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Forward indifference valuation for dynamically incoming projects

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Abstract Classical indifference valuation, a widely studied approach in incomplete markets, uses critically the a priori knowledge of the characteristics (arrival, maturity, payoff structure) of the projects in consideration. This assumption, however, may not accommodate realistic scenarios in which projects, not initially anticipated, arrive at later times. To accommodate this, we employ forward indifference valuation criteria, which by construction are flexible enough to adapt to such "non-anticipated" cases while yielding time-consistent indifference prices. We consider and analyze in detail two representative cases: valuation adjustments due to incoming non-anticipated project and the relative forward indifference valuation of new projects in relation to existing ones.

Keywords Forward indifference valuation, Dynamically incoming projects, Indifference pricing, Relative forward indifference valuation, Predictable forward criteria, Adaptive forward criteria, Risk decomposition

2020 Mathematics Subject Classification 60H30, 91G30

1. Introduction

Indifference pricing is a popular, and widely studied, approach for the valuation and risk management of projects in incomplete markets. It produces the so-called indifference price by comparing the optimal investment behavior with and without the project in consideration. For this, two expected utility problems must be solved, and the indifference price is the amount that makes the agent indifferent between writing (buying) the project or not. Providing a complete bibliography of the literature is beyond the scope herein, so we only refer the reader to representative references (see, among others, [1, 2, 12]).

To properly define the underlying expected utility models, one needs to choose a trading horizon, a utility at this horizon, a market model describing the dynamics of the traded and the non-traded assets, and the projects to be valued. The investment horizon, say T, is typically chosen to be the maturity of the project, if a single project is considered, or of the longest maturity in case of multiple projects. It may, thus, be finite or infinite. The utility function is predominantly chosen to be of exponential type because not only this is the most tractable case, but it also yields wealth-independent indifference prices as is the case in complete markets. The model

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dynamics are also pre-chosen, at initial time, for the entire investment horizon, otherwise the backward induction to solve the expected utility models is not viable. Finally, also at initial time, a model for the upcoming project(s) to be valued is also needed, otherwise the optimization problem of the writer (buyer) is not well defined.

In many applications, however, the initial model choice turns out to be inaccurate and model revision is unavoidable. Clearly, any model changes within the classical setting will lead to time inconsistency and, as a result, to time inconsistent indifference prices. One may attempt to remedy this by working in a robust control framework, but even in this case the family of plausible models is chosen once, statically at initial time. In a different direction, filtering may be used but, once more, the filtering source is pre-chosen and may turn out to be inaccurate. Finally, one may work in an adaptive control framework but, as shown herein, this leads to time-inconsistent indifference prices. In addition to model selection, the classical indifference valuation approach cannot accommodate the valuation of incoming projects that were not incorporated at initial time. This is a considerable difficulty as, very frequently and especially in long horizons, new projects arrive and mature at times that were not known at initiation.

To accommodate these limitations, we employ the forward indifference valuation approach, developed, among others, in [4–11, 13]. We, first, analyze the case of non-anticipated (at initial time) model revision. We examine both the classical and the forward cases and show the discrepancies arising in the former. In contrast, we show that forward indifference valuation precludes such problems as it adapts in a time-consistent way to upcoming model changes and this semigroup-type structure is inherited to the forward prices.

We, in turn, examine the indifference valuation of incoming projects which arrive at later times, not initially accounted for. To keep the presentation simple, we only work with two projects, assuming that all characteristics of the second one, i.e. its arrival, maturity and profile, were not known when the valuation of the first project was put in place.

We consider two families of forward performance processes, the *predictable* and the *adaptive* family, and also compute the respective relative indifference price of the first project given the risk exposure of the second. We derive the related prices and study their properties. Among others, we show that although the two types of forward criteria have different measurability, they give rise to the same relative indifference price that is consistent with the initially settled price for the first project. In the same direction, we examine the relative forward indifference prices and examine various conditionally additive properties they acquire. The work extends the classical notion of relative indifference pricing as it allows for dynamically incoming new projects.

The paper is organized as follows. In Section 2, we consider the case of a new project arriving at a future time, not known before, and develop valuation adjustments due to this non-anticipated project by constructing the forward indifference prices under both predictive and adaptive forward criteria. In Section 3, we introduce the notion of relative forward prices and study their properties. We also analyze the related hedging policies, residual wealth, and residual risks. We conclude in Section 4.

2. Relative forward indifference valuation of existing project

Classical backward indifference valuation methodology is subject to project commitment, in that complete knowledge of project's profile is available at initial time. Herein, we examine how this can be relaxed developing the ideas of forward indifference valuation. We work with two projects and do not assume full knowledge of the second project at $t = 0$, neither deterministically nor randomly. Specifically, we assume that both the expiry and payoff functional of the second project are specified at its initiation time τ_1 with $0 < \tau_1 < T$. However, time τ_1 is known at $t = 0$ but not the payoff structure of the incoming project.

This model setup can describe more general scenarios in practice. For instance, a drug company may know when to start developing the second drug, but it is not clear today how long the process would take and what the risk would be for its development, as these characteristics may mostly depend on other factors, including the R&D outcome of the first drug, and they are more likely to be accurate when the second drug development officially starts.

We extend the single project valuation model of [3]. Therein, the investment universe consists of a riskless asset and two risky assets. Let the horizon $[0, T]$ be given. Assume for simplicity that the riskless asset is a bond offering zero interest rate, $B_t = 1$, $t \geqslant 0$.

The first risky asset is a stock that can be traded, whose price follows the log-normal diffusion,

$$\mathrm{d}S_s = \mu S_s \mathrm{d}s + \sigma S_s \mathrm{d}W_s^1, \tag{Stock}$$

with $S_t = S > 0$, and $0 \leqslant t \leqslant s \leqslant T$. We denote the Sharpe ratio $\lambda := \mu/\sigma$. The second asset is non-traded and its value is modeled by the diffusion process

$$\mathrm{d}Y_s = b(Y_s, s)\mathrm{d}s + a(Y_s, s)\mathrm{d}W_s, \tag{Y-factor}$$

with $Y_t = y \in \mathbb{R}$, and $0 \leqslant t \leqslant s \leqslant T$.

The two Brownian motions W^1, W are defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with the filtration $\{\mathcal{F}_t^{(S,Y)}\}_{t \geqslant 0}$ generated by (W^1, W) and satisfying the usual conditions. We assume that the correlation between W^1, W is $\rho \in (-1, 1)$, and that the deterministic functions $b(\cdot, \cdot)$, $a(\cdot, \cdot)$ are such that the stochastic differential equation for Y has a unique strong solution.

The wealth process of the plain investor (in the absence of any project) who chooses a self-financing portfolio process $\pi_s, 0 \leqslant t \leqslant s \leqslant T$, satisfies

$$\mathrm{d}X_s = \mu \pi_s \mathrm{d}s + \sigma \pi_s \mathrm{d}W_s^1, \tag{wealth}$$

with $X_t = x \in \mathbb{R}$, and $0 \leqslant t \leqslant s \leqslant T$. The set of admissible portfolios, denoted by $\mathcal{A}$, contains all $\mathcal{F}_s^{(S,Y)}$-progressively measurable processes π that also satisfy the integrability condition $\mathbb{E}\int_0^t \pi_s^2 \mathrm{d}s < \infty$, for any $t > 0$.

We next introduce the two projects. The first project is initiated at $t = 0$ with expiry at $t = T$ at which it pays $H(Y_T)$ with the function $H(\cdot) \in \mathcal{F}_0^{(S,Y)}$. It is known that a second project will arrive at $\tau_1 \in \mathcal{F}_0^{(S,Y)}$ and expire at $\tau_2 \in \mathcal{F}_{\tau_1}^{(S,Y)}$ with $0 < \tau_1 < \tau_2 \leqslant T$. It will pay $G(Y_{\tau_2})$ with payoff functional $G(\cdot) \in \mathcal{F}_{\tau_1}^{(S,Y)}$. Both payoff functionals $H(\cdot)$ and $G(\cdot)$ are bounded.

The agents involved (i.e., both the plain investor and the writer) are assumed to have exponential utility $U_T(x) = -e^{-\gamma x}$ at time $t = 0$, but they are allowed to update this terminal criterion (and intermediate criteria) based on the arrival of the second project. Hence, this $\mathcal{F}_0^{(S,Y)}$-measurable terminal utility $U_T(x) = -e^{-\gamma x}$ would only be used for the valuation of the first project before $t = \tau_1$; after this time, both investors would revise their respective performance criterion in a consistent way, with the knowledge of the existence of the second unanticipated project.

