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G-forward performance process and representation of homothetic case via ergodic quadratic G-BSDE

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Abstract We introduce a new type of robust forward criterion under model uncertainty, called the G-forward performance process, which extends the classical notion of forward performance process to the G-expectation framework. We then derive the representations of homothetic G-forward performance processes in a single stochastic factor model with uncertainty, building on the well-posedness of ergodic and infinite horizon backward stochastic differential equations driven by G-Brownian motion (G-BSDEs) with quadratic generators.

Keywords G-forward performance process, Infinite horizon G-BSDE, Ergodic G-BSDE

2020 Mathematics Subject Classification 60H10, 60H30, 91G10, 91G80

1. Introduction

The main aim of this paper is to propose a novel concept of G-forward performance process, which is a robust forward performance criterion in consideration of model uncertainty, and construct several homothetic examples in a single stochastic factor model via ergodic and infinite horizon G-BSDEs with quadratic generators.

Expected utility maximization has been a popular topic in the area of financial mathematics and been investigated by various authors in the settings of classical probability space (see, for example, [6, 21, 24, 42]). The problem is formulated as

$$\mathbb{E}^{\mathbb{P}}\left[u_T(X_T^{\pi})\right] \to \max,$$

namely, the key idea is to maximize the expected value of the terminal wealth X_T^{π} measured by a given utility function u_T, which reflects the investor's attitude towards risk, at some pre-specified terminal time $T > 0$ over all admissible trading policies π. Here $\mathbb{P}$ is the historical probability measure and $\mathbb{E}^{\mathbb{P}}$ is the corresponding linear expectation.

However, there are some shortcomings of this approach. First, the utility function u_T is

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deterministic and fixed at the terminal time, which means that one cannot adjust his/her preference during the trading. Then the value function is derived "backwards" by dynamic programming principle while the market evolves "forward" in time. Furthermore, the classical utility maximization approach can only evaluate the investment up to the pre-chosen terminal time T. Forward performance processes, introduced and developed by Musiela and Zariphopoulou in [31–35], complement the classical static utilities and serve as a dynamic criterion parameterized by time and wealth. They are defined for all time and constructed by the forward counterpart of dynamic programming principle: for a given utility $U(t,x)$ at time t, the forward performance process for arbitrary future time $T \geqslant t$ is chosen to satisfy: for each admissible strategy π,

$$\mathbb{E}^{\mathbb{P}}\left[U\left(T, X_T^{\pi}\right) | \mathcal{F}_t, X_t^{\pi}=x\right] \leqslant U(t,x),$$

and there exists an optimal strategy π^* such that

$$\mathbb{E}^{\mathbb{P}}\left[U\left(T, X_T^{\pi^*}\right) | \mathcal{F}_t, X_t^{\pi^*}=x\right] = U(t,x).$$

This property is called the self-generation. Various works were devoted to the representations and characterizations of forward performance processes. For example, Žitković [44] characterized the self-generation property via duality theory; a class of stochastic partial differential equations related to forward performance processes were obtained and investigated by Musiela and Zariphopoulou [35] and El Karoui and Mrad et al. [8–10], respectively; Widder's theorem has been applied to construct time-monotonic forward performance processes by Avanesyan, Shkolnikov and Sicar [3]; discrete-time case has also been studied recently by [1, 2, 29]. It is worth mentioning that Liang and Zariphopoulou [30] showed that ergodic and infinite horizon BSDEs are effective tools to construct homothetic (exponential, power and logarithmic) forward performance processes in stochastic factor models, and, building on which, [4] obtained the representation for the forward entropic risk measure.

In addition to the aforementioned drawbacks, the model can be ambiguous for investors, namely, it is always difficult to specify a certain probability measure in the real market. This type of uncertainty is known as Knightian uncertainty, introduced by Knight [26]. Motivated by this, Källblad, Obłój and Zariphopoulou [23] proposed the notion of robust forward criteria, which are pairs of dynamic utilities and penalty functions, and established the dual characterization of them. Subsequently, Chong and Liang [5] studied the optimal forward investment and consumption problem under uncertainty stemming from coefficients for risky stocks. Combining ergodic BSDE representation and stochastic game approach, Li, Li and Liang [27] constructed a class of robust forward performance process in stochastic factor model.

