,*

$$\widetilde{Y}_t = \operatorname*{ess\,inf}_{\tau \in \overline{\mathcal{T}}_{t,T}} \operatorname*{ess\,sup}_{\sigma \in \mathcal{T}_{t,T}} \mathbb{E}[\Theta(\sigma,\tau) \,|\, \mathcal{F}_t] = \operatorname*{ess\,sup}_{\sigma \in \mathcal{T}_{t,T}} \operatorname*{ess\,inf}_{\tau \in \overline{\mathcal{T}}_{t,T}} \mathbb{E}[\Theta(\sigma,\tau) \,|\, \mathcal{F}_t], \tag{4.17}$$

where

$$\Theta(\sigma,\tau) := \zeta_\sigma \mathbb{1}_{\{\tau > \sigma\}} + (\eta \vee \zeta)_\tau \mathbb{1}_{\{\tau \leqslant \sigma\}}.$$

Notice that the pair $(\zeta, \mathcal{T})$ corresponds to the player who is the maximizer in the Dynkin game introduced in Definition 4.12, whereas the pair $(\eta, \overline{\mathcal{T}})$ corresponds to the minimizer and only the stopping decisions of the minimizer are constrained to the random set $\overline{S}$.

We conclude this work by examining the second penalization scheme for reflected GBSDEs where the process ζ is still the lower obstacle but the role of the process η differs from the previous result and thus the limiting process $\widetilde{Y}$ is expected to represent the value of the constrained Dynkin game. The following result provides a link between the value of the constrained Dynkin game with data $(\zeta, \mathcal{T} \,|\, \eta, \overline{\mathcal{T}})$ and a sequence of solutions to reflected GBSDEs. In particular, it follows from Theorem 4.13 that the value process introduced in Definition 4.12 is well defined under the assumptions of the theorem.

Theorem 4.13 *Let Assumptions 4.1 and 4.4 be satisfied and the $\mathbb{F}$-optional and bounded process ζ (resp., the $\mathbb{F}$-optional and bounded process η) be right-upper-semicontinuous (resp., right-continuous). Consider the sequence of unique solutions $(\widetilde{Y}^n, \widetilde{M}^n, \widetilde{K}^n)$ to the reflected GBSDEs*

$$\widetilde{Y}^n_\tau = \zeta_T - \int_{]\tau,T]} n(\widetilde{Y}^n_s - \eta_s)^+ \, \mathrm{d}A_s - (\widetilde{M}^n_T - \widetilde{M}^n_\tau) + \widetilde{K}^n_T - \widetilde{K}^n_\tau,$$

where $\widetilde{K}^n$ satisfies the Skorokhod conditions with the lower obstacle ζ. If $\zeta_T = \eta_T$, then the sequence $\widetilde{Y}^n$ converges monotonically to the process $\widetilde{Y}$ given by equation (4.17).

Proof Let $(\widetilde{Y}^n, \widetilde{M}^n, \widetilde{K}^n)$ be the unique solution in $\mathcal{S}^2 \times \mathcal{M}^2_0 \times \overline{\mathcal{K}}$ to the reflected GBSDE (see Proposition 4.7)

$$\widetilde{Y}^n_\tau = \zeta_T - \int_{]\tau,T]} n(\widetilde{Y}^n_s - \eta_s)^+ \, \mathrm{d}A_s - (\widetilde{M}^n_T - \widetilde{M}^n_\tau) + \widetilde{K}^n_T - \widetilde{K}^n_\tau, \tag{4.18}$$

where $\widetilde{Y}^n \geqslant \zeta$ and the Skorokhod conditions are satisfied by an $\mathbb{F}$-adapted, nondecreasing process $\widetilde{K}^n$. We note that the sequence $\widetilde{Y}^n$ is monotonically decreasing as n tends to ∞ (see Corollary 4.9) and the limit $\widetilde{Y} = \lim_{n\to\infty} \widetilde{Y}^n$ exists.