We begin with the evaluation of the first project. During period $[0, \tau_1)$, both the writer and the plain investor do not have the knowledge of the profile of the second project, so they can only price the first project under terminal utility $U_T(x) = -e^{\gamma x}$. Therefore, a classical indifference valuation is performed for this period.

We recall the classical value function processes V^0 for the plain investor and V^{P_1} for the writer over $[0,\tau_1)$,

$$V^0(x,t)=\operatorname*{esssup}_{\mathcal{A}}\mathbb{E}\left[-e^{-\gamma X_T}\big|\mathcal{F}_t^{(S,Y)},\ X_t=x\right], \tag{1}$$

$$V^{P_1}(x,t)=\operatorname*{esssup}_{\mathcal{A}}\mathbb{E}\left[-e^{-\gamma(X_T-H(Y_T))}\big|\mathcal{F}_t^{(S,Y)},\ X_t=x\right], \tag{2}$$

respectively. Well known classical results give (see, for example, [3])

$$V^0(x,t)=-e^{-\gamma x-\frac{1}{2}\lambda^2(T-t)},$$

and

$$V^{P_1}(x,t)=u^{P_1}(x,Y_t,t),$$

with

$$u^{P_1}(x,y,t):=-e^{-\gamma x}\Big(\mathbb{E}_{\mathbb{Q}}\big[e^{\gamma(1-\rho^2)H(Y_T)-\frac{1}{2}(1-\rho^2)\lambda^2(T-t)}\big|Y_t=y\big]\Big)^{\frac{1}{1-\rho^2}},$$

for $0\leqslant t<\tau_1$. Herein, $\mathbb{Q}$ is the minimal relative entropy martingale measure with respect to $\mathbb{P}$. Namely, $\mathbb{Q}$ is defined on $\mathcal{F}_T^{(S,Y)}$ as

$$\mathbb{Q}(A)=\mathbb{E}_{\mathbb{P}}\Big[\exp\Big(-\lambda W_T^1-\frac{1}{2}\lambda^2T\Big)\mathbb{1}_A\Big],\quad A\in\mathcal{F}_T^{(S,Y)},$$

and is the minimizer, among all martingale measures Q, of

$$\min_Q\mathcal{H}(Q|\mathbb{P})=\min_Q\mathbb{E}_{\mathbb{P}}\Big[\frac{\mathrm{d}Q}{\mathrm{d}\mathbb{P}}\ln\frac{\mathrm{d}Q}{\mathrm{d}\mathbb{P}}\Big].$$

At $t=\tau_1$, the writer has preserved her individual optimality under the risk exposure of the first project up to $V^{P_1}(x,\tau_1)$, and the plain investor has achieved $V^0(x,\tau_1)$. Both investors also observe the arrival of the second project and its profile, i.e, they acquire the knowledge of τ_2, $G(\cdot)\in\mathcal{F}_{\tau_1}^{(S,Y)}$, and reevaluate the first project during the life-span of the second project following a relative indifference valuation approach. Specifically, during $[\tau_1,\tau_2]$, the goal for the writer is to construct a valuation criterion, say $U^{P_1|P_2}(x,\tau_2)$, such that consistency along the optimality of investment in first project is preserved, but now under the additional liability to pay $G(Y_{\tau_2})$ at $t=\tau_2$. In particular, at $t=\tau_1$, she solves

$$V^{P_1}(x,\tau_1)=\operatorname*{esssup}_{\mathcal{A}}\mathbb{E}\big[U^{P_1|P_2}(X_{\tau_2}-G(Y_{\tau_2}),\tau_2)\big|\mathcal{F}_{\tau_1}^{(S,Y)},\ X_{\tau_1}=x\big],\quad \text{a.s.} \tag{3}$$

Similarly, the plain investor would also undertake the liability $G(Y_{\tau_2})$ of the second project but without the liability $H(Y_T)$ of the first. The goal is then to determine the valuation criterion $U^0(x,\tau_2)$ to maintain intertemporal consistency. At $t=\tau_1$, the plain investor solves

$$V^0(x,\tau_1)=\operatorname*{esssup}_{\mathcal{A}}\mathbb{E}\left[U^0(X_{\tau_2}-G(Y_{\tau_2}),\tau_2)\big|\mathcal{F}_{\tau_1}^{(S,Y)},\ X_{\tau_1}=x\right],\quad \text{a.s.} \tag{4}$$

Once the two utility functions $U^{P_1|P_2}$ and U^0 are determined, we in turn define the value function processes similarly as in (2) and (1) for $\tau_1\leqslant t\leqslant\tau_2$, conditionally on $\mathcal{F}_{\tau_1}^{(S,Y)}$, namely,

$$V^{P_1|P_2}(x,t;\omega):=\operatorname*{esssup}_{\mathcal{A}}\mathbb{E}\big[U^{P_1|P_2}(X_{\tau_2}-G(Y_{\tau_2}),\tau_2)\big|\mathcal{F}_{\tau_1}^{(S,Y)},\ X_t=x\big], \tag{5}$$

and

$$V^{P_2}(x,t;\omega):=\operatorname*{esssup}_{\mathcal{A}}\mathbb{E}\big[U^0(X_{\tau_2}-G(Y_{\tau_2}),\tau_2)\big|\mathcal{F}_{\tau_1}^{(S,Y)},\ X_t=x\big]. \tag{6}$$

The forward indifference price of the first project relative to the second project during period $[\tau_1, \tau_2]$ would be naturally defined as the "break-even" process $H_t^{P_1|P_2}$, $\tau_1 \leqslant t \leqslant \tau_2$, that satisfies

$$V^{P_2}(X_t - H_t^{P_1|P_2}, t) = V^{P_1|P_2}(X_t, t), \quad \text{a.s.} \tag{7}$$

The problem now boils down to determining the respective forward criterion $U^{P_1|P_2}$ and U^0 for the writer and the plain investor, such that the consistency conditions (3) and (4) hold. Notice that, as in the continuous time framework for the forward performance processes, the consistent forward criterion in general is not unique. In the sequel, we consider two types of forward performance criteria. The first ones are predictable, $U^{P_1|P_2}(x, \tau_2)$, $U^0(x, \tau_2) \in \mathcal{F}_{\tau_1}^{(S,Y)}$, while the second criteria are adaptive, in that $U^{P_1|P_2}(x, \tau_2)$, $U^0(x, \tau_2) \in \mathcal{F}_{\tau_2}^{(S,Y)}$. As we shall see, different class of forward criteria leads to different relative indifference prices, however, time-inconsistency and pricing discrepancy are excluded in both cases.

2.1 Predictable forward criteria and relative indifference valuation

We first consider the predictable criteria and the associated relative indifference price of the first project over $[\tau_1, \tau_2]$. To this end, suppose the writer adopts forward criteria of the separable form $U^{P_1|P_2}(x, \tau_2) = -e^{-\gamma x + F_{\tau_2}}$, for some $F_{\tau_2} \in \mathcal{F}_{\tau_1}^{(S,Y)}$.

Theorem 2.1 *The unique predictable forward performance criterion of separable form for the writer of P_1 given P_2 is*

$$U^{P_1|P_2}(x, \tau_2) = -\exp\left(-\gamma x - \frac{1}{2}\lambda^2(T - \tau_2) + \frac{1}{1-\rho^2}\ln\frac{E_{\mathbb{Q}}[e^{\gamma(1-\rho^2)H(Y_T)}|Y_{\tau_1}]}{E_{\widetilde{\mathbb{Q}}}[e^{\gamma(1-\rho^2)G(Y_{\tau_2})}|Y_{\tau_1}]}\right) \in \mathcal{F}_{\tau_1}^{(S,Y)}. \tag{8}$$

