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Forward robust portfolio selection: The binomial case

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Abstract We introduce a new approach for optimal portfolio choice under model ambiguity by incorporating predictable forward preferences in the framework of Angoshtari et al. [2]. The investor reassesses and revises the model ambiguity set incrementally in time while, also, updating his risk preferences forward in time. This dynamic alignment of preferences and ambiguity updating results in time-consistent policies and provides a richer, more accurate learning setting. For each investment period, the investor solves a worst-case portfolio optimization over possible market models, which are represented via a Wasserstein neighborhood centered at a binomial distribution. Duality methods from Gao and Kleywegt [10]; Blanchet and Murthy [8] are used to solve the optimization problem over a suitable set of measures, yielding an explicit optimal portfolio in the linear case. We analyze the case of linear and quadratic utilities, and provide numerical results.

Keywords Forward robust portfolio selection, Binomial case, Optimal portfolio, Forward performance processes, Linear utilities, Quadratic utilities, Robust forward performance criteria

2020 Mathematics Subject Classification 91G10, 60H30

1. Introduction

The classical optimal portfolio problem is composed of three fundamental modeling ingredients: an investment horizon $T > 0$, a performance criterion $\mathcal{U}_T(\cdot)$ applied at the end of this horizon, and a market model, represented as a probability distribution ν_T, which describes the dynamics of tradable assets over the investment horizon $[0, T]$. We, then, call $(T, \mathcal{U}_T, \nu_T)$ the *investment triplet*. With an investment triplet specified, standard methods, such as dynamic programming or the Pontryagin maximum principle, can be used to solve for an optimal portfolio $\pi^*(\cdot)$. The classical setup leads to many interesting mathematical and financial insights and has theoretical appeal. However, it is limited to exogenously specified investment triplets which are known explicitly (i.e., chosen statically) by the investor at time $t = 0$. In reality, however, the investor is subject to uncertainty in each element of the investment triplet, and this presents various challenges to the classical dynamic optimization paradigm.

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A popular approach for incorporating uncertainty about the market model into portfolio optimization is through techniques from Distributionally Robust Optimization (DRO); see, for example, Obłój and Wiesel [23]; Blanchet et al. [6]; Huang et al. [15]; Bardakci and Lagoa [4]; Bielecki et al. [5]. In this setting, the investor considers *Knightian uncertainty*, i.e., the distribution of the asset may belong to a set of possible probability distributions, known as the *ambiguity set*; see Rahimian and Mehrotra [25] for a review of DRO. Then, the investor selects a portfolio which maximizes the worst-case expected utility with respect to the probability measures in the ambiguity set, resulting in a max-min optimization problem. In Gilboa and Schmeidler [11], the decision-theoretic foundations of a risk and model ambiguity averse agent are described axiomatically, justifying the max-min optimization. A key component of DRO is the construction of a *suitable ambiguity set* over which the investor's minimization should take place. In particular, the ambiguity set must be large enough to incorporate non-trivial deviations from the reference measure yet small enough such that the underlying optimization problem is still tractable and that only realistic distributions are considered Maccheroni et al. [18]; Hansen and Sargent [13].

To construct ambiguity sets, the Kullback-Leibler (KL) divergence is used often because of its interpretability and tractability. However, these KL ambiguity sets generally lack expressivity, because they contain only measures which are absolutely continuous with respect to the reference measure. When the reference measure is assumed to be binomial, the KL ambiguity set around this measure consists only of binomial measures, which is particularly restrictive. In this paper, we consider binomial measures, and instead of KL ambiguity sets, we consider ambiguity sets which are neighborhoods around a reference measure in the Wasserstein space. The Wasserstein distance, originating from the theory of optimal transportation Villani et al. [29], is desirable for the definition of ambiguity sets, because it is non-parametric and well-motivated from statistical theory Esfahani and Kuhn [9]; Pflug and Wozabal [24]. What Wasserstein ambiguity sets gain over KL ambiguity sets in terms of expressivity they lose in terms of tractability, optimization over a Wasserstein ambiguity set is an infinite-dimensional problem. However, in Gao and Kleywegt [10]; Blanchet and Murthy [8], the authors prove a strong duality result for the optimization over Wasserstein ambiguity sets, making the problem finite-dimensional. In this paper, we make extensive use of these duality results in order to solve for a robust optimal portfolio.

While DRO incorporates ambiguity into the market model, the investor might not a priori know the investment horizon, and, furthermore, her risk preferences may change dynamically with time. Empirical evidence suggests that unforeseen events may cause risk preferences to change Hanaoka et al. [12]. The desire to incorporate flexibility about the investment horizon and utility function led to an innovative reformulation of the classical investment problem by Musiela and Zariphopoulou in which the investment horizon and the utility function are built *forward* in time. This reformulation centers on the pioneering *forward performance criteria*, introduced and developed in Musiela and Zariphopoulou [19–22], in which the investor begins with an initial performance criterion $\mathcal{U}_0(\cdot)$ and updates it forward in time parsimoniously with the evolution of the market. Continuous-time forward performance processes have been studied extensively since their inception, as in Avanesyan et al. [3]; Žitković [30]; Shkolnikov et al. [26] among many others, and only Källblad et al. [16]; He et al. [14] consider forward performance under model ambiguity. Discrete-time forward performance processes have seen considerably less attention, with notable exceptions including Angoshtari et al. [2]; Angoshtari [1]; Strub and Zhou [28]; Liang et al. [17], however, none investigate forward utility under model ambiguity.

To the best of our knowledge, this paper is the first to incorporate uncertainty in all three elements of the investment triplet in the discrete-time setting. To this end, we introduce *robust predictable forward performance criteria*. In our model, trading of a single risky asset and a riskless asset occurs at a sequence of increasing times $\{t_n\}_{n\in\mathbb{N}}$. At each time t_n, the investor models the returns of the risky asset at time t_{n+1} using a binomial distribution as a reference measure which, we note, may change period-to-period. The investor is uncertain of the reference measure in each period and considers a Wasserstein ambiguity set around it.

At time t_0, the investor is endowed with an initial performance criterion $\mathcal{U}_0(\cdot)$, and solves for both a time t_1 utility $\mathcal{U}_1(\cdot)$ and an optimal portfolio π_0^* which satisfy a single period formulation of the robust portfolio optimization problem. A robust predictable forward performance process is a sequence of performance criteria $\mathcal{U}_n(\cdot)$ built forward in time and measurable with respect to the filtration generated by the evolution of market variables, which satisfies a multi-period formulation of the robust portfolio optimization problem. To construct a robust predictable forward performance process, we show, as in Angoshtari et al. [2], that solving the multi-period robust control problem reduces to solving a conditionally-defined single period robust control problem.

The rest of the paper is structured as followed. In Section 2, we outline our model in the general case and introduce the robust predictable forward performance criteria. In Section 3, we present the robust forward investment problem in the case of linear utility and solve the problem explicitly when the investor is subject to portfolio constraints. In Section 4 we present the case of quadratic utility and solve the problem implicitly up to the solution of a polynomial. In this case, we provide numerical solutions of the optimal portfolio. To conclude, we present the general robust forward investment problem, and comment on the difficulties in establishing a general theory when utilizing Wasserstein uncertainty sets.

