Probability, Uncertainty and Quantitative Risk
Vol. 9, No. 2, 2024, 235–262

https://www.aimsciences.org/PUQR
DOI:10.3934/puqr.2024012

# Optimal liquidation with dynamic parameter updating: A forward approach

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**Abstract** We propose a forward approach to study the performance of liquidation strategies under sequential model parameter updates. The forward liquidation program consists of pasting forward in time and in a time-consistent fashion a series of optimal liquidation problems. They are triggered at the parameter shift instances, thus entirely eliminating model error, and last at most till the next parameter update. However, due to the nature of the model dynamics, solutions may cease to exist in finite time, even before the subsequent parameter update. Furthermore, forward liquidation strategies may never lead to full liquidation, even though they maximize the average utility of revenue and always preserve time-consistency. In juxtaposition, the traditional approach delivers full liquidation at the sought horizon but encounters considerable model error, generates value erosion, and is time-inconsistent.

**Keywords** Optimal liquidation, Dynamic parameter updating, Forward approach, Full liquidation, Time consistency, Backward approach, Expected utility model

**2020 Mathematics Subject Classification** 60H30, 91G30, 93C95, 93E20, 93E35

## 1. Introduction

Forward performance criteria were introduced by M. Musiela and the second author in [23] and [24], and subsequently further developed by them and others, in order to accommodate several limitations of the classical expected utility problems related to pre-chosen horizon, utility and model dynamics. These new utility criteria offer substantial flexibility in terms of rolling horizons, evolving risk preferences and dynamically adjusted model dynamics, while remaining time consistent at all times. They are built on the dynamic programming principle in that compiled with the wealth process, forward utilities processes are (local) supermartingales along admissible strategies and become (local) martingales along an optimum policy. The traditional (backward) value function processes are special cases of forward utilities.

Received 7 June 2024; Accepted 15 June 2024; Early access 21 June 2024
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The literature on forward performance criteria has considerably expanded since their introduction. Various papers have examined, among others, forward utilities in general semimartingale markets, stochastic factor models, model ambiguity, robust control, risk measures, certainty equivalents, risk sharing, contract theory, competition, mean field games, pension fund management, as well as their connection with ergodic control, infinite horizon BSDE, ergodic BSDE, ill-posed PDE and SPDE (see, among others, [2, 3, 5, 6, 9–11, 17, 18, 20, 21, 25–29, 32, 33, 38] and others; we, also, refer the interested reader to the recent review article by M. Musiela [22].

In all existing works in the forward literature, with the exception of [20], the updating of model dynamics and preferences is perfectly aligned with the frequency at which trading takes place. Furthermore, because of the nature of optimal investment problems, the focus has been on building forward criteria that are well defined for all times. This paper takes a different route and contributes in a number of new directions. Firstly, the underlying model comes from optimal liquidation in which price impact is encountered for, which puts the solution approach outside the ones that have been used so far in optimal investments. Secondly, because of the nature of the controlled dynamics, forward solutions do not exist for all times and, thus, analysis till the first blow-up time needs to be developed. Thirdly, and most importantly, we allow for sequential, piece-wise constant changes in the temporary price impact parameter. As a result, changes in the model dynamics happen discretely as opposed to trading that takes place continuously.

The underlying optimization framework is based on the expected utility model of separable exponential type, proposed in [30] and [31], with singular terminal data. Therein, the model dynamics (cf. (4) and (5)) depend on both the temporary and permanent price impact parameters, $\lambda$ and $\gamma$, which were taken to be constants, chosen at initial time $t=0$. There is substantial literature pointing out that, while $\gamma$ varies slowly, parameter $\lambda$ changes considerably through the sought liquidation horizon (see, among others, [4, 7, 8, 12–14, 16, 19] and the more recent works, [15] and [37]; providing a detailed bibliography for the vast area of optimal liquidation is beyond the scope of this work, for our aim is to use a specific model to study the use of forward criteria within it).

Herein, we explore how forward performance criteria can be built that take into account, from the one hand, the effects of model dynamics with price impact and, from the other, the sequential, discrete in time, changes in parameter $\lambda$, assuming that $\gamma$ remains constant throughout. We elaborate on the two parts next, starting with the latter. We assume that the temporary price impact parameter $\lambda$ is initially (at $t=0$) equal to $\lambda_0$, and afterwards, takes values $\lambda_1, ..., \lambda_{N-1}$, with the parameter shift occurring at times $T_1, ..., T_{N-1}$, respectively. We, also, assume that $T_{N-1} < T$, with $\lambda_{N-1}$ being the active parameter value in the last interval $[T_{N-1}, T]$, where $T \leqslant \infty$ is the final, and possibly full-liquidation horizon. The critical assumption in this toy model is that each pair $(\lambda_i, T_{i+1})$ becomes known only at time $T_i$, $i = 1, ..., N-1$. For example, at initial time $t=0$, the agent knows the value $\lambda_0$ and the "confidence interval" $[0, T_1]$, the longest interval in which value $\lambda_0$ is deemed accurate. The updated value $\lambda_1$ will be only revealed at $T_1$, together with time $T_2$, with $[T_1, T_2]$ being now the new "confidence interval" for the new value. We may think of the pair $(\lambda_i, T_{i+1})$ as an "$\mathcal{F}_i$-measurable" modeling input (with $\{\mathcal{F}_t\}_{t \geqslant 0}$ being an underlying filtration), but we make no assumption about an underlying model that generates these parameter values and their updating times. We, also, do not refer to the "confidence" intervals in any statistical sense. We, only, consider this rather simple, if not simplistic, framework of sequential, piece-wise constant in time, changes in the model, and investigate how forward performance criteria may be applied to evaluate the various liquidation strategies and, also, to potentially offer a complimentary framework to the traditional analysis aligned with adaptive control. As we will see, even in this rather simple modeling framework, the underlying mathematical questions are interesting on their own right.

The forward approach we propose consists of pasting, in $[0,T]$, *sequentially and forward in time* stochastic optimization problems,

$$\mathcal{P}_{0,T} := \mathcal{P}_0 \circ \mathcal{P}_1 \circ \ldots \circ \mathcal{P}_{N-1} \tag{1}$$

with the following characteristics. Each problem $\mathcal{P}_i$ is born at time $T_i$, when the accurate parameter value $\lambda_i$ becomes known, provided that its predecessor $\mathcal{P}_{i-1}$ has not expired strictly before $T_i$. It is cast in a forward setting and its initial condition taken to be equal to the terminal condition of $\mathcal{P}_{i-1}$. This preserves *time consistency* while going from one parameter regime to the next, forward in time. Furthermore, it *excludes* any *model error*, as each new problem $\mathcal{P}_i$ is triggered exactly when the *accurate* value $\lambda_i$ becomes known. On the other hand, as a forward problem, $\mathcal{P}_i$ is ill-posed and, therefore, it may not be viable for all times. We denote its solvability time by $\mathcal{T}_i$.

Parameter $\lambda_i$ remains valid from $T_i$ till $T_{i+1}$ when the next value $\lambda_{i+1}$ is revealed. If $T_i + \mathcal{T}_i < T_{i+1}$, then problem $\mathcal{P}_i$ is not viable beyond $T_i + \mathcal{T}_i$ and the forward liquidation schedule $\mathcal{P}_{0,T}$ terminates all together at $T_i + \mathcal{T}_i$, making all subsequent parameter adjustments obsolete. If, on the other hand, $T_i + \mathcal{T}_i \geqslant T_{i+1}$, the next problem in the sequence (1), $\mathcal{P}_{i+1}$, is born at $T_{i+1}$ and the construction continues in the same iterative manner, always forward in time. Like $\mathcal{P}_i$, problem $\mathcal{P}_{i+1}$ is set as a forward problem with initial, at $T_{i+1}$, condition being the terminal value of its predecessor $\mathcal{P}_i$.

As mentioned above, $\mathcal{P}_i$ is ill-posed and, thus, it may not be viable for the entire interval $[T_i, T_{i+1}]$. The model dynamics with price impact put the analysis outside existing results in forward utilities and a new approach needs to be developed. Within the separable class of criteria we consider herein, we study the solvability of $\mathcal{P}_i$ by analyzing the characteristic curves of an underlying Hamilton-Jacobi equation and determining their longest invertibility time. This, in turn, yields the *solvability time* $\mathcal{T}_i$ of problem $\mathcal{P}_i$. An inherent complication is that if $T_i + \mathcal{T}_i < T_{i+1}$, which forces the forward program to terminate before $T_{i+1} < T$, any goal to fully liquidate at $T$ is not achieved. Furthermore, depending on the interplay of various model inputs, it is not always the case that full liquidation actually occurs, in that full liquidation might not be optimal under forward criteria.

Generally speaking, the solution patterns depend crucially on *both* the choice of the initial condition and the relative sizes of the upcoming parameter values. Since we work in the framework of [30] (and [31]) we choose an initial condition which is separable and of exponential type (cf. (15)), and seek to preserve its form from one forward problem to the next, which offers substantial tractability. In summary, working in the forward framework does not encounter any model misspecification error as each new problem is perfectly aligned with the incoming information about changes in $\lambda$. It, also, offers by construction, time consistent solutions through forward in time "model pasting". On the other hand, there are two drawbacks. Firstly, forward problems are ill-posed so they may loose solvability in finite time, and in particular before the next updating time. Secondly, forward liquidation strategies, while they optimize the average forward utility of inventory and revenue, may not lead to full liquidation after all. As we will see, this happens predominantly when the parameter $\lambda$ becomes very large. One may then argue that while forward strategies could fail to deliver full liquidation, they may be, after all, superior in the presence of unfavorable market conditions.

The traditional (backward) analysis has its own advantages and disadvantages. The agent is strictly committed to fully liquidate at $T$ no matter what changes, favorable or unfavorable, occur in parameter $\lambda$ as time advances.

As a result, at each time $T_i$, $i = 0, 1, ..., N-1$, he has to solve an expected utility problem for the *entire* remaining horizon $[T_i, T]$ as in [31] and, thus, he faces the rather heavy constraint to have to pre-choose a model for this horizon, even though the model is known only for $[T_i, T_{i+1})$. Having a model specified for the whole horizon $[T_i, T]$ is needed in order for the backward utility problem to be solved even in $[T_i, T_{i+1})$. We further elaborate on this next.

At time $t = 0$, the agent knows the value $\lambda_0$ and its validity interval $[0, T_1]$, but she is entirely agnostic of any upcoming values and model update times. For simplicity, let us assume that the entire stream $T_1, ..., T_{N-1}$ is known at $t = 0$. Then, she must choose a model for $[T_1, T]$ which amounts to choosing proxies $\hat{\lambda}_1, ..., \hat{\lambda}_{N-1}$ for the respective intervals $[T_1, T_2), .., [T_{N-1}, T)$ which, of course, might not be accurate. To comply with backward induction, the agent will first solve a full liquidation problem, denoted by $\mathcal{L}\left(\hat{\lambda}_{N-1}; T_{N-1}, T\right)$, a replica of the expected utility problem in [31], with singular terminal condition (7). In turn, the initial condition of $\mathcal{L}\left(\hat{\lambda}_{N-1}; T_{N-1}, T\right)$ at $T_{N-1}$ will serve as the terminal utility of an optimal liquidation problem, denoted by $\mathcal{S}\left(\hat{\lambda}_{N-2}; T_{N-2}, T_{N-1}\right)$. Because the terminal condition of the latter is not any more of singular form, $\mathcal{S}\left(\hat{\lambda}_{N-2}; T_{N-2}, T_{N-1}\right)$ is not a full liquidation problem (which is intuitively expected, after all). Working backwards, the agent will have to solve a sequence of such problems, moving from $\mathcal{S}\left(\hat{\lambda}_{i+1}; T_{i+1}, T_{i+2}\right)$ to its predecessor $\mathcal{S}\left(\hat{\lambda}_i; T_i, T_{i+1}\right)$. It is, then, obvious that each optimal strategy for $\mathcal{S}(\lambda_0; 0, T_1), ..., \mathcal{S}\left(\hat{\lambda}_{N-2}; T_{N-2}, T_{N-1}\right)$ and $\mathcal{L}\left(\hat{\lambda}_{N-1}; T_{N-1}, T\right)$, respectively, is heavily polluted across $[0, T]$ by model error. This is the case even for the very first problem $\mathcal{S}(\lambda_0; 0, T_1)$ in which the pair $(\lambda_0, T_1)$ is known and, thus, the model is accurate on $[0, T_1]$. On the other hand, the terminal condition of $\mathcal{S}(\lambda_0; 0, T_1)$ is distorted by the incremental model misspecification moving backwards in time.

In analogy to (1), we may think of the pre-committed agent having to solve a sequence of backward liquidation programs, $\mathcal{L}_{0,T}, ..., \mathcal{L}_{T_{N-1},T}$. Each of them, say $\mathcal{L}_{T_i,T}$, is triggered at $T_i$ and consists of first solving a full liquidation problem $\mathcal{L}\left(\hat{\lambda}^i_{N-1}; T_{N-1}, T\right)$ for the last period $[T_{N-1}, T]$ and, subsequently, a family of optimal liquidation (but not of full liquidation) problems $\mathcal{S}\left(\hat{\lambda}^i_{N-2}; T_{N-2}, T_{N-1}\right), ..., \mathcal{S}(\lambda_i; T_i, T_{i+1})$, moving backwards in time. With the exception of the first problem $\mathcal{S}(\lambda_i; T_i, T_{i+1})$, within program $\mathcal{L}_{T_i,T}$, for all other problems the agent *must* assume proxies of the parameter values $\hat{\lambda}^i_{i+1}, ..., \hat{\lambda}^i_{N-1}$. We stress that $\mathcal{L}_{T_i,T}$ only lasts for $t \in [T_i, T_{i+1}]$, since at $T_{i+1}$ the new value $\lambda_{i+1}$ is announced and program $\mathcal{L}_{T_{i+1},T}$ is triggered.