The relative forward indifference price process of the first project is

$$H_t^{P_1|P_2} = h(Y_t, t; \omega) := \frac{1}{\gamma(1-\rho^2)}\ln\mathbb{E}_{\mathbb{Q}}\big[e^{\gamma(1-\rho^2)H(Y_T)}|Y_{\tau_1}\big], \quad \text{a.s.}, \tag{9}$$

for $\tau_1 \leqslant t \leqslant \tau_2$, where $\mathbb{Q}$ is the minimal relative entropy martingale measure with respect to $\mathbb{P}$, and $\widetilde{\mathbb{Q}}$, conditionally on $\mathcal{F}_{\tau_1}^{(S,Y)}$, is defined on $\mathcal{F}_{\tau_2}^{(S,Y)}$ as

$$\widetilde{\mathbb{Q}}(A) = \mathbb{E}_{\mathbb{P}}\Big[\exp\Big(-\lambda(W_{\tau_2}^1 - W_{\tau_1}^1) - \frac{1}{2}\lambda^2(\tau_2 - \tau_1)\Big)\mathbb{1}_A\Big], \quad A \in \mathcal{F}_{\tau_2}^{(S,Y)}.$$

Proof Applying the distortion transformation (cf.[3])

$$V^{P_1|P_2}(x, t; \omega) = -e^{-\gamma x}v(Y_t, t; \omega)^{\frac{1}{1-\rho^2}},$$

we rewrite (5) as

$$-e^{-\gamma x}v(y, t; \omega)^{\frac{1}{1-\rho^2}} = \operatorname*{ess\,sup}_{\mathcal{A}}\mathbb{E}\big[U^{P_1|P_2}(X_T - G(Y_{\tau_2}), \tau_2)\big|\mathcal{F}_{\tau_1}^{(S,Y)},\ X_t = x,\ Y_t = y\big], \quad \text{a.s.}$$

The function $v(y, t; \omega)$ solves a.s. the (random) linear parabolic equation

$$v_t + \frac{1}{2}a^2(y, t)v_{yy} + \big(b(y, t) - \rho\lambda a(y, t)\big)v_y = \frac{1}{2}(1 - \rho^2)\lambda^2 v \tag{10}$$

for $\tau_1 < t < \tau_2$ with terminal condition

$$v(y, \tau_2; \omega) = e^{\gamma(1-\rho^2)G(y) + (1-\rho^2)F_{\tau_2}} \in \mathcal{F}_{\tau_1}^{(S,Y)}.$$

Conditionally on $\mathcal{F}_{\tau_1}^{(S,Y)}$, the solution to (10) has the Feynman-Kac representation for $\tau_1 \leqslant t \leqslant \tau_2$,

$$v(y,t;\omega)=\mathbb{E}_{\widetilde{\mathbb{Q}}}\big[e^{\gamma(1-\rho^2)G(Y_{\tau_2})+(1-\rho^2)F_{\tau_2}-\frac{1}{2}(1-\rho^2)\lambda^2(\tau_2-t)}\big|\mathcal{F}^{(S,Y)}_{\tau_1},\ Y_t=y\big],\quad \text{a.s.} \tag{11}$$

Conditionally on $\mathcal{F}^{(S,Y)}_{\tau_1}$, the measure $\widetilde{\mathbb{Q}}$ is defined on $\mathcal{F}^{(S,Y)}_{\tau_2}$ as

$$\frac{\mathrm{d}\widetilde{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\Big|_{\tau_2}=e^{-\lambda(W^1_{\tau_2}-W^1_{\tau_1})-\frac{1}{2}\lambda^2(\tau_2-\tau_1)},$$

where $\widetilde{W}_s:=W_s-W_{\tau_1}+\rho\lambda(s-\tau_1)$ is a standard Brownian motion for $\tau_1\leqslant s\leqslant\tau_2$. Therefore,

$$v(y,\tau_1)=\mathbb{E}_{\widetilde{\mathbb{Q}}}\big[e^{\gamma(1-\rho^2)G(Y_{\tau_2})+(1-\rho^2)F_{\tau_2}-\frac{1}{2}(1-\rho^2)\lambda^2(\tau_2-\tau_1)}\big|\,\mathcal{F}^{(S,Y)}_{\tau_1},Y_{\tau_1}=y\big].$$

On the other hand, the consistency condition (3) suggests that

$$v(y,\tau_1)=\mathbb{E}_{\mathbb{Q}}\big[e^{\gamma(1-\rho^2)H(Y_T)-\frac{1}{2}(1-\rho^2)\lambda^2(T-\tau_1)}\big|Y_{\tau_1}=y\big]. \tag{12}$$

Under the predictability assumption that $U^{P_1|P_2}(x,\tau_2)=-e^{-\gamma x+F_{\tau_2}}$, $F_{\tau_2}\in\mathcal{F}^{(S,Y)}_{\tau_1}$, the above gives

$$F_{\tau_2}=-\frac{1}{2}\lambda^2(T-\tau_2)+\frac{1}{1-\rho^2}\ln\frac{\mathbb{E}_{\mathbb{Q}}[e^{\gamma(1-\rho^2)H(Y_T)}|Y_{\tau_1}]}{\mathbb{E}_{\widetilde{\mathbb{Q}}}[e^{\gamma(1-\rho^2)G(Y_{\tau_2})}|Y_{\tau_1}]}\in\mathcal{F}^{(S,Y)}_{\tau_1}. \tag{13}$$

Hence, the predictable forward criterion for the writer at the expiry of the second project is

$$U^{P_1|P_2}(x,\tau_2)=-\exp\left(-\gamma x-\frac{1}{2}\lambda^2(T-\tau_2)+\frac{1}{1-\rho^2}\ln\frac{E_{\mathbb{Q}}[e^{\gamma(1-\rho^2)H(Y_T)}|Y_{\tau_1}]}{E_{\widetilde{\mathbb{Q}}}[e^{\gamma(1-\rho^2)G(Y_{\tau_2})}|Y_{\tau_1}]}\right) \tag{14}$$

and $U^{P_1|P_2}(x,\tau_2)\in\mathcal{F}^{(S,Y)}_{\tau_1}$. The next step is to determine the predictable forward criterion $U^0(x,\tau_2)$ for the plain investor who only pays $G(Y_{\tau_2})$ at $t=\tau_2$. Using similar arguments, we obtain

$$U^0(x,\tau_2)=-\exp\left(-\gamma x-\frac{1}{2}\lambda^2(T-\tau_2)-\frac{1}{1-\rho^2}\ln\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{\gamma(1-\rho^2)G(Y_{\tau_2})}|Y_{\tau_1}\right]\right), \tag{15}$$

with $U^0(x,\tau_2)\in\mathcal{F}^{(S,Y)}_{\tau_1}$. To derive the relative indifference price $H^{P_1|P_2}_t$ for the first project given the second one over $[\tau_1,\tau_2]$, we use the distortion transformation and the (relative) indifference price definition (7), and obtain

$$-e^{-\gamma(x-h(y,t;\omega))}\widetilde{v}(y,t;\omega)^{\frac{1}{1-\rho^2}}=-e^{-\gamma x}v(y,t;\omega)^{\frac{1}{1-\rho^2}},\quad \text{a.s.}, \tag{16}$$

where we have assumed that $H^{P_1|P_2}_t=h(Y_t,t;\omega)$, due to the scaling property of the exponential utility. Then, it follows from (16) that, for $\tau_1\leqslant t\leqslant\tau_2$,

$$\begin{aligned}h(y,t;\omega)&=\frac{1}{\gamma(1-\rho^2)}\ln\frac{v(y,t;\omega)}{\widetilde{v}(y,t;\omega)}=\frac{1}{\gamma(1-\rho^2)}\ln e^{(1-\rho^2)(F_{\tau_2}-\widetilde{F}_{\tau_2})}\\&=\frac{1}{\gamma(1-\rho^2)}\ln\mathbb{E}_{\mathbb{Q}}\big[e^{\gamma(1-\rho^2)H(Y_T)}|Y_{\tau_1}\big],\quad \text{a.s.}\end{aligned} \tag{17}$$

□

We notice that after $t=\tau_1$, i.e., after the arrival/initiation time of the second project, the consistent price for the first project over $[\tau_1,\tau_2]$ remains unchanged (conditionally on $\mathcal{F}^{(S,Y)}_{\tau_1}$) under the predictable assumption of the utility functions for the two investors. The price of the first project would stay at the level exactly before the arrival of the second project. Such invariant extension of the valuation problem over $[0,\tau_1]$ maintains pricing consistency before and after the appearance of a new project. As we will see in the next section, even under a different class of forward performance processes that are not predictable, the same conditionally invariant indifference price can be derived to exclude time-inconsistency and price discrepancy.