2. The model and the robust forward performance criteria

2.1 Review of the single period robust portfolio problem

We start by recalling the classical single period optimal portfolio problem with and without model uncertainty. An investor seeks to allocate capital between a risky asset and a riskless bond over a fixed time horizon $T > 0$ to maximize her expected utility of terminal wealth $\mathcal{U}_T : \mathbb{R} \to \mathbb{R}$. Thus, she seeks a portfolio π_0^* which maximizes

$$\pi_0^* \in \underset{\pi_0 \in A_{0,T}(x_0)}{\arg\sup} \; \mathbb{E}^{\nu}\left[\mathcal{U}_T(X_T^{\pi_0})\right], \tag{1}$$

where the investor's terminal wealth $X_T^{\pi_0}$ is defined as

$$X_T^{\pi_0} = x_0 + \pi_0(R_T - 1).$$

Herein, $x_0 \in \mathbb{R}$ is the investor's initial capital, and π_0 is the allocation to the risky asset at time $t = 0$. The random variable $R_T \sim \nu$ is the risky asset's returns at time T, with distribution $\nu \in P(\mathbb{R}^+)$, where $P(\mathbb{R}^+)$ denotes the set of probability measures on $\mathbb{R}^+$.

The set of admissible portfolios $\mathcal{A}_{0,T}$ is defined as

$$\mathcal{A}_{0,T}(x_0) = \{\pi \in \mathbb{R} : L_0 x_0 \leqslant \pi \leqslant U_0 x_0\},$$

for some constants $L_0, U_0 \in \mathbb{R}$. Throughout this paper, the interest rate is taken to be zero. The optimal portfolio π_0^* which maximizes (1) yields an optimal value function of initial wealth $\mathcal{U}_0(x_0)$,

$$\mathcal{U}_0(x_0) = \mathbb{E}^{\nu}\left[\mathcal{U}_T(X_T^{\pi_0^*})\right].$$

The binomial model is simple, however it has substantial expressive power and has been studied at length; see, for example Shreve [27].

To consider a robust formulation of the classical portfolio optimization problem, we recast (1) as a DRO problem. In this setting, the investor is uncertain of the true distribution of the tradable asset ν, but has some estimate about this distribution, say, $\nu_0 \in P(\mathbb{R}^+)$. We call ν_0 the investor's *reference measure*. The investor is uncertain of her estimate ν_0 and considers that the true distribution of the asset could lie in set of *possible measures* $\mathcal{P} \subset P(\mathbb{R}^+)$ containing the reference measure ν_0, which we call the *ambiguity set*. In the robust portfolio optimization problem, the investor seeks a portfolio π_0^* which maximizes expected utility over the worst-case measure in the ambiguity set, which is written as

$$\pi_0^* \in \underset{\pi_0 \in A_{0,T}(x_0)}{\arg\sup} \inf_{\eta \in \mathcal{P}} \mathbb{E}^{\eta}\left[\mathcal{U}_T(X_T^{\pi_0})\right]. \tag{2}$$

For an admissible portfolio π_0^* which maximizes (2), we analogously define the optimal utility of initial wealth $\mathcal{U}_0(x_0)$ as

$$\mathcal{U}_0(x_0) = \inf_{\eta \in \mathcal{P}} \mathbb{E}^{\eta}\left[\mathcal{U}_T(X_T^{\pi_0^*})\right].$$

Tractability of (2) strongly depends on the set of admissible portfolios $\mathcal{A}_{0,T}$, the choice of the ambiguity set $\mathcal{P}$, and the form of the utility function $\mathcal{U}_T$.

Herein, we take the ambiguity set to be $\mathcal{P} = B_\delta(\nu_0)$, where $B_\delta(\nu_0)$ is a Wasserstein ball of radius δ centered at the reference measure ν_0. We recall that for two probability measures, say, $\eta, \zeta \in P(\mathbb{R}^+)$, their p-Wasserstein distance, associated to a cost function $c : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ for $p > 0$, is defined as

$$\mathrm{d}(\eta, \zeta) := \inf_{\gamma \in \Gamma(\eta,\zeta)} \left(\int_{\mathbb{R}^+ \times \mathbb{R}^+} c(x, y)^p \, \gamma(\mathrm{d}x, \mathrm{d}y) \right)^{1/p},$$

where $\Gamma(\eta, \zeta) = \{\gamma \in P(\mathbb{R}^+ \times \mathbb{R}^+) : \gamma(\cdot, \mathbb{R}^+) = \eta(\cdot), \gamma(\mathbb{R}^+, \cdot) = \zeta(\cdot)\}$ denotes the set of all couplings of η and ζ. When c is a distance metric, $\mathrm{d}(\cdot, \cdot)$ becomes a metric on $P(\mathbb{R}^+)$.

2.2 The robust forward problem

In addition to uncertainty about the distribution of the asset ν, the investor wants to maintain flexibility about her investment horizon T and her utility function $\mathcal{U}_T$. To allow for such flexibility in the robust problem (2), we first recast (2) as a multi-period, dynamic investment problem which is built forward in time. We assume the investor may trade the underlying asset at an exogenous sequence of times $\{t_n\}_{n \in \mathbb{N}}$, with time $t_0 = 0$.

First, we outline the first period problem. At time t_0, the investor is endowed exogenously with an *initial* utility function $\mathcal{U}_0$, representing her attitude towards wealth at initial time. Moreover, at time t_0, the investor observes the following quantities:

- A probability measure $\nu_0 \in P(\mathbb{R})$, denoting the investor's reference measure for the asset's distribution of returns R_1 at time t_1.

- A level of uncertainty about her reference measure $\delta_0 > 0$, which is the radius of the Wasserstein ball used as the investor's ambiguity set for the distribution of returns R_1.

• Portfolio constraints $L_0, U_0 \in \mathbb{R}$, defining the set of admissible portfolios for the first investment period $\mathcal{A}_{0,1}(x_0)$.

In the first period, the investor seeks a non-random function $\mathcal{U}_1$ such that for all portfolios $\pi_0 \in \mathcal{A}_{0,1}(x_0)$, the inequality

$$\mathcal{U}_0(x) \geqslant \operatorname*{ess\,inf}_{\eta \in B_{\delta_0}(\nu_0)} \mathbb{E}^\eta \left[\mathcal{U}_1(X_1^{\pi_0}) \,\middle|\, \delta_0, \nu_0, L_0, U_0 \right]$$

holds, and there exists $\pi_0^* \in \mathcal{A}_{0,1}(x_0)$ such that

$$\mathcal{U}_0(x) = \operatorname*{ess\,inf}_{\eta \in B_{\delta_0}(\nu_0)} \mathbb{E}^\eta \left[\mathcal{U}_1(X_1^{\pi_0^*}) \,\middle|\, \delta_0, \nu_0, L_0, U_0 \right],$$

where X_1^π denotes the time t_1 wealth using portfolio π and starting at x.

