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Mean field and n-player games in Ito-diffusion markets under forward performance criteria

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Abstract We study n-player games of portfolio choice in general common Ito-diffusion markets under relative performance criteria and time monotone forward utilities. We, also, consider their continuum limit which gives rise to a forward mean field game with unbounded controls in both the drift and volatility terms. Furthermore, we allow for general (time monotone) preferences, thus departing from the homothetic case, the only case so far analyzed. We produce explicit solutions for the optimal policies, the optimal wealth processes and the game values, and also provide representative examples for both the finite and the mean field game.

Keywords Mean field game, n-player game, Ito-diffusion, Forward performance criteria

2020 Mathematics Subject Classification 91A16, 91G10, 93E20

1. Introduction

The paper contributes to the literature of forward performance processes and portfolio choice games, with both finite and infinite number of players, under relative performance concerns. It considers managers who invest in a common market and the value of their strategies is affected by the average performance of their peers. For the n-player games, we define *forward best-response* and *forward Nash equilibrium* strategies, and produce closed form solutions building on time monotone forward utilities. For the continuum limit, we propose a notion of a *forward mean field game*, which has common noise and non-compact controls in both the drift and the volatility of the controlled state process, and produces closed form solutions for its value and the optimal processes. For both the finite and the infinite case, we work with arbitrary preferences, departing from the exponential case which is the only one that has been so far analyzed in infinite domains.

Relative performance is, arguably, one of the most fundamental factors that influence the behavior of managers, in both the mutual and hedge fund industries. It has been extensively studied in the finance literature in specific settings, like in static or single-period models, models with risk neutral managers, cases with two managers, models with index benchmarking, and

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others (see [2] for a detailed review of the literature). The first continuous-time models were introduced in lognormal markets in [3] for two players with power utilities, and later in [9] for n players and exponential utilities. The work was subsequently generalized by Lacker and the author in [19] where both the power and exponential utilities were examined in lognormal incomplete markets and for inhomogeneous agents. They were, also, the first to introduce the related mean field game (MFG) using a probabilistic definition. This MFG turned out to be of both common and idiosyncratic noise, and with non-compact controls in both the drift and the volatility of the state controlled process. For such games a unified theory does not exist to date. However, the homotheticity of the utilities allowed for significant reduction and tractability. This homotheticity has been, also, assumed in all subsequent works, which built on and extended the framework of [19] (see, among others, [4, 10, 12, 13, 18, 34]). To our knowledge, the only work which handled both general utilities and general competition couplings is the recent work of Souganidis and the author [32].

Despite the modeling advances, there are various shortcomings in the existing literature. The portfolio choice models are cast as expected utility ones in an arbitrary but fixed, investment horizon. As a result, one needs to pre-specify both the model dynamics and the investment horizon. In practice, however, it is almost impossible to choose the model accurately, especially if the horizon is not adequately short. Furthermore, in the best-response case, even the policies of the competitors (and/or, alternatively, the dynamics of a benchmark) are assumed to be known for the entire investment period, which might not be a very realistic assumption. These considerations do not only come from theoretical, conceptual arguments. Recent empirical works point out to dependencies of the observed policies to factors that cannot be explained in the existing settings; see, for example, [16] and [17] where the effects of the current phase of the market on the managers' behavior are discussed. We refer the reader to [2] for further details on the empirical literature and critique of the classical setting in portfolio management under competition.

Motivated by these limitations, Geng in [11] and subsequently together with Anthropelos and the author in [2] proposed, in a two-player setting, a new framework adding substantial flexibility with regards to model choice, investment horizons and policies of the competitors/benchmarks. This approach is based on the so-called *forward performance criteria*, which were introduced by Musiela and the author in [25] (see, also, [26]), and were further studied by them and others in the context of investment choice, intermediate consumption, indifference valuation, entropic risk measures, optimal liquidation, model uncertainty, robust control, and others; providing a complete bibliography is beyond the scope here and we only list representative papers ([1, 7, 8, 14, 15, 20, 22, 27, 30, 31, 33, 35, 37]; see, also, the recent review article by Musiela [24]). They extend the classical expected utility framework, for they allow for both "real-time" adaptive updating of model dynamics, flexible horizons and stochastically evolving risk preferences, while they remain consistent across times. They are built on the classical Dynamic Programming Principle and define a process, the forward performance criterion, with the following properties: compiled with the state wealth process generated by any admissible strategy, the forward process is a (local) supermartingale and there exists an optimal policy that generates a wealth process which, when compiled with the forward utility process, yields a (local) martingale. Under relative performance, the state wealth process is replaced by suitable relative performance metrics and the definitions are extended appropriately. Besides [11] and [2], forward criteria in similar games have been used in [5] and [6] for lognormal markets and special utilities, and more recently in [21] for predictable forward preferences of power type.

We generalize the results of [2] in several directions. We also work with Ito-diffusion markets but we depart from both the two-player setting and the forward homothetic preferences. In addition, we consider the continuum limit and propose a notion of a forward mean field game. We focus on the so-called asset diversification case, in that all players invest in a common market. On the other hand, we assume that the players are inhomogeneous in terms of their wealth, competition parameters and risk preferences. We take the latter to belong to the general class of time monotone forward utilities. As mentioned above, the only work in which general preferences have been considered is in [32] for the classical setting but for lognormal markets. For mere simplicity, we assume there is only one stock of Ito-diffusion dynamics and a riskless asset, as the multi-stock case can be studied similarly. Finally, we assume that the players' wealth process is unbounded and the competition is of linear type, leaving the half-domain as well as the nonlinear competition (studied in [32]) cases for future research.

We derive both the forward best-response and the forward Nash equilibrium policies, extending the class of forward time monotone criteria to incorporate linear competition. We summarize some of the main results next.

We establish that the forward best-response policy, $\breve{\pi}_{i,t},\ t \geqslant 0,$ of player $i,\ i = 1, ..., n,$ when her competitors follow (arbitrary) policies $\pi_{j,t},\ t \geqslant 0,\ j = 1, ..., n,\ j \neq i,$ is given by

$$\breve{\pi}_{i,t} = \frac{1}{1 - \frac{\theta_i}{n}} \alpha_{i,t}^{*,x_i - \theta_i \hat{x}} + \frac{\theta_i}{1 - \frac{\theta_i}{n}} \frac{1}{n} \sum_{j=1, j \neq i}^{n} \pi_{j,t}. \tag{1}$$

The constant θ_i is the competition parameter of player i, x_i her initial wealth and $\hat{x}$ the average initial wealth of all players. For player i, the process $\alpha_{i,t}^{*,x_i - \theta_i \hat{x}},\ t \geqslant 0,$ represents her optimal policy in the absence of competition, starting at $x_i - \theta_i \hat{x}$, and following time monotone forward performance criteria. We note that, in contrast to the classical utility setting, the forward best-response strategy does *not* require full knowledge of the competitors' policies in the entire investment horizon (see, for example, the analogous policy under exponential utility in an Ito-diffusion market in [13] which depends heavily on the competitor's policy for the entire trading horizon).

The forward Nash equilibrium policy, $\pi_{i,t}^{N},\ t \geqslant 0,\ i = 1, ..., n,$ is given by

$$\pi_{i,t}^{*,N} = \alpha_{i,t}^{*,x_i - \theta_i \hat{x}} + \frac{\theta_i}{1 - \hat{\theta}} \frac{1}{n} \sum_{j=1}^{n} \alpha_{j,t}^{*,x_j - \theta_j \hat{x}}, \tag{2}$$

where $\hat{\theta}$ is the average competition parameter $(\hat{\theta} \neq 1)$. The policies $\alpha_{i,t}^{*,x_i - \theta_i \hat{x}},\ t \geqslant 0,\ i = 1, ..., n,$ entering in (1) and (2) are given by

$$\alpha_{i,t}^{*,x_i - \theta_i \hat{x}} = \frac{\lambda_t}{\sigma_t} h_{i,x}(h_i^{(-1)}(x_i - \theta_i \hat{x}, 0) + A_t + M_t, A_t),$$

where $(\lambda, \sigma, A, M)_{t \geqslant 0}$ comes from the market (see (6) and (14)) while the function $h_i(x,t)$ is solely specified by the risk preferences of the player. It solves the ill-posed heat equation (16) and is uniquely represented as

$$h_i(x,t) = \int_{\mathbb{R}} \frac{e^{yx - \frac{1}{2}y^2 t} - 1}{y} \mathrm{d}\nu_i(y), \tag{3}$$

(modulo a generic additive constant) where ν_i is the *"personalized" risk preference measure* of player i.

To define the limiting game as the number of players goes to infinity, we follow the approach of [19] and consider the triplets $\zeta_i = (x_i, \theta_i, \nu_i),\ i = 1, ..., n,$ which model the initial wealth, competition parameter and risk preferences measure of player i. We, in turn, assume that the related empirical measure converges weakly to a measure $\mathbb{P}_0$, the law of the type of the representative player which is modeled via a random variable $Z = (\Xi, \Theta, \mathcal{N})$, taking values (ξ, θ, ν) (and being independent of the Brownian motion driving the stock price). We propose a notion of a forward mean field game (see Definition 6), extending the approach in [19] to accommodate the (time monotone) forward setting.

We produce the related MFG equilibrium policies $\pi_t^{*,MFG}$ and the optimal processes $X_t^{*,MFG}$, $t \geqslant 0$. Specifically, if the representative agent starts, say at $(\xi, \theta, \nu) \in Z$, the MFG equilibrium policy is given by

$$\pi_t^{*,MFG} = \alpha_t^{*,\xi - \theta \mathbb{E}_0[\Xi]} + \frac{\theta}{1 - \mathbb{E}_0 [\Theta]} \mathbb{E}_0 \left[\alpha_t^{*,\xi - \theta \mathbb{E}_0[\Xi]} \right], \tag{4}$$

where $\mathbb{E}_0$ is the expectation under $\mathbb{P}_0$ and $\alpha_t^{*,\xi - \theta \mathbb{E}_0[\Xi]}$ the optimal policy under no competition and starting at the initial random condition $\xi - \theta \mathbb{E}_0 [\Xi]$, given by

$$\alpha_t^{*,\xi - \theta \mathbb{E}_0[\Xi]} = \frac{\lambda_t}{\sigma_t} h_x (h^{(-1)} (\xi - \theta \mathbb{E}_0 [\Xi], 0) + A_t + M_t, A_t),$$

with $h(x,t) = \int_{\mathbb{R}} \frac{e^{yx - \frac{1}{2} y^2 t} - 1}{y} \mathrm{d}\nu(y)$ (cf. (3)) and $h^{(-1)}$ being its spatial inverse. Furthermore, the associated optimal wealth process is given by

$$X_t^{*,MFG} = x_t^{*,\xi - \theta \mathbb{E}_0[\Xi]} + \frac{\theta}{1 - \mathbb{E}_0 [\Theta]} \mathbb{E}_0 \left[x_t^{*,\xi - \theta \mathbb{E}_0[\Xi]} \right],$$

with $x_t^{*,\xi - \theta \mathbb{E}_0[\Xi]}$ being the optimal wealth process under no competition and starting at $\xi - \theta \mathbb{E}_0 [\Xi]$.

We also construct the corresponding forward best-response, Nash and mean field criteria, denoted, respectively, by $\breve{U}_i(x_1, ..., x_n, t; \pi_{-i,t})$, $U_i^N(x_1, ..., x_n, t)$, $i = 1, ..., n$, and $U^{MFG}(\xi, m, t)$, $t \geqslant 0$. They are given by the processes

$$\breve{U}_i(x_1, ..., x_n, t) = U_i^N(x_1, ..., x_n, t) = u_i (x_i - \theta_i \hat{x}, A_t), \tag{5}$$

and

$$U^{MFG}(\xi, m, t) = u (\xi - \theta \mathbb{E}_0 [\Xi], A_t),$$

where A_t is as in (14), and functions u_i and u given in (19) with measures ν_i and ν being, respectively, used in (17).