2.2 Adaptive forward criteria and relative indifference valuation

We work with forward criteria that are adaptive, $U^{P_1|P_2}(x,\tau_2)$, $U^0(x,\tau_2) \in \mathcal{F}^{(S,Y)}_{\tau_2}$. The main arguments will follow closely those in the previous section, except that we consider factor form forward criteria $U^{P_1|P_2}(x,\tau_2) = -e^{-\gamma x + F(Y_{\tau_2},\tau_2)}$ for the writer, and $U^0(x,\tau_2) = -e^{-\gamma x + \widetilde{F}(Y_{\tau_2},\tau_2)}$ for the plain investor, where $F(y,\tau_2;\omega)$ and $\widetilde{F}(y,\tau_2;\omega)$ are now both $\mathcal{F}^{(S,Y)}_{\tau_1}$-measurable random functions. For this family of forward criteria, we have the following result whose proof is presented in the Appendix.

Theorem 2.2 *The writer's adaptive forward performance criterion in factor form is given by*

$$U^{P_1|P_2}(x,\tau_2) = -e^{-\gamma(x+G(Y_{\tau_2}))-\frac{1}{2}\lambda^2(T-\tau_2)}\Big(\mathbb{E}_{\mathbb{Q}}\big[e^{\gamma(1-\rho^2)H(Y_T)}\big|Y_{\tau_1}\big]\Big)^{\frac{1}{1-\rho^2}} \in \mathcal{F}^{(S,Y)}_{\tau_2}. \tag{18}$$

The relative forward indifference price process of the first project is

$$H_t^{P_1|P_2} = h(Y_t,t;\omega) := \frac{1}{\gamma(1-\rho^2)} \ln \mathbb{E}_{\mathbb{Q}}\big[e^{\gamma(1-\rho^2)H(Y_T)}|Y_{\tau_1}\big], \quad \text{a.s.,} \tag{19}$$

for $\tau_1 \leqslant t \leqslant \tau_2$, where $\mathbb{Q}$ is the minimal relative entropy martingale measure with respect to $\mathbb{P}$.

Proof See Appendix A. □

3. Relative forward indifference valuation of the incoming project

In this section, we discuss, under the forward approach, the relative indifference valuation of the second project given the first. As before, it is assumed that the second project has a deterministic initiation time $\tau_1, 0 < \tau_1 < T$, an expiry $\tau_2 \in \mathcal{F}^{(S,Y)}_{\tau_1}$ and a payoff $G(Y_{\tau_2})$ with bounded $G(\cdot) \in \mathcal{F}^{(S,Y)}_{\tau_1}$. The case we are mainly interested in is when $\tau_2 > T$ a.s., where T is the expiry of the first project whose payoff is $H(Y_T)$; the other case, when the second project expires before the first project, is easier to handle and is not examined herein.

At time τ_1, the second project is introduced and the investor learns both its expiry and payoff structure. It is, then, necessary to extend the current log-normal model at $t = \tau_1$ to cover the life-span of the new project for the purpose of (relative) indifference valuation. For this, we assume that conditionally on $\mathcal{F}^{(S,Y)}_{\tau_1}$, the extended model over $[T,\tau_2]$ still follows the log-normal dynamics but with the Sharpe ratio $\lambda_1 \in \mathcal{F}^{(S,Y)}_{\tau_1}$. We may take $\lambda_1 = \lambda$ but we choose to have $\lambda_1 \in \mathcal{F}^{(S,Y)}_{\tau_1}$ for more generality. For simplicity, we assume that the model for the non-traded asset Y would remain the same after extension to $[T,\tau_2]$. A summary of the model inputs is given in Figure 1.

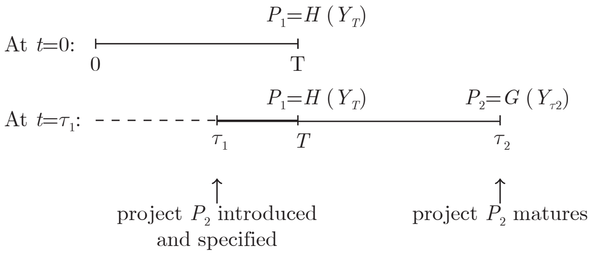

Figure 1 Model inputs for relative indifference valuation of the second project

To price the second project P_2 relative to the existing P_1, we regard the plain investor as the agent under the liability of first project. The writer then becomes the investor who holds both the first and second projects. Conditionally on $\mathcal{F}_{\tau_1}^{(S,Y)}$, we look for an extended forward performance criterion $U(x,\tau_2)$ under which the optimality of the benchmark performance, i.e., the performance of the investor holding only the first project, can be preserved over $[T,\tau_2]$. Specifically, we are looking for U such that, for $T \leqslant t \leqslant \tau_2$,

$$-e^{-\gamma(x-H(y))} = \operatorname*{esssup}_{\mathcal{A}} \mathbb{E}\left[U(X_{\tau_2},\tau_2)\middle|\mathcal{F}_{\tau_1}^{(S,Y)}, X_T = x, Y_T = y\right], \quad \text{a.s.} \tag{20}$$

Herein, the left hand side is the (benchmark) performance of the investor with exponential utility $U(x) = -e^{-\gamma x}$ at $t = T$, under the liability of the first project only. Once such a consistent forward evaluation criterion is constructed, the relative indifference price of the second project over period $[\tau_1,\tau_2]$ is then naturally the classical indifference price, conditionally on $\mathcal{F}_{\tau_1}^{(S,Y)}$, in that the writer is indifferent with and without the second project under $U(x,\tau_2)$. Throughout this section, we focus on the class of forward criteria that are of the factor form (hence adaptive),

$$U(x,\tau_2) = -e^{-\gamma x + F(Y_{\tau_2},\tau_2)}, \tag{21}$$

for some $F(y,\tau_2) \in \mathcal{F}_{\tau_1}^{(S,Y)}$, with $x,\ y \in \mathbb{R}$. The following theorem states the main result of this section.

Theorem 3.1 *Suppose that the ill-posed (random) parabolic equation*

$$h_t + \frac{1}{2}a^2(y)h_{yy} + \big(b(y) - \rho\lambda_1 a(y)\big)h_y = 0, \quad T < t < \tau_2,$$

with $h(y,T;\omega) = e^{\gamma(1-\rho^2)H(y)+\frac{1}{2}(1-\rho^2)\lambda_1^2(\tau_2-T)} \in \mathcal{F}_{\tau_1}^{(S,Y)}$, *has a nonnegative classical solution* $h(y,t;\omega)$, $T \leqslant t \leqslant \tau_2$, *a.s. Then, conditionally on* $\mathcal{F}_{\tau_1}^{(S,Y)}$, *the relative forward indifference price of the second project relative to the first one is given by*

$$H_t^{2|1} = \frac{1}{\gamma(1-\rho^2)} \ln \frac{\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{(1-\rho^2)\left(\gamma G(Y_{\tau_2})+F(Y_{\tau_2},\tau_2)\right)}\middle|Y_T\right]\middle|Y_t\right]}{\mathbb{E}_{\mathbb{Q}}\left[e^{(1-\rho^2)\gamma H(Y_T)}\middle|Y_t\right]} - \frac{\lambda_1^2}{2\gamma}(\tau_2 - T),$$

for $\tau_1 \leqslant t \leqslant T$, *and*

$$H_t^{2|1} = \frac{1}{\gamma(1-\rho^2)} \ln \frac{\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{(1-\rho^2)\left(\gamma G(Y_{\tau_2})+F(Y_{\tau_2},\tau_2)\right)}\middle|Y_t\right]}{\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{(1-\rho^2)F(Y_{\tau_2},\tau_2)}\middle|Y_t\right]},$$

for $T < t \leqslant \tau_2$, *where*

$$F(y,\tau_2;\omega) = \frac{1}{1-\rho^2}\ln h(y,\tau_2;\omega), \quad y \in \mathbb{R}. \tag{22}$$

Here, $\mathbb{Q}$ *is the minimal relative entropy martingale measure with respect to* $\mathbb{P}$, *and* $\widetilde{\mathbb{Q}}$, *conditionally on* $\mathcal{F}_{\tau_1}^{(S,Y)}$, *is defined on* $\mathcal{F}_{\tau_2}^{(S,Y)}$ *as*

$$\widetilde{\mathbb{Q}}(A) = \mathbb{E}_{\mathbb{P}}\Big[\exp\Big(-\lambda_1(W_{\tau_2}^1 - W_T^1) - \frac{1}{2}\lambda_1^2(\tau_2 - T)\Big)\mathbb{1}_A\Big], \quad A \in \mathcal{F}_{\tau_2}^{(S,Y)}.$$