The proxies $\hat{\lambda}^i_{i+1}, ..., \hat{\lambda}^i_{N-1}$ differ, in general, from the true values $\lambda_{i+1}, ..., \lambda_{N-1}$, and may also change in the next $\mathcal{L}_{T_{i+1},T}$; to be precise, $\left(\hat{\lambda}^i_{i+2}, ..., \hat{\lambda}^i_{N-1}\right) \neq \left(\hat{\lambda}^{i+1}_{i+2}, ..., \hat{\lambda}^{i+1}_{N-1}\right)$.

Schematically, we may write, in analogy to (1), the backward liquidation program with sequential parameter updating, as the family of liquidation sub-programs $\mathcal{L}_{0,T}, ..., \mathcal{L}_{T_{N-1},T}$,

$$\begin{cases} \mathcal{L}_{0,T} = \mathcal{L}\left(\hat{\lambda}^0_{N-1}; T_{N-1}, T\right) \circ \mathcal{S}\left(\hat{\lambda}^0_{N-2}; T_{N-2}, T_{N-1}\right) \circ ... \circ \mathcal{S}\left(\lambda_0; 0, T_1\right), \\ \mathcal{L}_{T_1,T} = \mathcal{L}\left(\hat{\lambda}^1_{N-1}; T_{N-1}, T\right) \circ \mathcal{S}\left(\hat{\lambda}^1_{N-2}; T_{N-2}, T_{N-1}\right) \circ ... \circ \mathcal{S}\left(\lambda_1; T_1, T_2\right), \\ \qquad\qquad\qquad\qquad ... \\ \mathcal{L}_{T_i,T} = \mathcal{L}\left(\hat{\lambda}^i_{N-1}; T_{N-1}, T\right) \circ \mathcal{S}\left(\hat{\lambda}^i_{N-2}; T_{N-2}, T_{N-1}\right) \circ ... \circ \mathcal{S}\left(\lambda_i; T_i, T_{i+1}\right), \\ \qquad\qquad\qquad\qquad ... \\ \mathcal{L}_{T_{N-1},T} = \mathcal{L}\left(\lambda_{N-1}; T_{N-1}, T\right). \end{cases} \tag{2}$$

In the Appendix, we consider the case of a single parameter change. This yields only two members in the set above, namely, $\mathcal{L}_{0,T} = \mathcal{L}\left(\hat{\lambda}^0_1; T_1, T\right) \circ \mathcal{S}\left(\lambda_0; 0, T_1\right)$ and $\mathcal{L}^{N-1}_{T_{N-1},T} = \mathcal{L}(\lambda_1; T_{N-1}, T)$; the proxy $\hat{\lambda}^0_1$ is chosen at $t = 0$ for problem $\mathcal{L}\left(\hat{\lambda}^0_1; T_1, T\right)$.

In summary, the forward and backward optimal liquidation programs have distinct advantages and disadvantages. The forward approach is built on time consistency, follows accurately the parameter updates and provides an intuitively pleasing optimization approach that pastes forward in time the related optimal liquidation sequential problems. On the other hand, while each model is, by construction, accurate at the time it is being introduced, the related optimization problems are ill-posed and, as such, they might not have solutions beyond their solvability time. As a result, the forward liquidation program may terminate before $T$ and, in addition, it might be that the final forward optimal inventory is not zero, thus failing to achieve full liquidation all together.

In contrast, the backward liquidation program consists of well-posed problems and, eventually, full liquidation is achieved at $T$. However, this may happen at considerable loss of value. Indeed, there may be substantial model error which dilutes the sequential model dynamics and, in turn, the value functions, the optimal strategies and the optimal processes. This model erosion is compoundly amplified due to sequential backward induction in which model errors "pile up" going backwards in time, moving from one well-posed problem to its predecessor in time. Furthermore, each "first in the upcoming queue" wrong model never truly materializes as its dynamics become obsolete as soon as the next parameter shift takes place. These problematic situations are the direct result of the strong adherence to fully liquidate at $T$ and the complete lack of knowledge of the upcoming model values while, at the same time, full model specification is needed for each remaining horizon.

As mentioned earlier, the aim herein is not to defend one approach against the other as, in many aspects, they are not comparable. Rather, we aim at exposing their differences and similarities. A possible route to juxtapose the two approaches is to build meaningful notions of regret metrics. This is being currently investigated by the authors in [36] in a broader context and applications.

Overall, the paper offers new results in the forward performance literature by allowing for liquidation dynamics with price impact and for model updates at a different frequency than the one of trading. Furthermore, the parameter changes happen in a very general, model-free way, which truly highlights the flexibility forward criteria offer in such settings.

The paper is structured as follows. In Section 2, we first review the full liquidation expected utility problem $\mathcal{L}\left(\lambda_0; 0, T\right)$ in [31] and $\mathcal{L}\left(\lambda_0; 0, \infty\right)$ in [30], where a single model (single parameter value) is assumed. We, then, build its forward analogue, $\mathcal{P}$ which depends on $\lambda_0$ and a parameter $k$ entering in its initial condition. We, then, establish that each classical single-model full liquidation problem $\mathcal{L}\left(\lambda_0; 0, T\right)$ or $\mathcal{L}\left(\lambda_0; 0, \infty\right)$ can be thought as a forward optimal liquidation one under a suitable choice of $k$. The opposite, however, is not always true which reconfirms that terminal non-zero inventory may occur optimally in forward problems. In Section 3, we present the forward liquidation program, produce its solution and discuss the optimal policies. In Section 4, we discuss the continuous time case and produce a solution for the coordinated variation case. We conclude in Section 5. For completeness, we also highlight the main findings for the backward problem. To ease the presentation, we only consider a single parameter shift and provide the key calculations in the Appendix.

## 2. Core optimal liquidation problems: the backward and forward approach

We start with two distinct optimal liquidation problems which will serve as the building blocks in the upcoming analysis. The first was solved in [30] and [31] (for finite and infinite liquidation horizon, respectively) while the second is new and formulated in the forward setting. We review the former and explicitly solve the latter. For both problems, we take the set of admissible policies to be deterministic but this is without loss of generality. Indeed, in the settings in [30] and [31] this is well documented. In the forward analysis, the arguments are a bit more complex but, as the analysis shows, working with deterministic policies suffices for our approach herein. We, also, study the connection of the two problems. We establish that the backward one is a special case of the forward problem in that, within the same stock price model, each backward problem can be cast as a forward one with a suitable initial condition; however, the opposite is not always true which highlights that terminal non-zero inventory can be optimal in the forward setting.

### 2.1 Classical (backward) optimal liquidation problem

We review the optimal liquidation problem in [31]. Assume an arbitrary, full liquidation time $T<\infty$. The stock price process $P_t$, $t\in[0,T]$, solves

$$P_t = P_0 + \sigma_0 W_t + \gamma_0(X_t - X_0) + \lambda_0 \dot{X}_t, \tag{3}$$

$P_0>0$ and $W_t$, being a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$, with $\{\mathcal{F}_t\}_{t\geqslant 0}$ being the natural filtration satisfying the usual conditions. Without loss of generality, it is assumed that $\sigma_0 = 1$. The parameters $\lambda_0, \gamma_0$ model, respectively, the *temporary* and *permanent* price impact and are known positive constants.

The inventory process $X_t$, $t\in[0,T]$, models the amount of stock shares held at time $t$, is absolutely continuous and solves

$$X_t^{\xi} = x - \int_0^t \xi_s \mathrm{d}s, \tag{4}$$

with initial inventory $X_0 = x > 0$. The control $\xi$ represents the rate of liquidation. The revenue process $R_t$, $t\in[0,T]$, is given by

$$R_t^{\xi} = \int_0^t \xi_s P_s \mathrm{d}s = \Big(P_0 x - \frac{\gamma_0}{2}x^2\Big) - \lambda_0 \int_0^t \xi_s^2 \mathrm{d}s + \int_0^t X_s^{\xi} \mathrm{d}W_s. \tag{5}$$

The set of admissible policies $\mathcal{A}_{[0,T]}$ consists of deterministic policies satisfying the full liquidation requirement $\int_0^T \xi_s \mathrm{d}s = x$, $\xi_t \geqslant 0$, $t\in[0,T]$. Furthermore, $\int_0^T \xi_s^2 \mathrm{d}s < \infty$ and the associated process $X_t^{\xi}$ is bounded uniformly in $t$, with upper and lower bounds possibly depending on the control $\xi$ (see [30]).

The manager is risk averse and seeks, from the one hand, to maximize the expected utility of terminal revenue and, from the other, to fully liquidate by $T$, starting with inventory $x>0$ and revenue $r_0 = P_0 x - \frac{\gamma_0}{2} x^2$. To solve this, the authors in [31] proposed the following expected utility problem, formulated for arbitrary $r \in \mathbb{R}$. We will be using the spatial domain notation $\mathbb{D} = \mathbb{R}^+ \times \mathbb{R}$.

**Problem** $\mathcal{L}(\lambda_0; 0, T)$ : *For* $(x, r) \in \mathbb{D}$, *solve*

$$V(x, r, 0; T) := \sup_{\mathcal{A}_{[0,T]}} \mathbb{E}\left[ v\left(X_T^\xi, R_T^\xi\right) \middle| X_0^\xi = x, R_0^\xi = r \right], \tag{6}$$

*with terminal datum*

$$v(x, r) = \begin{cases} -e^{-r}, & \text{if} \quad x = 0, \\ -\infty, & \text{if} \quad x > 0. \end{cases} \tag{7}$$

The related Hamilton-Jacobi-Bellman (HJB) equation is given by

$$V_t + \frac{1}{2} x^2 V_{rr} + \max_\xi \left(-\lambda_0 \xi^2 V_r - \xi V_x\right) = 0, \tag{8}$$

$(x, r, t) \in \mathbb{D} \times [0, T]$ with $V(x, r, T; T) = v(x, r)$. Direct calculations yield

$$V(x, r, t; T) = -\exp\left(-r + \sqrt{\frac{\lambda_0}{2}} x^2 \coth \frac{T - t}{\sqrt{2\lambda_0}}\right), \tag{9}$$

and for the optimal feedback control,

$$\xi^*(x, r, t) = \frac{1}{\sqrt{2\lambda_0}} x \coth \frac{T - t}{\sqrt{2\lambda_0}}. \tag{10}$$

Therefore, the value in (6) is given by

$$V(x, r, 0; T) = -\exp\left(-r + \sqrt{\frac{\lambda_0}{2}} x^2 \coth \frac{T}{\sqrt{2\lambda_0}}\right). \tag{11}$$

Using (10) and (4), the optimal liquidation and inventory processes are, indeed, deterministic and given, for $t \in [0, T]$, by

$$X_t^{\xi^*} = x \frac{\sinh \frac{T - t}{\sqrt{2\lambda_0}}}{\sinh \frac{T}{\sqrt{2\lambda_0}}} \quad \text{and} \quad \xi_t^* = \frac{1}{\sqrt{2\lambda_0}} x \frac{\cosh \frac{T - t}{\sqrt{2\lambda_0}}}{\sinh \frac{T}{\sqrt{2\lambda_0}}}. \tag{12}$$

The optimal revenue process $R_t^{\xi^*}$ is, in turn, directly derived from (5) and the above.

From (12), it is easy to see that $X_T^{\xi^*} = 0$ and, thus, the optimal process $\xi^*$ indeed leads to full liquidation. It is worth noticing that this is "*forced*" through the *singular penalization* form of the terminal datum (7).

**Problem** $\mathcal{L}(\lambda_0; 0, \infty)$ : When full liquidation is sought for $T = \infty$, the problem was solved using a similar stochastic optimization approach in [30], yielding

$$V(x, r; 0) = -\exp\left(-r + \sqrt{\frac{\lambda_0}{2}} x^2\right) \quad \text{and} \quad \xi^*(x, r) = \frac{1}{\sqrt{2\lambda_0}} x.$$

The above follows easily from the infinite horizon version of (8). The optimal inventory process and liquidation strategy are given by

$$X_t^{\xi^*} = x e^{-\frac{t}{\sqrt{2\lambda_0}}} \quad \text{and} \quad \xi_t^* = \frac{1}{\sqrt{2\lambda_0}} x e^{-\frac{t}{\sqrt{2\lambda_0}}}.$$

This optimal constant liquidating strategy fully liquidates the initial inventory asymptotically, $\lim_{t\uparrow\infty} X_t^{\xi^*} = 0$.

### 2.2 Forward liquidation problem

We introduce a new problem, which will be the analogue of the backward full liquidation $\mathcal{L}(\lambda_0; 0, T)$, $T \leqslant \infty$, which we cast in the forward setting. The definition of the forward performance criterion is along the lines of the one introduced by M. Musiela and the second author in [23] and [24]. Specifically, we seek a process, depending on the inventory, revenue and time, which is a supermartingale when compiled with them on arbitrary control processes and becomes a martingale at an optimum. As mentioned in the introduction, one of the new elements herein is the solvability time, which yields the horizon within which the forward problem is well defined. In all existing works in the forward literature so far, the focus was on identifying conditions such that the underlying problem - optimal investment and consumption - is well defined for all times. Herein, however, one of the scopes is to study the interplay between full liquidation times, solvability times and times at which the model characteristics change. We further elaborate on this below and in the next section.