When the personalized measures are multiples of the Dirac measure at the origin, $\nu_i = r_i \delta_0$, $r_i > 0$, $i = 1, ..., n$, the optimal policies are independent of the wealth argument for both the n-player and the continuum limit game. Indeed, in this case, $h_i(x,t) = r_i x$ and $h(x,t) = rx$, $x \in \mathbb{R}$, $t \geqslant 0$, and, thus, $\alpha_{i,t}^{*,x_j - \theta_j \hat{x}} = \frac{\lambda_t}{\sigma_t} r_i$ and $\alpha_t^{*,\xi - \theta \mathbb{E}_0[\Xi]} = \frac{\lambda_t}{\sigma_t} r$. Therefore,

$$\pi_{i,t}^N = \frac{\lambda_t}{\sigma_t} \left(r_i + \frac{\theta_i}{1 - \hat{\theta}} \frac{1}{n} \sum_{j=1}^n r_i \right) \quad \text{and} \quad \pi_t^{*,MFG} = \frac{\lambda_t}{\sigma_t} \left(r + \frac{\theta}{1 - \mathbb{E}_0 [\Theta]} \mathbb{E}_0 [r] \right).$$

For lognormal markets, $\lambda_t = \lambda$, $\sigma_t = \sigma$, the above policies degenerate, respectively, to the constants $\pi_{i,t}^N = \frac{\lambda}{\sigma} \left(r_i + \frac{\theta_i}{1 - \hat{\theta}} \frac{1}{n} \sum_{j=1}^n r_i \right)$ and the random variables $\pi_t^{*,MFG} = \frac{\lambda}{\sigma} \left(r + \frac{\theta}{1 - \mathbb{E}_0[\Theta]} \mathbb{E}_0 [r] \right)$ (see [5] for the asset diversification case). Besides the exponential case, we also consider the double exponential case (introduced in [29] in the forward setting; see, also, [36] and the so called SAHARA utilities in [33]), and solve it explicitly.

We conclude mentioning that we consider the competition framework, as reflected on the sign of the θ_i and θ parameters, only for convenience. Because the relative performance term is linear in the average wealth of the players, one may also consider the *homophilous* case by reversing the sign. The technical results do not change but the qualitative behavior of the policies does, as we briefly mention in Section 2. The homophilous case was examined in [13] for exponential utilities in general incomplete markets, but we do not analyze it here.

The paper is organized as follows. In Section 2 we introduce the Ito-diffusion market, the n-player game and the notions of forward best-response and forward Nash criteria. We provide closed form solutions and discuss their structure. For the reader's convenience, we also review the single-agent (no competition) case. In Section 3, we propose a notion of a forward mean field game and construct the optimal policies and its value. In Section 4 we provide representative examples and conclude in Section 5.

2. The forward n-player game

We introduce a game of n players who invest in a common market that consists of a riskless asset (taken to be the numeraire and offering zero interest rate) and a risky stock with price S_t, $t \geqslant 0$, solving

$$\mathrm{d}S_t = b_t S_t \mathrm{d}t + \sigma_t S_t \mathrm{d}W_t, \quad S_0 > 0, \tag{6}$$

where $W_t,\ t \geqslant 0$, is a standard Brownian motion in $(\Omega, \mathcal{F}, \mathbb{P})$ with natural filtration $\left\{\mathcal{F}_t^W\right\}_{t\geqslant 0}$. The market coefficients b_t, σ_t are nonnegative $\mathcal{F}_t^W$-adapted processes, assumed to satisfy $b_t, \sigma_t \in (K_1, K_2),\ t \geqslant 0$, for some positive constants K_1, K_2. This condition, introduced for mere simplicity, may be relaxed. Furthermore, the financial model may be readily extended to the case of many stocks and even allowing for certain degeneracies in their volatility matrix (see, for example, the general Ito-diffusion model in [29]). We denote the Sharpe ratio process by $\lambda_t = \frac{b_t}{\sigma_t}$, $\lambda_t \in \left(\frac{K_1}{K_2}, \frac{K_2}{K_1}\right), t \geqslant 0$.

The generic player i, $i = 1, ..., n$, uses self-financing strategies $\pi_{i,t},\ t \geqslant 0$, which represent the amount invested in the stock and belong to the common across agents admissibility set

$$\mathcal{A} = \left\{\pi_i : \pi_{i,t}\ \in \mathcal{F}_t^W \text{ and } \mathbb{E}\int_0^t \sigma_s^2 \pi_{i,s}^2 \mathrm{d}s < \infty,\ t \geqslant 0\right\}. \tag{7}$$

Her state wealth process $X_{i,t},\ t \geqslant 0,\ i = 1, ..., n$, solves

$$\mathrm{d}X_{i,t} = b_t \pi_{i,t} \mathrm{d}t + \sigma_t \pi_{i,t} \mathrm{d}W_t, \quad X_i = x_i \in \mathbb{R}, \tag{8}$$

where both $\pi_{i,t}$ and $X_{i,t}$ are expressed in discounted (by the riskless asset) units.

Notation: We will be denoting the solvability domain by $\mathbb{D} = \mathbb{R} \times \mathbb{R}_+$ and, whenever appropriate, we will be using the self-evident notation $X_{i,t}^{\pi_i}$ and $X_{i,t}^{x_i}$.

Whenever appropriate, we will be also denoting $z_{i:k} := (z_i, z_{i+1}, .., z_{k-1}, z_k)$, $0 \leqslant i \leqslant k$, and $z_{-i} := (z_1, ..., z_{i-1}, z_{i+1}, ..., z_n)$ and $z_{-i} = (z_{1;i-1}, z_{i+1:n})$. In particular, we will denote the generic policies of all players excluding player i, by

$$\pi_{-i,t} := (\pi_{1,t}, ..., \pi_{i-1,t}, \pi_{i+1,t}, ..., \pi_{n,t}). \tag{9}$$

We will be also writing, with a slight abuse of notation, $\pi_{-i,t} \in \mathcal{A}$ when each component in (9) is admissible, $\pi_j \in \mathcal{A},\ j \neq i$.

Next, we introduce the notions of *forward best-response* and *forward Nash equilibrium*. They were first introduced for two players in Ito-diffusion markets by [11] and subsequently further developed and extended in [2] (see, also, [5]). We will be using the superscripts " $\smile$ " and " N " to refer to best-response and Nash equilibria quantities, respectively.

Definition 1 *Let* $i, j = 1, ..., n$ *and* $\pi_{-i,t} \in \mathcal{A}$*. An* $\mathcal{F}_t^W$-adapted *process* $\breve{U}_i(x_{1:n}, t; \pi_{-i,t})$, $x_{1:n} \in \mathbb{R}^n$, $t \geqslant 0$, *is called a forward best-response (to* $\pi_{-i,t}$*) criterion for player* i, *if the following conditions hold:*

i) *For each* $x_{-i} \in \mathbb{R}^{n-1}$ *and* $t \geqslant 0$, *the mapping* $x \rightarrow \breve{U}_i\left(x_{1:i-1}, x, x_{i+1:n}, t; \pi_{-i,t}\right)$ *is strictly increasing and strictly concave.*

ii) *For any* $\pi_i \in \mathcal{A}$, $\breve{U}_i\left(X_{1,t}^{\pi_1}, .., , X_{i,t}^{\pi_i}, ..., X_{n,t}^{\pi_n}, t; \pi_{-i}\right)$, $t \geqslant 0$, *is a (local) supermartingale, with* $X_{i,t}^{\pi_i}$ *and* $X_{j,t}^{\pi_j}$, $j \neq i$, *solving (8) with* π_i *and* π_j *being, respectively, used.*

iii) *There exists policy* $\breve{\pi}_i \in \mathcal{A}$, *such that* $\breve{U}_i\left(X_{1,t}^{\pi_1}, ..., X_{i,t}^{\breve{\pi}_i}, ..., X_{n,t}^{\pi_N}, t; \pi_{-i}\right)$, $t \geqslant 0$, *is a (local) martingale, with* $X_{i,t}^{\breve{\pi}_i}$ *solving (8) with* $\breve{\pi}_{i,t}$ *being used.*

Definition 2 *A forward Nash equilibrium consists of* n *pairs of* $\mathcal{F}_t^W$-adapted *processes,* $(U_i^N(x_{1:n}, t), \pi_{i,t}^N)$, $t \geqslant 0$, $i = 1, ..., n$, *with the following properties:*

i) *The control processes* $\pi_i^N \in \mathcal{A}$.

ii) *For each* $x_{-i} \in \mathbb{R}^{n-1}$ *and* $t \geqslant 0$, *the mapping* $x \rightarrow U_i^N(x_{1:i-1}, x, x_{i+1:n}, t)$ *is strictly concave and strictly increasing.*

iii) *For any* $\pi_i \in \mathcal{A}$, $U_i^N\left(X_{1,t}^{\pi_1^N}, ..., X_{i,t}^{\pi_i}, ..., X_{n,t}^{\pi_n^N}, t\right)$, $t \geqslant 0$, *is a (local) super-martingale.*

iv) *There exists* $\pi_i^N \in \mathcal{A}$ *such that* $U_i^N\left(X_{1,t}^{\pi_1^N}, ..., X_{i,t}^{\pi_i^N}, ..., X_{n,t}^{\pi_n^N}, t\right)$, $t \geqslant 0$, *is a (local) martingale.*

We highlight two important differences between the classical and the forward settings.

Firstly, Definition 1 proposes a notion of best-response policies for each agent *without* requiring knowledge of her competitors' policies *for all* future times. Note that this is not case in the classical setting. Indeed, in the latter, in order to define the best-response problem in a horizon, say $[0, T]$, the policies $\pi_{-i,t}$ need to be pre-specified for each $t \in [0, T]$, otherwise the underlying stochastic optimization problem cannot be solved. In the forward framework, however, this is not a requirement.

Secondly, in order to define both the forward best-response criterion and the forward Nash equilibrium, one does not need to prespecify the model dynamics for all upcoming times. This is not the case in the classical setting where model pre-commitment is ubiquitous in order to properly define the underlying expected utility problems on $[0, T]$. Even if one relaxes this assumption and works with a family of models, still this plausible family is pre-assigned at initial time. For further details and discussion on the differences between the forward and the traditional setting with relative performance concerns, we refer the reader to [2].

The set of processes expected to satisfy Definitions 1 and 2 is rather large and its full specification is beyond the scope of this paper. Herein, we focus on the case of linear competition and on forward criteria that build on time monotone forward utilities in the absence of competition. The latter is a rich enough family of forward performance processes and, furthermore, gives considerable tractability. For the reader's convenience, we briefly review this class next and refer to [29] for further details.

2.1 Review of the single-player (no competition) problem

We consider a single agent who invests in stock (6) and the riskless account using self-financing strategies $\alpha \in \mathcal{A}$, generating wealth process x_t^α, $t \geqslant 0$, solving

$$\mathrm{d}x_t^\alpha = b_t \alpha_t \mathrm{d}t + \sigma_t \alpha_t \mathrm{d}W_t, \quad x_0 = x \in \mathbb{R}. \tag{10}$$

Definition 3 *An* $\mathcal{F}_t^W$-adapted *process* $U(x,t)$ *is a forward performance criterion if the following conditions hold:*

i) *For each* $t \geqslant 0$, *the mapping* $x \to U(x,t)$ *is strictly increasing and strictly concave.*

ii) *For each* $\alpha \in \mathcal{A}$, $U(x_t^\alpha, t)$, $t \geqslant 0$, *is a (local) supermartingale, where* x_t^α *solves (10) with* α *being used.*

iii) *There exists* $\alpha^* \in \mathcal{A}$ *such that* $U(x_t^{\alpha^*}, t)$, $t \geqslant 0$, *is a (local) martingale, where* $x_t^{\alpha^*}$ *solves (10) with* α^* *being used.*

In [28] it was shown that if an $\mathcal{F}_t^W$-adapted (and strictly increasing and strictly concave in its spatial argument) process $U(x,t)$ satisfies the stochastic PDE

$$\mathrm{d}U(x,t) = \frac{\left(\lambda_t U_x(x,t) + a_x(x,t)\right)^2}{U_{xx}(x,t)}\mathrm{d}t + a(x,t)\mathrm{d}W_t, \tag{11}$$

for a suitable $\mathcal{F}_t^W$-adapted volatility process $a\left(x,t\right)$, then it is a forward performance criterion; additional integrability and smoothness conditions are needed, in analogy to classical verification results.

The forward volatility process $a\left(x,t\right)$ is the novel element herein and mainly differentiates the forward approach from the classical one. In the latter, the volatility of the value function process is part of the solution of the expected utility problem but, in the forward case, the utility volatility is a player-specific input. The characterization of suitable volatility processes for which the forward SPDE (11) has a desirable solution remains open. Results for volatility processes related to forward criteria in Markovian models may be found, among others, in [22] and [30].

Next, we focus on forward processes within the *zero volatility* class, $a(x,t) \equiv 0$, $(x,t) \in \mathbb{D}$, which was extensively analyzed in [29]. It was established that such forward criteria are uniquely represented by processes of the form

$$U(x,t) = u(x, A_t), \tag{12}$$

where $u : \mathbb{D} \to \mathbb{R}$ is a deterministic, strictly increasing and strictly concave, function satisfying

$$u_t = \frac{1}{2}\frac{u_x^2}{u_{xx}}, \tag{13}$$

and A_t given by

$$M_t = \int_0^t \lambda_s \mathrm{d}W_s \ \text{ and } A_t = \langle M \rangle_t = \int_0^t \lambda_s^2 \mathrm{d}s. \tag{14}$$

We will be calling processes M_t and A_t the *market input.* It is common to all players and does not depend on their preferences.