Proof See Appendix B. □

3.1 Risk decomposition under relative forward indifference valuation

We construct the decomposition of the relative indifference price, the residual optimal wealth process and the residual risk process for relative forward indifference valuation of the second project given the first one. To this end, we first introduce

$$Z_{\tau_2} := \frac{1}{\gamma}\left(F(Y_{\tau_2},\tau_2) - \frac{\lambda_1^2}{2}(\tau_2 - T)\right), \tag{23}$$

where $F(y,\tau_2) \in \mathcal{F}^{(S,Y)}_{\tau_1}$, for $y \in \mathbb{R}$, is given in (22). Then, the forward consistency equation (20) can rewrite as

$$-e^{-\gamma(x-H(y))} = \operatorname*{esssup}_{\mathcal{A}} \mathbb{E}\left[-e^{-\gamma(X_{\tau_2}-Z_{\tau_2})+\frac{\lambda_1^2}{2}(\tau_2-T)}\middle|\mathcal{F}^{(S,Y)}_{\tau_1}, X_T = x, Y_T = y\right], \quad \text{a.s.} \tag{24}$$

We may now regard the quantity Z_{τ_2} as the future "reincarnation" of the first project after its expiry $t = T$, in the sense that the optimality of the performance of the investor who only holds the first project can be maintained through $[T,\tau_2]$, if she were to pay the virtual payoff Z_{τ_2} at $t = \tau_2$ instead of paying the actual payoff $H(Y_T)$ at expiry $t = T$. Note that the payoff Z_{τ_2} in equation (24) is evaluated under the forward criterion $U^0(x,\tau_2) = -e^{-\gamma x+\frac{\lambda_1^2}{2}(\tau_2-T)}$, which is consistent with the criterion $U^0(x,T) = -e^{-\gamma x}$ at $t = T$. In fact, it is easy to recognize that the criterion $U^0(x,\tau_2)$ is the extended forward criterion from $U^0(x,T)$ for an investor who only invests in the stock and the bond without undertaking any liability from either the first or the second project. We can rewrite the relative forward indifference pricing formula in Theorem 3.1 using the introduced virtual payoff Z_{τ_2}, i.e., conditionally on $\mathcal{F}^{(S,Y)}_{\tau_1}$,

$$H_t^{2|1} = \frac{1}{\gamma(1-\rho^2)}\ln\frac{\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{\gamma(1-\rho^2)\left(G(Y_{\tau_2})+Z_{\tau_2}\right)}\middle|Y_T\right]\middle|Y_t\right]}{\mathbb{E}_{\mathbb{Q}}\left[e^{\gamma(1-\rho^2)H(Y_T)}\middle|Y_t\right]},$$

for $\tau_1 \leqslant t \leqslant T$, and

$$H_t^{2|1} = \frac{1}{\gamma(1-\rho^2)}\ln\frac{\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{\gamma(1-\rho^2)\left(G(Y_{\tau_2})+Z_{\tau_2}\right)}\middle|\ Y_t\right]}{\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{\gamma(1-\rho^2)Z_{\tau_2}}\middle|\ Y_t\right]},$$

for $T < t \leqslant \tau_2$.

Next, we define the optimal wealth processes for the writer who values the second project in relation to the first one, and the benchmark investor who holds only the first project. Let $\Pi^{2|1,W^*}$ and $\Pi^{2|1,*}$ be their respective optimal control processes by following the relative forward indifference valuation procedure. Then, conditionally on $\mathcal{F}^{(S,Y)}_{\tau_1}$, the writer's optimal wealth satisfies

$$\mathrm{d}X_s^{2|1,W^*} = \mu_s\Pi_s^{2|1,W^*}\mathrm{d}s + \sigma_s\Pi_s^{2|1,W^*}\mathrm{d}W_s^1, \quad t \leqslant s \leqslant \tau_2, \tag{25}$$

with initial condition $X_t^{2|1,W^*} = x + h^{2|1}(y,t;\omega)$, for $\tau_1 \leqslant t \leqslant \tau_2$. Herein, the random function $h^{2|1}(y,t;\omega)$, conditionally on $\mathcal{F}^{(S,Y)}_{\tau_2}$, is the relative indifference price

$$h^{2|1}(y,t;\omega) = \frac{1}{\gamma(1-\rho^2)}\ln\frac{\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{\gamma(1-\rho^2)\left(G(Y_{\tau_2})+Z_{\tau_2}\right)}\middle|Y_T\right]\middle|Y_t = y\right]}{\mathbb{E}_{\mathbb{Q}}\left[e^{\gamma(1-\rho^2)H(Y_T)}\middle|Y_t = y\right]}, \tag{26}$$

for $\tau_1 \leqslant t \leqslant T$, and

$$h^{2|1}(y,t;\omega) = \frac{1}{\gamma(1-\rho^2)}\ln\frac{\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{\gamma(1-\rho^2)\left(G(Y_{\tau_2})+Z_{\tau_2}\right)}\middle|\ Y_t = y\right]}{\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{\gamma(1-\rho^2)Z_{\tau_2}}\middle|\ Y_t = y\right]}, \tag{27}$$

for $T < t \leqslant \tau_2$. Moreover, $\mu_s = \mu$, $\sigma_s = \sigma$ for $\tau_1 \leqslant s \leqslant T$ and $\mu_s = \mu_1 \in \mathcal{F}^{(S,Y)}_{\tau_1}$, $\sigma_s = \sigma_1 \in \mathcal{F}^{(S,Y)}_{\tau_1}$ for $T < s \leqslant \tau_2$.

Similarly, the optimal wealth for the benchmark investor follows

$$\mathrm{d}X_s^{2|1,*} = \mu_s\Pi_s^{2|1,*}\mathrm{d}s + \sigma_s\Pi_s^{2|1,*}\mathrm{d}W_s^1, \quad t \leqslant s \leqslant \tau_2, \tag{28}$$

with initial condition $X_t^{2|1,*} = x$, for $\tau_1 \leqslant t \leqslant \tau_2$. Motivated by the similar definition for the single project indifference valuation in [3], we introduce the residual optimal wealth process and the residual risk process associated to the relative forward indifference valuation of the second project given the first project.

Definition 3.1 *Let the relative forward indifference price of the second project given the first one be* $H_t^{2|1}$ *for* $\tau_1 \leqslant t \leqslant \tau_2$, *and the optimal wealth processes for the writer and the benchmark investor be, respectively, (25) and (28). The residual optimal wealth process for the relative indifference valuation of the second project given the first one is defined as*

$$L_s^{2|1} := X_s^{2|1,W^*} - X_s^{2|1,*}, \quad t \leqslant s \leqslant \tau_2, \quad L_t^{2|1} := h^{2|1}(y,t;\omega),$$

for $\tau_1 \leqslant t \leqslant \tau_2$, *and the residual risk process as*

$$R_s^{2|1} = L_s^{2|1} - H_s^{2|1}, \quad t \leqslant s \leqslant \tau_2, \quad R_t^{2|1} = 0,$$

for $\tau_1 \leqslant t \leqslant \tau_2$.

We can similarly define the processes $L^{1,2}$ and $R^{1,2}$ for the total payoff $G(Y_{\tau_2}) + Z_{\tau_2}$, under the classical forward criterion $U^0(x,\tau_2) = -e^{-\gamma + \frac{\lambda_1^2}{2}(\tau_2 - T)}$, and regard this problem as the problem for determining the (non-relative) indifference price of the two projects together, with the payoff of the first project being replaced by its future reincarnation Z_{τ_2} at $t = \tau_2$.

Under the same extended forward criterion $U^0(x,\tau_2)$, we define the processes L^1 and R^1 associated to the problem of pricing only the first project under its future "virtual" payoff Z_{τ_2}, without the liability of the second project. Proposition 3.1 below yields a decomposition for the risk processes. It shows that the residual risk for both projects under the (non-relative) criterion $U^0(x,\tau_2)$ can be decomposed into the risk coming from the first project under $U^0(x,\tau_2)$ and the risk from the second project in relation to the first project under the relative forward criterion $U(x,\tau_2)$.