Step 1 We will first prove that

$$\widetilde{Y}_t \geqslant \operatorname*{ess\,inf}_{\tau\in\overline{\mathcal{T}}_{t,T}} \operatorname*{ess\,sup}_{\sigma\in\mathcal{T}_{t,T}} \mathbb{E}\left[\zeta_\sigma \mathbb{1}_{\{\tau>\sigma\}} + (\zeta\vee\eta)_\tau \mathbb{1}_{\{\tau\leqslant\sigma\}} \,|\, \mathcal{F}_t\right]. \tag{4.19}$$

To establish (4.19), for any fixed t and n, we define $\bar{\tau}_t^n := \inf\{s\in[t,T] : \widetilde{L}_s^n - \widetilde{L}_t^n > 0\}$ where $\widetilde{L}_t^n := \int_{]0,t]} n(\widetilde{Y}_s^n - \eta_s)^+ \,\mathrm{d}A_s$. Since the process $\widetilde{L}^n$ is continuous, the graph of the stopping time $\bar{\tau}_t^n$ is contained in $\overline{S}\cap[t,T]$, which implies that $\bar{\tau}_t^n \in \overline{\mathcal{T}}_{t,T}$. Suppose, on the contrary, that $\bar{\tau}_t^n \notin \overline{\mathcal{T}}_{t,T}$. Then the event $\{\bar{\tau}_t^n < T\}\cap\{\bar{\tau}_t^n \in \overline{S}^c\}$ has a positive probability and, for any fixed $\omega\in\{\bar{\tau}_t^n < T\}\cap\{\bar{\tau}_t^n\in\overline{S}^c\}$, there exists $\delta=\delta(\omega)>0$ such that $A_{\bar{\tau}_t^n+\delta} = A_{\bar{\tau}_t^n}$. However, this contradicts the definition of $\bar{\tau}_t^n$ since $\widetilde{L}^n$ is absolutely continuous with respect to $\widetilde{\Gamma}$ and thus $\widetilde{L}^n_{\bar{\tau}_t^n+\delta} = \widetilde{L}^n_{\bar{\tau}_t^n}$.

The continuity of A entails that $\widetilde{Y}^n_{\bar{\tau}_t^n+} \geqslant \liminf_{s\downarrow\bar{\tau}_t^n} \eta_s$ on $\{\bar{\tau}_t^n < T\}$ and, consequently, using also (4.18) and the right-continuity of η we deduce that $\widetilde{Y}^n_{\bar{\tau}_t^n} = \widetilde{Y}^n_{\bar{\tau}_t^n+} + \Delta^+\widetilde{K}^{n,g}_{\bar{\tau}_t^n} \geqslant \widetilde{Y}^n_{\bar{\tau}_t^n+} \geqslant \eta_{\bar{\tau}_t^n}$. In addition, we have $\widetilde{Y}^n_{\bar{\tau}_t^n} \geqslant \zeta_{\bar{\tau}_t^n}$ since $(\widetilde{Y}^n, \widetilde{M}^n, \widetilde{K}^n)$ solves the reflected GBSDE (4.18). We conclude that $\widetilde{Y}^n_{\bar{\tau}_t^n} \geqslant (\zeta\vee\eta)_{\bar{\tau}_t^n}$ on $\{\bar{\tau}_t^n < T\}$ and, manifestly, $\widetilde{Y}_T^n = \zeta_T$.

We now take an arbitrary stopping time $\sigma\in\mathcal{T}_{t,T}$ and define $\nu := \bar{\tau}_t^n \wedge \sigma$ so that $\widetilde{Y}^n$ is a strong supermartingale on $[\![t,\nu]\!]$ since $\widetilde{L}_\nu^n = \widetilde{L}_t^n$. Then $\widetilde{Y}_\nu^n \geqslant \zeta_\nu$ on $E_1 := \{\bar{\tau}_t^n \geqslant \sigma\}$ and $\widetilde{Y}_\nu^n \geqslant (\zeta\vee\eta)_\nu$ on $E_2 := \{\bar{\tau}_t^n < \sigma\}$. Consequently, for any $\sigma\in\mathcal{T}_{t,T}$,

$$\widetilde{Y}_t^n \geqslant \mathbb{E}[\widetilde{Y}_\nu^n \,|\, \mathcal{F}_t] \geqslant \mathbb{E}\left[\zeta_\sigma \mathbb{1}_{\{\bar{\tau}_t^n>\sigma\}} + (\zeta\vee\eta)_{\bar{\tau}_t^n} \mathbb{1}_{\{\bar{\tau}_t^n\leqslant\sigma\}} \,|\, \mathcal{F}_t\right] \tag{4.20}$$