From (13) and the strict concavity of u, we deduce that $u(x,t)$ is strictly decreasing in time, for each $x \in \mathbb{R}$. This, together with the fact that A_t is strictly increasing, yields that the forward utility $U(x, A_t)$ is *time monotone.*

Central role in the construction of $u(x,t)$ plays a space-time harmonic function $h : \mathbb{D} \to \mathbb{R}$, defined through the transformation

$$u_x(h(x,t),t) = e^{-x+\frac{1}{2}t}. \tag{15}$$

It solves the ill-posed heat equation

$$h_t + \frac{1}{2} h_{xx} = 0, \tag{16}$$

and, for each $t \geqslant 0$, is strictly increasing in x. This class of solutions admits an "if and only if " characterization via a non-negative Borel measure, ν. Specifically, as shown in [29], $h(x,t)$ *must* be of the form

$$h(x,t) = \int_{\mathbb{R}} \frac{e^{yx - \frac{1}{2} y^2 t} - 1}{y} \mathrm{d}\nu\left(y\right) + C, \tag{17}$$

where the measure $\nu \in \mathcal{B}^+\left(\mathbb{R}\right)$ with

$$\mathcal{B}^+\left(\mathbb{R}\right) = \left\{ \nu \in \mathcal{B}\left(\mathbb{R}\right) : \forall B \in \mathcal{B},\ \nu\left(B\right) \geqslant 0 \text{ and } \int_{\mathbb{R}} e^{yx} \mathrm{d}\nu\left(y\right) < \infty,\ x \in \mathbb{R} \right\}, \tag{18}$$

and C being an immaterial constant. It was further established in [29] that in order to have $Range(h) = (-\infty, \infty)$, which is needed in order to have the wealth process in the entire $\mathbb{R}$, the measure ν must have the additional properties $\nu\left(\{0\}\right) > 0$, or $\nu \in B_0^+\left(\mathbb{R}\right)$, or $\nu \in B_+^+\left(\mathbb{R}\right)$ and $\int_{0^+}^{\infty} \frac{1}{y} \mathrm{d}\nu\left(y\right) = \infty$, or $\nu \in B_-^+\left(\mathbb{R}\right)$ and $\int_{-\infty}^{0^-} \frac{1}{y} \mathrm{d}\nu\left(y\right) = -\infty$.

Using (15) and (17), we obtain the representation

$$u(x,t) = -\frac{1}{2} \int_0^t e^{-h^{(-1)}(x,s) + \frac{s}{2}} h_x \left(h^{(-1)}(x,s), s \right) \mathrm{d}s + \int_0^x e^{-h^{(-1)}(z,0)} \mathrm{d}z, \tag{19}$$

where $h^{(-1)} : \mathbb{D} \to \mathbb{R}$ is the spatial inverse of h, which is well defined due to the strict spatial monotonicity of the latter. We note that one may prove that the familiar Inada conditions, $\lim_{x \downarrow -\infty} u_x(x,t) = \infty$ and $\lim_{x \uparrow \infty} u_x(x,t) = 0$, for each $t \geqslant 0$, follow directly from the properties of measure ν.

If the integrability condition $\int_{\mathbb{R}} e^{yx + \frac{1}{2} y^2 t} \mathrm{d}\nu\left(y\right) < \infty,\ (x,t) \in \mathbb{D}$, holds then the optimal control policy $\alpha_t^{*,x}$, $t \geqslant 0$, associated with (12), is given by

$$\alpha_t^* = \frac{\lambda_t}{\sigma_t} h_x(h^{(-1)}(x,0) + A_t + M_t, A_t). \tag{20}$$

Because $\lambda_t, \sigma_t > 0$ and $h\left(x,t\right)$ is strictly increasing in its spatial argument, it is always the case that $\alpha_t^* > 0,\ t \geqslant 0$, i.e. it is always optimal to invest a positive amount in the stock. Policy α_t^* generates the optimal wealth process

$$x_t^{\alpha^*} = h(h^{-1}(x,0) + A_t + M_t, A_t). \tag{21}$$

From definition (15), representation (12) and (17), we see that the pair (A_t, ν) are, essentially, the *defining elements* in constructing time monotone forward performance criteria. Indeed, once ν is initially $(t = 0)$ chosen, the function $h(x,t)$ and, thus, $u(x,t)$, are fully specified for all future times. In turn, the stochastic time change $t \to A_t,\ t \geqslant 0$, in $u(x,t)$ produces the time monotone criterion $U(x,t)$ (cf. (12)). Furthermore, the processes M_t and A_t, together with $h_x(x,t)$ and $h^{(-1)}(x,0)$, give the optimal processes α_t^* and $x_t^{\alpha^*}$ in (20) and (21), respectively.

At $t = 0$, (15) yields that the initial inverse marginal utility[1] $I(x,0) := \left(u_x\right)^{(-1)}(x,0)$ is given explicitly by

$$I(x,0) := \left(u_x\right)^{(-1)} \left(e^{-x}\right) = \int_{\mathbb{R}} \frac{x^{-y} - 1}{y} \mathrm{d}\nu(y), \tag{22}$$

[1] For general results on inverse marginal utilities in the classical setting, see the recent work [23].

where we took, for simplicity, $C = 0$ in (17). One may then characterize the agent's initial preferences by his initial inverse marginal utility directly through this underlying measure ν. This feature will be particularly useful later on in the n-game to describe each agent i, $i = 1, ..., n$, through her "personalized" risk preference measure ν_i.

In summary, time monotone forward criteria are built combining a static, chosen at initial time, personalized measure ν and the stochastically evolving market input process A_t. The optimal allocation and optimal wealth processes also require the market martingale M_t.

Another key function in this construction is the dynamic risk tolerance function $r : \mathbb{D} \to \mathbb{R}_+$,

$$r(x,t) := -\frac{u_x(x,t)}{u_{xx}(x,t)},$$

which, in view of (15), yields the representation

$$r(x,t) = h_x(h^{(-1)}(x,t),t). \tag{23}$$

In turn, the optimal control process α_t^*, $t \geqslant 0$, is given by

$$\alpha_t^* = \frac{\lambda_t}{\sigma_t} r(x_t^{\alpha^*}, t),$$

where $x_t^{\alpha^*}$ solves (cf. (10))

$$\mathrm{d}x_t^{\alpha^*} = \lambda_t^2 r(x_t^{\alpha^*}, t)\mathrm{d}t + \lambda_t r(x_t^{\alpha^*}, t)\mathrm{d}W_t, \quad x_0 = x \in \mathbb{R}.$$

Therefore, instead of using the inverse marginal utility $I(x,0)$ as a modeling input, we may alternatively employ the initial risk tolerance $r\,(x,0)$ which is, after all, more intuitive. In this case, $h(x,0)$ is recovered through the ODE $r(h(x,0),0) = h_x(x,0)$ and, from $h(x,0)$, one may extract the underlying measure using (17) for $t = 0$. The examples in Section 4 highlight this modeling perspective.

In general, identifying the underlying measure ν using the information coming from the inverse marginal $I(x,0)$ (or from $h(x,0)$ or the risk tolerance $r(x,0)$) might not be an easy task. This is a question interesting in its own right and is being currently investigated by the author and others.

2.2 Solving the n-player game for general forward preferences and linear couplings

In the classical setting of terminal expected utility, continuous time n-player games of competition in optimal portfolio management were firstly considered in [9] for exponential preferences and lognormal markets. They modelled competition linearly, in the sense that the utility of the generic player is affected not only by her terminal wealth but, also, by the average wealth of her competitors. These results were later extended in [19] for both the finite and continuum limit, and for both exponential and power utilities. Several other papers appeared afterwards for homothetic utilities as mentioned in the introduction. More recently, Souganidis and the author in [32] further generalized some of the existing results to the case of both general preferences and general couplings in a finite horizon setting in lognormal markets. They assumed a common complete market and homogeneous players with regards to both their terminal utility and the coupling function. They derived the master equation and produced closed form solutions using the value function in the absence of competition and a quasilinear PDE that the dynamic coupling function turns out to solve.

Herein, we follow some of the steps in [32] but extend the work to the forward setting, allowing for Ito-diffusion dynamics as well as for inhomogeneous players and personalized competition parameters. We work with relative forward performance criteria of the form

$$U_i(x_1,...,x_n,t) = V_i\left(x_i - \frac{\theta_i}{n}\sum_{j=1}^{n} x_j, t\right), \quad i=1,...,n, \tag{24}$$

for a suitable process $V_i(z,t) \in \mathcal{F}_t^W$. The parameter θ_i measures the i^{th} player's competitive preferences towards the average wealth of her peers.

Assumption 1: For $i=1,...,n$, the social competition parameters θ_i satisfy $\theta_i \in [0,1]$ with $\Pi_{i=1}^n \theta_i \neq 1$.

We introduce the average values

$$\hat{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \quad \text{and} \quad \hat{\theta} = \frac{1}{n}\sum_{i=1}^{n}\theta_i. \tag{25}$$

From the above assumption, we have that $\hat{\theta} \neq 1$.

Next, we assume that in the absence of competition, each player i, $i=1,...,n$, follows a time monotone forward criterion, i.e. she is endowed with forward preferences of the form $u_i(x, A_t)$, with u_i satisfying (13), rewritten below for convenience,

$$u_{i,t} = \frac{1}{2}\frac{u_{i,x}^2}{u_{i,xx}}. \tag{26}$$

Equivalently, we assume that each agent has a personalized risk preference measure $\nu_i \in \mathcal{B}^+(\mathbb{R})$ (cf. (18)) which generates the individual space-time harmonic function $h_i : \mathbb{D} \to \mathbb{R}$, $i=1,...,n$,

$$h_i(x,t) = \int_{-\infty}^{\infty} \frac{e^{yx-\frac{1}{2}y^2 t} - 1}{y} \mathrm{d}\nu_i(y). \tag{27}$$

In turn, the functions $u_i : \mathbb{D} \to \mathbb{R}$, $i=1,...,n$, are obtained as

$$u_i(x,t) = -\frac{1}{2}\int_0^t e^{-h_i^{(-1)}(x,s)+\frac{s}{2}} h_{i,x}\left(h_i^{(-1)}(x,s), s\right)\mathrm{d}s + \int_0^x e^{-h_i^{(-1)}(z,0)}\mathrm{d}z. \tag{28}$$

Next, we model the linear competition among the players by assuming that at initial time, $t=0$, player i, $i=1,...,n$, has *initial relative forward* criterion of the form

$$\begin{aligned} U_i(x_1,...,x_n,0) &= V_i(x_i - \theta_i\hat{x}, 0) \\ &= u_i(x_i - \theta_i\hat{x}, 0) = \int_0^{x_i-\theta_i\hat{x}} e^{-h_i^{(-1)}(z,0)}\mathrm{d}z. \end{aligned} \tag{29}$$

The above form might look non-intuitive as the linear coupling $\theta_i\hat{x}$ includes the initial state x_i of the i^{th} player. This is done only for technical convenience, as we explain in the sequel, after Proposition 5. The solution approach is the same, however the results using the above formulation facilitate the passing to the limit (which is, after all, not affected by the formulation we choose).

Proposition 4 *For $i=1,...,n$, let measures $\nu_i \in \mathcal{B}^+(\mathbb{R})$, functions $h_i(x,t)$ and $u_i(x,t)$ as in (27) and (28), and the market input processes A_t and M_t as in (14). Assume that the rest of the players $j=1,...,i-1,i+1,...,n$, follow arbitrary policies $\pi_{-i,t}$, $t \geqslant 0$, in $\mathcal{A}$. The process*

$$\breve{U}_i(x_1,...,x_n,t;\pi_{-i,t}) = u_i(x_i - \theta_i\hat{x}, A_t),$$

$t \geqslant 0$ and $\hat{x}$ as in (25), is a forward best-response performance criterion for player i. The associated forward best-response policy $\breve{\pi}_{i,t}$, $t \geqslant 0$, is given by

$$\breve{\pi}_{i,t} = \frac{1}{1-\frac{\theta_i}{n}}\alpha_{i,t}^{*,x_i-\theta_i\hat{x}} + \frac{\theta_i}{1-\frac{\theta_i}{n}}\frac{1}{n}\sum_{j=1,j\neq i}^{n}\pi_{j,t}, \tag{30}$$

where

$$\alpha_{i,t}^{*,x_i-\theta_i\hat{x}} = \frac{\lambda_t}{\sigma_t}h_{i,x}\left(h_i^{(-1)}(x_i-\theta_i\hat{x},0)+A_t+M_t,A_t\right).$$

The forward best-response wealth process $\breve{X}_{i,t}$, $t \geqslant 0$, is given by

$$\breve{X}_{i,t} = \frac{1}{1-\frac{\theta_i}{n}}x_{i,t}^{*,x_i-\theta_i\hat{x}} + \frac{\theta_i}{1-\frac{\theta_i}{n}}\frac{1}{n}\sum_{j=1,j\neq i}^{n}X_{j,t}^{\pi_j}, \tag{31}$$

where

$$x_{i,t}^{*,x_i-\theta_i\hat{x}} = h_i\left(h_i^{(-1)}(x_i-\theta_i\hat{x},0)+A_t+M_t,A_t\right),$$

and $X_{j,t}^{\pi_j}$, $t \geqslant 0$, $j=1,...,n$, $j\neq i$, solves (8) with $\pi_{j,t}$ being used.