Proposition 3.1 *Let the incremental optimal hedging strategy for the relative indifference valuation under the forward criterion* $U(x,\tau_2)$ *be*

$$\Delta\Pi_t^{2|1,*} = \Pi_t^{2|1,W^*} - \Pi_t^{2|1,*}, \quad \tau_1 \leqslant t \leqslant \tau_2,$$

and similarly define $\Delta\Pi^{1,2,*}$, $\Delta\Pi^{1,*}$ *for the forward indifference valuation problems under the (non-relative) forward criterion* $U^0(x,\tau_2)$, *respectively. Let, also, the associated indifference prices be given by* $H^{2|1}$, $H^{1,2}$ *and* H^1. *Then, conditionally on* $\mathcal{F}_{\tau_1}^{(S,Y)}$, *we have, a.s. and for* $\tau_1 \leqslant t \leqslant \tau_2$,

$$\Delta\Pi_t^{1,2,*} = \Delta\Pi_t^{1,*} + \Delta\Pi_t^{2|1,*}, \quad H_t^{1,2} = H_t^1 + H_t^{2|1},$$

and

$$L_t^{1,2} = L_t^1 + L_t^{2|1}, \quad R_t^{1,2} = R_t^1 + R_t^{2|1}.$$

Proof We first focus on the valuation problems over $[T,\tau_2]$. It is shown in the Appendix that the value function $V^W(x,t)$ of the writer under the relative forward criterion $U(x,\tau_2)$ is given by (45). Conditionally on $\mathcal{F}_{\tau_1}^{(S,Y)}$, the associated HJB equation yields the optimal control policy

$$\pi^{2|1,W^*}(x,y,t;\omega) = \rho\frac{a(y)}{\sigma_1}\frac{1}{\gamma(1-\rho^2)}\frac{\partial}{\partial y}\ln v^{1,2,W^*}(y,t;\omega) + \frac{\mu_1}{\gamma\sigma_1^2} = \rho\frac{a(y)}{\sigma_1}h_y^{1,2}(y,t;\omega) + \frac{\mu_1}{\gamma\sigma_1^2},$$

for $T \leqslant t \leqslant \tau_2$, where

$$v^{1,2,W^*}(y,t;\omega) := \mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{(1-\rho^2)\left(\gamma G(Y_{\tau_2})+F(Y_{\tau_2},\tau_2)\right)-\frac{1}{2}(1-\rho^2)\lambda_1^2(\tau_2-t)}\big|\mathcal{F}^{(S,Y)}_{\tau_1},\ Y_t = y\right]$$

and

$$h^{1,2}(y,t;\omega) := \frac{1}{\gamma(1-\rho^2)}\ln \mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{\gamma(1-\rho^2)\left(G(Y_{\tau_2})+Z_{\tau_2}\right)}\big|\mathcal{F}^{(S,Y)}_{\tau_1},\ Y_t = y\right].$$

We, also, have that the benchmark investor's value function given by (43) with the associated optimal control policy is given by

$$\pi^{2|1,*}(x,y,t;\omega) = \rho\frac{a(y)}{\sigma_1}h^1_y(y,t;\omega) + \frac{\mu_1}{\gamma\sigma_1^2},$$

with

$$h^1(y,t;\omega) := \frac{1}{\gamma(1-\rho^2)}\ln \mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{\gamma(1-\rho^2)Z_{\tau_2}}\big|\mathcal{F}^{(S,Y)}_{\tau_1},\ Y_t = y\right],$$

for $T \leqslant t \leqslant \tau_2$. It, hence, follows that

$$\Delta\Pi^{2|1,*}_t = \rho\frac{a(Y_t)}{\sigma_1}\left(h^{1,2}_y(Y_t,t;\omega) - h^1_y(Y_t,t;\omega)\right).$$

On the other hand, the (non-relative) indifference valuation of the two projects with the payoff $G(Y_{\tau_2}) + Z_{\tau_2}$ under the (non-relative) forward criterion $U^0(x,\tau_2)$ can be solved following standard arguments. We obtain

$$H^{1,2}_t = \frac{1}{\gamma(1-\rho^2)}\ln \mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{\gamma(1-\rho^2)\left(G(Y_{\tau_2})+Z_{\tau_2}\right)}\big|\mathcal{F}^{(S,Y)}_{\tau_1},\ Y_t\right],$$

as well as the hedging policy $\pi^{1,2,W^*} = \pi^{2|1,W^*}$ for the writer with both projects under $U^0(x,\tau_2)$. The benchmark Merton investor under the criterion $U^0(x,\tau_2)$ has the optimal policy given by $\pi^{1,2,*}(x,y,t;\omega) = \frac{\mu_1}{\gamma\sigma_1^2}$, for $T \leqslant \tau_1 \leqslant \tau_2$. We, hence, deduce that

$$\Delta\Pi^{1,2,*}_t = \rho\frac{a(Y_t)}{\sigma_1}h^{1,2}_y(Y_t,t;\omega).$$

Finally, we can compute the (non-relative) indifference price of the first project under the extended forward criterion $U^0(x,\tau_2)$, again, following the standard argument to get

$$H^1_t = \frac{1}{\gamma(1-\rho^2)}\ln \mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{\gamma(1-\rho^2)Z_{\tau_2}}\big|\mathcal{F}^{(S,Y)}_{\tau_1},\ Y_t\right],$$

and the incremental hedging strategy

$$\Delta\Pi^{1*}_t = \rho\frac{a(Y_t)}{\sigma_1}h^1_y(Y_t,t;\omega).$$

It, thus, follows directly that $\Delta\Pi^{1,2,*}_t = \Delta\Pi^{1,*}_t + \Delta\Pi^{2|1,*}_t$, a.s., and $H^{1,2}_t = H^1_t + H^{2|1}_t$, a.s. By the definition of the residual optimal wealth processes $L^{2|1}$, $L^{1,2}$, L^1, the linearity of the wealth dynamics and the additive property $\Delta\Pi^{1,2,*}_t = \Delta\Pi^{1,*}_t + \Delta\Pi^{2|1,*}_t$, we have $\mathrm{d}L^{1,2}_s = \mathrm{d}L^1_s + \mathrm{d}L^{2|1}_s$, for $t \leqslant s \leqslant \tau_2$, with the initial condition $L^{1,2}_t = L^1_t + L^{2|1}_t$, due to the additive property $h^{1,2}(y,t) = h^1(y,t) + h^{2|1}(y,t)$. This proves that $L^{1,2}_t = L^1_t + L^{2|1}_t$, a.s., for $T \leqslant t \leqslant \tau_2$.

The additivity of the residual risk processes follows from that of the residual optimal wealth processes and that of the indifference price processes. The analysis over the interval $[\tau_1, T]$ is similar.

□

Remark 3.1 *The additive decomposition of price, residual optimal wealth and residual risk actually holds in a similar spirit when it comes to the classical backward setting. Indeed, the classical relative indifference valuation comprises a special scenario under the more general forward framework, when the full profile of the second project is completely known at* $t=0$ *and the terminal utility is also known at* $t=0$*. A result similar to Proposition 3.1 can be derived for the backward setting where complete information about the projects is available and is left to the interested reader.*

4. Conclusions

We studied indifference valuation under forward exponential performance criteria in markets with a traded and a non-traded asset, under dynamic setting with new incoming projects that are not initially known. We studied the differences and similarities between the classical and the forward approaches, and computed the related prices and hedging strategies. We, also, introduced the concept of relative forward indifference prices and examined their symmetry and additivity properties.