from which we deduce that

$$\begin{aligned}\widetilde{Y}_t^n &\geqslant \operatorname*{ess\,sup}_{\sigma\in\mathcal{T}_{t,T}} \mathbb{E}\left[\zeta_\sigma \mathbb{1}_{\{\bar{\tau}_t^n>\sigma\}} + (\zeta\vee\eta)_{\bar{\tau}_t^n} \mathbb{1}_{\{\bar{\tau}_t^n\leqslant\sigma\}} \,|\, \mathcal{F}_t\right] \\ &\geqslant \operatorname*{ess\,inf}_{\tau\in\overline{\mathcal{T}}_{t,T}} \operatorname*{ess\,sup}_{\sigma\in\mathcal{T}_{t,T}} \mathbb{E}\left[\zeta_\sigma \mathbb{1}_{\{\tau>\sigma\}} + (\zeta\vee\eta)_\tau \mathbb{1}_{\{\tau\leqslant\sigma\}} \,|\, \mathcal{F}_t\right].\end{aligned}$$

Finally, the sequence $\widetilde{Y}^n$ is decreasing and $\widetilde{Y} = \lim_{n\to\infty}\widetilde{Y}^n$ so that we obtain (4.19).

Step 2 In this step, we will establish the inequality

$$\widetilde{Y}_t \leqslant \operatorname*{ess\,sup}_{\sigma\in\mathcal{T}_{t,T}} \operatorname*{ess\,inf}_{\tau\in\overline{\mathcal{T}}_{t,T}} \mathbb{E}\left[\zeta_\sigma \mathbb{1}_{\{\tau>\sigma\}} + (\zeta\vee\eta)_\tau \mathbb{1}_{\{\tau\leqslant\sigma\}} \,|\, \mathcal{F}_t\right] \tag{4.21}$$

by showing that, for any $\varepsilon > 0$, there exists $\bar{\sigma}_t \in \mathcal{T}_{t,T}$, which may depend on ε, such that for an arbitrary $\tau\in\overline{\mathcal{T}}_{t,T}$ we have

$$\widetilde{Y}_t \leqslant \mathbb{E}\left[\zeta_{\bar{\sigma}_t} \mathbb{1}_{\{\tau>\bar{\sigma}_t\}} + (\zeta\vee\eta)_\tau \mathbb{1}_{\{\tau\leqslant\bar{\sigma}_t\}} \,|\, \mathcal{F}_t\right] + \varepsilon. \tag{4.22}$$

For a fixed t and $\varepsilon>0$, we define $\bar{\sigma}_t^n := \inf\{s\in[t,T] : \widetilde{Y}_s^n \leqslant \zeta_s + \varepsilon\}$. Recall that the sequence $\widetilde{Y}^n$ is monotonically decreasing as n tends to ∞ and $\widetilde{Y} = \lim_{n\to\infty}\widetilde{Y}^n$ so that $\bar{\sigma}_t^n \geqslant \bar{\sigma}_t^{n+1}$. We define an $\mathbb{F}$-stopping time $\bar{\sigma}_t := \lim_{n\to\infty}\bar{\sigma}_t^n$. From the lower bound in (4.20) we know that $\widetilde{Y}_t^n \geqslant 0$, while the comparison theorem for reflected GBSDEs gives, for every $n\in\mathbb{N}$,

$$\widetilde{Y}_t \leqslant \widetilde{Y}_t^n \leqslant X_t = \zeta_T - (M_T - M_t) + K_T - K_t = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}[\zeta_\tau \,|\, \mathcal{F}_t] \leqslant c_\zeta, \tag{4.23}$$

where (X, M, K) is a solution to the reflected BSDE implicit in (4.23) with the lower obstacle ζ (see Section 4.1) and thus the second equality is due to the well-known relationship between a solution to the reflected BSDE with null generator and the value process of an optimal stopping problem with the bounded reward process ζ.