In this paper, we propose a new class of robust forward performance processes under model uncertainty, called G-forward performance processes, using the notion of G-expectation space $(\Omega, \mathbb{L}_G^1(\Omega), \hat{\mathbb{E}})$, which is a dynamically consistent sublinear expectation space proposed by Peng [37, 38]. The canonical process on this space is the so-called G-Brownian motion, and the corresponding G-Itô's calculus theory is also established (see, among others, [7, 15, 16, 20, 28, 39]). It was shown by Denis, Hu and Peng [7] that the G-expectation can be written as an upper expectation over a family of linear expectations, namely,

$$\hat{\mathbb{E}}[X] = \sup_{\mathbb{P}\in\mathcal{P}_G} \mathbb{E}^{\mathbb{P}}[X] \text{ for } X \in \mathbb{L}_G^1(\Omega),$$

where $\mathcal{P}_G$ is a weakly compact family of probability measures on $(\Omega, \mathcal{B}(\Omega))$ which can be mutually singular. The G-expectation framework, in turn, provides a powerful tool in finance to deal with volatility uncertainty (see [11, 12]).

Herein, we first introduce a financial market where the stocks are driven by a non-degenerate G-Brownian motion. Different from the classical dynamics of the stocks, the $\mathrm{d}t$ terms are replaced by the terms of quadratic variation of G-Brownian motion $\mathrm{d}\langle B\rangle_t$, which provide the uncertainty of the drift. In this market, an ambiguity-averse investor aims to optimize his/her investment across arbitrary time horizon under worst-case scenario, which motivates the definition of the G-forward performance processes. In other words, G-forward performance process is a "forward" dynamic investment preference in consideration of worst-case scenario. The key point is to replace the self-generation property under classical linear expectation by the one under G-expectation, utilizing the notion of G-(sub)martingale.

Inspired by Liang and Zariphopoulou [30], we use ergodic and infinite horizon quadratic G-BSDEs to represent homothetic G-forward performance processes in a stochastic factor model. To do this, we first need to establish the well-posedness of infinite horizon Markovian forward-backward stochastic differential equations driven by G-Brownian motion (G-FBSDEs) as well as ergodic G-BSDEs, both with quadratic generators. The main idea comes from the truncation approach in [30]. We first consider a truncated (in z) infinite horizon G-FBSDE which satisfies standard Lipschitz condition and hence admits a unique solution according to [19]. Then we verify that the Z-component of the solution is indeed bounded as we expected, so the truncated G-FBSDE coincides with the original one. This step is more complicated than the classical case in physical probability space (see [14, 30]) because of G-expectation setup, and we overcome this by utilizing nonlinear stochastic analysis techniques. Based on the well-posedness of infinite horizon quadratic G-BSDEs, we construct a solution of ergodic quadratic G-BSDE through a standard approximation approach (see [13, 18, 19]).

Building on above abstract results, we verify some specific ergodic and infinite horizon quadratic G-BSDEs and use them to construct some examples of homothetic (exponential, power and logarithmic) G-forward performance processes in a market with a single stock whose dynamics are affected by a single stochastic factor, with the help of G-Girsanov transformation. We also compare our results with the representations in [30] by reducing the G-expectation to a linear expectation.

The main contribution of this paper is that we study the application of G-BSDE theory in optimal forward investment problem under model uncertainty, including volatility uncertainty. More precisely, we initiate the study of the robust forward performance process under G-expectation framework and give some examples via G-BSDE theory. As a byproduct, we also prove the well-posedness of Markovian infinite horizon G-FBSDEs as well as ergodic G-BSDEs with quadratic coefficients.

The rest of this paper is organized as follows. Section 2 gives some preliminaries of G-expectation framework, and we introduce the financial market model with uncertainty as well as the definition of G-forward performance process in Section 3. In Section 4, we show the well-posedness of general infinite horizon and ergodic quadratic G-BSDEs in Markovian case and, based on which, construct three types of homothetic G-forward performance processes in single stock and single stochastic factor model as examples. Section 5 concludes the paper.