With the performance criterion $\mathcal{U}_1$ and optimal portfolio π_0^* computed, the investor commits to π_0^* in the first investment period, and realizes wealth $X_1^{\pi_0^*}$ at time t_1. At time t_1, the investor also observes a new reference measure ν_1, confidence level δ_1, and portfolio constraints L_1, U_1, forming a set of admissible portfolios $\mathcal{A}_{1,2}(X_1^{\pi_0^*})$ for the second period. The investor then repeats the single period problem described above, using $\mathcal{U}_1$ as her *initial* utility. That is, conditionally on the information known at time t_1, the investor solves for a function $\mathcal{U}_2$ such that, for all $\pi_1 \in \mathcal{A}_{1,2}(X_1^{\pi_0^*})$,

$$\mathcal{U}_1(X_1^{\pi_0^*}) \geqslant \operatorname*{ess\,inf}_{\eta \in B_{\delta_1}(\nu_1)} \mathbb{E}^\eta \left[\mathcal{U}_2(X_2^{\pi_1}) \,\middle|\, X_1^{\pi_0^*}, \delta_1, \nu_1, L_1, U_1 \right],$$

and there exists $\pi_1^* \in \mathcal{A}_{1,2}(X_1^{\pi_0^*})$ such that

$$\mathcal{U}_1(X_1^{\pi_0^*}) = \operatorname*{ess\,inf}_{\eta \in B_{\delta_1}(\nu_1)} \mathbb{E}^\eta \left[\mathcal{U}_2(X_2^{\pi_1^*}) \,\middle|\, X_1^{\pi_0^*}, \delta_1, \nu_1, L_1, U_1 \right].$$

With $\mathcal{U}_2$ that satisfies the above relation, the investor computes an optimal portfolio π_1^*. The single period problem is, in turn, repeated to construct a sequence of (random) functions $\mathcal{U}_n$ and corresponding optimal portfolios π_{n-1}^*.

Clearly the computation of the time t_n utility function $\mathcal{U}_n$ and portfolio π_{n-1}^* depends only on the variables observed at time t_{n-1}. That is, if we define a filtration $\mathcal{F}_n := \sigma\left(\nu_k, \delta_k, L_k, U_k \,|\, k \leqslant n\right)$ then $\mathcal{U}_n$ is *predictable* and π_n^* is adapted with respect to $\mathcal{F}_n$. This leads to the definition of a *predictable robust forward performance process.*

Definition 1 *Fix a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$. *Consider an increasing set of times* $\{t_n\}_{n\in\mathbb{N}}$, *and a sequence of random variables* $(\nu_n, \delta_n, L_n, U_n)_{n\in\mathbb{N}}$ *with* $\nu_n : \Omega \to P(\mathbb{R}^+)$, $\delta_n : \Omega \to \mathbb{R}^+$, *and* $L_n, U_n : \Omega \to \mathbb{R}$ *which are finite a.s. For* $x \in \mathbb{R}$, *let*

$$\mathcal{A}_{n,n+1}(x) = \{\pi \in \mathbb{R} : L_n x \leqslant \pi \leqslant U_n x\}.$$

Let the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ *be such that* $\mathcal{F}_n = \sigma\left(\nu_k, \delta_k, L_k, U_k \,|\, k \leqslant n\right)$. *A family of random functions* $(\mathcal{U}_n)_{n\in\mathbb{N}}$ *is called a robust predictable forward performance process with respect to the filtration* $(\mathcal{F}_n)_{n\in\mathbb{N}}$ *if the following conditions hold:*

• $\mathcal{U}_0$ *is a deterministic utility function and* $\mathcal{U}_n : \Omega \times \mathbb{R} \to \mathbb{R}$ *is such that* $\mathcal{U}_n(\cdot, x)$ *is* $\mathcal{F}_{n-1}$ *measurable for all* $x \in \mathbb{R}$,

● *For any initial wealth* $x_0 > 0$ *and any sequence of admissible portfolios* $(\pi_0, \pi_1, ...)$*, the following inequality holds*:

$$\mathcal{U}_{n-1}(X_{n-1}^{\pi_{n-2}}) \geqslant \operatorname*{ess\,inf}_{\eta \in B_{\delta_{n-1}}(\nu_{n-1})} \mathbb{E}^{\eta}\left[\mathcal{U}_n(X_n^{\pi_{n-1}})|\mathcal{F}_{n-1}\right],$$

● *For any initial wealth* $x_0 > 0$*, there exists a sequence of admissible portfolios* $(\pi_0^*, \pi_1^*, \dots)$ *such that*

$$\mathcal{U}_{n-1}(X_{n-1}^{\pi_{n-2}^*}) = \operatorname*{ess\,inf}_{\eta \in B_{\delta_{n-1}}(\nu_{n-1})} \mathbb{E}^{\eta}\left[\mathcal{U}_n(X_n^{\pi_{n-1}^*})|\mathcal{F}_{n-1}\right].$$

In what follows, we investigate specific cases of robust, predictable forward performance processes when we restrict the reference measure to be binomial and the utility functions to be linear in Section 3 and quadratic in Section 4.

3. Linear case

We investigate the properties of robust predictable forward performance criteria when the reference measure is assumed binomial and the utility functions are restricted to be linear. That is, assume we have processes $(\nu_n, \delta_n, L_n, U_n)_{n\in\mathbb{N}}$ as above, with

$$\nu_n = p_n\delta_{u_n} + (1-p_n)\delta_{d_n},$$

for some processes $(p_n, u_n, d_n)_{n\in\mathbb{N}}$ with $p_n \in [0,1]$ and $d_n < 1 < u_n$. We also assume that the investor is endowed with an initial utility of the form

$$\mathcal{U}_0(x) = C_0\, x, \quad x \in \mathbb{R},$$

for some constant $C_0 > 0$. We look for forward performance criteria $\mathcal{U}_n$ of the linear form

$$\mathcal{U}_n(x) = C_n\, x, \quad x \in \mathbb{R},$$

for some $C_n \in \mathcal{F}_{n-1}$ with $C_n > 0$ a.s. The robust forward performance problem is then to find a utility process $(C_n\, x)_{n\in\mathbb{N}}$ and optimal portfolio process $(\pi_n^*)_{n\in\mathbb{N}}$, with $\pi_n^* \in \mathcal{A}_{n,n+1}(X_n^{\pi_{n-1}^*})$ for all $n \in \mathbb{N}$, such that for any sequence of admissible portfolios $(\pi_0, \pi_1, ...)$,

$$C_{n-1}X_{n-1}^{\pi_{n-2}} \geqslant \operatorname*{ess\,inf}_{\eta \in B_{\delta_{n-1}}(\nu_{n-1})} \mathbb{E}\left[C_nX_n^{\pi_{n-1}}|\mathcal{F}_{n-1}\right],$$

and

$$C_{n-1}X_{n-1}^{\pi_{n-2}^*} = \operatorname*{ess\,inf}_{\eta \in B_{\delta_{n-1}}(\nu_{n-1})} \mathbb{E}\left[C_nX_n^{\pi_{n-1}^*}|\mathcal{F}_{n-1}\right].$$

Next, we solve period-by-period the robust forward optimization problem for each $n \in \mathbb{N}$. For each trading period, the problem reduces to a single-period optimization problem amenable to the method developed by Gao and Kleywegt [10] who solved a similar problem using a suitable dual formulation.