From the assumption that ζ is right-upper-semicontinuous we deduce that $\widetilde{Y}^n_{\bar\sigma^n_t} \leqslant \zeta_{\bar\sigma^n_t} + \varepsilon$ where the inequality is trivially satisfied on the event $\{\bar\sigma^n_t = T\}$. Since $\widetilde{Y}^n$ is a làdlàg process, it would be possible to have sample paths satisfying: $\widetilde{Y}^n_{\bar\sigma^n_t} > \zeta_{\bar\sigma^n_t} + \varepsilon$ and there exists $\delta > 0$ such that $\widetilde{Y}^n \leqslant \zeta + \varepsilon$ on $\rrbracket \bar\sigma^n_t, \bar\sigma^n_t + \delta\rrbracket$. However, by the right-upper-semicontinuity of ζ, this would imply the inequalities $\widetilde{Y}^n_{\bar\sigma^n_t+} \leqslant \zeta_{\bar\sigma^n_t} + \varepsilon$ and $\Delta^+ \widetilde{K}^{n,g}_{\bar\sigma^n_t} > 0$. This would lead to a contradiction since, from the Skorokhod condition for $\widetilde{K}^{n,g}$, the inequality $\Delta^+ \widetilde{K}^{n,g}_{\bar\sigma^n_t} > 0$ implies that $\widetilde{Y}^n_{\bar\sigma^n_t} = \zeta_{\bar\sigma^n_t} < \zeta_{\bar\sigma^n_t} + \varepsilon < \widetilde{Y}^n_{\bar\sigma^n_t}$. In view of these considerations, we conclude that $\widetilde{Y}^n > \widetilde{Y}^n - \varepsilon > \zeta$ on $\llbracket t, \bar\sigma^n_t \llbracket$ and $\widetilde{Y}^n_- > \widetilde{Y}^n_- - \varepsilon \geqslant \bar\zeta$ on $\rrbracket t, \bar\sigma^n_t\rrbracket$. Together with the Skorokhod condition for the process $\widetilde{K}^n$, this gives

$$\widetilde{K}^n_{\bar\sigma^n_t} - \widetilde{K}^n_t = \int_{]t,\bar\sigma^n_t]} \mathrm{d}\widetilde{K}^{n,r}_s + \int_{[t,\bar\sigma^n_t[} \mathrm{d}\widetilde{K}^{n,g}_{s+} = 0. \tag{4.24}$$

If we take $\nu := \tau \wedge \bar\sigma^n_t$ where $\tau \in \overline{\mathcal{T}}_{t,T}$ is arbitrary, then

$$\begin{aligned}
\widetilde{Y}^n_t &= \mathbb{E}\left[\widetilde{Y}^n_\nu - \int_{]t,\nu]} n(\widetilde{Y}^n_s - \eta_s)^+ \,\mathrm{d}A_s + \widetilde{K}^n_\nu - \widetilde{K}^n_t \,\Big|\, \mathcal{F}_t\right] \\
&\leqslant \mathbb{E}\left[\widetilde{Y}^n_\nu | \,\mathcal{F}_t\right] = \mathbb{E}\left[\widetilde{Y}^n_{\bar\sigma^n_t} \mathbb{1}_{\{\tau > \bar\sigma^n_t\}} + \widetilde{Y}^n_\tau \mathbb{1}_{\{\tau \leqslant \bar\sigma^n_t\}} \,|\, \mathcal{F}_t\right] \\
&\leqslant \mathbb{E}\left[\zeta_{\bar\sigma^n_t} \mathbb{1}_{\{\tau > \bar\sigma^n_t\}} + (\widetilde{Y}^n \vee \zeta \vee \eta)_\tau \mathbb{1}_{\{\tau \leqslant \bar\sigma^n_t\}} \,|\, \mathcal{F}_t\right] + \varepsilon \\
&\leqslant \mathbb{E}\left[\zeta_{\bar\sigma^n_t} \mathbb{1}_{\{\tau > \bar\sigma^n_t\}} + (\widetilde{Y} \vee \zeta \vee \eta)_\tau \mathbb{1}_{\{\tau \leqslant \bar\sigma_t\}} \,|\, \mathcal{F}_t\right] + \mathbb{E}\left[|\widetilde{Y}^n_\tau - \widetilde{Y}_\tau| \,|\, \mathcal{F}_t\right] + C\mathbb{E}\left[\mathbb{1}_{\rrbracket \bar\sigma_t, \bar\sigma^n_t\rrbracket}(\tau) |\, \mathcal{F}_t\right] + \varepsilon,
\end{aligned}$$

where on the event $E_1 := \{\tau > \bar\sigma^n_t\}$ we have used the inequality $\widetilde{Y}^n_{\bar\sigma^n_t} \leqslant \zeta_{\bar\sigma^n_t} + \varepsilon$ while on the event $E_2 := \{\tau \leqslant \bar\sigma^n_t\}$ we have used the trivial inequality $\widetilde{Y}^n_\tau \leqslant \widetilde{Y}^n_\tau \vee \zeta_\tau \vee \eta_\tau$. By considering the limit superior in n and using the conditional reverse Fatou lemma (see Theorem 2 in [43]), together with the right-upper-semicontinuity of ζ and the monotone convergence theorem, we obtain