As in the backward case, the agent starts at $t = 0$ with model dynamics (4) and (5). For notational flexibility in the upcoming general case in Section 3, we use the generic notation $\lambda$, instead of $\lambda_0$. We denote the forward policies by $\zeta$ and write the dynamics

$$\mathrm{d}X_t^{\zeta} = -\zeta_t \mathrm{d}t \quad \text{and} \quad \mathrm{d}R_t^{\zeta} = -\lambda \zeta_t^2 \mathrm{d}t + X_t^{\zeta} \mathrm{d}W_t, \tag{13}$$

with $X_0 = x,\ R_0 = r,\ (x, r) \in \mathbb{D}$. The (forward) *admissible control* set $\mathcal{A}$ consists of deterministic functions $\zeta$, with $\zeta_t \geqslant 0,\ t \in [0, T^{\zeta})$, where

$$T^{\zeta} := \inf\left\{ t > 0 : \int_0^t \zeta_s \mathrm{d}s = x \right\}, \tag{14}$$

with $\int_0^{T^{\zeta}} \zeta_s^2 \mathrm{d}s < L(\zeta)$, for some $L(\zeta) > 0$, possibly depending on the policy.

**Problem** $\mathcal{P}(\lambda, k; 0)$: Let $\lambda > 0$ and the inventory and revenue processes $X_t^{\zeta}$ and $R_t^{\zeta}$ satisfy (13). Let $k > 0$ and $u : \mathbb{D} \longrightarrow \mathbb{R}^-$ given by

$$u(x, r) = -e^{-r + kx^2}. \tag{15}$$

Find the maximal deterministic time $T(\lambda, k; 0) \geqslant 0$ and a deterministic function $U(x, r, t) : \mathbb{D} \times [0, T(\lambda, k; 0)) \to \mathbb{R}^-$ of the separable exponential form

$$U(x, r, t) = -e^{-r + h(x,t)}, \tag{16}$$

for a suitable function $h \in C^{1,1}(\mathbb{R}^+ \times [0, T(\lambda_0, k_0; 0))$, with the following properties:

i) $U(x, r, 0) = u(x, r)$ and $U(x, r, t)$ is decreasing in $x$, for $(x, r, t) \in \mathbb{D} \times [T(\lambda, k; 0))$,

ii) for any $\zeta \in \mathcal{A}$, the process $U\left(X_t^{\zeta}, R_t^{\zeta}, t\right)$ is a (local) supermartingale, for $t \in [0, T(\lambda, k; 0) \wedge T^{\zeta})$,

iii) there exists $\zeta^* \in \mathcal{A}$ such that the process $U(X_t^{\zeta^*}, R_t^{\zeta^*}, t)$ is a (local) martingale, for $t \in [0, T(\lambda, k; 0) \wedge T^{\zeta})$.

In problem $\mathcal{P}(\lambda, k; 0)$, while the manager trades in a similar market environment as in the backward problem, she does *not* pre-determine a time at which full liquidation may occur. Rather, she chooses an initial datum of form (15) and seeks the longest time such that the conditions (i)−(iii) are satisfied.

We will be calling $T(\lambda, k; 0)$ the *solvability time* of problem $\mathcal{P}(\lambda, k; 0)$. We stress that $T(\lambda, k; 0)$ may be *finite*, a direct consequence of the ill-posed nature of forward performance criteria. We also stress that $T(\lambda, k; 0)$ is *not required* to be a full liquidation time.

While this problem might, for now, look artificial, it will become the building block for the forward approach. As a first step, we show that both backward problems $\mathcal{L}(\lambda_0; 0, T)$ and $\mathcal{L}(\lambda_0; 0, \infty)$ are merely *special cases* of $\mathcal{P}(\lambda, k; 0)$ with $\lambda_0 = \lambda$ and suitable choices of $k$ in the initial condition (15). On the other hand, the opposite is not always true, as we discuss in Remark 5.

We solve $\mathcal{P}(\lambda, k; 0)$ for arbitrary constants $\lambda, k > 0$. We introduce the key parameter,

$$m := k\sqrt{\frac{2}{\lambda}} > 0, \tag{17}$$

and the auxiliary functions $F, G : \mathbb{R}^+ \to \mathbb{R}$,

$$\begin{cases} F(t; m, \lambda) := \cosh \dfrac{t}{\sqrt{2\lambda}} - m \sinh \dfrac{t}{\sqrt{2\lambda}}, \\[2ex] G(t; m, \lambda) := \cosh \dfrac{t}{\sqrt{2\lambda}} - \dfrac{1}{m} \sinh \dfrac{t}{\sqrt{2\lambda}}. \end{cases} \tag{18}$$

**Lemma 1** *Functions $F$ and $G$ in (18) satisfy, for $m > 0$,*

$$F(t; m, \lambda) = G\left(t; \frac{1}{m}, \lambda\right).$$

*Furthermore, $F(0; m, \lambda) = G(0; m, \lambda) = 1$,*

$$G\left(\sqrt{\frac{\lambda}{2}} \ln \frac{1+m}{1-m}; m, \lambda\right) = 0, \quad F\left(\sqrt{\frac{\lambda}{2}} \ln \frac{1+m}{1-m}; m, \lambda\right) > 0, \quad m \in (0,1),$$

*and*

$$F\left(\sqrt{\frac{\lambda}{2}} \ln \frac{1+m}{1-m}; m, \lambda\right) = 0, \quad m > 1.$$

We establish the following auxiliary result for the candidate function $h(x,t)$ in (16).

**Lemma 2** *Let $\lambda, k > 0$ and $m = k\sqrt{\frac{2}{\lambda}}$ (cf. (17)). Define*

$$T(\lambda, k; 0) := \begin{cases} \sqrt{\dfrac{\lambda}{2}} \ln \dfrac{1+m}{1-m}, & m < 1, \\[1ex] \infty, & m = 1, \\[1ex] \sqrt{\dfrac{\lambda}{2}} \ln \dfrac{1+m}{m-1}, & m > 1. \end{cases} \tag{19}$$

*Then, for $t \in [0, T(\lambda, k))$, the Hamilton-Jacobi equation*

$$h_t - \frac{1}{4\lambda} h_x^2 + \frac{1}{2} x^2 = 0, \tag{20}$$

*with*

$$h(x, 0) = kx^2, \quad x > 0 \quad \textit{and} \quad h(0, t) = 0, \tag{21}$$

*has a unique non-decreasing in $x$ solution,*

$$h(x,t)=kx^2\frac{G(t;m,\lambda)}{F(t;m,\lambda)}, \tag{22}$$

*with $F$ and $G$ as in (18). Furthermore, for $x\geqslant 0$,*

$$\lim_{t\uparrow T(\lambda,k;0)} h(x,t)=\begin{cases}0, & m<1,\\ kx^2, & m=1,\\ \infty,\ x>0, & \\ 0,\quad x=0, & m>1.\end{cases} \tag{23}$$

**Proof** We solve equation (20) using the method of characteristics. These curves, denoted by $X(t)$ and $P(t)$ satisfy

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t}=-\frac{1}{2\lambda}P(t),\quad \frac{\mathrm{d}P(t)}{\mathrm{d}t}=-X(t), \tag{24}$$

and

$$\frac{\mathrm{d}h(X(t),t)}{\mathrm{d}t}=-\frac{1}{4\lambda}P^2(t)-\frac{1}{2}X^2(t), \tag{25}$$

with $P(t)=h_x(X(t),t)$, $X(0)=x$. Therefore, for $t\geqslant 0$,

$$X(t)=C_1e^{\frac{t}{\sqrt{2\lambda}}}+C_2e^{-\frac{t}{\sqrt{2\lambda}}}\quad\text{and}\quad P(t)=\sqrt{2\lambda}\left(-C_1e^{\frac{t}{\sqrt{2\lambda}}}+C_2e^{-\frac{t}{\sqrt{2\lambda}}}\right). \tag{26}$$

The initial condition (21) yields $P(0)=h_x(x,0)=2kx$ and, thus,

$$C_1=\frac{x}{2}\left(1-k\sqrt{\frac{2}{\lambda}}\right)\quad\text{and}\quad C_2=\frac{x}{2}\left(1+k\sqrt{\frac{2}{\lambda}}\right), \tag{27}$$

which implies that, for $t\geqslant 0$,

$$X(t)=x\left(\cosh\frac{t}{\sqrt{2\lambda}}-m\sinh\frac{t}{\sqrt{2\lambda}}\right)=xF(t;m,\lambda). \tag{28}$$

Therefore, for $t\geqslant 0$, the function $h(x,t)$ satisfies

$$\begin{aligned}h(X(t),t)&=h(x,0)-\int_0^t\left(\frac{1}{4\lambda}P^2(s)+\frac{1}{2}X^2(s)\right)\mathrm{d}s\\ &=kx^2+\sqrt{\frac{\lambda}{2}}\left(C_2^2e^{-\sqrt{\frac{2}{\lambda}}t}-C_1^2e^{\sqrt{\frac{2}{\lambda}}t}\right)+\sqrt{\frac{\lambda}{2}}\left(C_1^2-C_2^2\right)\\ &=x^2\left(k\cosh\left(\sqrt{\frac{2}{\lambda}}t\right)-\sqrt{\frac{\lambda}{2}}\left(\frac{1}{2}+\frac{k^2}{\lambda}\right)\sinh\left(\sqrt{\frac{2}{\lambda}}t\right)\right).\end{aligned}$$

Next, we seek the longest time $T(\lambda,k;0)$ such that a well-defined solution $h(x,t)$ exists, for each $x\geqslant 0$ and $t\in[0,T(\lambda,k;0))$, which is, in addition, nondecreasing in $x$ and satisfies $h(0,t)=0$. For this, we first need to invert the characteristic curve (28), insuring that for each $X(t)>0$, $t\in[0,T(\lambda,k;0))$, there exists a unique $x>0$ that satisfies (28).

We look at the following cases:

i) If $m=1$, then $F(t;1,\lambda)=e^{-\frac{t}{\sqrt{2\lambda}}}$ while, if $m<1$, $F(t;m,\lambda)>e^{-\frac{t}{\sqrt{2\lambda}}}$. Thus, for $m\leqslant 1$, $F(t;m,\lambda)>0$, $t\geqslant 0$ and therefore, (28) can be inverted for all times $t>0$.

ii) If $m>1$, curve (28) can be inverted only up to the first zero of $F(t;m,\lambda)$, which occurs at the (finite) time, say $T_1$, given by

$$T_1 = \sqrt{2\lambda}\operatorname{arc\,coth} m = \sqrt{\frac{\lambda}{2}} \ln \frac{m+1}{m-1}.$$

Therefore, if we define (with a slight abuse of notation) time $T_1(\lambda, k; 0)$ as

$$T_1(\lambda, k; 0) := \infty, \text{ if } m \leqslant 1 \quad \text{and} \quad T_1(\lambda, k; 0) := \sqrt{\frac{\lambda}{2}} \ln \frac{m+1}{m-1}, \text{ if } m > 1, \tag{29}$$

we deduce that a well-defined solution is given, for $t \in [0, T_1(\lambda, k; 0))$, by

$$\begin{aligned} h(x,t) &= x^2 \frac{k \cosh \sqrt{\frac{2}{\lambda}} t - \sqrt{\frac{\lambda}{2}} \left( \frac{1}{2} + \frac{k^2}{\lambda} \right) \sinh \sqrt{\frac{2}{\lambda}} t}{\left( \cosh \frac{t}{\sqrt{2\lambda}} - k \sqrt{\frac{2}{\lambda}} \sinh \frac{t}{\sqrt{2\lambda}} \right)^2} \\ &= kx^2 \frac{\left( \cosh \frac{t}{\sqrt{2\lambda}} - \frac{1}{m} \sinh \frac{t}{\sqrt{2\lambda}} \right)}{\left( \cosh \frac{t}{\sqrt{2\lambda}} - m \sinh \frac{t}{\sqrt{2\lambda}} \right)} = kx^2 \frac{G(t; m, \lambda)}{F(t; m, \lambda)}. \end{aligned}$$

Note, however, that the above function might not yield a solution $h(x,t)$ which is increasing in $x$, for each $t \in [0, T_1(\lambda, k; 0))$. To this end, let

$$T_2(\lambda, k; 0) := \sqrt{\frac{\lambda}{2}} \ln \frac{1+m}{1-m}, \text{ if } m < 1 \quad \text{and} \quad T_2(\lambda, k; 0) := \infty, \text{ if } m \geqslant 1. \tag{30}$$

Then $F(t; m, \lambda) > 0$ and $G(t; m, \lambda) > 0$, for all $t \in [0, T_2(\lambda, k; 0))$. Combining (29) and (30), we easily conclude.

To show uniqueness, we assume that there are two solutions that are non-decreasing in $x$, say $h \in C^{1,1}(\mathbb{R}^+ \times [0, T))$ and $\tilde{h} \in C^{1,1}\left(\mathbb{R}^+ \times \left[0, \widetilde{T}\right)\right)$, with $\widetilde{T} > T$, satisfying $h(x, 0) = \tilde{h}(x, 0) = kx^2$, $x > 0$, $h(0, t) = \tilde{h}(0, t)$, $t \in [0, T)$. Then, $H := h - \tilde{h}$ satisfies, for $(x, t) \in \mathbb{R}^+ \times [0, T)$,

$$H_t - \frac{1}{4}\left(h_x^2 - \tilde{h}_x^2\right) = H_t - \frac{1}{4} H_x \left(h_x + \tilde{h}_x\right) = 0,$$

with $H(x, 0) = 0$ and $H(0, t) = 0$. For its characteristics, we would then have $\frac{\mathrm{d}X(t)}{\mathrm{d}t} = -\frac{h_x(X(t),t) + \tilde{h}_x(X(t),t)}{4\lambda_0}$, with $h_x(X(t), t) + \tilde{h}_x(X(t), t) \geqslant 0$. Using routine arguments we conclude.