Proof Let $\pi_{-i,t}$, $t \geqslant 0$, in $\mathcal{A}$. It suffices to show that the process $u_i\left(\left(1-\frac{\theta_i}{n}\right)X_{i,t}^{\pi_i} - \frac{\theta_i}{n}\sum_{j=1,j\neq i}^{n}X_{j,t}^{\pi_j}, A_t\right)$ is a (local) supermartingale for any $\pi_i \in \mathcal{A}$ and becomes a (local) martingale for $\breve{\pi}_{i,t}$, $t \geqslant 0$, as in (30). To this end, let $\pi_i \in \mathcal{A}$, and introduce the process

$$Z_{i,t} := (1-\frac{\theta_i}{n})X_{i,t}^{\pi_i} - \frac{\theta_i}{n}\sum_{j=1,j\neq i}^{n}X_{j,t}^{\pi_j}, \quad Z_{i,0} = z_{i,0} := x - \theta_i\hat{x}.$$

From (8) we have that $Z_{i,t}$ solves

$$\mathrm{d}Z_{i,t} = b_t\alpha_{i,t}\mathrm{d}t + \sigma_t\alpha_{i,t}\mathrm{d}W_t,$$

with $\alpha_{i,t} := (1-\frac{\theta_i}{n})\pi_{i,t} - \frac{\theta_i}{n}\sum_{j=1,j\neq i}^{n}\pi_{j,t}$. Because both $\pi_{i,t}$ and $\pi_{j,t} \in \mathcal{A}$, $j=1,...,n$, $j\neq i$, we have that $\alpha_{i,t} \in \mathcal{A}$ as well.

From the results in the single player case, we deduce that the process $u_i(Z_{i,t}, A_t)$, $t \geqslant 0$, is a (local) supermartingale for any $\alpha_{i,t} \in \mathcal{A}$, and, thus, for any $\pi_{i,t} \in \mathcal{A}$ given that for $j=1,...,n$, $j\neq i$, the policies $\pi_{j,t} \in \mathcal{A}$ are arbitrary but fixed. Furthermore, from (20) we deduce that the related optimal policy is given by

$$\alpha_{i,t}^{*,z_{i,0}} = \frac{\lambda_t}{\sigma_t}h_{i,x}\left(h_i^{(-1)}(z_{i,0},0)+A_t+M_t,A_t\right).$$

Therefore, the best-response policy of player i must satisfy

$$\left(1-\frac{\theta_i}{n}\right)\breve{\pi}_{i,t} - \frac{\theta_i}{n}\sum_{j=1,j\neq i}^{n}\pi_{j,t} = \frac{\lambda_t}{\sigma_t}h_{i,x}\left(h_i^{(-1)}(x_i-\theta_i\hat{x},0)+A_t+M_t,A_t\right), \tag{32}$$

and (30) follows. Using once more the results of the single agent case, we deduce that the control process $\alpha_{i,t}^{*,z_{i,0}}$ generates the state process $x_{i,t}^{*,x_i-\theta_i\hat{x}}$, $t \geqslant 0$. The rest of the proof follows. □

We next present that forward Nash equilibrium policies and the related criterion.

Proposition 5 *For $i = 1, ..., n,$ let measures $\nu_i \in \mathcal{B}^+(\mathbb{R})$, functions $h_i(x,t)$ and $u_i(x,t)$ as in (27) and (28), and $\alpha^{*,x}_{i,t}$, $t \geqslant 0$, being the optimal process for the single player problem (cf. (20)). Then, the control process $\pi^N_{i,t}$, $t \geqslant 0$, $i = 1, ..., N$, given by*

$$\pi^N_{i,t} = \alpha^{*,x_i-\theta_i\hat{x}}_{i,t} + \frac{\theta_i}{1-\hat{\theta}}\frac{1}{n}\sum_{j=1}^{n}\alpha^{*,x_j-\theta_j\hat{x}}_{j,t}, \tag{33}$$

is a forward Nash equilibrium policy. It generates the forward Nash wealth process

$$X^N_{i,t} = x^{*,x_i-\theta_i\hat{x}}_{i,t} + \frac{\theta_i}{1-\hat{\theta}}\frac{1}{n}\sum_{j=1}^{n}x^{*,x_j-\theta_j\hat{x}}_{j,t}. \tag{34}$$

The pairs

$$\left(u_i\left(x_i-\theta_i\hat{x}, A_t\right), \pi^N_{i,t}\right), \quad i = 1, ..., n, \tag{35}$$

constitute a forward Nash criterion.

Proof From (30) we deduce that the forward Nash equilibrium policy $\left(\pi^N_{1,t} : \pi^N_{n,t}\right)$, $t \geqslant 0$, must satisfy

$$\pi^N_{i,t} - \frac{\theta_i}{n}\sum_{j=1}^{n}\pi^N_{j,t} = \alpha^{*,x_i-\theta_i\hat{x}}_{i,t}.$$

Summing up for $i = 1, ..., n$ gives $\left(1-\hat{\theta}\right)\sum_{j=1}^{n}\pi^N_{j,t} = \sum_{j=1}^{n}\alpha^{*,x_j-\theta_j\hat{x}}_{j,t}$. Using that $\hat{\theta} \neq 1$, we obtain

$$\sum_{j=1}^{n}\pi^N_{j,t} = \frac{1}{1-\hat{\theta}}\sum_{j=1}^{n}\alpha^{*,x_j-\theta_j\hat{x}}_{j,t},$$

and the rest of the proof follows. □

Discussion: i) The forward best-response policy of player i consists of first decomposing his initial wealth as $x_i = x_i^0 + \breve{x}_i$, with

$$x_i^0 = \frac{1}{1-\frac{\theta_i}{n}}\left(x_i - \theta_i\hat{x}\right) \quad \text{and} \quad \breve{x}_i = x_i - x_i^0 = \frac{\theta_i}{1-\frac{\theta_i}{n}}\frac{1}{n}\sum_{j=1, j\neq i}^{n}x_j,$$

and, in turn, apply respectively policies

$$\breve{\pi}^0_{i,t} = \frac{1}{1-\frac{\theta_i}{n}}\alpha^{*,x_i-\theta_i\hat{x}}_{i,t} \quad \text{and} \quad \breve{\pi}^1_{i,t} = \frac{\theta_i}{1-\frac{\theta_i}{n}}\frac{1}{n}\sum_{j=1, j\neq i}^{n}\pi_{j,t}.$$

We recall that policy $\alpha^{*,x_i-\theta_i\hat{x}}_{i,t}$ is optimal for player i who starts at $x_i - \theta_i\hat{x}$ and does not face any competition (cf. (20)). The forward best-response to $\pi_{-1,t}$ allocation $\breve{\pi}_{i,t}$ is, thus, decomposed as

$$\breve{\pi}_{i,t} = \breve{\pi}^0_{i,t} + \breve{\pi}^1_{i,t}. \tag{36}$$

Policy $\breve{\pi}^0_{i,t}$ generates wealth $\frac{1}{1-\frac{\theta_i}{n}}x^{*,x_i-\theta_i\hat{x}}_{i,t}$ while policy $\breve{\pi}^1_{i,t}$ yields $\frac{\theta_i}{1-\frac{\theta_i}{n}}\frac{1}{n}\sum_{j=1,j\neq i}^{n}X^{\pi_j}_{j,t}$. Therefore, the forward best-response wealth process of player i is given by

$$\breve{X}_{i,t} = \frac{1}{1-\frac{\theta_i}{n}}x^{*,x_i-\theta_i\hat{x}}_{i,t} + \frac{\theta_i}{1-\frac{\theta_i}{n}}\frac{1}{n}\sum_{j=1, j\neq i}^{n}X^{\pi_j}_{j,t}.$$

Clearly, $\breve{\pi}_{i,t}$, $t \geqslant 0$, adjusts to current changes of the competitors' policies $\pi_{j,t}, j = 1, ..., N$,

$j \neq i$, and its specification does *not* require knowledge of the competitors' policies for all times. This is not the case in the classical paradigm. For example, in a similar Ito-diffusion market considered in [13], the analogous best-response policy must solve the expected utility problem

$$v_i(x_1, ..., x_n, t; \pi_{-i,t}) = \sup_{\pi_i} \mathbb{E}\left[-\exp\left(-\gamma_i \left(X_{i,T} - \frac{\theta_i}{n-1} \sum_{j=1, j\neq i}^{n} X_{j,T}^{\pi_j} \right)\right) \middle| X_{1:n,t} = x_{1:n} \right],$$

where we used the condensed notation $X_{1:n,t} = x_{1:n}$ to denote $X_{j,t} = x_j,\ j = 1, ..., n$. Clearly, to properly define the above problem, the terminal term $\sum_{j=1, j\neq i}^{N} X_{j,T}^{\pi_j}$ must be fully known, which directly requires knowledge of each $\pi_{j,t},\ j \neq i$, for all $0 \leqslant t \leqslant s \leqslant T$.

Depending on the direction and size of the $\breve{\pi}_{j,t},\ j \neq i$, the forward best-response policy $\breve{\pi}_{i,t}$ may be positive, negative or zero. However, its first component is always positive, $\breve{\pi}_{i,t}^0 > 0$, $t \geqslant 0$.

ii) The forward Nash equilibrium policy π_t^N also decomposes in two components, namely, $\pi_t^N = \pi_{0,t}^N + \pi_{1,t}^N$ with

$$\pi_{0,t}^N = \alpha_{i,t}^{*,x_i - \theta_i \hat{x}} \quad \text{and} \quad \pi_{1,t}^N = \frac{\theta_i}{1-\hat{\theta}} \frac{1}{n} \sum_{j=1}^{n} \alpha_{j,t}^{*,x_j - \theta_j \hat{x}}.$$

The second term involves the average of all $\alpha_{j,t}^{*,x_i - \theta_i \hat{x}}$, $j = 1, ..., n$, including $\alpha_{i,t}^{*,x_i - \theta_i \hat{x}}$ and is well defined only if $1 - \hat{\theta} \neq 0$, which justifies the second condition in Assumption 1.

iii) In both the forward best-response and forward Nash equilibrium cases, the forward criteria coincide and are equal to $u_i\left(x_i - \theta_i \hat{x}, A_t\right), t \geqslant 0$. For the reader familiar with indifference valuation or, equivalently, arbitrage-free pricing as the market herein is complete, this is expected. Indeed, the term $-\theta_i \hat{x}$ represents the arbitrage-free price, at initial time, of $\breve{C}_{i,t} := -\frac{\theta_i}{n}\left(\breve{X}_{i,t} + \sum_{j=1, j\neq i}^{n} X_{j,t} \right)$ as well as of $C_{i,t}^N := -\frac{\theta_i}{n} \sum_{j=1}^{n} X_{j,t}^N$. This, however, does not yield any contradiction as the two liabilities do not coincide, $\breve{C}_{i,t} \neq C_{i,t}^N, t > 0$. This difference is, also, reflected on the related policies.

iv) We may alternatively consider the case that the linear coupling does not include the player herself, i.e. we may look for forward processes of the form $V_i(x - \frac{\theta_i}{n-1} \sum_{j=1, j\neq i}^{n} x_j, t)$ instead of (24). The results are directly modified. Specifically, setting $\breve{x}_{-i} := \frac{1}{n-1} \sum_{j=1, j\neq i}^{n} x_j$ and working as in the proof of Proposition 4, we deduce that the forward best-response strategy for player i, denoted now by $\breve{\pi}_{i,t},\ i = 1, ..., n$, is given by

$$\breve{\pi}_{i,t} = \alpha_t^{*,x_i - \theta_i \breve{x}_{-i}} + \frac{\theta_i}{n-1} \sum_{j=1, j\neq i}^{n} \pi_{j,t},$$

where $\alpha_t^{*,x_i - \theta_i \breve{x}_{-i}} = \frac{\lambda_t}{\sigma_t} h_{i,x}\left(h_i^{(-1)}(x_i - \theta_i \breve{x}_{-i}, 0) + A_t + M_t, A_t \right)$.