Proposition 2 *Let* $\mathcal{F}_n$ *be as in Definition 1, and suppose that* $\nu_n = p_n\delta_{u_n} + (1-p_n)\delta_{d_n}$*. Further, assume that for each* $n \in \mathbb{N}$*,* $0 < \delta_n < \mu_n$ *where* $\mu_n = p_nu_n + (1-p_n)d_n$*. Then, for any* $C > 0$*, the following holds*

$$\operatorname*{ess\,inf}_{\eta \in B_{\delta_n}(\nu_n)} \mathbb{E}^{\eta}\left[CX_{n+1}^{\pi_n}\,|\,\mathcal{F}_n\right] = \mathbb{E}^{\nu_n}\left[CX_{n+1}^{\pi_n}\,|\,\mathcal{F}_n\right] - \delta_nC|\pi_n|.$$

Proof We write

$$\operatorname*{ess\,inf}_{\eta\in B_{\delta_n}(\nu_n)} \mathbb{E}^\eta\left[CX^{\pi_n}_{n+1}\,|\,\mathcal{F}_n\right] = -\operatorname*{ess\,sup}_{\eta\in B_{\delta_n}(\nu_n)} \mathbb{E}^\eta\left[-C(X^{\pi_{n-1}}_n + \pi_n(R_{n+1}-1)))\,|\,\mathcal{F}_n\right].$$

Next, we apply the results Gao and Kleywegt [10], conditioning on $\mathcal{F}_n$. Given information $\mathcal{F}_n$, their arguments can be readily adapted to deduce that

$$\begin{aligned}
&\operatorname*{ess\,sup}_{\eta\in B_{\delta_n}(\nu_n)} \mathbb{E}^\eta\left[-C(X^{\pi_{n-1}}_n + \pi_n(R_{n+1}-1))\,|\,\mathcal{F}_n\right]\\
&= \operatorname*{ess\,inf}_{\lambda\geqslant 0}\left(\lambda\delta_n - \mathbb{E}^{\nu_n}\left[\inf_{z\in\mathbb{R}^+}\lambda\mathrm{d}(z,R_{n+1}) + C(X^{\pi_{n-1}}_n+\pi_n(z-1))\,\Big|\,\mathcal{F}_n\right]\right)\\
&= \operatorname*{ess\,inf}_{\lambda\geqslant 0}\left(\lambda\delta_n - p_n\Big(\operatorname*{ess\,inf}_{z\in\mathbb{R}^+}\lambda|z-u_n| + C(X^{\pi_{n-1}}_n+\pi_{n+1}(z-1))\Big)\right.\\
&\qquad\left. -(1-p_n)\Big(\operatorname*{ess\,inf}_{z\in\mathbb{R}^+}\lambda|z-d_n| + C(X^{\pi_{n-1}}_n+\pi_n(z-1))\Big)\right)\\
&= \operatorname*{ess\,inf}_{\lambda\geqslant 0}\left(\lambda\delta_n - C(X^{\pi_{n-1}}_n - \pi_n) - p_n\inf_{z\geqslant 0}\Big(\lambda|z-u_n| + C\pi_n z\Big)\right.\\
&\qquad\left. -(1-p_n)\inf_{z\geqslant 0}\Big(\lambda|z-d_n| + C\pi_n z\Big)\right).
\end{aligned} \tag{3}$$

Setting $K := \mathbb{E}^{\nu_n}\left[-CX^{\pi_n}_{n+1}\,|\,\mathcal{F}_n\right]$ and D to be the right hand side of (3), we deduce that, almost surely,

$$\begin{aligned}
D-K = &\operatorname*{ess\,inf}_{\lambda\geqslant 0}\left(\lambda\delta_n - p_n\inf_{z\geqslant 0}\Big(\lambda|z-u_n| + C\pi_n(z-u_n)\Big)\right.\\
&\left. -(1-p_n)\inf_{z\geqslant 0}\Big(\lambda|z-d_n| + C\pi_n(z-d_n)\Big)\right).
\end{aligned}$$

Notice that if $\lambda\geqslant C|\pi_n|$, then both inner infimum terms are zero. Hence, almost surely, the following inequality holds almost surely

$$D-K\leqslant C|\pi_n|\delta_n.$$

Next, we look at the cases $\pi_n>0$ and $\pi\leqslant 0$ separately. Suppose $\pi_n>0$ and consider the case $0\leqslant\lambda\leqslant C\pi_n$. Note that

$$\lambda|z-u_n| + C\pi_n(z-u_n) = \begin{cases}(C\pi_n+\lambda)(z-u_n), & z\geqslant u_n,\\ (C\pi_n-\lambda)(z-u_n), & z<u_n.\end{cases} \tag{4}$$

This quantity is minimized when $z=0$, so, almost surely,

$$\begin{aligned}
D-K &= \operatorname*{ess\,inf}_{0<\lambda<C\pi_n}\Big(\lambda\delta_n - p(\lambda u_n - C\pi_n u_n) - (1-p)(\lambda d - C\pi_n d)\Big)\\
&= \operatorname*{ess\,inf}_{0<\lambda<C\pi_n}\Big(\lambda\delta_n - \lambda\mu_n + C\pi_n\mu_n\Big)\\
&= \operatorname*{ess\,inf}_{0<\lambda<C\pi_n}\Big(\lambda(\delta_n-\mu_n) + C\pi_n\mu_n\Big).
\end{aligned}$$

When $\delta_n > \mu_n$, we have $D-K = C\pi_n\mu_n$, almost surely while when $\delta_n < \mu_n$ we have $D-K = C\pi_n\delta_n$ almost surely.

Next, suppose that $\pi_n \leqslant 0$. Consider first the case $0 \leqslant \lambda \leqslant -C\pi_n$. Note that (4) is minimized when $z = +\infty$, i.e.,

$$\inf_{z\geqslant 0}\Big(\lambda|z-u_n| + C\pi_n(z-u_n)\Big) = -\infty.$$

Hence, when $\delta_n < \mu_n$, $D-K = C|\pi_n|\delta_n$ almost surely. Recalling the definitions of D and K, we have

$$\operatorname*{ess\,inf}_{\eta\in B_{\delta_n}(\nu_n)} \mathbb{E}^{\eta}\left[CX^{\pi_n}_{n+1}\,|\,\mathcal{F}_n\right] = \mathbb{E}^{\nu_n}\left[CX^{\pi_n}_{n+1}\,|\,\mathcal{F}_n\right] - \delta_n C|\pi_n|,$$

which completes the proof. □

Proposition 2 shows that the minimization over the infinite-dimensional ambiguity set can be rewritten as the expectation of a *modified* objective with respect to the *reference measure*.

3.1 The optimal portfolio and robust forward performance criteria

We first calculate the optimal portfolio of the single period robust problem assuming that the utility parameter is fixed, say, some $C > 0$. That is, we seek $\pi^* = \pi^*(C)$ such that

$$\pi^* \in \operatorname*{arg\,sup}_{\pi\in\mathcal{A}_{n,n+1}(X_n^{\pi_{n-1}})} \left(\mathbb{E}^{\nu_n}\left[CX^{\pi}_{n+1}\,|\,\mathcal{F}_n\right] - \delta_n C|\pi|\right),$$

for wealth profile $X_n^{\pi_{n-1}}$ at time t_n. Note that

$$\begin{aligned}
&\sup_{\pi\in\mathcal{A}_{n,n+1}(X_n^{\pi_{n-1}})} \left(\mathbb{E}^{\nu_n}\left[C(X_n^{\pi_{n-1}} + \pi(R_{n+1}-1))\right] - \delta_n C|\pi|\right)\\
&= \sup_{\pi\in A_{0,1}(X_n^{\pi_{n-1}})} \begin{cases} CX_n^{\pi_{n-1}} + C\pi(\mu_n - 1 - \delta_n), & \pi > 0,\\ CX_n^{\pi_{n-1}} + C\pi(\mu_n - 1 + \delta_n), & \pi < 0.\end{cases}
\end{aligned}$$

In other words, we are maximizing a piecewise linear function of π on the bounded interval $L_nX_n^{\pi_{n-1}} \leqslant \pi \leqslant U_nX_n^{\pi_{n-1}}$. Therefore, the maximal value will occur at an endpoint of the interval.