Appendix

A Proof of Theorem 2.2

Proof Under the factor form assumption on the forward performance criterion, we derive from (5) that $v(y,t;\omega)$ solves the linear parabolic PDE (10) with $v(y,\tau_2;\omega)=e^{\gamma(1-\rho^2)G(y)+(1-\rho^2)F(y,\tau_2)}\in\mathcal{F}_{\tau_1}^{(S,Y)}$ a.s. Conditionally on $\mathcal{F}_{\tau_1}^{(S,Y)}$, the Feynman-Kac representation of its solution is

$$v(y,t;\omega)=\mathbb{E}_{\widetilde{\mathbb{Q}}}\big[e^{\gamma(1-\rho^2)G(Y_{\tau_2})+(1-\rho^2)F(Y_{\tau_2},\tau_2)-\frac{1}{2}(1-\rho^2)\lambda^2(\tau_2-t)}\big|\mathcal{F}_{\tau_1}^{(S,Y)},\ Y_t=y\big],\quad \text{a.s.},\tag{29}$$

for $\tau_1\leqslant t\leqslant\tau_2$, where the measure $\widetilde{\mathbb{Q}}$ is defined as in the predictable forward criteria case. Condition (3) for the writer then becomes

$$\begin{aligned}&\mathbb{E}_{\widetilde{\mathbb{Q}}}\big[e^{\gamma(1-\rho^2)G(Y_{\tau_2})+(1-\rho^2)F(Y_{\tau_2},\tau_2)-\frac{1}{2}(1-\rho^2)\lambda^2(\tau_2-\tau_1)}\big|\,\mathcal{F}_{\tau_1}^{(S,Y)},Y_{\tau_1}=y\big]\\&=\mathbb{E}_{\mathbb{Q}}\big[e^{\gamma(1-\rho^2)H(Y_T)-\frac{1}{2}(1-\rho^2)\lambda^2(T-\tau_1)}\big|Y_{\tau_1}=y\big],\quad\text{a.s.}\end{aligned}\tag{30}$$

We can directly verify that

$$F(Y_{\tau_2},\tau_2)=-\bigg(\gamma G(Y_{\tau_2})+\frac{1}{2}\lambda^2(T-\tau_2)\bigg)+\frac{1}{1-\rho^2}\ln\mathbb{E}_{\mathbb{Q}}\left[e^{\gamma(1-\rho^2)H(Y_T)}\big|Y_{\tau_1}\right]\tag{31}$$

would satisfy the consistency condition (30). Hence, the writer's adaptive forward criterion at the expiry of the second project is

$$U^{P_1|P_2}(x,\tau_2)=-e^{-\gamma(x+G(Y_{\tau_2}))-\frac{1}{2}\lambda^2(T-\tau_2)}\Big(\mathbb{E}_{\mathbb{Q}}\big[e^{\gamma(1-\rho^2)H(Y_T)}\big|Y_{\tau_1}\big]\Big)^{\frac{1}{1-\rho^2}}\in\mathcal{F}_{\tau_2}^{(S,Y)}.\tag{32}$$

To derive the plain investor's adaptive forward criterion $U^0(x,\tau_2)\in\mathcal{F}_{\tau_2}^{(S,Y)}$, we seek $U^0(x,\tau_2)=-e^{-\gamma x+\widetilde{F}(Y_{\tau_2},\tau_2)}$ with $\widetilde{F}(y,\tau_2)\in\mathcal{F}_{\tau_1}^{(S,Y)}$. Then, the consistency requirement (4) for the plain investor gives

$$\mathbb{E}_{\widetilde{\mathbb{Q}}}\big[e^{\gamma(1-\rho^2)G(Y_{\tau_2})+(1-\rho^2)\widetilde{F}(Y_{\tau_2},\tau_2)-\frac{1}{2}(1-\rho^2)\lambda^2(\tau_2-\tau_1)}\big|\mathcal{F}_{\tau_1}^{(S,Y)},Y_{\tau_1}=y\big]=e^{-\frac{1}{2}(1-\rho^2)\lambda^2(T-\tau_1)}.\tag{33}$$

One can then verify that

$$\widetilde{F}(Y_{\tau_2},\tau_2) = -\gamma G(Y_{\tau_2}) - \frac{1}{2}\lambda^2(T-\tau_2) \in \mathcal{F}^{(S,Y)}_{\tau_2}. \tag{34}$$

Therefore, the plain investor's forward criterion is given by

$$U^0(x,\tau_2) = -e^{-\gamma(x+G(Y_{\tau_2}))-\frac{1}{2}\lambda^2(T-\tau_2)} \in \mathcal{F}^{(S,Y)}_{\tau_2}. \tag{35}$$

The relative indifference price of the first project, given the second project over $[\tau_1,\tau_2]$, follows from (16), namely,

$$-e^{-\gamma(x-h(y,t;\omega))}\widetilde{v}(y,t;\omega)^{\frac{1}{1-\rho^2}} = -e^{-\gamma x}v(y,t;\omega)^{\frac{1}{1-\rho^2}}, \quad \text{a.s.},$$

where

$$\widetilde{v}(y,t;\omega) = \mathbb{E}_{\widetilde{\mathbb{Q}}}\big[e^{\gamma(1-\rho^2)G(Y_{\tau_2})+(1-\rho^2)\widetilde{F}(Y_{\tau_2},\tau_2)-\frac{1}{2}(1-\rho^2)\lambda^2(\tau_2-t)}\big|\mathcal{F}^{(S,Y)}_{\tau_1}, Y_t = y\big],$$

and

$$v(y,t;\omega) = \mathbb{E}_{\widetilde{\mathbb{Q}}}\big[e^{\gamma(1-\rho^2)G(Y_{\tau_2})+(1-\rho^2)F(Y_{\tau_2},\tau_2)-\frac{1}{2}(1-\rho^2)\lambda^2(\tau_2-t)}\big|\mathcal{F}^{(S,Y)}_{\tau_1},\ Y_t = y\big],$$

respectively. It then follows that

$$\begin{aligned} h(y,t;\omega) &= \frac{1}{\gamma(1-\rho^2)}\ln\frac{v(y,t;\omega)}{\widetilde{v}(y,t;\omega)} \\ &= \frac{1}{\gamma(1-\rho^2)}\ln\mathbb{E}_{\mathbb{Q}}\big[e^{\gamma(1-\rho^2)H(Y_T)}\big|Y_{\tau_1}\big], \quad \text{a.s.}, \end{aligned} \tag{36}$$

for $\tau_1 \leqslant t \leqslant \tau_2$, which yields the same (conditionally) constant relative indifference price of the first project over $[\tau_1,\tau_2]$ as in the predictable forward criteria setting,

$$H^{P_1|P_2}_t = h(Y_t,t) = \frac{1}{\gamma(1-\rho^2)}\ln\mathbb{E}_{\mathbb{Q}}\big[e^{\gamma(1-\rho^2)H(Y_T)}\big|Y_{\tau_1}\big].$$

□

B Proof of Theorem 3.1

Proof To determine the forward criterion $U(x,\tau_2)$, specifically to determine $F(Y_{\tau_2},\tau_2)$ in (21), we define the value function for $T \leqslant t \leqslant \tau_2$ as

$$V(x,t) = \operatorname*{esssup}_{\mathcal{A}}\,\mathbb{E}\left[U(X_{\tau_2},\tau_2)\big|\mathcal{F}^{(S,Y)}_{\tau_1}, X_t = x\right], \tag{37}$$

and apply the representation

$$V(x,t;\omega) = -e^{-\gamma x}v(Y_t,t;\omega)^{\frac{1}{1-\rho^2}}.$$

Standard arguments (see, for example, [3]) implies that $V(x,t)$ in (37) solves a (random) HJB with $v(y,t)$ being the solution to the (random) linear parabolic equation

$$v_t + \frac{1}{2}a^2(y)v_{yy} + \big(b(y)-\rho\lambda_1 a(y)\big)v_y = \frac{1}{2}(1-\rho^2)\lambda_1^2 v, \quad \text{a.s.}, \quad T<t<\tau_2 \tag{38}$$

with the terminal condition $v(y,\tau_2) = e^{(1-\rho^2)F(y,\tau_2)} \in \mathcal{F}^{(S,Y)}_{\tau_1}$. Conditionally on $\mathcal{F}^{(S,Y)}_{\tau_1}$, we have

$$v(y,t) = \mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{(1-\rho^2)F(Y_{\tau_2},\tau_2)-\frac{1}{2}(1-\rho^2)\lambda_1^2(\tau_2-t)}\big|\mathcal{F}^{(S,Y)}_{\tau_1},\ Y_t = y\right], \quad \text{a.s.},$$

where the measure $\widetilde{\mathbb{Q}}$, conditionally on $\mathcal{F}^{(S,Y)}_{\tau_1}$, is defined on $\mathcal{F}^{(S,Y)}_{\tau_2}$ as

$$\frac{\mathrm{d}\widetilde{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}^{(S,Y)}_{\tau_2}} = e^{-\lambda_1(W^1_{\tau_2}-W^1_T)-\frac{1}{2}\lambda_1^2(\tau_2-T)}.$$

Under measure $\widetilde{\mathbb{Q}}$, the process $\widetilde{W}_s = W_s - W_T + \rho\lambda_1(s-T)$, $T \leqslant s \leqslant \tau_2$, is a standard Brownian motion with $\widetilde{W}_T = 0$. Furthermore, the process Y, conditionally on $\mathcal{F}^{(S,Y)}_{\tau_1}$, has dynamics

$$\mathrm{d}Y_s = \big(b(Y_s,s) - \rho\lambda_1 a(Y_s,s)\big)\mathrm{d}s + a(Y_s,s)\mathrm{d}\widetilde{W}_s, \tag{39}$$

with $Y_t = y \in \mathbb{R}$, for $T \leqslant t \leqslant s \leqslant \tau_2$, under measure $\widetilde{\mathbb{Q}}$.