It remains to show (23). The case $m = 1$ is trivial. If $m < 1$, then

$$\lim_{t \uparrow T(\lambda, k; 0)} h(x,t) = \lim_{t \uparrow \sqrt{\frac{\lambda}{2}} \ln \frac{1+m}{1-m}} kx^2 \frac{G(t; m, \lambda)}{F(t; m, \lambda)},$$

and using Lemma 1 we conclude. The case $m > 1$, follows similarly. □

The next result states that within the class of deterministic separable functions (16), the forward liquidation problem $\mathcal{P}(\lambda, k; 0)$ has, for *each pair* $\lambda, k > 0$, a *unique* solution pair $(U(x, r, t), T(\lambda, k; 0))$, which we construct explicitly.

**Proposition 3** *Let $\lambda, k > 0$ and $m = k\sqrt{\frac{2}{\lambda}}$ (cf. (17)). Let also $F$ and $G$ be as in (18), $T(\lambda, k; 0)$ as in (19) and $h(x,t)$ as in (22). The following assertions hold :*

i) *Problem $\mathcal{P}(\lambda, k; 0)$ has a solution pair $(U(x, r, t), T(\lambda, k; 0))$ with $U(x, r, t)$ given, for $(x, r) \in \mathbb{D}$ and $t \in [0, T(\lambda, k; 0))$, by*

$$U(x, r, t) = -e^{-r + h(x,t)}. \tag{31}$$

*This solution is unique in the class of separable functions of form (16). Furthermore, for each* $(x,r)\in\mathbb{D}$,

$$\lim_{t\uparrow T(\lambda,k;0)} U(x,r,t)=\begin{cases}-e^{-r}, & m<1,\\ -e^{-r+kx^2}, & m=1,\\ -\infty,\quad x>0, & \\ -e^{-r},\ x=0, & m>1.\end{cases} \tag{32}$$

ii) *The optimal policy* $\zeta^*$ *and optimal inventory* $X^{\zeta^*}$ *are given, respectively, for* $t\in[0,T(\lambda,k;0))$*, by*

$$\zeta_t^*=\frac{1}{2\lambda}h_x\left(X_t^{\zeta^*},t\right)=\frac{k}{\lambda}x\frac{G(t;m,\lambda)}{F(t;m,\lambda)}, \tag{33}$$

*and*

$$X_t^{\zeta^*}=xF(t;m,\lambda). \tag{34}$$

iii) *For each* $x>0$,

$$\lim_{t\uparrow T(\lambda,k;0)} X_t^{\zeta^*}=\begin{cases}x\sqrt{(1-m)(1+m)}, & m<1,\\ 0, & m\geqslant 1.\end{cases} \tag{35}$$

*Therefore, if* $m\geqslant 1$*, the optimal policy* $\zeta^*$ *is also a full liquidation policy at solvability time* $T(\lambda,k;0)$.

**Proof** It follows trivially that $U(x,r,0)=u(x,r)$. To show properties (ii) and (iii), we work as follows. Let $\zeta\in\mathcal{A}$. Then, for $t\in[0,T(\lambda,k;0))$, the functions $h(x,s)$ and $U(x,r,s)$ are well defined for $s\in[0,t]$. Ito's formula yields, with $T^\zeta$ as in (14), gives

$$\begin{aligned}
&U(X^\zeta_{t\wedge T^\zeta},R^\zeta_{t\wedge T^\zeta},t\wedge T^\zeta)\\
&=u(r,x)+\int_0^{t\wedge T^\zeta}\left(U_t\left(X_s^\zeta,R_s^\zeta,s\right)-\zeta_sU_x\left(X_s^\zeta,R_s^\zeta,s\right)\right)\mathrm{d}s\\
&\quad+\int_0^{t\wedge T^\zeta}\left(-\lambda\zeta_s^2U_r\left(X_s^\zeta,R_s^\zeta,s\right)+\frac12\left(X_s^\zeta\right)^2U_{rr}\left(X_s^\zeta,R_s^\zeta,s\right)\right)\mathrm{d}s+\int_0^{t\wedge T^\zeta}X_s^\zeta U_r(X_s^\zeta,R_s^\zeta,s)\mathrm{d}W_s\\
&=\int_0^{t\wedge T^\zeta}\left(\zeta_s-\frac{1}{2\lambda}h_x\left(X_s^\zeta,s\right)\right)^2U\left(X_s^\zeta,R_s^\zeta,s\right)\mathrm{d}s+\int_0^{t\wedge T^\zeta}X_s^\zeta U_r(X_s^\zeta,R_s^\zeta,s)\mathrm{d}W_s\\
&=-\int_0^{t\wedge T^\zeta}\left(\zeta_s-\frac{1}{2\lambda}h_x\left(X_s^\zeta,s\right)\right)^2e^{-R_s^\zeta+h\left(X_s^\zeta,s\right)}\mathrm{d}s+\int_0^{t\wedge T^\zeta}X_s^\zeta U_r(X_s^\zeta,R_s^\zeta,s)\mathrm{d}W_s,
\end{aligned} \tag{36}$$

where we used (31) and (20). Next, we show that the process

$$\int_0^{t\wedge T^\zeta}X_s^\zeta U_r(X_s^\zeta,R_s^\zeta,s)\mathrm{d}W_s=\int_0^{t\wedge T^\zeta}e^{-R_s^\zeta+h(X_s^\zeta,s)}X_s^\zeta\mathrm{d}W_s$$

is a genuine martingale, for $t\in[0,T(\lambda,k;0))$. To this end, first note that

$$\mathbb{E}\left[\int_0^t\left(e^{-R_s^\zeta+h(X_s^\zeta,s)}X_s^\zeta\right)^2\mathrm{d}s\right]\leqslant x^2\mathbb{E}\left[\int_0^t e^{-2R_s^\zeta+2kx^2\frac{G(s;m,\lambda)}{F(s;m,\lambda)}}\mathrm{d}s\right].$$

Furthermore, if $m\geqslant 1$, (18) yields $\frac{G(s;m,\lambda)}{F(s;m,\lambda)}\leqslant\frac{G(t;m,\lambda)}{F(t;m,\lambda)}$ while, if $m<1$, $\frac{G(s;m,\lambda)}{F(s;m,\lambda)}\leqslant\frac{G(0;m,\lambda)}{F(0;m,\lambda)}=1$.

Therefore, it suffices to show that $\mathbb{E}\left[\int_0^{t\wedge T^\zeta}e^{-2R_s^\zeta}\mathrm{d}s\right]<\infty$. From the admissibility of $\zeta$, there exist constants $L(\zeta)$, $K(\zeta)>0$, such that $\int_0^{T^\zeta}\zeta_s^2\mathrm{d}s<L(\zeta)$ and $\mathbb{E}\left[\int_0^{T^\zeta}(X_s^\zeta)^2\mathrm{d}s\right]<K(\zeta)$. Hence, using dynamics (13) we obtain, for $t\in[0,T(\lambda,k;0))$,

$$\begin{aligned}
\mathbb{E}\left[\int_0^{t\wedge T^\zeta}e^{-2R_s^\zeta}\mathrm{d}s\right]&=\mathbb{E}\left[\int_0^{t\wedge T^\zeta}\exp\left(-2r-2\int_0^s X_u^\zeta\mathrm{d}W_u+2\lambda\int_0^s\zeta_u^2du\right)\mathrm{d}s\right]\\
&\leqslant e^{-2r+2\lambda L(\zeta)}\mathbb{E}\left[\int_0^{t\wedge T^\zeta}e^{-2\int_0^s X_u^\zeta\mathrm{d}W_u}\mathrm{d}s\right]\\
&=e^{-2r+2\lambda L(\zeta)}\int_0^t\mathbb{E}\left[e^{-2\int_0^{s\wedge T^\zeta}X_u^\zeta\mathrm{d}W_u}\right]\mathrm{d}s\\
&\leqslant e^{-2r+2\lambda L(\zeta)}\int_0^t e^{2K(\zeta)}\mathrm{d}s=e^{-2r+2\lambda L(\zeta)+2K(\zeta)}t<e^{-2r+2\lambda L(\zeta)+2K(\zeta)}T(\lambda,k),
\end{aligned}$$

where we have used that $\int_0^{s\wedge T^\zeta}X_u^\zeta\mathrm{d}W_u$, $s\in[0,t]$, is a square integrable martingale with quadratic variation at most $K(\zeta)$. We easily conclude.

Next, let, for $t\in[0,T(\lambda,k;0))$ the feedback policy $\zeta_t^*=\frac{1}{2\lambda}h_x(X_t^*,t)>0$. Then, (13), (18) and (22) give

$$\mathrm{d}X_t^{\zeta^*}=-\frac{k}{\lambda}X_t^{\zeta^*}\frac{\cosh\dfrac{t}{\sqrt{2\lambda}}-\dfrac{1}{m}\sinh\dfrac{t}{\sqrt{2\lambda}}}{\cosh\dfrac{t}{\sqrt{2\lambda}}-m\sinh\dfrac{t}{\sqrt{2\lambda}}}\mathrm{d}t,\quad X_0^*=x,\tag{37}$$

and solution (34) follows. In turn, $\zeta_t^*$ is given by the deterministic function in (33). Note that for $t\in[0,T(\lambda,k;0))$, all involved quantities are well defined. We, then, easily deduce that this policy is admissible and its optimality follows from (36).

We next look at $\lim_{t\uparrow T(\lambda,k;0)}X_t^{\zeta^*}$. If parameters $\lambda,k$ are such that the associated $m<1$, then (34), (18) and (19) give

$$\begin{aligned}
\lim_{t\uparrow T(\lambda,k;0)}X_t^{\zeta^*}&=x\left(\cosh\left(\tanh^{(-1)}m\right)-m\sinh\left(\tanh^{(-1)}m\right)\right)\\
&=x\left(1-m^2\right)\cosh\left(\tanh^{(-1)}m\right)=x\left(1-m^2\right)\cosh\left(\ln\sqrt{\frac{1+m}{1-m}}\right)\\
&=\frac{1}{2}x\left(1-m^2\right)\left(\sqrt{\frac{1+m}{1-m}}+\sqrt{\frac{1-m}{1+m}}\right)=x\sqrt{(1-m)(1+m)}>0.
\end{aligned}\tag{38}$$

If $m=1$, then $T(\lambda,k;0)=\infty$ and (34) and (19) give $\lim_{t\uparrow\infty}X_t^{\zeta^*}=\lim_{t\uparrow\infty}xe^{-\frac{k}{\lambda}t}=0$ and, thus, $T^{\zeta^*}=T(\lambda,k;0)=\infty$.

Finally, if $m>1$, then (34) and (19) yield that $\lim_{t\uparrow T(\lambda,k;0)}X_t^{\zeta^*}=0$, and that $T^{\zeta^*}=T(\lambda,k;0)<\infty$. Therefore, the optimal forward policy achieves full liquidation at time $T(\lambda,k;0)$. □

**Corollary 4** *For $x>0$ and $t\in[0,T(\lambda,k;0))$, we have*

$$h_t(x,t)\gtrless 0\ \textit{if}\ m\gtrless 1\quad\textit{and}\quad h_t(x,t)=0\ \textit{if}\ m=1.$$

*Moreover, the optimal liquidation strategy $\zeta^*$ satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}\zeta_t^*=\frac{1}{2}k\frac{m^2-1}{F^2(t;m,\lambda)}.$$

*Therefore, if $m>1$ (resp .$m<1$), $\zeta_t^*$ is strictly increasing (decreasing) in time. If $m=1$, the optimal policy is constant, $\zeta_0^*=x\frac{k}{\lambda}\frac{G(t;1,\lambda)}{F(t;1,\lambda)}=x\frac{k}{\lambda}$.*

## 2.3 Reconciling the backward and forward core liquidation problems

The forward liquidation problem $\mathcal{P}(\lambda,k;0)$ is solvable for any parameter pair $(\lambda,k)$, up to solvability time $T(\lambda,k;0)$, which is *endogenous* derived. This is expected given that $\mathcal{P}(\lambda,k;0)$ is an ill-posed problem. Furthermore, the related optimal liquidation strategies might *not* liquidate fully if the model input $(\lambda,k)$ is such that the critical parameter $m=k\sqrt{\frac{2}{\lambda}}<1$. This may, for example, happen if $\lambda$ is very large.

The backward liquidation problem $\mathcal{L}(\lambda_0;0,T)$ is, also, solvable for any pair $(\lambda_0,T)$ but with the full liquidation time $T$ being *exogenously* given. The optimal strategy is always well defined and achieves full liquidation exactly at $T$ (and not earlier; this is a direct consequence of choosing the singular terminal datum $v(x,r)$ in (7)).

We now connect the two problems $\mathcal{P}(\lambda,k;0)$ and $\mathcal{L}(\lambda_0;0,T)$. We establish that for *each* backward problem $\mathcal{L}(\lambda_0;0,T)$, $T\leqslant\infty$, there exists a *unique* parameter $k_0$ such that the forward liquidation problem $\mathcal{P}(\lambda_0,k_0;0)$ has the same optimal solution and, thus, is itself a full liquidation problem. The inverse, however, may not always be true as we discuss in Remark 5 below.