We consider the following three cases: the investor can only short, can only buy, and can short or buy. To ease notation, set

$$J_n(\pi) := \begin{cases} CX_n^{\pi_{n-1}} + C\pi(\mu_n - 1 - \delta_n), & \pi > 0,\\ CX_n^{\pi_{n-1}} + C\pi(\mu_n - 1 + \delta_n), & \pi < 0.\end{cases}$$

3.1.1 Short selling only

Let $L_nX_n^{\pi_{n-1}} \leqslant \pi \leqslant U_nX_n^{\pi_{n-1}} \leqslant 0$. As noted above, the optimal portfolio will either be L_nx_n or U_nx_n. Therefore, to determine the optimal portfolio, we need to check the sign of $J(L_nX_n^{\pi_{n-1}}) - J(U_nX_n^{\pi_{n-1}})$. Note that

$$\begin{aligned}
&J(L_nX_n^{\pi_{n-1}}) - J(U_nX_n^{\pi_{n-1}})\\
&= CX_n^{\pi_{n-1}} + CL_nX_n^{\pi_{n-1}}(\mu_n - 1 + \delta_n) - CX_n^{\pi_{n-1}} - CU_nX_n^{\pi_{n-1}}(\mu_n - 1 + \delta_n)\\
&= CX_n^{\pi_{n-1}}(\mu_n - 1 + \delta_n)(L_n - U_n).
\end{aligned}$$

The assumption $L_nX_n^{\pi_{n-1}} \leqslant \pi \leqslant U_nX_n^{\pi_{n-1}} \leqslant 0$, yields $L_n \leqslant U_n$, when $X_n^{\pi_{n-1}} > 0$. Analogously, $U_n \leqslant L_n$, when $X_n^{\pi_{n-1}} < 0$. The optimal portfolio in this case is summarized in Table 1.

Table 1 Optimal portfolio when $L_nX_n^{\pi_{n-1}} \leqslant U_nX_n^{\pi_{n-1}} \leqslant 0$

Market properties	$X_n^{\pi_{n-1}} < 0$	$X_n^{\pi_{n-1}} > 0$
$\mu_n - 1 + \delta_n > 0$	$U_nX_n^{\pi_{n-1}}$	$U_nX_n^{\pi_{n-1}}$
$\mu_n - 1 + \delta_n < 0$	$L_nX_n^{\pi_{n-1}}$	$L_nX_n^{\pi_{n-1}}$

3.1.2 Buying only

Let $0 \leqslant L_nX_n^{\pi_{n-1}} \leqslant \pi \leqslant U_nX_n^{\pi_{n-1}}$. We have that

$$J(L_nX_n^{\pi_{n-1}}) - J(U_nX_n^{\pi_{n-1}}) = CX_n^{\pi_{n-1}}(\mu - 1 - \delta)(L_n - U_n).$$

Proceeding in a similar manner as above, we obtain the optimal portfolio summarized in Table 2.

Table 2 Optimal portfolio when $0 \leqslant L_nX_n^{\pi_{n-1}} \leqslant U_nX_n^{\pi_{n-1}}$

Market properties	$X_n^{\pi_{n-1}} < 0$	$X_n^{\pi_{n-1}} > 0$
$\mu_n - 1 - \delta_n > 0$	$U_nX_n^{\pi_{n-1}}$	$U_nX_n^{\pi_{n-1}}$
$\mu_n - 1 - \delta_n < 0$	$L_nX_n^{\pi_{n-1}}$	$L_nX_n^{\pi_{n-1}}$

3.1.3 Short selling and buying

Let $L_nX_n^{\pi_{n-1}} \leqslant 0 \leqslant U_nX_n^{\pi_{n-1}}$. Suppose first that $X_n^{\pi_{n-1}} > 0$. One has

$$\begin{aligned} J(L_nX_n^{\pi_{n-1}}) - J(U_nX_n^{\pi_{n-1}}) &= CL_nX_n^{\pi_{n-1}}(\mu_n - 1 + \delta_n) - CU_nX_n^{\pi_{n-1}}(\mu_n - 1 - \delta_n) \\ &= CX_n^{\pi_{n-1}}(L_n\mu_n - L_n + L_n\delta_n - U_n\mu_n + U_n + U_n\delta_n) \\ &= CX_n^{\pi_{n-1}}\left((L_n - U_n)\mu_n + U_n - L_n + (U_n + L_n)\delta_n\right). \end{aligned}$$

Using once more that when $X_n^{\pi_{n-1}} > 0$ (resp. < 0), one has $L_n \leqslant U_n$ (resp. $L_n \leqslant U_n$), we easily obtain the optimal portfolio depicted in Table 3.

Table 3 Optimal portfolio when $L_nX_n^{\pi_{n-1}} \leqslant 0 \leqslant U_nX_n^{\pi_{n-1}}$

Market properties	$X_n^{\pi_{n-1}} < 0$	$X_n^{\pi_{n-1}} > 0$
$\mu_n > 1 + \frac{(U_n + L_n)}{U_n - L_n}\delta$	$L_nX_n^{\pi_{n-1}}$	$U_nX_n^{\pi_{n-1}}$
$\mu_n < 1 + \frac{(U_n + L_n)}{U_n - L_n}\delta$	$U_nX_n^{\pi_{n-1}}$	$L_nX_n^{\pi_{n-1}}$

For the reader's convenience, we provide the optimal portfolio for all three cases of the single period optimization problem below.

We may now construct the unique robust predictable forward performance process for the linear case. We first solve for the performance process in the first period and then iterate. For this, it suffices to find C_1 such that the equality

$$C_0x = \sup_{\pi_0 \in \mathcal{A}_{0,1}(x)} \inf_{\eta \in B_{\delta_0}(\nu_0)} \mathbb{E}^{\eta}\left[C_1X_1^{\pi_0} \,|\, \mathcal{F}_0\right]$$

holds for each $x \in \mathbb{R}$. This can be rewritten as

$$C_0x = \sup_{\pi_0 \in \mathcal{A}_{0,1}(x)} \mathbb{E}^{\nu_0}\left[C_1X_1^{\pi_0} \,|\, \mathcal{F}_0\right] - \delta_0|\pi_0|.$$