2. Preliminaries

We recall some basic notion and results of G-expectation theory, which will be frequently used in the sequel. The readers may refer to [15, 16, 19, 20, 38, 40] for more details.

Consider the space $\Omega := \{\omega \in C_0([0,\infty);\mathbb{R}^d), \omega_0 = 0\}$ equipped with the distance

$$\rho\left(\omega^1 - \omega^2\right) := \sum_{N=1}^{\infty} 2^{-N}\Big[\Big(\max_{t\in[0,N]} \left|\omega_t^1 - \omega_t^2\right|\Big) \wedge 1\Big],$$

where $C_0([0,\infty);\mathbb{R}^d)$ is the space of $\mathbb{R}^d$-valued continuous functions on $[0,\infty)$. Let $B_t(\omega) = \omega_t$ be the canonical process, $\mathbb{P}_0$ the Wiener measure and $\mathbb{F} := \{\mathcal{F}_t\}_{t\geqslant 0}$ the natural filtration generated by B.

For each $t > 0$, let

$$L_{ip}(\Omega_t) := \left\{\varphi(B_{t_1\wedge t}, \dots, B_{t_n\wedge t}), n \geqslant 1, t_1, \dots, t_n \in [0,\infty), \varphi \in C_{b,Lip}(\mathbb{R}^{d\times n})\right\},$$

and $L_{ip}(\Omega) := \bigcup_{t\geqslant 0} L_{ip}(\Omega_t)$, where $\Omega_t := \{\omega_{\cdot\wedge t}, \omega \in \Omega\}$ and $C_{b,Lip}(\mathbb{R}^{d\times n})$ is the space of all bounded and Lipschitz continuous functions on $\mathbb{R}^{d\times n}$.

Let $G : \mathbb{S}(d) \to \mathbb{R}$ be a monotonic and sublinear function, where $\mathbb{S}(d)$ denotes the space of all $d \times d$ symmetric matrices. Then there exists a bounded, convex and closed set $\Theta \subset \mathbb{R}^{d\times d}$ such that

$$G(A) = \frac{1}{2}\sup_{\gamma\in\Theta} \operatorname{tr}\left[\gamma\gamma^\top A\right] \text{ for } A \in \mathbb{S}(d). \tag{2.1}$$

Peng [37, 38] constructed a consistent sublinear expectation space $(\Omega, L_{ip}(\Omega), \hat{\mathbb{E}})$, called G-expectation space, via a special PDE called G-heat equation. In this space, the canonical process $B = (B^i)_{i=1}^d$ is called a d-dimensional G-Brownian motion. In this paper, we only consider the non-degenerate G-Brownian motion, namely, there exists a constant $\underline{\sigma} > 0$ such that

$$G(A) - G(B) \geqslant \frac{1}{2}\underline{\sigma}^2 \operatorname{tr}[A - B] \text{ for any } A \geqslant B. \tag{2.2}$$

For $p \geqslant 1$, denote by $\mathbb{L}_G^p(\Omega)$ (resp., $\mathbb{L}_G^p(\Omega_t)$) the completion of $L_{ip}(\Omega)$ (resp., $L_{ip}(\Omega_t)$) under the norm $(\hat{\mathbb{E}}[|\cdot|]^p)^{1/p}$, and $(\hat{\mathbb{E}}_t)_{t\geqslant 0}$ can be continuously extended to $\mathbb{L}_G^1(\Omega)$.