### 2.3.1 Full liquidation in finite time

Let $\lambda_0,T$ with $0<\lambda_0,T<\infty$ be arbitrary but fixed. Introduce the constant

$$k_0:=\sqrt{\frac{\lambda_0}{2}}\coth\frac{T}{\sqrt{2\lambda_0}},\tag{39}$$

and consider the forward liquidation problem $\mathcal{P}(\lambda_0,k_0;0)$. Then, (17) gives

$$m_0=\coth\frac{T}{\sqrt{2\lambda_0}}>1,\tag{40}$$

and, in turn, from (19) we have

$$T(\lambda_0,k_0;0)=\sqrt{2\lambda_0}\ln\sqrt{\frac{1+m_0}{1-m_0}}=T.\tag{41}$$

Therefore, the full liquidation time $T$ of the classical problem $\mathcal{L}(\lambda_0;0,T)$ coincides with the solvability time $T(\lambda_0,k_0;0)$ of the forward problem $\mathcal{P}(\lambda_0,k_0;0)$. Furthermore, for $t\in[0,T)$, (33) and (39) give,

$$\begin{aligned}\zeta_t^*&=\frac{k_0}{\lambda_0}x\frac{\cosh\frac{t}{\sqrt{2\lambda_0}}-\sinh\frac{t}{\sqrt{2\lambda_0}}}{\cosh\frac{t}{\sqrt{2\lambda_0}}-\coth\frac{T}{\sqrt{2\lambda_0}}\sinh\frac{t}{\sqrt{2\lambda_0}}}\\&=\frac{k_0}{\lambda_0}x\frac{\coth\frac{T}{\sqrt{2\lambda_0}}\cosh\frac{t}{\sqrt{2\lambda_0}}-\sinh\frac{t}{\sqrt{2\lambda_0}}}{\coth\frac{T}{\sqrt{2\lambda_0}}\left(\cosh\frac{t}{\sqrt{2\lambda_0}}-\coth\frac{T}{\sqrt{2\lambda_0}}\sinh\frac{t}{\sqrt{2\lambda_0}}\right)}\\&=\frac{1}{\sqrt{2\lambda_0}}x\frac{\cosh\frac{T-t}{\sqrt{2\lambda_0}}}{\sinh\frac{T}{\sqrt{2\lambda_0}}}=\xi_t^*.\end{aligned}$$

Obviously, the optimal inventory processes of the two problems coincide,

$$X_t^{\xi^*} = X_t^{\varsigma^*}, \quad t \in [0, T]$$

and with full liquidation at terminal time, $X_T^{\xi^*} = X_{T(\lambda_0, k_0; 0)}^{\varsigma^*} = 0$. Notice that full liquidation at $T(\lambda_0, k_0; 0)$ for problem $\mathcal{P}(\lambda_0, k_0; 0)$ is expected since the parameter $m_0$ above satisfies $m_0 > 1$. The latter essentially implies that $\lambda_0$ is relatively small and, thus, favorable for full liquidation.

Tedious but direct calculations also show that, for $t \in [0, T)$, $U(x, r, t) = V(x, r, t; T)$ and $\lim_{t \uparrow T(\lambda_0, k; 0)} U(x, r, t) = \lim_{t \uparrow T} V(x, r, t) = v(x, r)$ (cf. (7)). Furthermore, the optimal revenue processes in the two problems coincide.

What we have shown is that the traditional optimal full liquidation problem, at finite horizon $T$, can be cast as a forward one, namely,

$$\mathcal{L}(\lambda_0; 0, T) \approx \mathcal{P}\left(\lambda_0, \sqrt{\frac{\lambda_0}{2}} \coth \frac{T}{\sqrt{2\lambda_0}}; 0\right). \tag{42}$$

**Remark 5** *The choice of the constant $k_0$ in (39) is not the only one that yields $T(\lambda_0, k_0; 0) = T$. Indeed, for*

$$k_0' = \sqrt{\lambda_0} \tanh \frac{T}{\sqrt{2\lambda_0}},$$

*we also have $T(\lambda_0, k_0'; 0) = T$. However, in this case, $m_0' = \tanh \frac{T}{\sqrt{2\lambda_0}} < 1$ and, as we have seen in (38), the optimal policy for $P(\lambda_0, k_0'; 0)$ does not lead to full liquidation. Therefore, problems $\mathcal{P}(\lambda_0, k_0'; 0)$ and $\mathcal{L}(\lambda_0; 0, T)$ do not have the same solution.*

*This highlights the fact that while every backward problem $\mathcal{L}(\lambda_0; 0, T)$ can be written as a forward problem $\mathcal{P}(\lambda_0, k_0; 0)$ for a suitable $k_0$, the opposite does not always hold.*

### 2.3.2 Full liquidation in infinite horizon

Let $\lambda_0$ be arbitrary but fixed, and $T = \infty$. Choose $k_0 := \sqrt{\frac{\lambda_0}{2}}$ and consider the forward liquidation problem $\mathcal{P}\left(\lambda_0, \sqrt{\frac{\lambda_0}{2}}; 0\right)$. Then (17) gives $m_0 = 1$ and, thus, $T(\lambda_0, k_0; 0) = T = \infty$. In turn, for $t \geqslant 0$,

$$\zeta_t^* = x \frac{1}{\sqrt{2\lambda_0}} = x \frac{k_0}{\lambda_0} = \xi_t^*.$$

Furthermore, $h(x, t) = \sqrt{\frac{\lambda_0}{2}} x^2$ and, thus, $U(x, r, t) = -\exp\left(-r + \sqrt{\frac{\lambda_0}{2}} x^2\right)$.

Working as in the previous case, we deduce that problems $\mathcal{L}(\lambda_0; 0, \infty)$ and $\mathcal{P}\left(\lambda_0, \sqrt{\frac{\lambda_0}{2}}; 0\right)$ have the same solution. In analogy to (42) we denote this as

$$\mathcal{L}(\lambda_0; 0, \infty) \approx \mathcal{P}\left(\lambda_0, \sqrt{\frac{\lambda_0}{2}}; 0\right). \tag{43}$$

Note that, contrary to the previous case of finite full liquidation horizon, there is always a *unique* choice of the constant $k$ that equates times, $T(\lambda_0, k; 0) = T = \infty$ and, furthermore, this is also (in the limit) a full liquidation time.

So far, we have studied the technical conditions that the initial condition (15) must satisfy in order for a well-defined solution for problem $\mathcal{P}(\lambda_0, k_0; 0)$ to exist. Departing from technical characterizations, it is interesting from the modeling perspective to investigate what are the suitable classes of initial conditions for the forward setting. We do not have a complete answer for this yet so we provide some preliminary thoughts on what could determine the initial forward datum.

i) *Preferences for a targeted average performance*: The initial condition may be taken to coincide with a specific performance the client would like to achieve on the average through time.

ii) *Preference for a liquidation horizon:* The forward agent may have preferences for a pre-chosen liquidation time. This will affect the selection of $k_0$ in (39) and, in turn, the initial condition.

iii) *Preference for a trading pattern:* The forward agent might have a preferred initial trading pattern $\zeta_0^*$ or, more broadly, a preferred pattern across different time regimes.

iv) *Comparative metrics:* The forward agent might want to compare the performance of the liquidation strategies in the classical setting with those in the forward framework. In this case, it would be natural to choose the initial condition to coincide with the value function, at $t = 0$, of the backward full liquidation problem.

## 3. Optimal liquidation under forward performance criteria and sequential model parameter updating

We propose a multi-period framework for optimal liquidation strategies following forward performance criteria, under sequential adjustments of the temporary price impact parameter $\lambda$. We fix, as in the backward case, a horizon $[0, T]$, $T \leqslant \infty$, and assume that the permanent price impact parameter $\gamma_0$ remains constant throughout. In contrast, parameter, $\lambda$, takes values $\lambda_0$, $\lambda_1, ..., \lambda_{N-1}$ at times $t = 0, T_1, ..., T_{N-1}$. Each pair $(\lambda_i, T_{i+1})$, $i = 1, ..., N-2$, becomes known at time $T_i$. As mentioned in the Introduction, we will build a family (cf. (1)) of sequential, forward in time problems,

$$\mathcal{P}_{0,T} := \mathcal{P}_0 \circ \mathcal{P}_1 \circ ... \circ \mathcal{P}_{N-1},$$

each born at the related revision time and with time-consistent pasting from one problem to the next, provided the previous problem remained viable. We make this precise next.

For each problem $\mathcal{P}_i$ we use subscript $i$ for the associated forward criterion, $U_i(x, r, t)$, generic liquidation control $\zeta_i$, the state processes $X_{i,t}^{\zeta_i}$ and $R_{i,t}^{\zeta_i}$ and the optimal control process $\zeta_{i,t}^*$.

**Problem** $\mathcal{P}_0 = \mathcal{P}(\lambda_0, k_0; 0)$ : At time $t = 0$, both $\lambda_0$ and $T_1$ are known. The agent starts with inventory $x$ and revenue $r$, $(x, r) \in \mathbb{D}$, and chooses initial forward criterion of separable form (16),

$$U_0(x, r, 0) = u(x, r) = -e^{-r+kx^2},$$

for some parameter $k > 0$. She solves the first forward liquidation problem $\mathcal{P}_0 = \mathcal{P}(\lambda_0, k_0; 0)$.

Let $T(\lambda_0, k; 0)$ be the related solvability time and $T_1$ the upcoming parameter update time. Let $m_0 := k\sqrt{\frac{2}{\lambda_0}}$ (cf. (17)). Using the results of the previous section, we look at the following cases:

*Case 1:* $T(\lambda_0, k; 0) < T_1 < \infty$.

This case is viable only if $m_0 \neq 1$. The parameter revision (from $\lambda_0$ to $\lambda_1$) per se is irrelevant since the forward liquidation problem $\mathcal{P}(\lambda_0, k; 0)$ is only well defined for times strictly less than $T_1$. Liquidation terminates at $T(\lambda_0, k; 0)$ and the final inventory is given by (35). Specifically,

If $m_0 < 1$, there is non-zero optimal inventory left at $T(\lambda_0, k; 0)$, given by

$$X_{0,T(\lambda_0,k;0)}^{\zeta_0^*} = x\sqrt{(1 - m_0)(1 + m_0)}. \tag{44}$$

If $m_0 > 1$, full liquidation does occurs at $T(\lambda_0, k; 0)$, $X_{0,T(\lambda_0,k;0)}^{\zeta_0^*} = 0$.

*Case 2:* $T(\lambda_0, k; 0) = T_1$

As above, this case is viable only if $m_0 \neq 1$. If $m_0 < 1$, there is non-zero inventory as in (44), which will serve as the initial condition for the inventory process in the next problem $\mathcal{P}_2$.

If $m_0 > 1$, full liquidation occurs, $X^{\zeta_0^*}_{0,T(\lambda_0,k;0)} = X^{\zeta_0^*}_{0,T_1} = 0$, and the problem terminates.

*Case 3:* $T_1 < T(\lambda_0, k; 0)$.

If $m_0 < 1$, then no full liquidation occurs in $[0, T(\lambda_0, k; 0))$ and, thus, neither in $[0, T_1)$. Then, from (13), (22), (33) and (34), we deduce that, for $t \in [0, T_1)$,

$$X^{\zeta_0^*}_{0,t} = xF(t; m_0, \lambda_0) > 0 \quad \text{and} \quad R^{\zeta_0^*}_{0,t} = r - \lambda_0 \int_0^t \left(\zeta^*_{0,s}\right)^2 \mathrm{d}s + \int_0^t X^{\zeta_0^*}_{0,s} \mathrm{d}W_s, \tag{45}$$

and

$$\zeta^*_{0,t} = \frac{k}{\lambda_0} x \frac{G(t; m_0, \lambda_0)}{F(t; m_0, \lambda_0)}. \tag{46}$$

The forward criterion is given by

$$U_0(x, r, t) = -\exp\left(-r + kx^2 \frac{G(t; m_0, \lambda_0)}{F(t; m_0, \lambda_0)}\right),$$

with functions with $F, G$ as in (18) with parameters $m_0$ and $\lambda_0$.

At $T_1$, the forward criterion $U_0(x, r, T_1)$ can be, thus, written as

$$U_0(x, r, T_1) = -e^{-r + k_1 x^2} \quad \text{with} \quad k_1 := k \frac{G(T_1; m_0, \lambda_0)}{F(T_1; m_0, \lambda_0)}, \tag{47}$$

preserving the separable form (16), but now with an updated coefficient $k_1$. Furthermore, there is non-zero inventory at $T_1$, $X^{\zeta_0^*}_{0,T_1} = xF(T_1; m_0, \lambda_0)$ and $R^{\zeta_0^*}_{0,T_1}$ is given by (45) with $\zeta^*_{0,t}$ as in (46).

**Problem** $\mathcal{P}_1 = \mathcal{P}(\lambda_1, k_1; T_1)$ : At $T_1$, the agent learns value $\lambda_1$, which is the *accurate* updated value of $\lambda$. If there is any inventory left, she introduces the new forward liquidation problem $\mathcal{P}_1$ in complete analogy to $\mathcal{P}(\lambda_0, k; 0)$. Specifically, she considers problem $\mathcal{P}_1 = \mathcal{P}(\lambda_1, k_1; T_1)$ with initial datum $U_0(x, r, T_1)$ (cf. (47)) and starting at $x = X^{\zeta_0^*}_{0,T_1}$ and $r = R^{\zeta_0^*}_{0,T_1}$.

Note that this new forward problem $\mathcal{P}(\lambda_1, k_1; T_1)$, introduced at time $T_1$, entirely captures not only the accurate, changed at "real-time", value $\lambda_1$ but, also, incorporates the past performance via its initial condition $U_0(x, r, T_1)$ as well as the initial conditions $x = X^{\zeta_0^*}_{0,T_1}$ and $r = R^{\zeta_0^*}_{0,T_1}$ which depend on the initial model input $(\lambda_0, k; 0)$, chosen at initial time $t = 0$ for $[0, T_1)$.