For initial wealth $x_0 > 0$ and the optimal portfolio π_0^* calculated as in Table 4, we deduce

$$C_1 = \frac{C_0x_0}{x_0 + \pi_0^*(\mu_0 - 1) - \delta_0|\pi_0^*|}.$$

Table 4 Summary of optimal portfolio π_n^*

Portfolio constraints	Market properties	$X_n^{\pi_{n-1}} < 0$	$X_n^{\pi_{n-1}} > 0$
$L_n X_n^{\pi_{n-1}} \leqslant U_n X_n^{\pi_{n-1}} \leqslant 0$	$\mu_n - 1 + \delta_n > 0$	$U_n X_n^{\pi_{n-1}}$	$U_n X_n^{\pi_{n-1}}$
	$\mu_n - 1 + \delta_n < 0$	$L_n X_n^{\pi_{n-1}}$	$L_n X_n^{\pi_{n-1}}$
$0 \leqslant L_n X_n^{\pi_{n-1}} \leqslant U_n X_n^{\pi_{n-1}}$	$\mu_n - 1 - \delta_n > 0$	$U_n X_n^{\pi_{n-1}}$	$U_n X_n^{\pi_{n-1}}$
	$\mu_n - 1 - \delta_n < 0$	$L_n X_n^{\pi_{n-1}}$	$L_n X_n^{\pi_{n-1}}$
$L_n X_n^{\pi_{n-1}} \leqslant 0 \leqslant U_n X_n^{\pi_{n-1}}$	$\mu_n > 1 + \dfrac{(U_n + L_n)}{U_n - L_n}\delta_n$	$L_n X_n^{\pi_{n-1}}$	$U_n X_n^{\pi_{n-1}}$
	$\mu_n < 1 + \dfrac{(U_n + L_n)}{U_n - L_n}\delta_n$	$U_n X_n^{\pi_{n-1}}$	$L_n X_n^{\pi_{n-1}}$

With the performance process for time t_1 calculated, we iterate forward in time. To this end, assume that the performance process C_n has been calculated for time t_n with corresponding portfolio π_{n-1}^*. Then, π_n^* can be calculated as in Table 4 and C_{n+1} can be calculated

$$C_{n+1} = \frac{C_n X_n^{\pi_{n-1}^*}}{X_n^{\pi_{n-1}^*} + \pi_n^*(\mu_n - 1) - \delta_n |\pi_n^*|}.$$

4. Quadratic case

We assume that the robust forward performance criteria are of quadratic form

$$\mathcal{U}_{n+1}(x) = A_{n+1}x - \frac{B_{n+1}}{2}x^2,$$

for $A_{n+1}, B_{n+1} \in \mathcal{F}_n$, with $B_{n+1} > 0$ a.s. for all $n \in \mathbb{N}$, and

$$\mathcal{U}_0(x) = A_0 x - \frac{B}{2}x^2,$$

for $A_0 \in \mathbb{R}$ and $B_0 > 0$. We assume that the ambiguity set $B_\delta(x)$ is a ball of radius δ using the 2-Wasserstein distance with $d(x, y) = |x - y|$. We first calculate an explicit formula as in Proposition 2. The following calculations are similar to those in Blanchet et al. [7], and, for this, we highlight the main steps.

Proposition 3 *Let $\mathcal{F}_n$ be as in Definition 1, and suppose $\nu_n = p_n\delta_{u_n} + (1 - p_n)\delta_{d_n}$. Then, for $A_{n+1}, B_{n+1} \in \mathcal{F}_n$ and $B_{n+1} > 0$ a.s.,*

$$\inf_{\eta \in B_{\delta_n}(\nu_n)} \mathbb{E}^\eta \left[A_{n+1} X_{n+1}^{\pi_n} - \frac{B_{n+1}}{2}(X_{n+1}^{\pi_n})^2 \,|\, \mathcal{F}_n \right]$$
$$= \mathbb{E}^{\nu_n} \left[A_{n+1} X_{n+1}^{\pi_n} - \frac{B_{n+1}}{2}(X_{n+1}^{\pi_n})^2 \right] - \delta_n \sqrt{\mathbb{E}^{\nu_n} \left[B_{n+1}\pi_n^2 \tilde{R}_{n+1} - \pi_n \left(A_{n+1} + B_{n+1} X_n^{\pi_{n-1}} \right) \right]^2}$$
$$- \frac{\delta^2 B_{n+1}\pi_n^2}{2},$$

where $\tilde{R}_{n+1} = R_{n+1} - 1$, and *$R_{n+1}$ denotes the asset returns at time t_{n+1}.*

Proof To ease notation, we first derive a dual formulation for $\inf\limits_{\eta \in B_\delta(\nu)} \mathbb{E}^\eta \left[MZ - \frac{N}{2}Z^2 \right]$, for $M \in \mathbb{R}$, $N > 0$ and $Z \sim \eta$. Using Theorem 1 from Gao and Kleywegt [10], we have

$$\inf_{\eta \in B_\delta(\nu)} \mathbb{E}^\eta \left[MZ - \frac{N}{2}Z^2 \right] = - \inf_{\lambda \geqslant 0} \left(\lambda\delta^2 - \mathbb{E}^\nu \left[\inf_{z \in \mathbb{R}} \lambda |z - Z|^2 + MZ - \frac{N}{2}z^2 \right] \right).$$

If $0 \leqslant \lambda \leqslant N/2$, the infimum over λ is $+\infty$, which yields

$$\inf_{\eta\in B_\delta(\nu)} \mathbb{E}^\eta\left[MZ - \frac{N}{2}Z^2\right] = -\inf_{\lambda>N/2}\left(\lambda\delta^2 - \mathbb{E}^\nu\left[\inf_{z\in\mathbb{R}} \lambda|z-Z|^2 + MZ - \frac{N}{2}z^2\right]\right). \tag{5}$$

Taking the first order conditions in z yields that the optimum occurs at

$$z^* = \frac{2\lambda Z - M}{2(\lambda - N/2)},$$

and evaluating (5) at z^* yields

$$\begin{aligned}
&\inf_{\eta\in B_\delta(\nu)} \mathbb{E}^\eta\left[MZ - \frac{N}{2}Z^2\right]\\
&= -\inf_{\lambda>N/2}\left(\lambda\delta^2 - \mathbb{E}^\nu\Big[\left(\lambda - \frac{N}{2}\right)\left(\frac{2\lambda Z - M}{2(\lambda - N/2)}\right)^2 - \frac{(2\lambda Z - M)^2}{2(\lambda - N/2)} + \lambda Z^2\Big]\right)\\
&= -\inf_{\lambda>N/2}\left(\lambda\delta^2 - \mathbb{E}^\nu\left[\frac{-\frac{1}{2}(2\lambda Z - M)^2}{2(\lambda - N/2)} + \lambda Z^2\right]\right)\\
&= -\inf_{\lambda>N/2}\left(\lambda\delta^2 - \mathbb{E}^\nu\left[\frac{(2ZM - NZ^2)\lambda - \frac{M^2}{2}}{2\lambda - N}\right]\right).
\end{aligned} \tag{6}$$

In turn, the optimal λ^* is given by

$$\lambda^* = \frac{1}{2}\left[\sqrt{\frac{\mathbb{E}^\nu\left[NZ - M\right]^2}{\delta^2}} + N\right],$$

and evaluating (6) at λ^* yields

$$\begin{aligned}
&\inf_{\eta\in B_\delta(\nu)} \mathbb{E}^\eta\left[MZ - \frac{N}{2}Z^2\right]\\
&= -\frac{1}{2}\left[\sqrt{\frac{\mathbb{E}^\nu\left[(NZ-M)^2\right]}{\delta^2}} + N\right]\delta^2 - \left(\frac{N\mathbb{E}^\nu(Z^2)}{2} - \mathbb{E}^\nu(Z)M\right)\left[1 + \frac{N\delta}{\sqrt{\mathbb{E}^\nu\left[(NZ-M)^2\right]}}\right]\\
&\quad - \frac{M^2\delta}{2\sqrt{\mathbb{E}^\nu\left[(NZ-M)^2\right]}}.
\end{aligned}$$