To find the forward Nash equilibrium policy, denoted by $\left(\breve{\pi}_{1,t}^N, ..., \breve{\pi}_{i,t}^N, ..., \breve{\pi}_{n,t}^N\right)$, we use the above equation repeatedly to deduce that

$$\left(1 + \frac{\theta_i}{n-1}\right) \breve{\pi}_{i,t}^N - \theta_i \frac{n}{n-1} \left(\frac{1}{n} \sum_{j=1}^{n} \breve{\pi}_{j,t}^N \right) = \alpha_t^{*,x_i - \theta_i \breve{x}_{-i}}.$$

Thus, $\breve{\pi}_{i,t}^N = \frac{1}{1+\frac{\theta_i}{n-1}} \alpha_t^{*,x_i - \theta_i \breve{x}_{-i}} + \frac{n\theta_i}{n-1+\theta_i} \left(\frac{1}{n} \sum_{j=1}^{n} \breve{\pi}_{j,t}^N \right)$ and, therefore,

$$\frac{1}{n}\sum_{j=1}^{n}\pi_{j,t}^{N}=\frac{1}{1-\sum_{j=1}^{n}\frac{n\theta_j}{n-1+\theta_j}}\sum_{j=1}^{n}\frac{1}{1+\frac{\theta_j}{n-1}}\alpha_t^{*,x_j-\theta_j\check{x}_{-j}}.$$

Therefore, the forward Nash equilibrium is given by the pairs $\left(u_i\left(x_i-\theta_i\check{x}_{-i},A_t\right),\pi_{i,t}^{N}\right),\ t\geqslant 0$, $i=1,...,n$, where

$$\check{\pi}_{i,t}^{N}=\frac{1}{1+\frac{\theta_i}{n-1}}\alpha_t^{*,x_i-\theta_i\check{x}_{-i}}+\frac{n\theta_i}{n-1+\theta_i}\frac{1}{1-\sum_{j=1}^{n}\frac{n\theta_j}{n-1+\theta_j}}\sum_{j=1}^{n}\frac{1}{1+\frac{\theta_j}{n-1}}\alpha_t^{*,x_j-\theta_j\check{x}_{-j}}.$$

As expected, when the number of players goes to infinity, the two formulations will give the same limiting results.

3. A notion of forward mean field game for general preferences and linear couplings

We propose a limiting game, which we call *forward mean field game (MFG)* as the number of players goes to infinity. We build this by combining notions in Definition 2 and the probabilistic approach in [19] in the classical setting.

The first step is to *properly represent* each player so we have a meaningful representation in the continuum limit. Given the commonality of the market environment, it is natural to consider that players are "personalized" by their initial wealth, personal competition parameter and initial utility[2]. For the latter, we recall from the analysis in Section 2 that initial utilities are entirely characterized by the measure that defines the auxiliary space-time harmonic functions. In the n-player game, this triplet together with the market input in (14) completely characterized the forward criteria as well as the optimal policies and optimal wealth processes, and for both the best response and the Nash equilibrium cases. It is, then, reasonable to represent each player, say player $i,\ i=1,...,n$, by the triplet

$$\zeta_i=(x_i,\ \theta_i,\ \nu_i),\ x_i\in\mathbb{R},\ \theta_i\in[0,1]\quad\text{and}\quad\nu_i\in\mathcal{B}^{+}(\mathbb{R}),$$

with $\mathcal{B}^{+}(\mathbb{R})$ as in (18). In turn, we consider the empirical distribution of the players' types,

$$m_n=\frac{1}{n}\sum_{i=1}^{n}\delta_{\zeta_i},$$

and assume that it converges weakly to the law $\mathbb{P}_0$ of a random variable Z taking values $\zeta=(\xi,\theta,\nu),\ \xi\in\mathbb{R},\ \theta\in[0,1]$ and $\nu\in\mathcal{B}^{+}(\mathbb{R})$. We take Z to be defined in a probability space $(\Omega_0,\mathcal{G},\mathbb{P}_0)$ and assume that $\mathcal{G}$ is independent of filtration $\left\{\mathcal{F}_t^W\right\}_{t\geqslant 0}$.

Next, we recall that the forward Nash equilibrium policies are given, for $i=1,...,n$, by

$$\pi_{i,t}^{N}=\alpha_{i,t}^{*,x_i-\theta_i\hat{x}}+\frac{\theta_i}{1-\hat{\theta}}\frac{1}{n}\sum_{j=1}^{n}\alpha_{j,t}^{*,x_j-\theta_j\hat{x}},$$

where $\alpha_{j,t}^{*,x_j-\theta_j\hat{x}}=\frac{\lambda_t}{\sigma_t}h_{j,x}\left(h_j^{(-1)}(x_j-\theta_j\hat{x},0)+A_t+M_t,A_t\right)$ (cf. (20)). We would thus like to introduce a limiting game in which the optimal policy of the representative agent of generic type

[2] In [19], players also invest in their own assets (asset specialization) so their personalization type includes this dimension.

(ξ, θ, ν) will take, for $t \geqslant 0$, the intuitively analogous form

$$\pi_t^{*,MFG} = \alpha_t^{*,\xi-\theta\bar{\xi}} + \frac{\theta}{1-\bar{\theta}} \mathbb{E}_0 \left[\alpha_t^{*,\xi-\theta\bar{\xi}} \right], \tag{37}$$

where

$$\alpha_t^{*,\xi-\theta\bar{\xi}} = \frac{\lambda_t}{\sigma_t} h_x \left(h^{(-1)}(\xi - \theta\bar{\xi}, 0) + A_t + M_t, A_t \right),$$

$h(\xi, t) = \left. h(x, t) \right|_{x=\xi}$, with

$$h(x,t) = \int_{-\infty}^{\infty} \frac{e^{yx - \frac{1}{2}y^2 t} - 1}{y} \mathrm{d}\nu(y), \tag{38}$$

and $\bar{\theta}$ and $\bar{\xi}$ are the averages under $\mathbb{P}_0$,

$$\bar{\theta} = \mathbb{E}_0 [\Theta] \text{ and } \bar{\xi} = \mathbb{E}_0 [\Xi]. \tag{39}$$

Then, the process $\mathbb{E}_0 \left[\alpha_t^{*,\xi-\theta\bar{\xi}} \right]$, $t \geqslant 0$, is $\mathcal{F}_t^W$-adapted and given by

$$\begin{aligned} \mathbb{E}_0 \left[\alpha_t^{*,\xi-\theta\bar{\xi}} \right] &= \mathbb{E}_0 \left[h_x \left(h^{(-1)} \left(\xi - \theta\bar{\xi}, 0 \right) + A_t + M_t, A_t \right) \right] \\ &= \int_{\Omega_0} \int_{-\infty}^{\infty} e^{y\left(h^{(-1)}(\xi - \theta\bar{\xi}, 0) + A_t + M_t \right) - \frac{1}{2}y^2 A_t} \mathrm{d}\nu(y) \mathrm{d}\mathbb{P}_0. \end{aligned} \tag{40}$$

We introduce the set of forward MFG admissible policies $\pi_t, t \geqslant 0$,

$$\mathcal{A}^{MFG} = \left\{ \pi : \pi_t \in \mathcal{G} \vee \mathcal{F}_t^W, \ t \geqslant 0, \text{ and, for each fixed } \zeta_0, \ \pi(\zeta_0) \in \mathcal{A} \right\}, \tag{41}$$

with $\mathcal{A}$ as in (7). The wealth process of the representative agent X_t^π, $t \geqslant 0$, solves

$$\mathrm{d}X_t^\pi = b_t \pi_t \mathrm{d}t + \sigma_t \pi_t \mathrm{d}W_t, \quad X_0 = \xi. \tag{42}$$

For $h(x,t)$ as in (38), we introduce for $(x,t) \in \mathbb{D}$,

$$u(x,t) = -\frac{1}{2} \int_0^t e^{-h^{(-1)}(x,s) + \frac{s}{2}} h_x \left(h^{(-1)}(x,s), s \right) \mathrm{d}s + \int_0^x e^{-h^{(-1)}(z,0)} \mathrm{d}z. \tag{43}$$

Assumption 2: Let Z defined in $(\Omega_0, \mathcal{G}, \mathbb{P}_0)$ be the representative player's type. The $\mathcal{F}_t^W$-adapted processes

$$m_t := \mathbb{E}_0 \left[h \left(h^{(-1)} \left(\xi - \theta\bar{\xi}, 0 \right) + A_t + M_t, A_t \right) \right]$$

and

$$n_t := \mathbb{E}_0 \left[h_x \left(h^{(-1)} \left(\xi - \theta\bar{\xi}, 0 \right) + A_t + M_t, A_t \right) \right]$$

are well defined, and $\mathbb{E}_{\mathbb{P}} \int_0^t n_s^2 \mathrm{d}s < \infty$, for $t \geqslant 0$.

We now introduce the notion of forward mean field games we develop herein.

Definition 6 *Let* $\zeta = (\xi, \theta, \nu) \in Z$ *be arbitrary but fixed. The pair of processes* $\left(U^{MFG}(\xi,t), \pi_t^{*,MFG}\right)$, $t \geqslant 0$, *solves a forward mean field game if the following conditions hold:*

i) $U^{MFG}(\xi,t)$ *is* $\mathcal{G} \vee \mathcal{F}_t^W$-adapted *and* $\pi_t^{*,MFG} \in \mathcal{A}^{MFG}$ *and, for each* $t \geqslant 0$, *the mapping* $\xi \to U^{MFG}(\xi,t)$ *is strictly concave and strictly increasing.*

ii) *At* $t=0$, $U^{MFG}\left(\xi - \theta\bar{\xi}, 0\right) = u(\xi - \theta\bar{\xi}, 0)$ *with* $\bar{\xi}$ *as in (39), and* $u(\xi,0)$ *as in (43) with the measure* ν *being used.*

iii) *There exists an* $\mathcal{F}_t^W$-adapted *process* $\bar{X}_t$, $t \geqslant 0$, *such that, for each* $\pi_t \in \mathcal{A}^{MFG}$ *the process* $U^{MFG}\left(X_t - \theta\bar{X}_t, t\right)$, $t \geqslant 0$, *is a* $\mathbb{P}$ *-(local) supermartingale and there exists* $\pi_t^{*,MFG} \in \mathcal{A}^{MFG}$ *such that* $U^{MFG}\left(X_t^{*,MFG} - \theta\bar{X}_t, t\right)$, $t \geqslant 0$, *is a* $\mathbb{P}$ *-(local) martingale, with processes* X_t^{π} *and* $X_t^{*,MFG}$ *solving (42) with* π_t *and* $\pi_t^{*,MFG}$ *being, respectively, used.*

iv) *The processes* $X_t^{*,MFG}$ *and* $\bar{X}_t$ *satisfy, for* $t \geqslant 0$,

$$\bar{X}_t = \mathbb{E}_{\mathbb{P}\times\mathbb{P}_0}\left[X_t^{*,MFG} \middle| \mathcal{F}_t^W \right]. \tag{44}$$

We will be referring to the process $U^{MFG}(\xi,t)$, $\xi \in \mathbb{R}$, $t \geqslant 0$ as the *forward MFG performance criterion* and to $\pi_t^{*,MFG}$, $t \geqslant 0$, as a *forward MFG equilibrium policy.*

A similar notion of MFG policies was produced in [19] and directly adopted in [5] but only for $\mathcal{G}$-measurable policies. Such simple strategies turned out to be adequate therein due to the combination of model lognormality and the homotheticity of utilities (exponential and power). Here, however, we consider a much broader class of both market dynamics and forward utilities and this is, naturally, reflected on the enhanced measurability of the solution. We further discuss this in the next section where we present representative examples.