To solve problem $\mathcal{P}(\lambda_1, k_1; T_1)$, we work as in the first period and choose the parameter $m_1$,

$$\begin{aligned} m_1 &:= k_1 \sqrt{\frac{2}{\lambda_1}} = k_0 \frac{G(T_1; m_0, \lambda_0)}{F(T_1; m_0, \lambda_0)} \sqrt{\frac{2}{\lambda_1}} \\ &= k_0 \frac{\cosh \dfrac{T_1}{\sqrt{2\lambda_0}} - \dfrac{1}{m_0} \sinh \dfrac{T_1}{\sqrt{2\lambda_0}}}{\cosh \dfrac{T_1}{\sqrt{2\lambda_0}} - m_0 \sinh \dfrac{T_1}{\sqrt{2\lambda_0}}} \sqrt{\frac{2}{\lambda_1}}, \end{aligned} \tag{48}$$

where we used (47).

We look at the following cases:

A. Let $m_1 < 1$. The solvability time $T(\lambda_1, k_1; T_1)$ of $\mathcal{P}(\lambda_1, k_1; T_1)$ is given (in analogy to (19)) by

$$T(\lambda_1, k_1; T_1) = \sqrt{2\lambda_1} \ln \sqrt{\frac{1 + m_1}{1 - m_1}}.$$

We introduce time $\tilde{T}_1$,

$$\tilde{T}_1 := T_1 + T(\lambda_1, k_1; T_1) = T_1 + \sqrt{2\lambda_1} \ln \sqrt{\frac{1+m_1}{1-m_1}}. \tag{49}$$

Then, for $t \in \left[T_1, \tilde{T}_1\right)$, the forward criterion $U_1(x, r, t)$ is given by

$$U_1(x, r, t) = -\exp\left(-r + k_1 x^2 \frac{G(t - T_1; m_1, \lambda_1)}{F(t - T_1; m_1, \lambda_1)}\right)$$

which, by construction, satisfies the *time consistency* property

$$U_1(x, r, T_1) = U_0(x, r, T_1). \tag{50}$$

Furthermore, the optimal inventory $X_{1,t}^{\zeta_1^*}$ and liquidation strategy $\zeta_{1,t}^*$ are given, for $t \in \left[T_1, \tilde{T}_1\right)$, by

$$X_{1,t}^{\zeta_1^*} = xF(T_1; m, \lambda_0) F(t - T_1; m_1, \lambda_1),$$

and

$$\zeta_{1,t}^* = X_{1,T_1}^{\zeta_1^*} \frac{k_1}{\lambda_1} \frac{G(t - T_1; m_1, \lambda_1)}{F(t - T_1; m_1, \lambda_1)} = x \frac{k_1}{\lambda_1} F(T_1; m, \lambda_0) \frac{G(t - T_1; m_1, \lambda_1)}{F(t - T_1; m_1, \lambda_1)}.$$

Combining the above and (35), we deduce that there is a non-zero optimal inventory left at $\tilde{T}_1$, given by

$$X_{1,\tilde{T}_1}^{\zeta_1^*} = xF(T_1; m, \lambda_0) \sqrt{(1 - m_1)(1 + m_1)} > 0.$$

B. Let $m_1 = 1$. The solvability time satisfies $T(\lambda_1, k_1; T_1) = \infty$ and, for $t \in [T_1, \infty)$,

$$X_{1,t}^{\zeta_1^*} = xF(T_1; m, \lambda_0) F(t - T_1; 1, \lambda_1),$$

and

$$\zeta_{1,t}^* = \frac{k_1}{\lambda_1} X_{1,T_1}^{\zeta_1^*} = xF(T_1; m, \lambda_0) F(t - T_1; 1, \lambda_1).$$

C. Let $m_1 > 1$. The solvability time is given by $T(\lambda_1, k_1; T_1) = \sqrt{2\lambda_1} \ln \sqrt{\frac{1+m}{m-1}}$. Setting

$$\tilde{T}_1 := T_1 + T(\lambda_1, k_1; \tau_1) = T_1 + \sqrt{2\lambda_1} \ln \sqrt{\frac{1+m_1}{m_1 - 1}},$$

and working as in the proof of Proposition 3, we deduce that the optimal inventory and liquidation policy are given, for $t \in \left[T_1, \tilde{T}_1\right)$, by

$$X_{1,t}^{\zeta_1^*} = xF(T_1; m, \lambda_0) F\left(t - \tilde{T}_1; m_1, \lambda_1\right),$$

and

$$\zeta_{1,t}^* = X_{T_1}^{\zeta^*} \frac{k_1}{\lambda_1} \frac{G(t - T_1; m_1, \lambda_1)}{F(t - T_1; m_1, \lambda_1)} = xF(T_1; m, \lambda_0) \frac{k_1}{\lambda_1} \frac{G(t - T_1; m_1, \lambda_1)}{F(t - T_1; m_1, \lambda_1)}.$$

We deduce that *full liquidation* occurs at $\tilde{T}_1$, i.e., $X_{1,\tilde{T}_1}^{\zeta^*} = 0$.

Next, we calculate the forward criterion $U_1(x, r, t)$, $t \in [T_1, T_1 + T(\lambda_1, k_1; T_1))$. By construction, $U_1(x, r, T_1)$ is given in (50). For $t \in (T_1, T_1 + T(\lambda_1, k_1; T_1))$, we work as in Proposition 3, and seek a criterion of the form

$$U_1(x,r,t) = -e^{-r+h_1(x,t)},$$

where $h_1$ solves, for $t \in [T_1, T_1 + T(\lambda_1, k_1; T_1))$ the Hamilton-Jacobi equation

$$h_{1,t} - \frac{1}{4\lambda_1} h_{1,x}^2 + \frac{1}{2} x^2 = 0,$$

with initial condition $h_1(x, T_1) = k_1 x^2$, $x \geqslant 0$ and $h_1(0, T_1) = 0$.

Let $m_1 := k_1 \sqrt{\frac{2}{\lambda_1}}$. Then, working as in Lemma 2, we deduce that

$$h_1(x,t) = k_1 x^2 \frac{G(t - T_1; m_1, \lambda_1)}{F(t - T_1; m_1, \lambda_1)},$$

for $t \in [T_1, T_1 + T(\lambda_1, k_1; T_1))$, and the solvability time $T(\lambda_1, k_1; T_1)$ derived for the various cases above.

At time $t = T_1$, the agent also needs to choose the next model revision time $T_2$.

If $m_1 \geqslant 1$ and $T_2$ is chosen such that $T_2 \geqslant T_1 + T(\lambda_1, k_1; T_1)$, then full liquidation occurs at $T_1 + T(\lambda_1, k_1; T_1)$, $X^{\zeta_1^*}_{1, T_1 + T(\lambda_1, k_1; T_1)} = 0$, and the liquidation program stops. Notice that $m_1 \geqslant 1$ implies that $\lambda_1$ is relatively small, while $T_2$ being large indicates that the agent is confident that the current market condition with small temporary price impact would last. It is, hence, intuitively reasonable to fully complete the liquidation program in such long-standing favorable market conditions.

If $m_1 < 1$, the agent may choose $T_2 < T_1 + T(\lambda_1, k_1; T_1)$ and, therefore, the forward liquidation program continues, with remaining inventory

$$X^{\zeta_1^*}_{1,T_2} = X^{\zeta_1^*}_{1,T_1} F(T_2 - T_1; m_1, \lambda_1) = xF(T_1; m_0, \lambda_0) F(T_2 - T_1; m_1, \lambda_1) > 0.$$

This assumption is reasonable, since $m_1 < 1$ corresponds to a relatively large $\lambda_1$ that indicates adverse market condition for liquidation. The agent typically would not commit to such $\lambda_1$ for a long time, but rather, revise it before the solvability time.

**Problem** $\mathcal{P}(\lambda_{N-1}, k_{N-1}; T_{N-1})$ : The forward liquidation program continues till $T_{N-1}$, provided non-zero optimal inventory still exists at that time. At $T_{N-1}$, the last forward liquidation problem $\mathcal{P}(\lambda_{N-1}, k_{N-1}; T_{N-1})$ is introduced with $\lambda_{N-1}$ acquired at time $T_{N-1}$. Let,

$$k_{N-1} := k \prod_{i=1}^{N-1} \frac{G(T_i; m_{i-1}, \lambda_{i-1})}{F(T_i; m_{i-1}, \lambda_{i-1})}.$$

Working as above, we may end up with optimal full liquidation strictly before $T$, or finish at $T$ but with non-zero inventory left, all depending on the quantity $m_{N-1} = k_{N-1} \sqrt{\frac{2}{\lambda_{N-1}}}$. Notice that $k_{N-1}$ contains all past, accurately realized parameter value updates $\lambda_0, \lambda_1, ..., \lambda_{N-2}$, all realized before the last period and, furthermore, it interacts with $\lambda_{N-1}$ and $T - T_{N-1}$ in this last period to jointly determine the feasibility of full liquidation before or at expiration horizon $T$.

## 4. The continuous time case

We conclude presenting some preliminary, mostly formal, arguments for developing a forward performance methodology when changes in the parameter $\lambda$ take place continuously. The analogues of (4) and (5) would then be

$$\mathrm{d}X_t^\zeta = -\zeta_t \mathrm{d}t \quad \text{and} \quad \mathrm{d}R_t^\zeta = -\lambda_t \zeta_t^2 \mathrm{d}t + \sigma_t X_t^\zeta \mathrm{d}W_t, \tag{51}$$

with $X_0^\zeta = x \in \mathbb{R}^+$, and $R_0^\zeta = r \in \mathbb{R}$, where $W$ is a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\{\mathcal{F}_t\}_{t \geqslant 0}$

the related filtration satisfying the usual conditions. We take the processes $\sigma_t, \lambda_t > 0,\ t \geqslant 0$ to be continuous and $\mathcal{F}_t$-adapted. The policies $\zeta_t$ are continuous, $\zeta_t \geqslant 0$, $\mathcal{F}_t$-adapted and such that the state processes above are well defined and $X_t^{\zeta} \geqslant 0$, for $t \in [0, \tau_1(\omega)]$, for some $\tau_1^{\zeta}(\omega)$.

In analogy to its definition in the context of optimal investment in continuous time (see, for example, [23, 26]), a forward performance process $U(x,r,t),\ t \in [0, \tau(\omega))$, with $\tau(\omega) \leqslant \tau_1(\omega)$ and $(x,r) \in \mathbb{D}$, is an $\mathcal{F}_t$-progressively measurable process, such that the mapping $r \mapsto U(x,r,t)$ is strictly increasing and strictly convex, for each $(x,t)$, and $x \mapsto U(x,r,t)$ is decreasing for each $(r,t)$. Furthermore, the process $U(X_t^{\zeta}, R_t^{\zeta}, t)$ is a (local) supermartingale and there exists a process $\zeta^*$ such that $U(X_t^{\zeta_t^*}, R_t^{\zeta_t^*}, t)$ is a (local) martingale. We do not pre-impose a full liquidation time which, as we saw in the previous cases, might not even exist. We are, also, not specific about times $\tau_1(\omega)$ and $\tau(\omega)$, which may be infinite (as in the case of forward criteria in optimal investment) or finite (in analogy to the solvability times in the previous sections). We note that the assumption $\lambda_t \in \mathcal{F}_t$ is only for simplicity, as richer measurability can be easily accommodated.

We consider a candidate forward processes $U(x,r,t)$ with Ito-diffusion representation

$$\mathrm{d}U(x,r,t) = b(x,r,t)\mathrm{d}t + a(x,r,t)\mathrm{d}W_t,$$

and $U(x,r,0) = u(x,r)$ as in (16). The forward volatility process $a(x,r,t)$ is taken to be $\mathcal{F}_t$-adapted and continuously differentiable in the variable $r$, for each $(x,t)$. Assuming that $U(x,r,t)$ is smooth enough so that the Itô-Ventzell formula can be applied to $U(X_t^{\zeta}, R_t^{\zeta}, t)$ for each admissible policy $\zeta$, we obtain

$$\begin{aligned}\mathrm{d}U(X_t^{\zeta}, R_t^{\zeta}, t) = &\Big(b(X_t^{\zeta}, R_t^{\zeta}, t) - \zeta_t U_x(X_t^{\zeta}, R_t^{\zeta}, t) - \lambda_t \zeta_t^2 U_r(X_t^{\zeta}, R_t^{\zeta}, t) \\ &+ \frac{1}{2}\sigma_t^2 X_t^2 U_{rr}(X_t^{\zeta}, R_t^{\zeta}, t) + \sigma_t X_t a_r(X_t^{\zeta}, R_t^{\zeta}, t)\Big)\mathrm{d}t \\ &+ \Big(\sigma_t X_t U_r(X_t^{\zeta}, R_t^{\zeta}, t) + a(X_t^{\zeta}, R_t^{\zeta}, t)\Big)\mathrm{d}W_t.\end{aligned}$$

The candidate optimal trading rate $\zeta^*$ takes the form

$$\zeta_t^* = -\frac{1}{2\lambda_t}\frac{U_x(X_t^{\zeta^*}, R_t^{\zeta^*}, t)}{U_r(X_t^{\zeta^*}, R_t^{\zeta^*}, t)} \geqslant 0. \tag{52}$$

Then, $U(x,r,t)$ is expected to satisfy the stochastic partial differential equation (SPDE)

$$\begin{aligned}\mathrm{d}U(x,r,t) = &\left(-\frac{1}{4\lambda_t}\frac{U_x(x,r,t)^2}{U_r(x,r,t)} - \sigma_t x^2\left(\frac{1}{2}\sigma_t U_{rr}(x,r,t) + a_r(x,r,t)\right)\right)\mathrm{d}t \\ &+ a(x,r,t)\mathrm{d}W_t,\end{aligned} \tag{53}$$

with the initial datum $U(x,r,0) = u(x,r)$. SPDE for forward performance processes were first developed in [25] and later developed by others (see, among others, [9] and [11]).