This can be simplified further to

$$\inf_{\eta\in B_\delta(\nu)} \mathbb{E}^\eta\left[MZ - \frac{N}{2}Z^2\right] = \mathbb{E}^\nu\left[MZ - \frac{N}{2}Z^2\right] - \delta\sqrt{\mathbb{E}^\nu(NZ-M)^2} - \frac{\delta^2 N}{2}. \tag{7}$$

To write (7) in terms of the wealth process $X_{n+1}^{\pi_n}$, note that

$$\begin{aligned}
A_{n+1}X_{n+1}^{\pi_n} - \frac{B_{n+1}}{2}(X_{n+1}^{\pi_n})^2 = {}& \left(A_{n+1}\pi_n + B_{n+1}\pi(X_n^{\pi_{n-1}} + \pi_n)\right)R_{n+1} - \frac{B_{n+1}\pi_n^2}{2}R_{n+1}^2\\
&+ A_{n+1}(X_n^{\pi_{n-1}} - \pi_n) - \frac{N}{2}(X_n^{\pi_{n-1}})^2 - B_{n+1}X_n^{\pi_{n-1}}\pi_n - \frac{\pi_n^2}{2}B_{n+1}.
\end{aligned}$$

Therefore, with $\mathcal{U}_{n+1}(X_{n+1}^{\pi_n}) = A_{n+1}X_{n+1}^{\pi_n} - \frac{B_{n+1}}{2}(X_{n+1}^{\pi_n})^2$, we have

$$
\begin{aligned}
&\inf_{\eta\in B_{\delta_n}(\nu_n)} \mathbb{E}^{\eta}\left[\mathcal{U}_{n+1}(X^{\pi_n}_{n+1})\,|\,\mathcal{F}_n\right] \\
&= \mathbb{E}^{\nu_n}\left[\left(A_{n+1}\pi_n + B_{n+1}\pi_n(X_n^{\pi_{n-1}}+\pi_n)\right)R_{n+1} - \frac{B_{n+1}\pi_n^2}{2}R_{n+1}^2\right] \\
&\quad - \delta\sqrt{\mathbb{E}^{\nu}\left[B_{n+1}\pi_n^2 R_{n+1} - \left(A_{n+1}\pi_n + B_{n+1}\pi_n(X_n^{\pi_{n-1}}+\pi_n)\right)\right]^2} - \frac{\delta_n^2 B_{n+1}\pi_n^2}{2} \\
&\quad + A_{n+1}(X_n^{\pi_{n-1}}-\pi_n) - \frac{B_{n+1}}{2}(X_n^{\pi_{n-1}})^2 - B_{n+1}x\pi - \frac{\pi_n^2}{2}B_{n+1}.
\end{aligned}
$$

Finally, we observe that

$$
\begin{aligned}
&\inf_{\eta\in B_{\delta_n}(\nu_n)} \mathbb{E}^{\eta}\left[\mathcal{U}_{n+1}(X^{\pi_n}_{n+1})\,|\,\mathcal{F}_n\right] \\
&= \mathbb{E}^{\nu_n}\left[A_{n+1}X^{\pi_n}_{n+1} - \frac{B_{n+1}}{2}(X^{\pi_n}_{n+1})^2\right] \\
&\quad - \delta_n\sqrt{\mathbb{E}^{\nu_n}\left[B_{n+1}\pi_n^2\tilde{R}_{n+1} - \pi_n\left(A_{n+1}+B_{n+1}X_n^{\pi_{n-1}}\right)\right]^2} - \frac{\delta^2 B_{n+1}\pi_n^2}{2},
\end{aligned}
$$

which concludes the proof. □

Depending on the market variables, the second term of the right hand side of Proposition 3 may not be concave. However, clearly if δ_n is small enough, then the entire right hand is concave, as the first and last terms are concave. In what follows, we then assume that δ_n is small enough to guarantee a unique solution π_n^*.

We now solve for time t_n optimal portfolio π_n^* for any utility parameters $A_{n+1}, B_{n+1} \in \mathcal{F}_{n+1}$, with $B_{n+1} > 0$ a.s. and any time t_n wealth profile $X_n^{\pi_{n-1}}$. Writing $\mu_n = \mathbb{E}^{\nu}\left[\tilde{R}_{n+1}\right]$, and $\sigma_n = \mathbb{E}^{\nu}\left[\tilde{R}^2_{n+1}\right]$, the optimal portfolio π_n^* must satisfy the first order conditions:

$$
\begin{aligned}
&(A_{n+1} - B_{n+1}X_n^{\pi_{n-1}})\mu_n - B_{n+1}(\sigma_n + \delta_n^2)\pi_n^* \\
&- \delta_n \frac{2B_{n+1}^2\sigma_n(\pi_n^*)^3 - 3(A_{n+1}+B_{n+1}X_n^{\pi_{n-1}})B_{n+1}\mu_n(\pi_n^*)^2 + (A_{n+1}+B_{n+1}X_n^{\pi_{n-1}})^2\pi_n^*}{\left(B_{n+1}^2\sigma(\pi_n^*)^4 - 2(A_{n+1}+B_{n+1}X_n^{\pi_{n-1}})B_{n+1}\mu(\pi_n^*)^3 + (A_{n+1}+B_{n+1}X_n^{\pi_{n-1}})^2(\pi_n^*)^2\right)^{1/2}} = 0.
\end{aligned}
\tag{8}
$$

The above function is rational in π_n^* and can be solved with standard numerical techniques, as seen in Figures 1 and 2.