Denis, Hu and Peng [7] showed that the G-expectation can be represented as an upper expectation:

$$\hat{\mathbb{E}}[X] = \sup_{\mathbb{P}\in\mathcal{P}_G} \mathbb{E}^{\mathbb{P}}[X] \text{ for } X \in \mathbb{L}_G^1(\Omega),$$

where $\mathcal{P}_G$ is a weakly compact family of probability measures on $(\Omega, \mathcal{B}(\Omega))$ with $\mathcal{B}(\Omega)$ being the Borel σ-algebra of Ω. We can, in turn, define the Choquet capacity associated to $\mathcal{P}_G$:

$$c(A) := \sup_{\mathbb{P}\in\mathcal{P}_G} \mathbb{P}(A) \text{ for } A \in \mathcal{B}(\Omega).$$

We call a set $A \in \mathcal{B}(\Omega)$ is polar if $c(A) = 0$ and a property holds quasi-surely (q.s.) if it holds outside a polar set. Herein, we do not distinguish two random variables X and Y if $X = Y$ q.s.

For a random variable X, we call X is quasi-continuous (q.c.) if, for each $\varepsilon > 0$, there exists an open set O satisfying $c(O) < \varepsilon$ such that $X|_{O^c}$ is continuous, and we call X has a q.c. version if there exists a q.c. random variable Y such that $X = Y$ q.s. Then we can introduce the following characterization for the space $\mathbb{L}_G^p(\Omega)$, which will be useful in the sequel when we verify the regularity of random variables.

Theorem 2.1 ([7, 17]). *For* $p \geqslant 1$,

$$\mathbb{L}_G^p(\Omega) = \{X \in \mathbb{L}^0(\Omega), \quad \lim_{N\to\infty} \hat{\mathbb{E}}\left[|X|^p \mathbf{1}_{|X|\geqslant N}\right] = 0 \text{ and } X \text{ has a q.c. version}\},$$

where $\mathbb{L}^0(\Omega)$ *denotes the space of* $\mathcal{B}(\Omega)$*-measurable random variables.*

Next we define the spaces of stochastic processes under G-framework.

Definition 2.2 (i) *Let* $\mathcal{H}_G^0(0,T)$ *be the collection of all processes of the form*

$$\eta_t(\omega) := \sum_{j=0}^{N-1} \xi_j(\omega) \mathbf{1}_{[t_j,t_{j+1})}(t), \quad t \in [0,T],$$

where $\{0 = t_0, \ldots, t_N = T\}$, $N \geqslant 1$ *is a given partition of* $[0,T]$, *and* $\xi_j \in L_{ip}(\Omega_{t_j})$ *for* $j = 1, \ldots, N-1$.

We denote by $\mathcal{H}_G^p(0,T)$ *the completion of* $\mathcal{H}_G^0(0,T)$ *under the norm* $\|\cdot\|_{\mathcal{H}_G^p} := (\hat{\mathbb{E}}[(\int_0^T |\cdot|^2 \mathrm{d}s)^{p/2}])^{1/p}$ *for each* $p \geqslant 1$ *and by* $\mathcal{H}_G^p(0,T;\mathbb{R}^d)$ *the space of all* d*-dimensional processes* $\eta = (\eta^i)_{i=1}^d$ *with* $\eta^i \in \mathcal{H}_G^p(0,T)$.

(ii) *Let* $\mathcal{S}_G^0(0,T)$ *be the collection of processes of the form*

$$h(t, B_{t_1 \wedge t}, \ldots, B_{t_n \wedge t}), \quad t \in [0,T],$$

where $t_1, \ldots, t_n \in [0,T]$ *and* $h \in C_{b,Lip}(\mathbb{R}^{1+d\times n})$.

We denote by $\mathcal{S}_G^p(0,T)$ *the completion of* $\mathcal{S}_G^0(0,T)$ *under the norm* $\|\cdot\|_{\mathcal{S}_G^p} := |\hat{\mathbb{E}}[\sup_{t\in[0,T]} |\cdot|^p]|^{1/p}$ *for each* $p \geqslant 1$.

(iii) *For processes defined on the infinite horizon* $[0,\infty)$, *we define the corresponding spaces by* $\mathcal{H}_G^p(0,\infty;\mathbb{R}^d) := \cap_{T\geqslant 0}\mathcal{H}_G^p(0,T;\mathbb{R}^d)$ *and* $\mathcal{S}_G^p(0,\infty) := \cap_{T\geqslant 0}\mathcal{S}_G^p(0,T)$.