In the zero volatility case, $a(x,r,t) \equiv 0,\ (x,r) \in \mathbb{D},\ t \geqslant 0$, the above equation reduces to

$$\mathrm{d}U(x,r,t) = -\left(\frac{1}{4\lambda_t}\frac{U_x(x,r,t)^2}{U_r(x,r,t)} + \frac{1}{2}\sigma_t^2 x^2 U_{rr}(x,r,t)\right)\mathrm{d}t. \tag{54}$$

We stress that, in contrast to the zero-volatility case arising in optimal investment (see [26]) where the related forward utility is always decreasing in time, this is not the case in (54).

Another observation is that when $\lambda_t \equiv \lambda$ and $\sigma_t \equiv \sigma$, for some constants $\lambda, \sigma > 0$, the solution to (54) is deterministic, and satisfies the HJB equation

$$U_t(x,r,t) + \frac{1}{4\lambda}\frac{U_x(x,r,t)^2}{U_r(x,r,t)} + \frac{1}{2}\sigma^2 x^2 U_{rr}(x,r,t) = 0, \tag{55}$$

with $U(x,r,0) = u(x,r)$ (cf. (15)), which is the ill-posed analogue of equation (8).

A general solution process for (54) is not available yet. However, we may consider a popular case for which the scaling assumptions make the problem tractable. Specifically, the authors in [1] studied the so-called *coordinated variation case*, i.e. when the process $\sigma_t^2\lambda_t$ is constant across time. As discussed therein, this is observed for periods when the largest fraction of the trading occurs. Without loss of generality, we assume that

$$\sigma_t^2\lambda_t = 1, \quad t \geqslant 0. \tag{56}$$

We can, then, build the continuous time analogue of the results in Proposition 3. We choose an initial datum of the separable exponential form,

$$U(x,r,0) = -\exp\left(-r + \frac{x^2}{\sqrt{2}}\coth\left(\frac{T}{\sqrt{2}}\right)\right), \tag{57}$$

parametrized by a constant $T > 0$; as in Section 2, $T$ may reflect the confidence of the agent about the horizon $[0,T]$ in which the constant variation property holds. Let $\tau(\omega) := \inf\{s > 0 : \int_0^s \frac{1}{\lambda_\rho}\mathrm{d}\rho = T\}$. Then, using arguments as in the proof of Proposition 2 we deduce that the process

$$U(x,r,t) = -\exp\left(-r + \frac{x^2}{\sqrt{2}}\coth\left(\frac{T - \int_0^t \frac{1}{\lambda_s(\omega)}\mathrm{d}s}{\sqrt{2}}\right)\right) \tag{58}$$

solves equation (53) for $t \in (0, \tau(\omega))$ and, furthermore, it is the unique solution in such separable form. Indeed, if we choose $U(x,r,t) = u(x,r,\int_0^t \sigma_s^2\mathrm{d}s)$, for some smooth, deterministic function $u(x,r,t)$, with $(x,r) \in \mathbb{D}$ and $t \in [0,\tau(\omega))$, satisfying (57), then equation (54) and the coordinated variation condition (56) yield that

$$u_t + \frac{1}{4}\frac{u_x^2}{u_r} + \frac{1}{2}x^2 u_{rr} = 0,$$

with initial condition (57). Within the separable family $u(x,r,t) = -e^{-r+h(x,t)}$, we obtain the unique solution

$$u(x,r,t) = -\exp\left(-r + \frac{x^2}{\sqrt{2}}\coth\left(\frac{T-t}{\sqrt{2}}\right)\right),$$

and the rest of the arguments follow. The associated optimal inventory process is given by

$$X_t^{\zeta^*} = x\exp\left(-\int_0^t \frac{1}{\sqrt{2}\lambda_s}\coth\left(\frac{T - \int_0^s \frac{1}{\lambda_u}\mathrm{d}u}{\sqrt{2}}\right)\mathrm{d}s\right). \tag{59}$$

It follows that the forward optimal strategy does not necessarily lead to full liquidation before or at the desired horizon $T$, which is implicitly set based on the information that (56) holds up to $T$. Indeed, if for some small $\varepsilon > 0$, it holds that $\int_0^t \frac{1}{\lambda_s(\omega)}\mathrm{d}s < T - \varepsilon$, $t > 0$, *a.s.*, it can be shown from (59) that there exists a positive constant $C$, such that $X_t^{\zeta^*} > C > 0$, $t > 0$, a.s. This condition may hold in a market with large price impact and, in this case, zero-volatility forward criteria may not yield full liquidation at the optimum. On the other hand, if process $\lambda_t$ is uniformly bounded from above and away from zero, then $\tau(\omega) < \infty$ (recall that $\tau(\omega) := \inf\{s > 0 : \int_0^s \frac{1}{\lambda_\rho}d\rho = T\}$) and full liquidation is optimal therein. In other words, with moderate market impact it is always possible to complete liquidation in finite time and maintain intertemporal consistency. However, the time at which this occurs, might not coincide with $T$.

## 5. Conclusions

We proposed an application of forward utilities to study the performance of liquidation strategies in a model with " real-time" sequential parameter adjustments. The modeling framework is based on the works on [30] and [31] who cast the full-liquidation problem as an exponential utility problem with singular terminal datum (which guarantees full liquidation by infinite penalization). The key difficulty stems from the fact that parameter adjustments and their associated validity horizons are generated in a model-free way and become known only at the time they change. This creates considerable difficulties when applying a traditional adaptive control kind of approach in which a model has to be guessed at initial time for all upcoming unknown parameter regimes. Once the first parameter change becomes known, a new full liquidation problem is activated with the parameter known in its first part, but remaining unknown for the rest. The process of guessing a model for this remaining period and solving the new problem is being repeated over and over. Naturally, model error is being accumulated even though full liquidation is ultimately accomplished.

Herein, we propose a complementary approach in which a series of optimal liquidation (but not necessarily of full liquidation at expiration) problems is used, with each one being generated every time the parameter is being adjusted. Each problem is cast in a forward manner, using the terminal value of its predecessor as it initial value, thus preserving time consistency across all times. It lasts till the minimum of its solvability time and the confidence time interval of the current parameter value. Depending on the initial condition we choose and the sequential interplay of parameter values and their confidence times, full liquidation may be achieved before the terminal horizon. Alternatively, the entire forward program expires without full liquidation being achieved. Possible failure of full liquidation should be viewed as a trade-off for time consistency and optimization of the expected forward utility of inventory and revenue, without the considerable value losses that occur in the classical setting due to compounding erroneous model choices.

There are various extensions that can be considered. The first direction is to extend the work to forward utilities which preserve the exponential separable form but have more general dependence on the inventory, $U(x,r,t) = -\exp(-r + g(x,t))$, $(x,r) \in \mathbb{D}$, for suitable functions $g$. This family will offer more flexibility, in particular in choosing more general initial conditions, $U(x,r,0) = -\exp(-r + g(x,0))$; we refer he reader to [35] for further details.

The second direction could be on how to build meaningful comparisons between the forward and the backward approaches. This is far from obvious as each approach has distinct, non comparable advantages and disadvantages. Indeed, the backward methodology always leads to full liquidation but in order to accomplish this, especially when the market conditions are unfavorable, significant losses of value can occur. Losses also occur because of the inherent model misspecification which compounds at each model choice iteration. Finally, time consistency is lost throughout. Alternatively, the forward approach may fail to provide full liquidation all together. On the other hand, it is always time consistent, follows the parameter changes accurately through time and provides non-eroded by wrong model choices optimized values. To compare the two approaches one needs to build rich enough stochastic regret metrics, which has not been done to date; some preliminary results may be found in [36].

The third direction, which may be the hardest, is how to incorporate the constraint of full liquidation at a given time while maintaining time consistency through the forward approach. Intuitively, forward criteria with volatility are expected to be considered but the related volatility process could be rather singular close to the full liquidation horizon. To our knowledge, such questions have not been examined to date.

## Appendix

### A. The backward approach in optimal liquidation with sequential parameter updates.

To juxtapose the forward approach we developed with the backward one, we highlight some results for the latter setting. We consider the case that the parameter changes only once as the calculations for the multi case are rather tedious. We, also, do not provide all needed technical assumptions as they are mainly variations of those in [31].

To this end, we assume that full liquidation is imposed at some $T<\infty$ and parameter $\lambda$ takes only two values, $\lambda_0$ and $\lambda_1$, and the update occurs at $T_1<T$. Both $\lambda_0$ and $T_1$ are predetermined at $t=0$, but $\lambda_1$ becomes known only at $T_1$. As we elaborate below, we will solve the backward liquidation program $\mathcal{L}_{0,T}$ as in (2), which has two components,

$$\begin{cases}\mathcal{L}_{0,T}=\mathcal{L}\left(\hat{\lambda};T_1,T\right)\circ\mathcal{S}\left(\lambda_0;0,T_1\right),\\ \mathcal{L}_{T_1,T}=\mathcal{L}\left(\lambda_1;T_1,T\right).\end{cases}$$

The pre-committed to full liquidation agent chooses a singular terminal datum as in (7) and seeks to solve an optimization problem $\mathcal{L}_{0,T}$ similar to (6) on $[0,T]$. Therefore, he needs to *pre-specify* at $t=0$ the model dynamics for the *entire* horizon $[0,T]$, which obviously cannot be done accurately as $\lambda_1$ is not known yet. Thus, he uses a proxy, say $\hat{\lambda}$, for $[T_1,T]$, which represents his best, at $t=0$, estimate for the upcoming new value of $\lambda$, $\lambda_1$. In general, of course, error model will occur since $\hat{\lambda}$ may not coincide with $\lambda_1$. The related model dynamics are, then, given by

$$\hat{X}^{\hat{\xi}_0}_{0,t}=x-\int_0^t\hat{\xi}_{0,s}\mathrm{d}s\quad\text{and}\quad\hat{R}_{0,t}=r-\lambda_0\int_0^t\hat{\xi}^2_{0,s}\mathrm{d}s+\int_0^t\hat{X}^{\hat{\xi}_0}_{0,s}\mathrm{d}W_s,\tag{60}$$

for $t\in[0,T_1)$, and

$$\hat{X}^{\hat{\xi}_1}_{1,t}=\hat{X}_{1,T_1}-\int_{T_1}^t\hat{\xi}_{1,t}\mathrm{d}t\quad\text{and}\quad\hat{R}^{\hat{\xi}_1}_{1,t}=\hat{R}_{1,T_1}-\hat{\lambda}\int_{T_1}^t\hat{\xi}^2_{1,s}\mathrm{d}s+\int_{T_1}^t\hat{X}^{\hat{\xi}_1}_{1,s}\mathrm{d}W_s,\tag{61}$$

for $t\in[T_1,T]$, with $\hat{X}_{1,T_1}:=\lim_{t\uparrow T_1}\hat{X}^{\hat{\xi}_0}_{0,t}$ and $\hat{R}_{1,T_1}:=\lim_{t\uparrow T_1}\hat{R}^{\hat{\xi}_0}_{0,t}$.

We use the subscripts 0 and 1 for periods $[0,T_1)$ and $[T_1,T]$, respectively. We, also, use the superscript "$\hat{}$" notation for all state processes in both (60) and (61), even though the model dynamics in (60) are correct. We do this because even for $[0,T_1)$, where the model is indeed correct, the optimal policies are being eroded by the erroneous model choice on $[T_1,T]$.

By backward induction, the agent must solve two distinct expected utility problems, on $[0,T_1]$ and $[T_1,T]$, respectively. For $[T_1,T]$, she will solve problem $\mathcal{L}\left(\hat{\lambda};T_1,T\right)$, the expected utility problem similar to (6) with full liquidation requirement at $T$. In turn, its value function at $T_1$, $\hat{V}_1\left(x,r,T_1\right)$, will serve as the terminal utility of the optimization problem she will solve on $[0,T_1]$. We will see that $\hat{V}_1\left(x,r,T_1\right)$ is not of singular type as in (7) and, thus, the optimization problem on $[0,T_1]$, denoted by $\mathcal{S}\left(\lambda_0;0,T_1\right)$, will not be a full liquidation problem.

We recall, as mentioned in the Introduction, that problem $\mathcal{L}\left(\hat{\lambda};T_1,T\right)$ is "virtual" in that it would never be materialized since as soon as time $T_1$ arrives, the agent will learn the true value $\lambda_1$ and dynamics (61) will be obsolete (unless it happens that $\hat{\lambda}$ coincides with $\lambda_1$, which is not, in general, the case). However, the solution of $\mathcal{L}\left(\hat{\lambda}; T_1, T\right)$ is needed, otherwise the problem of the pre-committed agent cannot be computed for $t \in [0, T_1]$.