$$\widetilde{Y}_t \leqslant \mathbb{E}\left[\zeta_{\bar\sigma_t} \mathbb{1}_{\{\tau > \bar\sigma_t\}} + (\widetilde{Y} \vee \zeta \vee \eta)_\tau \mathbb{1}_{\{\tau \leqslant \bar\sigma_t\}} \,|\, \mathcal{F}_t\right] + \varepsilon.$$

Next, we will show that $\widetilde{Y}$ can be omitted from the conditional expectation above. For any $\tau \in \overline{\mathcal{T}}_{t,T}$, equation (4.24) gives

$$\widetilde{Y}^n_\tau = \widetilde{Y}^n_{\bar\sigma^n_\tau} - \int_{]\tau,\bar\sigma^n_\tau]} n(\widetilde{Y}^n_s - \eta_s)^+ \,\mathrm{d}A_s - \int_{]\tau,\bar\sigma^n_\tau]} \mathrm{d}\widetilde{M}^n_s.$$

We now use similar arguments as in Step 2 in the proof of Theorem 4.11. We observe that $\widetilde{Y}^n_{\bar\sigma^n_\tau} \leqslant \zeta_{\bar\sigma^n_\tau} + \varepsilon$ and, for all $(\omega, s, y) \in \Omega \times [0,T] \times \mathbb{R}$,

$$(y - \eta_s)^+(\omega) \geqslant \mathbb{1}_{\llbracket 0,\bar\sigma_\tau\rrbracket}(s)(y - \eta_s)^+(\omega) \geqslant \mathbb{1}_{\llbracket 0,\bar\sigma_\tau\rrbracket}(s)(y - \eta_s)(\omega),$$

where the function $g(\omega, s, y) := \mathbb{1}_{\llbracket 0,\bar\sigma_\tau\rrbracket}(s)(\eta_s - y)(\omega)$ is nonincreasing in y, for every $(\omega, s) \in \Omega \times [0,T]$. By applying the comparison theorem for GBSDEs (see Proposition 3.6) on the interval $\llbracket \tau, \bar\sigma^n_\tau\rrbracket$, we see that $\widetilde{Y}^n \leqslant Y^n$ where (Y^n, M^n) solves the following linear BSDE

$$\begin{aligned}
Y^n_\tau &= \zeta_{\bar\sigma^n_\tau} + \varepsilon - \int_{]\tau,\bar\sigma^n_\tau]} n(Y^n_s - \eta_s)\mathbb{1}_{[0,\bar\sigma_\tau]}(s) \,\mathrm{d}A_s - \int_{]\tau,\bar\sigma^n_\tau]} \mathrm{d}M^n_s \\
&= \zeta_{\bar\sigma^n_\tau} + \varepsilon + \int_{]\tau,\bar\sigma^n_\tau]} n(\eta_s - Y^n_s) \,\mathrm{d}A^{\bar\sigma_\tau}_s - \int_{]\tau,\bar\sigma^n_\tau]} \mathrm{d}M^n_s.
\end{aligned}$$

Let us denote $A^n := nA$. Since $\tau \leqslant \bar{\sigma}_\tau \leqslant \bar{\sigma}^n_\tau \leqslant T$, Corollary 3.15 gives

$$\begin{aligned} Y^n_\tau &= \mathbb{E}\left[(\zeta_{\bar{\sigma}^n_\tau} + \varepsilon)\mathcal{E}_{\tau,\bar{\sigma}_\tau}(-A^n) + (\mathbb{1}_{\rrbracket \tau,\bar{\sigma}_\tau\rrbracket}\eta\mathcal{E}_{\tau,\cdot}(-A^n)\bullet A^n)_T\,|\mathcal{F}_\tau\right] \\ &\leqslant \mathbb{E}\left[\zeta_{\bar{\sigma}_\tau}\mathcal{E}_{\tau,\bar{\sigma}_\tau}(-A^n) + (\mathbb{1}_{\rrbracket \tau,\bar{\sigma}_\tau\rrbracket}\eta\mathcal{E}_{\tau,\cdot}(-A^n)\bullet A^n)_T\,|\mathcal{F}_\tau\right] + \mathbb{E}\left[\left(\zeta_{\bar{\sigma}^n_\tau} - \zeta_{\bar{\sigma}_\tau}\right)\mathcal{E}_{\tau,\bar{\sigma}_\tau}(-A^n)|\mathcal{F}_\tau\right] + \varepsilon, \end{aligned}$$