For G-Brownian motion $B = (B^i)_{i=1}^d$, define its *quadratic variation process matrix* by

$$\langle B \rangle_t := \left(\langle B \rangle_t^{ij}\right)_{i,j=1}^d,$$

with

$$\langle B \rangle_t^{ij} := \lim_{\mu(\pi_t^N)\to 0} \sum_{k=0}^{N} \left(B^i_{t^N_{k+1}} - B^i_{t^N_k}\right)\left(B^j_{t^N_{k+1}} - B^j_{t^N_k}\right) = B_t^i B_t^j - \int_0^t B_s^i \mathrm{d}B_s^j - \int_0^t B_s^j \mathrm{d}B_s^i,$$

where $\pi_t^N := \{0 = t_0^N, t_1^N, \ldots, t_N^N = t\}$, $N = 1,2,\ldots,$ be a sequence of partitions of $[0,t]$ and $\mu(\pi_t^N) := \max\{|t_{i+1}^N - t_i^N|, i = 0,1,\ldots,N-1\}$. $\langle B \rangle^{ij}$ is called the mutual variation process of B^i and B^j. According to [39, Corollary 3.5.8], we know that the quadratic variation of B satisfies

$$\langle B \rangle_t \in \left\{t\gamma\gamma^\top, \gamma \in \Theta\right\}, \quad t \geqslant 0,$$

and, for any $i,j = 1,\ldots,d$, there exists a $\bar{\sigma}_{ij} \geqslant 0$ such that $|\langle B \rangle_t^{ij}| \leqslant \bar{\sigma}_{ij}^2 t$. For simplicity, we denote

$$\bar{\sigma}_\Sigma^2 := \sum_{i,j=1}^d \bar{\sigma}_{ij}^2. \tag{2.3}$$

Next we introduce the important concept of G-martingale as well as the G-Girsanov transformation with respect to G-BMO martingale generator.

Definition 2.3 *A process* M *is called a* G*-submartingale (respectively,* G*-supermartingale) if* $M_t \in \mathbb{L}_G^1(\Omega_t)$ *for any* $t \geqslant 0$ *and, for any* $0 \leqslant s \leqslant t < \infty$,

$$\hat{\mathbb{E}}_s[M_t] \geqslant M_s \ (\text{respectively,} \hat{\mathbb{E}}_s[M_t] \leqslant M_s).$$

M *is called a* G*-martingale if it is* G*-supermartingale and* G*-submartingale at the same time, and it is called a symmetric* G*-martingale if* $-M$ *is also a* G*-martingale.*

Definition 2.4 ([22]) *A process* $\lambda \in \mathcal{H}_G^2(0,T;\mathbb{R}^d)$ *is called a G-BMO martingale generator if*

$$\|\lambda\|_{BMO_G}^2 := \sup_{\mathbb{P}\in\mathcal{P}_G} \|\lambda\|_{BMO(\mathbb{P})}^2 < \infty,$$

where

$$\|\lambda\|_{BMO(\mathbb{P})}^2 := \sup_{\tau\in\mathcal{T}_0^T} \left\| E_\tau^{\mathbb{P}}\left[\int_\tau^T |\lambda_t|^2\,\mathrm{d}t\right]\right\|_{\mathbb{L}^\infty(\mathbb{P})},$$

$\mathcal{T}_0^T$ *denotes the set of all* $[0,T]$*-valued* $\mathcal{F}$*-stopping times, and* $\|\xi\|_{\mathbb{L}^\infty(\mathbb{P})} := \inf\{M \geqslant 0, |\xi| \leqslant M \ \ \mathbb{P}\text{-a.s.}\}$.