Proposition 7 *Let* $(\xi,\theta,\nu) \in Z$, h *as in (38),* $\bar{\theta}$ *and* $\bar{\xi}$ *as in (39) and assume that* $\bar{\theta} \in (0,1)$. *Let processes* A_t *and* M_t, $t \geqslant 0$, *be as in (14), and* $\bar{X}_t$, $t \geqslant 0$, *defined by*

$$\bar{X}_t := \frac{1}{1-\bar{\theta}} \mathbb{E}_0\left[x_t^{*,\xi-\theta\bar{\xi}} \right], \tag{45}$$

where

$$x_t^{*,\xi-\theta\bar{\xi}} = h\left(h^{(-1)}\left(\xi - \theta\bar{\xi}, 0\right) + A_t + M_t, A_t \right).$$

A forward mean field equilibrium policy is given by

$$\begin{aligned} \pi_t^{*,MFG} &= \frac{\lambda_t}{\sigma_t} h_x\left(h^{(-1)}\left(\xi - \theta\bar{\xi}, 0\right) + A_t + M_t, A_t \right) \\ &\quad + \frac{\theta}{1-\bar{\theta}} \frac{\lambda_t}{\sigma_t} \mathbb{E}_0\left[h_x\left(h^{(-1)}\left(\xi - \theta\bar{\xi}, 0\right) + A_t + M_t, A_t \right) \right]. \end{aligned} \tag{46}$$

It generates the forward MFG wealth process $X_t^{*,MFG}$, $t \geqslant 0$,

$$X_t^{*,MFG} = x_t^{*,\xi-\theta\bar{\xi}} + \frac{\theta}{1-\bar{\theta}} \mathbb{E}_0\left[x_t^{*,\xi-\theta\bar{\xi}} \right]. \tag{47}$$

The related forward MFG performance criterion is given by

$$U^{MFG}(\xi,t) = u\left(\xi - \theta\bar{\xi}, A_t\right), \tag{48}$$

with $u(x,t)$ as in (43) with the measure ν being used.

Proof To simplify the notation, we set $N_t := A_t + M_t,\ t \geqslant 0$. Assuming for now that the candidate policy $\pi^{*,MFG}$ in (46) belongs to $\mathcal{A}^{MFG}$, we first identify the process that it generates. To this end, we have that

$$
\begin{aligned}
&\xi + \int_0^t b_s \pi_s^{*,MFG} \mathrm{d}s + \int_0^t \sigma_s \pi_s^{*,MFG} \mathrm{d}W_s \\
= &\xi + \int_0^t b_s \frac{\lambda_s}{\sigma_s} h_x\left(h^{(-1)}\left(\xi - \theta\bar{\xi}, 0\right) + N_s, A_s\right) \mathrm{d}s \\
&+ \int_0^t \sigma_s \frac{\lambda_s}{\sigma_s} h_x\left(h^{(-1)}\left(\xi - \theta\bar{\xi}, 0\right) + N_s, A_s\right) \mathrm{d}W_s \\
&+ \frac{\theta}{1-\bar{\theta}}\left(\int_0^t b_s \frac{\lambda_s}{\sigma_s} \mathbb{E}_0\left[h_x\left(h^{(-1)}\left(\xi - \theta\bar{\xi}, 0\right) + N_s, A_s\right)\right] \mathrm{d}s\right. \\
&\left.+ \int_0^t \sigma_s \frac{\lambda_s}{\sigma_s} \mathbb{E}_0\left[h_x\left(h^{(-1)}\left(\xi - \theta\bar{\xi}, 0\right) + N_s, A_s\right)\right] \mathrm{d}W_s\right).
\end{aligned}
$$

Direct calculations together with (20) yield that

$$
\begin{aligned}
&\int_0^t b_s \frac{\lambda_s}{\sigma_s} h_x\left(h^{(-1)}\left(\xi - \theta\bar{\xi}, 0\right) + N_s, A_s\right) \mathrm{d}s + \int_0^t \sigma_s \frac{\lambda_s}{\sigma_s} h_x\left(h^{(-1)}\left(\xi - \theta\bar{\xi}, 0\right) + N_s, A_s\right) \mathrm{d}W_s \\
= &\int_0^t b_s \alpha_s^{*,\xi-\theta\bar{\xi}} \mathrm{d}s + \int_0^t \sigma_s \alpha_s^{*,\xi-\theta\bar{\xi}} \mathrm{d}W_s = x_s^{*,\xi-\theta\bar{\xi}} - \left(\xi - \theta\bar{\xi}\right). \qquad (49)
\end{aligned}
$$

Furthermore, it follows from Assumption 2 and routine arguments that

$$
\begin{aligned}
&\int_0^t b_s \frac{\lambda_s}{\sigma_s} \mathbb{E}_0\left[h_x\left(h^{(-1)}\left(\xi - \theta\bar{\xi}, 0\right) + N_s, A_s\right)\right] \mathrm{d}s \\
&+ \int_0^t \sigma_s \frac{\lambda_s}{\sigma_s} \mathbb{E}_0\left[h_x\left(h^{(-1)}\left(\xi - \theta\bar{\xi}, 0\right) + N_s, A_s\right)\right] \mathrm{d}W_s \\
= &\mathbb{E}_0\left[\int_0^t b_s \frac{\lambda_s}{\sigma_s} h_x\left(h^{(-1)}\left(\xi - \theta\bar{\xi}, 0\right) + N_s, A_s\right) \mathrm{d}s\right. \\
&\left.+ \int_0^t \sigma_s \frac{\lambda_s}{\sigma_s} h_x\left(h^{(-1)}\left(\xi - \theta\bar{\xi}, 0\right) + N_s, A_s\right) \mathrm{d}W_s\right] \\
= &\mathbb{E}_0\left[x_t^{*,\xi-\theta\bar{\xi}} - \left(\xi - \theta\bar{\xi}\right)\right] = \mathbb{E}_0\left[x_t^{*,\xi-\theta\bar{\xi}}\right] - \bar{\xi}\left(1 - \bar{\theta}\right). \qquad (50)
\end{aligned}
$$

From (49) and (50) we then deduce that (46) generates $X_t^{*,MFG}$ in (47) since

$$
\begin{aligned}
&\xi + x_t^{*,\xi-\theta\bar{\xi}} - \left(\xi - \theta\bar{\xi}\right) + \frac{\theta}{1-\bar{\theta}}\left(\mathbb{E}_0\left[x_t^{*,\xi-\theta\bar{\xi}}\right] - \bar{\xi}\left(1-\bar{\theta}\right)\right) \\
= &x_t^{*,\xi-\theta\bar{\xi}} + \frac{\theta}{1-\bar{\theta}} \mathbb{E}_0\left[x_t^{*,\xi-\theta\bar{\xi}}\right].
\end{aligned}
$$

From the above analysis we also deduce the admissibility of $\pi_t^{*,MFG}$. Next we observe, that by independence and Assumption 2, we have

$$\begin{aligned}\mathbb{E}_{\mathbb{P}\times\mathbb{P}_0}\left[X_t^{*,MFG}\middle|\mathcal{F}_t^W\right] &= \mathbb{E}_{\mathbb{P}\times\mathbb{P}_0}\left[h\left(h^{(-1)}\left(\xi-\theta\bar{\xi},0\right)+N_t,A_t\right)\middle|\mathcal{F}_t^W\right]\\ &+\mathbb{E}_{\mathbb{P}\times\mathbb{P}_0}\left[\frac{\theta}{1-\bar{\theta}}\mathbb{E}_0\left[h\left(h^{(-1)}\left(\xi-\theta\bar{\xi},0\right)+N_t,A_t\right)\right]\middle|\mathcal{F}_t^W\right]\\ &=\mathbb{E}_0\left[h\left(h^{(-1)}\left(\xi-\theta\bar{\xi},0\right)+N_t,A_t\right)\right]+\frac{\bar{\theta}}{1-\bar{\theta}}\mathbb{E}_0\left[h\left(h^{(-1)}\left(\xi-\theta\bar{\xi},0\right)+N_t,A_t\right)\right]\\ &=\frac{1}{1-\bar{\theta}}\mathbb{E}_0\left[h\left(h^{(-1)}\left(\xi-\theta\bar{\xi},0\right)+N_t,A_t\right)\right]=\frac{1}{1-\bar{\theta}}\mathbb{E}_0\left[x^{*,\xi-\theta\bar{\xi}}\right]=\bar{X}_t,\end{aligned}$$

as it follows from (45) and (21), and thus requirement (ii) is satisfied.

Next, we show that for any arbitrary but fixed $\pi_t \in \mathcal{A}^{MFG}$, the process $u\left(X_t^\pi-\theta\bar{X}_t,A_t\right)$ with X_t^π and $\bar{X}_t$ as in (42) and (45) respectively, is a $\mathbb{P}$-(local) supermartingale.

Since (ξ,θ,ν) is arbitrary but fixed, X_t^π is the outcome of an $\mathcal{A}$-admissible policy in the single player setting in Section 2. Furthermore, as shown above, the process $\theta\bar{X}_t$ is generated by the policy $\frac{\theta}{1-\bar{\theta}}\frac{\lambda_s}{\sigma_s}\mathbb{E}_0\left[h_x\left(h^{(-1)}\left(\xi-\theta\bar{\xi},0\right)+N_s,A_s\right)\right]$, which is also in $\mathcal{A}$. Therefore, the difference $Y_t := X_t^\xi-\theta\bar{X}_t$ is generated by $\mathcal{A}$-admissible policies and, by Definition 3, we deduce that $u\left(X_t-\theta\bar{X}_t,A_t\right)$ is a $\mathbb{P}$-(local) supermartingale. It remains to show that $u\left(X_t^{*,MFG}-\theta\bar{X}_t,A_t\right)$ is a $\mathbb{P}$-(local) martingale. For this, we observe that

$$X_t^{*,MFG}-\theta\bar{X}_t = x_t^{*,\xi-\theta\bar{\xi}},$$

which, for fixed (ξ,θ,ν), is generated by the $\mathcal{A}$-admissible policy $\alpha_t^{*,\xi-\theta\bar{\xi}}=\frac{\lambda_t}{\sigma_t}h_x(h^{(-1)}(\xi-\theta\bar{\xi},0)+N_t,A_t)$. However, this policy is optimal for the single player who starts at $\xi-\theta\bar{\xi}$ and has (time monotone) forward criterion $u(x,A_t)$. We easily conclude. □

4. Examples

We present two representative examples using time monotone forward preferences firstly introduced in [29] (in the absence of competition). The first one yields the exponential class and is generated by risk preference measures that are proportional to the Dirac function δ_0. The second example gives the forward analogues of utilities with asymptotically linear risk tolerance (see also [36] and [33]) also known as SAHARA utility functions. We recall that the market input processes $A_t,\ M_t,\ t\geqslant 0$, are as in (14).

4.1 The exponential case

The results below generalize the ones in [5] for lognormal dynamics to the Ito-diffusion case, for the case of asset diversification. Among others, we show that while the optimal processes are wealth-independent, they are substantially more general than the static ones in [5].

4.1.1 Forward best-response case

We fix player i and assume that she has constant initial risk tolerance, $r_i(x,0)=r_i,\ x\in\mathbb{R}$, $r_i>0$, and that the rest of the players use arbitrary policies $\pi_{-i,t},\ t\geqslant 0$. Transformation (23) yields $h_{i,x}(h_i^{(-1)}(x,0),0)=r_i$, and thus $h_i(x,0)=r_ix,\ x\in\mathbb{R}$. This is equivalent to choosing the risk preference measure to be $\nu_i=r_i\delta_0$. We make no assumptions about the risk preferences of the other players.

From (17) we deduce that $h_i(x,t)=r_ix$, and in turn (19) yields the time monotone forward exponential solution

$$u_i(x,t) = -r_i e^{-\frac{x}{r_i}+\frac{t}{2}}, \quad (x,t) \in \mathbb{D},$$

or more generally $u_i(x,t) = K_i - r_i e^{-\frac{x}{r_i}+\frac{t}{2}}$, for a generic constant K_i. Then, the optimal policy and optimal wealth in the absence of competition are given by

$$\alpha_{i,t}^* = \frac{\lambda_t}{\sigma_t} r_i \text{ and } x_{i,t}^{\alpha^*} = x_i + r_i (A_t + M_t).$$

In turn, (30) yields the best-response policy of player i,

$$\breve{\pi}_{i,t}^{x_i} = \frac{1}{1-\frac{\theta_i}{n}} \frac{\lambda_t}{\sigma_t} r_i + \frac{\theta_i}{1-\frac{\theta_i}{n}} \frac{1}{n} \sum_{j=1, j\neq i}^{n} \pi_{j,t}.$$

The forward best-response wealth process $\breve{X}_{i,t}^{x_i}$, $t \geqslant 0$, is given by

$$\begin{aligned}\breve{X}_{i,t}^{x_i} &= \frac{1}{1-\frac{\theta_i}{n}} x_{i,t}^{*,x_i-\theta_i\hat{x}} + \frac{\theta_i}{1-\frac{\theta_i}{n}} \frac{1}{n} \sum_{j=1,j\neq i}^{n} X_{j,t}^{\pi_j}, \\ &= \frac{1}{1-\frac{\theta_i}{n}} \left(x_i - \theta_i \hat{x} + r_i (A_t + M_t) + \theta_i \frac{1}{n} \sum_{j=1,j\neq i}^{n} X_{j,t}^{\pi_j} \right),\end{aligned}$$

with $X_{j,t}^{x_j}$, $t \geqslant 0$, solving (8) with arbitrary policies $\pi_{j,t}$ being used. In general, the best-response policy $\breve{\pi}_{i,t}^{x_i}$ may be smaller or larger than the optimal policy $\frac{\lambda_t}{\sigma_t} r_i$ in the absence of competition. However, for $\theta_i \geqslant 0$, we always have $0 < \frac{1}{1-\frac{\theta_i}{n}} \frac{\lambda_t}{\sigma_t} r_i < \frac{\lambda_t}{\sigma_t} r_i$.