In complete analogy with the results in [31], we obtain that the solution $\hat{V}_1(x, r, t)$ of $\mathcal{L}\left(\hat{\lambda}; T_1, T\right)$ is given, for $(x, r) \in \mathbb{D}$ and $t \in [T_1, T]$, by

$$\hat{V}_1(x, r, t) = -\exp\left(-r + \sqrt{\frac{\hat{\lambda}}{2}} x^2 \coth\frac{T-t}{\sqrt{2\hat{\lambda}}}\right). \tag{62}$$

For the first period $[0, T_1]$, the agent then formulates optimization problem $\mathcal{S}(\lambda_0; 0, T_1)$ and solves

$$\begin{aligned} \hat{V}_0(x, r, t) &= \sup_{\hat{\xi}} \mathbb{E}\left[\hat{V}_1\left(\hat{X}^{\hat{\xi}_0}_{0,T_1}, \hat{R}^{\hat{\xi}_0}_{0,T_1}, T_1\right) \Big| \hat{X}^{\hat{\xi}_0}_{0,t} = x, \hat{R}^{\hat{\xi}_0}_{0,t} = r\right] \\ &= \sup_{\hat{\xi}} \mathbb{E}\left[-e^{-\hat{R}^{\hat{\xi}_0}_{0,T_1} + \left(\hat{X}^{\hat{\xi}_0}_{0,T_1}\right)^2 \sqrt{\frac{\hat{\lambda}}{2}} \coth\frac{T-T_1}{\sqrt{2\hat{\lambda}}}} \middle| \hat{X}^{\hat{\xi}_0}_{0,t} = x, \hat{R}^{\hat{\xi}_0}_{0,t} = r\right], \end{aligned} \tag{63}$$

where we set $\hat{X}^{\hat{\xi}_0}_{0,T_1} := \lim_{t\uparrow T_1} \hat{X}^{\hat{\xi}_0}_{0,t}$ and $\hat{R}^{\hat{\xi}_0}_{0,T_1} := \lim_{t\uparrow T_1} \hat{R}^{\hat{\xi}_0}_{0,t}$.

While (63) is not a full liquidation problem due to the non-singularity of $\hat{V}_1(x, r, T_1)$, it can nevertheless be solved using similar arguments using the separable structure of its solution. To this end, we look for a solution of the form

$$\hat{V}_0(x, r, t) = -e^{-r+h(x,t)} \quad \text{with} \quad h(x, t) = x^2 g(t), \quad t \in [0, T_1],$$

for some function $g$ to be determined. Then, for $t \in [0, T_1]$, $h$ will satisfy (20) with terminal condition $h(x, T_1) = \sqrt{\frac{\hat{\lambda}}{2}} x^2 \coth\frac{T-T_1}{\sqrt{2\hat{\lambda}}}$ and $h(0, t) = 0$. It follows directly that $g$ must solve

$$g'(t) = \frac{1}{\lambda_0} g^2(t) - \frac{1}{2}, \quad g(T_1) = \sqrt{\frac{\hat{\lambda}}{2}} \coth\frac{T-T_1}{\sqrt{2\hat{\lambda}}}. \tag{64}$$

We introduce the constant

$$c := \sqrt{\frac{\hat{\lambda}}{\lambda_0}} \coth\frac{T-T_1}{\sqrt{2\hat{\lambda}}} = \sqrt{\frac{2}{\lambda_0}} g(T_1). \tag{65}$$

We have the following cases:

*Case 1*: $c > 1$ or, equivalently, $g(T_1) > \sqrt{\frac{\lambda_0}{2}}$.

Setting $C_1 := T_1 + \sqrt{2\lambda_0} \coth^{(-1)}\left(\sqrt{\frac{\hat{\lambda}}{\lambda_0}} \coth\frac{T-T_1}{\sqrt{2\hat{\lambda}}}\right)$, we obtain

$$\begin{aligned} g(t) &= \sqrt{\frac{\lambda_0}{2}} \coth\frac{C_1 - t}{\sqrt{2\lambda_0}} \\ &= \sqrt{\frac{\lambda_0}{2}} \coth\left(\frac{T_1 - t}{\sqrt{2\lambda_0}} + \coth^{(-1)}\left(\sqrt{\frac{\hat{\lambda}}{\lambda_0}} \coth\frac{T-T_1}{\sqrt{2\hat{\lambda}}}\right)\right) \\ &= \sqrt{\frac{\lambda_0}{2}} \coth\left(\frac{T_1 - t}{\sqrt{2\lambda_0}} + \coth^{(-1)}\left(\sqrt{\frac{2}{\lambda_0}} g(T_1)\right)\right). \end{aligned}$$

Therefore, for $t \in [0, T_1]$,

$$\hat{V}_0\left(x, r, t\right) = -\exp\left(-r + x^2\sqrt{\frac{\lambda_0}{2}}\coth\frac{C_1 - t}{\sqrt{2\lambda_0}}\right),$$

and for the feedback optimal control function by $\hat{\xi}_0^*\left(x, t\right) = \frac{x}{\sqrt{2\lambda_0}}\coth\frac{C_1-t}{\sqrt{2\lambda_0}}$.

The optimal inventory and revenue processes are, thus, given, for $t \in [0, T_1]$, by

$$\begin{aligned}
\hat{X}_{0,t}^{\hat{\xi}_0^*} &= x\frac{\sinh\dfrac{C_1 - t}{\sqrt{2\lambda_0}}}{\sinh\dfrac{C_1}{\sqrt{2\lambda_0}}}, \\
\hat{R}_{0,t}^{\hat{\xi}_0^*} &= r - \lambda_0\int_0^t\left(\hat{\xi}_{0,s}^*\right)^2\mathrm{d}s + \int_0^t\hat{X}_{0,s}^{\hat{\xi}_0^*}\mathrm{d}W_s \\
&= r - \frac{x^2}{4\left(\sinh\dfrac{C^1}{\sqrt{2\lambda_0}}\right)^2}\left(t + \sqrt{\frac{\lambda_0}{2}}\left(\sinh\left(\sqrt{\frac{2}{\lambda_0}}C_1\right) - \sinh\left(\sqrt{\frac{2}{\lambda_0}}(C_1 - t)\right)\right)\right) \\
&\quad + \frac{x}{\sinh\dfrac{C^1}{\sqrt{2\lambda_0}}}\int_0^t\sinh\frac{C_1 - s}{\sqrt{2\lambda_0}}\mathrm{d}W_s.
\end{aligned}$$

The optimal liquidation process is given by

$$\hat{\xi}_{0,t}^* = \frac{x}{\sqrt{2\lambda_0}}\frac{\cosh\dfrac{C_1 - t}{\sqrt{2\lambda_0}}}{\sinh\dfrac{C_1}{\sqrt{2\lambda_0}}}.$$

*Case 2*: $c < 1$ or, equivalently, $g\left(T_1\right) < \sqrt{\frac{\lambda_0}{2}}$.

Setting $C_2 := T_1 + \sqrt{2\lambda_0}\tanh^{(-1)}\left(\sqrt{\frac{\hat{\lambda}}{\lambda_0}}\coth\frac{T-T_1}{\sqrt{2\hat{\lambda}}}\right)$, we obtain

$$\begin{aligned}
g\left(t\right) &= \sqrt{\frac{\lambda_0}{2}}\tanh\left(\frac{C_2 - t}{\sqrt{2\lambda_0}}\right) \\
&= \sqrt{\frac{\lambda_0}{2}}\tanh\left(\frac{T_1 - t}{\sqrt{2\lambda_0}} + \tanh^{(-1)}\left(\sqrt{\frac{\hat{\lambda}}{\lambda_0}}\coth\frac{T - T_1}{\sqrt{2\hat{\lambda}}}\right)\right) \\
&= \sqrt{\frac{\lambda_0}{2}}\tanh\left(\frac{T_1 - t}{\sqrt{2\lambda_0}} + \tanh^{(-1)}\left(\sqrt{\frac{2}{\lambda_0}}g\left(T_1\right)\right)\right).
\end{aligned}$$

Thus, for $t \in [0, T_1]$, the value function is given by

$$\hat{V}_0\left(x, r, t\right) = -\exp\left(-r + x^2\sqrt{\frac{\lambda_0}{2}}\tanh\frac{C_2 - t}{\sqrt{2\lambda_0}}\right),$$

and the optimal feedback control by $\hat{\xi}_0^*\left(x, t\right) = \frac{x}{\sqrt{2\lambda_0}}\tanh\frac{C_2-t}{\sqrt{2\lambda_0}}$. In turn, for $t \in [0, T_1)$, the optimal processes are given by

$$\hat{X}_{0,t}^{\hat{\xi}_0^*} = x\frac{\cosh\dfrac{C_2 - t}{\sqrt{2\lambda_0}}}{\sinh\dfrac{C_2}{\sqrt{2\lambda_0}}},$$

$$
\begin{aligned}
\hat{R}_{0,t}^{\hat{\xi}_0^*} &= r - \lambda_0 \int_0^t \left(\hat{\xi}_{0,s}^*\right)^2 \mathrm{d}s + \int_0^t \hat{X}_{0,s}^{\hat{\xi}_0^*} \mathrm{d}W_s \\
&= r + \frac{x^2}{4\left(\sinh \frac{C_2}{\sqrt{2\lambda_0}}\right)^2} \left(t - \sqrt{\frac{\lambda_0}{2}} \left(\sinh\left(\sqrt{\frac{2}{\lambda_0}} C_2\right) - \sinh\left(\sqrt{\frac{2}{\lambda_0}}(C_2 - t)\right)\right)\right) \\
&\quad + \frac{x}{\sinh \frac{C_2}{\sqrt{2\lambda_0}}} \int_0^t \cosh \frac{C_2 - s}{\sqrt{2\lambda_0}} \mathrm{d}W_s,
\end{aligned}
$$

and the optimal liquidation process by

$$
\hat{\xi}_{0,t}^* = \frac{x}{\sqrt{2\lambda_0}} \frac{\sinh \frac{C_2 - t}{\sqrt{2\lambda_0}}}{\sinh \frac{C_2}{\sqrt{2\lambda_0}}}.
$$

iii) $c = 1$, or equivalently, $g\left(T_1\right) = \sqrt{\frac{\lambda_0}{2}}$.

Then, for $t \in [0, T_1]$, the value function and optimal feedback policy are given by

$$
\hat{V}_0\left(x, r, t\right) = -e^{-r + x^2 \sqrt{\frac{\lambda_0}{2}}} \quad \text{and} \quad \hat{\xi}_0^*\left(x, t\right) = \frac{x}{\sqrt{2\lambda_0}}.
$$

The optimal processes are given by

$$
\hat{X}_{0,t}^* = x e^{-\frac{t}{\sqrt{2\lambda_0}}},
$$

and

$$
\hat{R}_{0,t}^{\hat{\xi}_0^*} = r + x^2 \sqrt{\frac{\lambda_0^3}{2}} \left(e^{-\sqrt{\frac{2}{\lambda_0}} t} - 1\right) + x \int_0^t e^{-\frac{s}{\sqrt{2\lambda_0}}} \mathrm{d}W_s,
$$

and the optimal liquidation process by

$$
\hat{\xi}_{0,t}^* = \frac{\hat{X}_{0,t}^{\hat{\xi}_0^*}}{\sqrt{2\lambda_0}}.
$$

When $T_1$ arrives, the model parameter is revised to $\lambda_1$ and the agent now uses the *accurately* revised model dynamics

$$
\mathrm{d}X_{1,t}^{\xi} = -\xi_t \mathrm{d}t \quad \text{and} \quad \mathrm{d}R_{1,t}^{\xi} = -\lambda_1 \xi_t^2 \mathrm{d}t + X_{1,t}^{\xi} \mathrm{d}W_t, \quad t \in [T_1, T], \tag{66}
$$

with initial conditions $X_{T_1} = \hat{X}_{0,T_1}^{\hat{\xi}_0^*}$ and $R_{T_1} = \hat{R}_{0,T_1}^{\hat{\xi}^*}$, the optimal inventory and revenue processes at $T_1$ generated from problem $\mathcal{S}\left(\lambda_0; 0, T_1\right)$. Naturally, both initial conditions $\hat{X}_{0,T_1}^{\hat{\xi}_0^*}$ and $\hat{R}_{0,T_1}^{\hat{\xi}^*}$ have already *inherited the model misspecification error* (choosing $\hat{\lambda}$ instead of $\lambda_1$ at $t = 0$) which enters in the terminal condition of problem $\mathcal{S}\left(\lambda_0; 0, T_1\right)$.

The agent now solves problem $\mathcal{L}\left(\lambda_1; T_1, T\right)$, yielding for $t \in [T_1, T]$,

$$
V_1\left(x, r, t\right) = -\exp\left(-r + \sqrt{\frac{\lambda_1}{2}} x^2 \coth \frac{T - t}{\sqrt{2\lambda_1}}\right), \tag{67}
$$

where $V_1\left(x, r, T\right) = v\left(x, r\right)$, $v$ as in (7). The related optimal processes are given by

$$\xi_{1,t}^{*} = \frac{1}{\sqrt{2\lambda_1}} \hat{X}_{0,T_1}^{\hat{\xi}_0^*} \coth \frac{T-t}{\sqrt{2\lambda_1}} \quad \text{and} \quad X_{1,t}^{*} = \hat{X}_{0,T_1}^{\hat{\xi}_0^*} \frac{\sinh \dfrac{T-t}{\sqrt{2\lambda_1}}}{\sinh \dfrac{T-T_1}{\sqrt{2\lambda_1}}}.$$

As expected, $X_{1,T}^{*} = 0$. Notice, however, that even though the agent considers an entirely new liquidation model on $[T_1, T]$, the initial wrong parameter assessment $\hat{\lambda}$ -instead of the true, in hindsight, value $\lambda_1$-still affects the solution of $\mathcal{L}(\lambda_1; T_1, T)$ through the initial conditions $\hat{X}_{0,T_1}^{\hat{\xi}_0^*}$ and $\hat{R}_{0,T_1}^{\hat{\xi}_0^*}$. For example, $\hat{X}_{0,T_1}^{\hat{\xi}_0^*}$ depends on $g(\cdot)$, which itself depends on the erroneous choice of $\hat{\lambda}$ through the terminal condition $g(T_1)$ above.

Further calculations can be carried out to compute in detail the related errors and refer the reader to Chapter 3 of [34]. We choose not to do it here as a more general study within a broader scope of building regret metrics is currently under preparation by the authors in [36].

## Acknowledgements

Thaleia Zariphopoulou would like to thank the University of Chicago for its hospitality during the Spring 2023 long-term program, during which most of this work was completed.

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