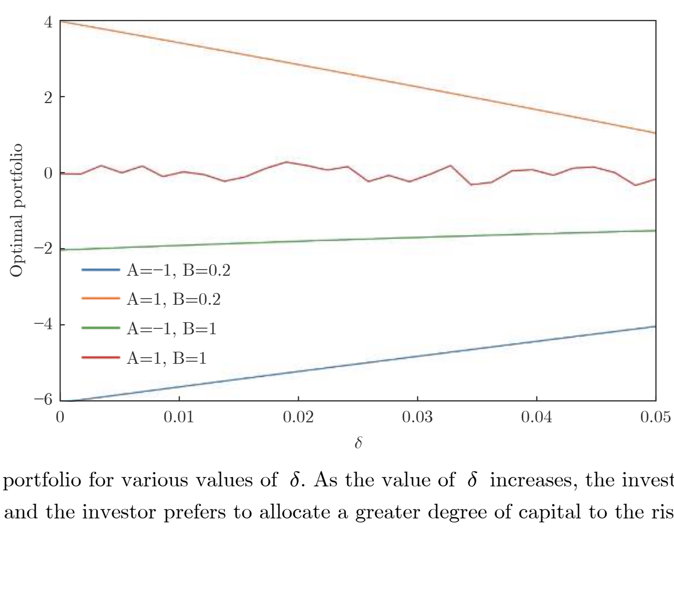
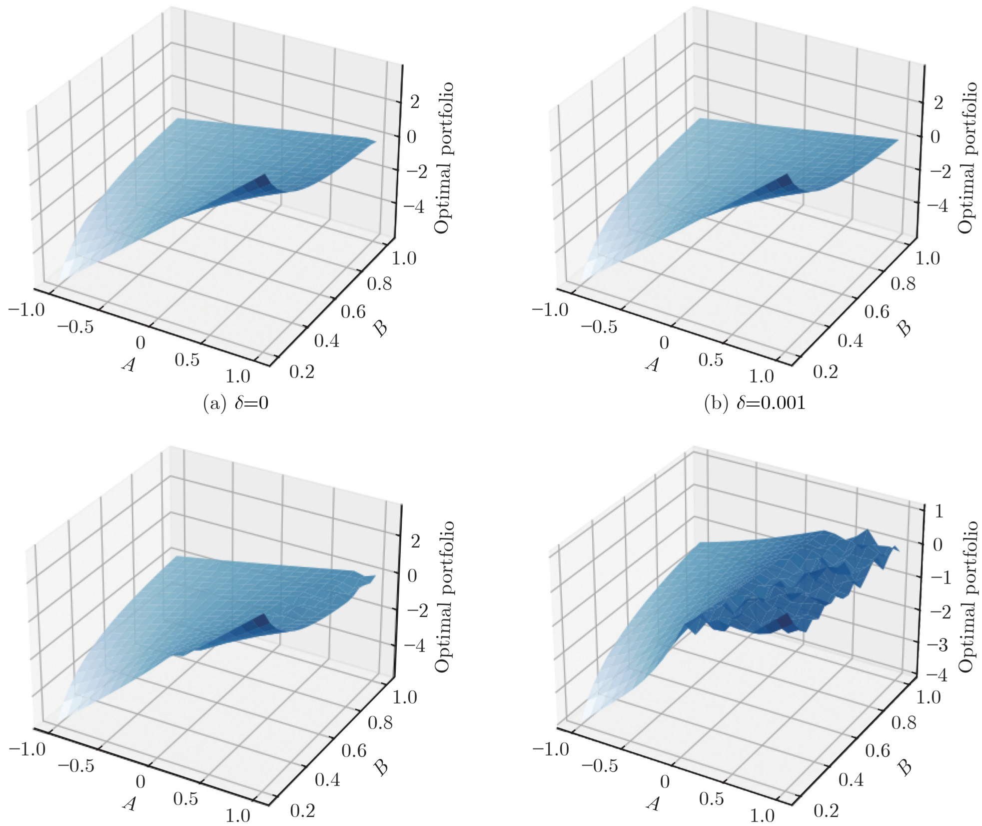


Figure 1 The optimal portfolio for various values of δ. As the value of δ increases, the investor's level of uncertainty increases, and the investor prefers to allocate a greater degree of capital to the riskless asset



(a) δ=0 (b) δ=0.001

(c) δ=0.01 (d) δ=0.05

Figure 2 The optimal portfolio for the single period problem for varying values of A, B, and δ. When $\delta = .05$, one observes numerical instability for large values of A

With the optimal portfolio π_n^* calculated as a function of A_{n+1} and B_{n+1}, we proceed as in the linear case to construct the entire robust predictable forward performance process. To this end, assume that at time t_0, the investor is endowed with an exogenous utility function $\mathcal{U}_0$ parameterized by A_0, B_0, an initial reference measure ν_0, and initial uncertainty δ_0 small enough such that (8) can be solved uniquely for some π_0^*. The performance process at time t_1 is then determined by solving

$$\begin{aligned}
&A_0 x_0 - \frac{B}{2} x_0^2 \\
&= A_1(x_0 + \pi_0^* \mu_0) - \frac{B_1}{2}\left(x_0^2 + 2\pi_0^* \mu_0 x_0 + (\pi_0^*)^2 \sigma_0\right) - \delta_0 (B_1^2 (\pi_0^*)^4 \sigma_0 - 2B_1(A_1 + B_1 x_0)(\pi_0^*)^3 \mu_0 \\
&\quad + (\pi_0^*)^2 (A_1 + B_1 x_0)^2)^{1/2} - \frac{\delta_0 B_1 (\pi_0^*)^2}{2},
\end{aligned}$$

for A_1, B_1, noting that π_0^* is a function of A_1 and B_1. In general, the above equation does not have a unique solution, and further work is needed. It can, however, be solved numerically to yield a solution (A_1, B_1) and the corresponding optimal portfolio π_0^*. As in the linear case, assume that at time t_n, the investor has wealth $X_n^{\pi_{n-1}^*}$ and utility $\mathcal{U}_n(x) = A_n x + \frac{B_n}{2} x^2$ for all $x \in \mathbb{R}$, for some $A_n, B_n \in \mathcal{F}_{n-1}$ with $B_n > 0$ a.s. Then $\mathcal{U}_{n+1}(x) = A_{n+1} x + \frac{B_{n+1}}{2} x^2$ must satisfy

$$
\begin{aligned}
&A_n X_n^{\pi^*_{n-1}} - \frac{B_n}{2}(X_n^{\pi^*_{n-1}})^2 \\
&= A_{n+1}(X_n^{\pi^*_{n-1}} + \pi_n^*\mu_n) - \frac{B_{n+1}}{2}\left((X_n^{\pi^*_{n-1}})^2 + 2\pi_n^*\mu_n X_n^{\pi^*_{n-1}} + (\pi_n^*)^2\sigma_n\right) \\
&\quad - \delta_n(B_{n+1}^2(\pi_n^*)^4\sigma_n - 2B_{n+1}(A_{n+1} + B_{n+1}X_n^{\pi^*_{n-1}})(\pi_n^*)^3\mu_n \\
&\quad + (\pi_n^*)^2(A_{n+1} + B_{n+1}X_n^{\pi^*_{n-1}})^2)^{1/2} - \frac{\delta_n B_{n+1}(\pi_0^*)^2}{2}.
\end{aligned}
$$

5. Conclusions

We introduced robust predictable forward performance processes and investigated two specific examples: linear and quadratic utilities. In both cases, the well-posedness of the period-by-period optimization problem was shown explicitly by calculating the exact form of the inner optimization over the relevant ambiguity set. In the general case, an explicit formula is generally unattainable, so well-posedness must be established directly. In Gao and Kleywegt [10], the authors provide a condition for the finiteness of the optimization problem, relating the cost function for the Wasserstein distance and the utility function. That is, for any $\nu \in P(\mathbb{R}^+)$, in order for

$$
\inf_{\eta \in B_\delta(\nu)} \mathbb{E}^\eta \left[\mathcal{U}(x + \pi(R-1))\right] < \infty,
$$

one must have

$$
\limsup_{z \to \infty} \frac{|\mathcal{U}(x + \pi(z-1)) - \mathcal{U}(x + \pi(\tilde{z}-1))|}{c(z, \tilde{z})} < \infty, \tag{9}
$$

for any $\tilde{z} \in \mathbb{R}^+$ where $c(\cdot,\cdot)$ is the cost function associated with the Wasserstein distance defining $B_\delta(\nu)$. For standard utility functions, say, exponential utility, selecting $c(\cdot,\cdot)$ as any Euclidean norm will yield (9) infinite, as clearly

$$
\limsup_{z \to \infty} \frac{e^{-a(x+\pi(\tilde{z}-1))} - e^{-a(x+\pi(z-1))}}{a|z - \tilde{z}|^p} = \infty,
$$

for any $a > 0$ and $p > 0$ when $\pi < 0$. Therefore, to investigate robust, predictable forward performances of exponential type, one must determine a suitable cost function $c(\cdot,\cdot)$ in order to ensure the problem is well-posed. For general utility functions, one could expand the definition of forward performance processes to include the cost function as well. However, the financial interpretation of non-standard cost functions is not immediately obvious. These questions are being investigated currently by the author.

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