where in the last inequality we have used the inequality $\mathcal{E}_{\tau,\bar{\sigma}_\tau}(-A^n) \leqslant 1$. The quantity $\mathcal{E}_{\tau,\bar{\sigma}_\tau}(-A^n)$ converges to $\mathbb{1}_{\{\tau=\bar{\sigma}_\tau\}}$ as n tends to ∞ and, by the subadditivity of the limit superior, the conditional reverse Fatou lemma and the dominated convergence theorem, we obtain

$$\begin{aligned} &\limsup_{n\to\infty} \mathbb{E}\left[\left(\zeta_{\bar{\sigma}^n_\tau} - \zeta_{\bar{\sigma}_\tau}\right)\mathcal{E}_{\tau,\bar{\sigma}_\tau}(-A^n)\,|\,\mathcal{F}_\tau\right] \\ &\leqslant \limsup_{n\to\infty} \mathbb{E}\left[\zeta_{\bar{\sigma}^n_\tau}\mathcal{E}_{\tau,\bar{\sigma}_\tau}(-A^n)\,|\,\mathcal{F}_\tau\right] - \lim_{n\to\infty} \mathbb{E}\left[\zeta_{\bar{\sigma}_\tau}\mathcal{E}_{\tau,\bar{\sigma}_\tau}(-A^n)\,|\,\mathcal{F}_\tau\right] \\ &\leqslant \mathbb{E}\left[\limsup_{n\to\infty}\left(\zeta_{\bar{\sigma}^n_\tau} - \zeta_{\bar{\sigma}_\tau}\right)\mathbb{1}_{\{\tau=\bar{\sigma}^n_\tau\}}\,|\,\mathcal{F}_\tau\right] \leqslant 0, \end{aligned}$$

where the last inequality holds since ζ is right-upper-semicontinuous along stopping times (see Remark B.3 in [31]). For any fixed $\varepsilon > 0$, we conclude from the subadditivity of the limit superior and Lemma 3.15 that

$$\widetilde{Y}_\tau \leqslant \zeta_\tau\mathbb{1}_{\{\tau=\bar{\sigma}_\tau\}} + \eta_\tau\mathbb{1}_{\{\tau<\bar{\sigma}_\tau\}} + \varepsilon \leqslant (\zeta\vee\eta)_\tau + \varepsilon,$$

and thus $\widetilde{Y}_\tau \leqslant (\zeta\vee\eta)_\tau$ for every stopping time τ in $\overline{\mathcal{T}}_{t,T}$, which gives the desired upper bound in (4.22).

Step 3 It is well known that the inequality

$$\operatorname*{ess\,inf}_{\tau\in\overline{\mathcal{T}}_{t,T}}\operatorname*{ess\,sup}_{\sigma\in\mathcal{T}_{t,T}}\mathbb{E}[\Theta(\sigma,\tau)\,|\,\mathcal{F}_t] \geqslant \operatorname*{ess\,sup}_{\sigma\in\mathcal{T}_{t,T}}\operatorname*{ess\,inf}_{\tau\in\overline{\mathcal{T}}_{t,T}}\mathbb{E}[\Theta(\sigma,\tau)\,|\,\mathcal{F}_t]$$

always holds and thus we obtain (4.17) by combining (4.19) with (4.21). □

Remark 4.14 *It is natural to conjecture that the process $\widetilde{Y}$ given by (4.17) can also be represented through a solution to a doubly reflected BSDE. Although that guess was not examined in the present work, let us point out that in the case where the processes η and ζ are càdlàg and $A_t = t$, it was demonstrated in Theorem 3.1 of Hamadène et al. [25] that the limit $\widetilde{Y}$ satisfies locally a doubly reflected BSDE with the lower and upper obstacles equal to ζ and η, respectively. In addition, Theorem 4.1 in [25] shows that if the obstacles are completely separated, in the sense that the strict inequality $\eta > \zeta$ holds, then the process $\widetilde{Y}$ is a solution to a doubly reflected BSDE.*

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