The best-response forward criterion of player i is given by

$$\breve{U}_{i,t}(x_1,...,x_n,t;\pi_{-i,t}) = -r_i \exp\left(-\frac{1}{r_i}(x_i - \theta_i \hat{x}) + \frac{1}{2} A_t\right),$$

with $\hat{x}$ as in (25).

4.1.2 Forward Nash equilibrium

We work with *inhomogeneous* players within the exponential class, assuming that player i, $i = 1,...,n$, has constant initial risk tolerance, $r_i(x,0) = r_i$, $x \in \mathbb{R}$, for some $r_i > 0$. In other words, each player's "personalized" risk preference measure is given by $\nu_i = r_i \delta_0$, $r_i > 0$. From (33), (34) and (35), we obtain the forward Nash equilibrium policies for player $i = 1,...,n$,

$$\pi_{i,t}^N = \frac{\lambda_t}{\sigma_t}\left(r_i + \frac{\theta_i}{1-\hat{\theta}} \hat{r}\right) \quad \text{with} \quad \hat{r} := \frac{1}{n}\sum_{j=1}^{n} r_j, \tag{51}$$

as well as the forward Nash wealth process and performance criterion,

$$X_{i,t}^N = x_i + \left(r_i + \frac{\theta_i}{1-\hat{\theta}}\hat{r}\right)(A_t + M_t), \tag{52}$$

and

$$U_i^N(x_1,...,x_n,t) = -r_i \exp\left(-\frac{1}{r_i}(x_i - \theta_i\hat{x}) + \frac{1}{2}A_t\right). \tag{53}$$

If the market is lognormal, $\lambda_t = \lambda$ and $\sigma_t = \sigma$, the above results yield the ones in [5] for the asset diversification case as well as the ones in the classical setting in [19]. Note, however, that once we depart from lognormal markets, this commonality disappears (see, for example, [16]) as additional terms appear in the equilibrium policies due to stochastic factors.

Each policy $\pi_{i,t}^{N}$, $t \geqslant 0$, inherits the time monotonicity of the ratio $\frac{\lambda_t}{\sigma_t}$ and is independent of all wealth arguments (personalized or aggregate). It is increasing in both the competition θ_i and the aggregate $\hat{\theta}$ parameters. It is, also, larger than the optimal policy $\alpha_{i,t}^{*,x_i}$ in the absence of competition.

Using (51) and (52), we may interpret the forward Nash policy as the one of a single player with initial wealth x_i and modified risk preference measure

$$\nu_i^N = \left(r_i + \frac{\theta_i}{1-\hat{\theta}} \hat{r} \right) \delta_0.$$

As in [19], we call the quantity $r_i + \frac{\theta_i}{1-\hat{\theta}}\hat{r}$ the *effective risk tolerance.*

4.1.3 Forward mean field game

The type of the representative player represented by the triplet $\zeta = (\xi, \theta, r)$, with $r > 0$ being the value of a random risk tolerance coefficient R and corresponding to a risk preference measure $\nu = r\delta_0$. Let $\bar{r} := \mathbb{E}_0[R]$. We then construct the MFG equilibrium policy, optimal wealth and forward criterion, namely,

$$\pi_t^{*,MFG} = \frac{\lambda_t}{\sigma_t}\left(r + \frac{\theta}{1-\bar{\theta}}\bar{r} \right),$$
$$X_t^{*,MFG} = \xi + \left(r + \frac{\theta}{1-\bar{\theta}}\bar{r} \right)(A_t + M_t),$$

and

$$U^{MFG}(\xi, t) = -r \exp\left(-\frac{1}{r}\left(\xi - \theta\bar{\xi}\right) + \frac{1}{2}A_t \right).$$

The above quantities have analogous properties to their forward Nash counterparts in terms of the player's type. Specifically, $\pi_t^{*,MFG}$ depends only on the stochastic market input $\frac{\lambda_t}{\sigma_t}$, the representative agent's risk tolerance parameter r and competition coefficient θ as well as their averages $\bar{r}$ and $\bar{\theta}$. It is increasing in each one of them and dominates the single agent policy in the absence of competition. It may be also viewed as the optimal policy of a single agent with effective risk tolerance

$$r^{MFG} := r + \frac{\theta}{1-\bar{\theta}}\bar{r},$$

starting at random wealth ξ, and following time monotone forward criterion of exponential type.

As mentioned in Section 3, $\pi_t^{*,MFG}$ is $\mathcal{G} \vee \mathcal{F}_t^W$-adapted and not just $\mathcal{G}$-measurable as in the lognormal Setting in the forward exponential model in [5] (see, also, [19] for the classical case). Similarly with the Nash equilibrium case, once we depart from the lognormal case, the forward MFG processes and values differ substantially from their classical setting counterparts (see, for example, [13]).

4.2 The symmetric double-exponential case

In the exponential case above, the risk preference measures were taken to be proportional to

Dirac measure centered at the origin. We now consider measures that are a multiple of two symmetric Diracs. We start with the simple case (54) and extend it to (55). Direct but tedious calculations show that as the parameters $k_i, i = 1, ..., n,$ and k vanish we recover the exponential case presented in the previous example.

4.2.1 Forward best-response case

We fix player i and assume she has risk preference measure

$$\nu_i = \frac{1}{2}\left(\delta_{-1} + \delta_1\right). \tag{54}$$

Then, (27) gives $h_i(x,0) = \sinh x$ and, thus, $h_{i,x}(x,0) = \cosh x$ and $h_i^{(-1)}(x,0) = \ln g(x)$, where $g(x) := x + \sqrt{1+x^2}$. Therefore, the initial risk tolerance $r_i(x,0)$ is given by (cf. (23))

$$r_i(x,0) = h_{i,x}\left(h_i^{(-1)}(x,0), 0\right) = \cosh\left(\ln g(x)\right) = \sqrt{1+x^2}.$$

Observe that $\lim_{|x|\uparrow\infty} \frac{r_i(x,0)}{|x|} = 1$ and that $x = 0$ is a global minimum, with $r_i(0,0) = 1$.

We easily deduce that $h_i(x,t) = e^{-\frac{t}{2}} \sinh x$ and $h_{i,x}(x,t) = e^{-\frac{t}{2}} \cosh x$, $t \geqslant 0$. Therefore, the optimal policy and optimal wealth processes in the absence of competition are given, respectively, by

$$\alpha_{i,t}^{*,x} = \frac{\lambda_t}{\sigma_t} e^{-\frac{1}{2}A_t} \cosh\left(\ln g(x) + A_t + M_t\right)$$

and

$$x_{i,t}^{*,x} = e^{-\frac{1}{2}A_t} \sinh\left(\ln g(x) + A_t + M_t\right).$$

If the rest of the players use policies $\pi_{-i,t}$, $t \geqslant 0$, the best-response policy of player i is given by

$$\begin{aligned}
\breve{\pi}_{i,t} &= \frac{1}{1-\frac{\theta_i}{n}} \frac{\lambda_t}{\sigma_t} \alpha_{i,t}^{*,x-\theta_i\hat{x}} + \frac{\theta_i}{1-\frac{\theta_i}{n}} \frac{1}{n} \sum_{j=1, j\neq i}^{n} \pi_{j,t} \\
&= \frac{1}{1-\frac{\theta_i}{n}} \frac{\lambda_t}{\sigma_t} e^{-\frac{1}{2}A_t} \cosh\left(\ln g(x-\theta_i\hat{x}) + A_t + M_t\right) + \frac{\theta_i}{1-\frac{\theta_i}{n}} \frac{1}{n} \sum_{j=1, j\neq i}^{n} \pi_{j,t}.
\end{aligned}$$

The forward best-response wealth process $\breve{X}_{i,t}$, $t \geqslant 0$, is given by

$$\breve{X}_{i,t} = \frac{1}{1-\frac{\theta_i}{n}} e^{-\frac{1}{2}A_t} \sinh\left(\ln g\left(x-\theta_i\hat{x}\right) + A_t + M_t\right) + \frac{\theta_i}{1-\frac{\theta_i}{n}} \frac{1}{n} \sum_{j=1, j\neq i}^{n} X_{j,t}^{\pi_j}.$$

4.2.2 Forward Nash equilibrium

To capture more generality, we extend the model by considering personalized risk preference measures of the parametric form

$$\nu_i = \frac{1}{2} r_i \left(\delta_{-k_i} + \delta_{k_i}\right), \ i = 1, ..., n, \quad \text{with } r_i, k_i > 0. \tag{55}$$

Then, (27) gives $h_i(x,0) = \frac{r_i}{k_i} \sinh(k_i x)$ and, thus, $h_{i,x}(x,0) = r_i \cosh(k_i x)$. We claim that such measure generates an initial risk tolerance of the form

$$r_i(x,0)=\sqrt{k_i^2x^2+r_i^2}.$$

Indeed, it suffices to show that $r_i^2(x,0)=k_i^2x^2+r_i^2$ or, equivalently, $h_{i,x}^2(x,0)=k_i^2h_i^2(x,0)+r_i^2$, where we used (23). But the latter follows directly from the forms of $h_{i,x}(x,0)$ and $h_i(x,0)$ and the properties of hyperbolic functions.

Parameters k_i, r_i play different role in the shape of the $r(x,0)$ in that k_i gives its asymptotic slope for large $|x|$, $\lim_{|x|\uparrow\infty}\frac{r_i(x,0)}{|x|}=k_i$, while r_i determines its minimum, $r_i=r_i(0,0)<r_i(x,0)$, $|x|>0$.

We easily deduce that, for $t>0$, $h_i(x,t)=\frac{r_i}{k_i}e^{-\frac{1}{2}k_i^2t}\sinh(k_ix)$ and $h_{i,x}(x,0)=r_ie^{-\frac{1}{2}k_i^2t}\cosh(k_ix)$. Using (23) once more we obtain that

$$r_i(x,t)=\sqrt{k_i^2x^2+r_i^2e^{-k_i^2t}}. \tag{56}$$

For each $x\in\mathbb{R}$, $r_i(x,t)$ is strictly decreasing in time with $\lim_{t\uparrow\infty}r_i(x,t)=k_i|x|$. Furthermore, let

$$g(x;r,k):=\frac{k}{r}x+\sqrt{1+\frac{k^2}{r^2}x^2},\quad x\in\mathbb{R},\quad r>0. \tag{57}$$

Then, for $x\in\mathbb{R}$, $h_i^{(-1)}(x,0)=\frac{1}{k_i}\ln g(x;r_i,k_i)$.

The optimal policy and optimal wealth process in the absence of competition are given by

$$\alpha_{i,t}^{*,x}=\frac{\lambda_t}{\sigma_t}r_ie^{-\frac{1}{2}k_i^2A_t}\cosh\left(\ln g(x;r_i,k_i)+k_i\left(A_t+M_t\right)\right)$$

and

$$x_{i,t}^{*,x}=\frac{r_i}{k_i}e^{-\frac{1}{2}k_i^2A_t}\sinh\left(\ln g(x;r_i,k_i)+k_i\left(A_t+M_t\right)\right).$$

Therefore, the forward Nash policy and the forward Nash wealth process are given, respectively, by

$$\begin{aligned}\pi_{i,t}^N&=\frac{\lambda_t}{\sigma_t}r_ie^{-\frac{1}{2}k_i^2A_t}\cosh\left(\ln g(x_i-\theta_i\hat{x};r_i,k_i)+k_i\left(A_t+M_t\right)\right)\\&\quad+\frac{\theta_i}{1-\hat{\theta}}\frac{\lambda_t}{\sigma_t}\frac{1}{n}\sum_{j=1}^n r_je^{-\frac{1}{2}k_j^2A_t}\cosh\left(\ln g(x_j-\theta_j\hat{x};r_j,k_j)+k_j\left(A_t+M_t\right)\right),\end{aligned}$$

and

$$\begin{aligned}X_{i,t}^N&=\frac{r_i}{k_i}e^{-\frac{1}{2}k_i^2A_t}\sinh\left(\ln g(x_i-\theta_i\hat{x};r_i)+k_i\left(A_t+M_t\right)\right)\\&\quad+\frac{\theta_i}{1-\hat{\theta}}\frac{1}{n}\sum_{j=1}^n r_je^{-\frac{1}{2}k_j^2A_t}\sinh\left(\ln g(x_j-\theta_j\hat{x};r_j)+k_j\left(A_t+M_t\right)\right).\end{aligned}$$

The forward Nash performance criterion is given by (48) with $u_i(x,t)$ as in (28), which can be explicitly calculated by direct but tedious arguments.