Then, the forward consistency condition (20) implies, a.s.,

$$-e^{-\gamma(x-H(y))} = -e^{-\gamma x}\left(\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{(1-\rho^2)F(Y_{\tau_2},\tau_2)-\frac{1}{2}(1-\rho^2)\lambda_1^2(\tau_2-t)}\big|\mathcal{F}^{(S,Y)}_{\tau_1},\ Y_T = y\right]\right)^{\frac{1}{1-\rho^2}},$$

which, in turn, yields

$$\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{(1-\rho^2)F(Y_{\tau_2},\tau_2)}\big|\mathcal{F}^{(S,Y)}_{\tau_1},\ Y_T = y\right] = e^{\gamma(1-\rho^2)H(y)+\frac{1}{2}(1-\rho^2)\lambda_1^2(\tau_2-T)}, \quad \text{a.s.} \tag{40}$$

The random function

$$h(y,t;\omega) := \mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{(1-\rho^2)F(Y_{\tau_2},\tau_2)}\big|\mathcal{F}^{(S,Y)}_{\tau_1},\ Y_t = y\right]$$

is a nonnegative solution to the random linear parabolic equation

$$h_t + \frac{1}{2}a^2(y)h_{yy} + \big(b(y) - \rho\lambda_1 a(y)\big)h_y = 0, \quad T < t < \tau_2, \tag{41}$$

with initial condition $h(y,T;\omega) = e^{\gamma(1-\rho^2)H(y)+\frac{1}{2}(1-\rho^2)\lambda_1^2(\tau_2-T)} \in \mathcal{F}^{(S,Y)}_{\tau_1}$, where we used that Y, conditionally on $\mathcal{F}^{(S,Y)}_{\tau_1}$, has dynamics given by (39) under the measure $\widetilde{\mathbb{Q}}$. Equation (41) is ill-posed and we refer to [10] for more detailed discussions of nonnegative solutions of such (random) ill-posed PDEs.

Next, having the nonnegative solution to (41), it follows easily that

$$F(y,\tau_2;\omega) = \frac{1}{1-\rho^2}\ln h(y,\tau_2;\omega), \tag{42}$$

for $y \in \mathbb{R}$. We are, thus, able to define the value functions for the plain investor and the writer over $[T,\tau_2]$, respectively. Indeed, the former is given by (37) as

$$V(x,t) = -e^{-\gamma x}\left(\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{(1-\rho^2)F(Y_{\tau_2},\tau_2)-\frac{1}{2}(1-\rho^2)\lambda_1^2(\tau_2-t)}\big|\mathcal{F}^{(S,Y)}_{\tau_1},\ Y_t\right]\right)^{\frac{1}{1-\rho^2}}, \tag{43}$$

whereas the latter is defined in a similar way, but with the liability of the second project taken into account,

$$V^W(x,t) = \operatorname*{esssup}_{\mathcal{A}} \mathbb{E}\left[U(X_{\tau_2} - G(Y_{\tau_2}),\tau_2)\big|\mathcal{F}^{(S,Y)}_{\tau_1}, X_t = x, Y_t\right]. \tag{44}$$

As before, the standard argument and the distortion transformation give rise to, for $T \leqslant t \leqslant \tau_2$,

$$V^W(x,t) = -e^{-\gamma x}\left(\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{(1-\rho^2)\left(\gamma G(Y_{\tau_2})+F(Y_{\tau_2},\tau_2)\right)-\frac{1}{2}(1-\rho^2)\lambda_1^2(\tau_2-t)}\big|\mathcal{F}^{(S,Y)}_{\tau_1},\ Y_t\right]\right)^{\frac{1}{1-\rho^2}}. \tag{45}$$

The relative indifference price of the second project over $[T,\tau_2]$ is then the conditional "break-even" price between the value functions $V(x,t)$ and $V^W(x,t)$. Indeed, it is the process $H^{2|1}_t$ that satisfies, conditionally on $\mathcal{F}^{(S,Y)}_{\tau_1}$,

$$V(X_t - H^{2|1}_t, t) = V^W(X_t,t), \quad \text{a.s.}, \quad T \leqslant t \leqslant \tau_2.$$

Further computations yield that, for $T \leqslant t \leqslant \tau_2$,

$$H_t^{2|1} = \frac{1}{\gamma(1-\rho^2)} \ln \frac{\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{(1-\rho^2)\left(\gamma G(Y_{\tau_2}) + F(Y_{\tau_2},\tau_2)\right)} \Big| \mathcal{F}_{\tau_1}^{(S,Y)},\ Y_t\right]}{\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{(1-\rho^2)F(Y_{\tau_2},\tau_2)} \Big| \mathcal{F}_{\tau_1}^{(S,Y)},\ Y_t\right]}, \tag{46}$$

with $F(y,\tau_2) \in \mathcal{F}_{\tau_1}^{(S,Y)}$ given by (42), being the nonnegative solution to the ill-posed problem (41).

To calculate the relative indifference price $H_t^{2|1}$ of the second project over period $[\tau_1, T]$, we still need to compare the optimal performance of the (benchmark) plain investor who holds only the first project and that of the writer who holds both the first and the second project. At $t = T$, the plain investor pays liability $H(Y_T)$ under the exponential utility $U(x) = -e^{-\gamma x}$, whereas the writer pays both $H(Y_T)$ and $H_T^{2|1}$, with the latter given by (46). The price $H_T^{2|1}$ can be seen as the time $t = T$ analogue of the terminal liability $G(Y_{\tau_2})$ under the extended forward criterion $U(x,\tau_2)$. Denote

$$\widehat{G}(Y_T) := \frac{1}{\gamma(1-\rho^2)} \ln \mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{(1-\rho^2)\left(\gamma G(Y_{\tau_2}) + F(Y_{\tau_2},\tau_2)\right)} \Big| \mathcal{F}_{\tau_1}^{(S,Y)},\ Y_T\right].$$

Then, from (46), we have

$$H_T^{2|1} = \widehat{G}(Y_T) - H(Y_T) - \frac{\lambda_1^2}{2\gamma}(\tau_2 - T).$$

The value function of the plain investor who is holding the single liability $H(Y_T)$ over $[\tau_1, T]$ under the exponential utility at $t = T$ follows from the results in [3],

$$V(x,t) = -e^{-\gamma x}\left(\mathbb{E}_{\mathbb{Q}}\left[e^{\gamma(1-\rho^2)H(Y_T) - \frac{1}{2}(1-\rho^2)\lambda^2(T-t)} \Big| Y_t\right]\right)^{\frac{1}{1-\rho^2}},$$

where the measure $\mathbb{Q}$ is defined on $\mathcal{F}_T^{(S,Y)}$ by

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big|_T = e^{-\lambda W_T^1 - \frac{1}{2}\lambda^2 T}.$$

The writer's optimization problem is similar but with liability $H(Y_T) + H_T^{2|1}$ at $t = T$; the value function is then given by

$$V^W(x,t) = -e^{-\gamma x}\left(\mathbb{E}_{\mathbb{Q}}\left[e^{\gamma(1-\rho^2)\widehat{G}(Y_T) - \frac{1}{2}(1-\rho^2)\lambda_1^2(\tau_2 - T) - \frac{1}{2}(1-\rho^2)\lambda^2(T-t)} \Big| Y_t\right]\right)^{\frac{1}{1-\rho^2}}.$$

The relative indifference price of the second project is similarly defined as the "break-even" price between the two value functions, i.e.,

$$V(X_t - H_t^{2|1}, t) = V^W(X_t, t),$$

for $\tau_1 \leqslant t \leqslant T$. A further computation leads to

$$H_t^{2|1} = \frac{1}{\gamma(1-\rho^2)} \ln \frac{\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{(1-\rho^2)\left(\gamma G(Y_{\tau_2}) + F(Y_{\tau_2},\tau_2)\right)} \Big| Y_T\right] \Big| Y_t\right]}{\mathbb{E}_{\mathbb{Q}}\left[e^{(1-\rho^2)\gamma H(Y_T)} \Big| Y_t\right]} - \frac{\lambda_1^2}{2\gamma}(\tau_2 - T). \tag{47}$$

□

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