We conclude pointing out that the form of $h_i(x,0)$ yields that the initial inverse marginal utility $I_i(x,0):\mathbb{R}_+\to\mathbb{R}$ satisfies $I_i(e^{-x},0)=\frac{r_i}{k_i}\sinh(k_ix)$. Therefore, it is given by the symmetric sum

$$I_i(x,0)=\frac{1}{2}\frac{r_i}{k_i}\left(x^{-k_i}-x^{k_i}\right).$$

In other words, for each $t\geqslant 0$, $I_i(x,0)$ is a completely monotonic function (since, after all, $h_i(x,t)$ is absolutely monotonic, for each $t\geqslant 0$); for general results on completely monotonic inverse marginal utilities see the recent work [23].

From the modeling perspective, it is more intuitive to specify the initial risk tolerance or the initial inverse marginal utility, instead of the risk preference measure. However, extracting the measure from arbitrary functions $r_i(x,0)$ and $I_i(x,0)$ might not be always tractable as in the symmetric case herein.

4.2.3 Forward mean field game

The representative player's type is parametrized by four random inputs, $\zeta=(\xi,\theta,(r,k))$, with (r,k) defining, in analogy to (56), her initial risk tolerance in the (k,r)-parametric form

$$r(\xi,0)=\sqrt{k^2\xi^2+r^2}.$$

Working as in the forward Nash equilibrium case, we readily deduce that the forward MFG policy is given by the process

$$\begin{aligned}\pi_{i,t}^{*,MFG}&=\frac{\lambda_t}{\sigma_t}re^{-\frac{1}{2}k^2A_t}\cosh\left(\ln g(\xi-\theta\bar{\xi};r,k)+k(A_t+M_t)\right)\\&\quad+\frac{\theta}{1-\bar{\theta}}\frac{\lambda_t}{\sigma_t}\mathbb{E}_0\left[re^{-\frac{1}{2}k^2A_t}\cosh\left(\ln g(\xi-\theta\bar{\xi};r,k)+k(A_t+M_t)\right)\right],\end{aligned}$$

with the (random) function $g(\xi;r,k)$ as in (57). It generates the MFG equilibrium wealth process

$$\begin{aligned}X_t^{*,MFG}&=\frac{r}{k}e^{-\frac{1}{2}k^2A_t}\sinh\left(\ln g(\xi-\theta\bar{\xi};r,k)+k(A_t+M_t)\right)\\&\quad+\frac{\theta}{1-\bar{\theta}}\mathbb{E}_0\left[\frac{r}{k}e^{-\frac{1}{2}k^2A_t}\sinh\left(\ln g(\xi-\theta\bar{\xi};r,k)+k(A_t+M_t)\right)\right],\end{aligned}$$

with $\bar{\xi}$ and $\bar{\theta}$ as in (39). As mentioned earlier, both $\pi_{i,t}^{*,MFG}$ and $X_t^{*,MFG}$ are $\mathcal{G}\vee\mathcal{F}_t$-adapted and not just $\mathcal{G}$-measurable (as in [5], or [19] in the classical setting).

5. Conclusions and extensions

We analyzed n-player games and their continuum limit under forward performance criteria when the interaction among players is generated by a multiple of the average wealth of their peers. For the n-player game, we extended the notions of forward best-response and forward Nash equilibrium, firstly introduced in [11] and [2]. We analyzed the asset diversification (common market) case for unbounded wealth domains and produced closed-form solutions, for both the optimal processes and the related values, building on time monotone forward performance criteria. In turn, we proposed a notion of forward mean field games and produced closed-form solutions which are the natural limits of their n-player game counterparts. Finally, we provided representative examples and recovered, as special cases, the results in [5] and [19] with lognormal dynamics and common investment opportunities.

For both the n-player and the mean field game, we worked with general forward risk preferences within the time monotone class, considerably extending all existing works with exponential preferences. Structurally, our solutions resemble the ones in [32] even though we work with entirely different utility settings and, furthermore, consider Ito-diffusion and not just lognormal markets.

Going beyond the linear average interaction case, it would be interesting to extend to the forward setting the results of [32] for general couplings. For this, however, one needs to derive a form of *forward master equation*, which is not yet done even for simple, lognormal market dynamics. Another generalization could be to work with forward criteria beyond the time monotone class and, in particular, to allow for forward volatilities for the agents' preferences. This could be first considered within the class of homothetic utilities and stochastic factor models where the volatility process is well understood (see, for example, [22]) and is generated from the stock dynamics. On the other hand, forward volatilities could be, also, used to model personalized attitude towards upcoming market changes, anticipation of competitor's actions and performance, model uncertainty, and others. Finally, the asset specialization case offers another research direction. It is expected that will give rise to a forward mean field game with both idiosyncratic and common noise, but it is not yet clear how tractable the problems will be.

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References

[1] Anthropelos, M., Forward exponential performances: Pricing and optimal risk sharing, SIAM Journal on Financial Mathematics, 2014, 5(1): 626–655.

[2] Anthropelos, M., Geng, T. and Zariphopoulou, T., Competition in fund management and forward relative performance criteria, SIAM Journal on Financial Mathematics, 2022, 13(4): 1271–1301.

[3] Basak, S. and Makarov, D., Strategic asset allocation in money management, Journal of Finance, 2014, 69(1): 179–217.

[4] Bo, L., Wang, S. and Zhou, C., A mean field game approach to optimal investment and risk control for competitive insurers, Insurance: Mathematics and Economics, 2024, 116: 202–2017.

[5] Dos Reis, G. and Platonov, V., Forward utilities and mean field games under relative performance concerns, In: Bernardin, C., Golse, F., Gonçalves, P., Ricci, V. and Soares, A. J.(eds.), From Particle Systems to Partial Differential Equations. Proceedings, ICPS, Springer Proceedings in Mathematics & Statistics, Springer, Cham, 2021, 352: 227–251.

[6] Dos Reis, G. and Platonov, V., Forward utilities and market adjustments in relative investment-consumption games of many players, SIAM Journal in Financial Mathematics, 2022, 13(3): 844–876.

[7] El Karoui, N., Hillairet, C. and Mrad, M., Consistent utility of investment and consumption: A forward-backward SPDE viewpoint, Stochastics, 2018, 90(6): 927–954.

[8] El Karoui, N. and Mrad, M., An exact connection between two solvable SDEs and a nonlinear utility stochastic PDE, SIAM Journal on Financial Mathematics, 2013, 4(1): 697–736.

[9] Espinosa, G.-E. and Touzi, N., Optimal investment under relative performance concerns, Mathematical Finance, 2015, 25(2): 221–257.

[10] Fu, G., Mean field portfolio games with consumption, Mathematics and Financial Economics, 2023, 17: 79–99.

[11] Geng, T., Essays on forward portfolio theory and financial time series modeling, Ph.D. Thesis, The University of Texas at Austin, 2017.

[12] Guo, X., Xu, R. and Zariphopoulou, T., Entropy regularization for mean field games with learning, Mathematics of Operations Research, 2022, 47(4): 3239–3260.

[13] Hu, R. and Zariphopoulou, T., N-player and mean-field games in Ito-diffusion markets with competitive and homophilous interaction, In: Yin, G. and Zariphopoulou, T.(eds.), Stochastic Analysis, Filtering, and Stochastic Optimization, Springer-Verlag, Cham, 2022: 209–237.

[14] Kallblad, S., Black's inverse investment problem and forward criteria with consumption, SIAM Journal on Financial Mathematics, 2020, 11(2): 494–525.

[15] Kallblad, S., Obloj, J. and Zariphopoulou, T., Dynamically consistent investment under model uncertainty: The robust forward case, Finance and Stochastics, 2018, 22: 879–918.

[16] Kempf, A. and Ruenzi, S., Tournaments in mutual fund families, Review of Financial Studies, 2008, 21(2): 1013–1036.

[17] Kempf, A., Ruenzi, S. and Thiele, T., Employment risk, compensation incentives and managerial risk taking: Evidence from the mutual fund industry, Journal of Financial Economics, 2009, 92(1): 92–108.

[18] Lacker, D. and Soret, A., Many-player games of optimal consumption and investment under relative performance concerns, Mathematics and Financial Economics, 2020, 14: 263–281.

[19] Lacker, D. and Zariphopoulou, T., Mean field and n-player games for optimal investment under relative performance criteria, Mathematical Finance, 2019, 29(4): 1003–1038.

[20] Liang, G., Strub, M. and Wang, Y., Predictable forward performance preferences: Infrequent evaluation and applications to human-machine interactions, Mathematical Finance, 2023, 33(4): 1248–1286.

[21] Liang, G., Strub, M. and Wang, Y., Predictable forward performance processes: Multi-agent and mean field games for portfolio management, arXiv: 2310.09830, 2023.

[22] Liang, G. and Zariphopoulou, T., Representation of homothetic forward performance processes in stochastic factor models via ergodic and infinite horizon BSDE, SIAM Journal on Financial Mathematics, 2017, 8(1): 344–372.

[23] Mostoviy, O., Sirbu, M. and Zariphopoulou, T., On the analyticity of the value function of optimal investment and stochastically dominant markets, Pure and Applied Functional Analysis, 2024, 9(3): 825–862.

[24] Musiela, M., Preface for " recent advances in forward performance processes", Probability, Uncertainty and Quantitative Risk, 2024, 9(1): 1–12.

[25] Musiela, M. and Zariphopoulou, T., Investments and forward utilities, Technical report, 2006.

[26] Musiela, M. and Zariphopoulou, T., Derivative pricing, investment management and the term structure of exponential utilities: The case of binomial model, In: Carmona, R.(ed.), Indifference Pricing, Princeton University Press, Princeton, 2009: 3–41.

[27] Musiela, M. and Zariphopoulou, T., Portfolio choice under dynamic investment performance criteria, Quantitative Finance, 2009, 9(2): 161–170.

[28] Musiela, M. and Zariphopoulou, T., Stochastic partial differential equations and portfolio choice, In: Chiarella, C. and Novikov, A.(eds.), Contemporary Quantitative Finance, Springer, Berlin, 2010: 195–215.

[29] Musiela, M. and Zariphopoulou, T., Portfolio choice under space-time monotone performance criteria, SIAM Journal on Financial Mathematics, 2010, 1: 326–365.

[30] Nadtochiy, S. and Tehranchi, M., Optimal investment for all time horizons and Martin boundary of space-time diffusions, Mathematical Finance, 2015, 27(2): 438–470.

[31] Nadtochiy, S. and Zariphopoulou, T., A class of homothetic forward investment performance process with non-

zero volatility, In: Kabanov, Y., Rutkowski, M. and Zariphopoulou, T.(eds.), Inspired by Finance: The Musiela Festschrift, Springer, Cham, 2014: 475–505.

[32] Souganidis, P. and Zariphopoulou, T., Mean field games with unbounded controlled common noise in portfolio management with relative performance criteria, Mathematics and Financial Economics, to appear, 2024.

[33] Strub, M. and Zhou, X. Y., Evolution of the Arrow-Pratt measure of risk tolerance for predictable forward utility processes, Finance and Stochastics, 2021, 25: 331–358.

[34] Tangpi, L. and Zhou, X., Optimal investment in large population of competitive and heterogeneous players, Finance and Stochastics, 2024, 28: 497–551.

[35] Waldon, H., Forward robust portfolio selection: The binomial case, Probability, Uncertainty and Quantitative Risk, 2024, 9(1): 107–122.

[36] Zariphopoulou, T. and Zhou, T., Investment performance measurement under asymptotically linear risk tolerance, In: Bensoussan, A. and Zhang, Q.(eds.), Handbook of Numerical Analysis, Elsevier, 2009, 15: 227–253.

[37] Zitkovic, G., A dual characterization of self-generation and exponential forward performances, The Annals of Applied Probability, 2009, 19(6): 2176–2210.