For a G-BMO martingale generator $\lambda \in \mathcal{H}_G^2(0,T;\mathbb{R}^d)$, we know that $\lambda \in BMO(\mathbb{P})$ for each $\mathbb{P}\in\mathcal{P}_G$, and, recalling Theorem 3.1 in [25] and Lemma 2.1 in [40], we can find a $q>1$ such that

$$\|\lambda\|_{BMO_G} \leqslant \Phi(q),$$

where $\Phi(q) := (1+\frac{1}{q^2}\ln\frac{2q-1}{2(q-1)})^{1/2}-1$. Then we define

$$\mathcal{E}_t\left(\int \lambda \mathrm{d}B\right) := \exp\left(\sum_{i=1}^d \int_0^t \lambda_s^i \mathrm{d}B_s^i - \frac{1}{2}\sum_{i,j=1}^d \int_0^t \lambda_s^i\lambda_s^j \mathrm{d}\langle B\rangle_s^{ij}\right), \quad t\in[0,T].$$

Lemma 2.5 ([22]). *Suppose* $\lambda \in \mathcal{H}_G^2(0,T;\mathbb{R}^d)$ *is a G-BMO martingale generator. Then we have*

$$\hat{\mathbb{E}}\left[\mathcal{E}_t\left(\int\lambda \mathrm{d}B\right)^q\right] < \infty \text{ and } \mathcal{E}_t\left(\int\lambda \mathrm{d}B\right) \in \mathbb{L}_G^1(\Omega_t) \text{ for } t\in[0,T]. \tag{2.4}$$

Moreover, $\mathcal{E}(\int\lambda \mathrm{d}B)$ *is a symmetric G-martingale.*

According to [22] and [36], for a G-BMO martingale generator $\lambda \in \mathcal{H}_G^2(0,T;\mathbb{R}^d)$, we define a d-dimensional process $B^\lambda = (B^{\lambda,i})_{i=1}^d$ by

$$B_t^{\lambda,i} := B_t^i - \sum_{j=1}^d \int_0^t \lambda_s^j \mathrm{d}\langle B\rangle_s^{ij} \text{ for } t\in[0,T]. \tag{2.5}$$

Then we can define a new G-expectation on the space $L_{ip}(\Omega_T)$ with respect to the G-BMO martingale generator λ by

$$\hat{\mathbb{E}}^\lambda[X] := \hat{\mathbb{E}}\left[\mathcal{E}_T\left(\int\lambda \mathrm{d}B\right)X\right] \text{ for } X\in L_{ip}(\Omega_T), \tag{2.6}$$

and its corresponding conditional G-expectation by

$$\hat{\mathbb{E}}_t^\lambda[X] := \hat{\mathbb{E}}_t\left[\mathcal{E}_T^t\left(\int\lambda \mathrm{d}B\right)X\right] \text{ for } X\in L_{ip}(\Omega_T) \text{ and } 0\leqslant t\leqslant T \tag{2.7}$$

with

$$\mathcal{E}_T^t\left(\int\lambda \mathrm{d}B\right) = \frac{\mathcal{E}_T\left(\int\lambda \mathrm{d}B\right)}{\mathcal{E}_t\left(\int\lambda \mathrm{d}B\right)}.$$

Denote by $\mathbb{L}_G^{\lambda,1}(\Omega_T)$ the completion of $L_{ip}(\Omega_T)$ under $\hat{\mathbb{E}}^\lambda[|\cdot|]$. Then $\hat{\mathbb{E}}^\lambda$ and $\hat{\mathbb{E}}_t^\lambda$ can be extended to the mappings on $\mathbb{L}_G^{\lambda,1}(\Omega_T)$ which are still denoted by $\hat{\mathbb{E}}^\lambda$ and $\hat{\mathbb{E}}_t^\lambda$.

Theorem 2.6 ([22]). *For a d-dimensional G-BMO martingale generator* $\lambda \in \mathcal{H}_G^2(0,T;\mathbb{R}^d)$, $B^\lambda = (B^{\lambda,i})_{i=1}^d$ *defined by (2.5) is a d-dimensional G-Brownian motion under* $\hat{\mathbb{E}}^\lambda$.

Remark 2.7 *We can check the following integrability results of G-BMO martingale generators: for a G-BMO martingale generator* $\lambda \in \mathcal{H}_G^2(0,T;\mathbb{R}^d)$ *with* $\|\lambda\|_{BMO_G} \leqslant \Phi(q)$ *for* $q>1$*, we have*

(i) $\|\lambda\|_{\mathcal{H}_G^p} < \infty$ *for any* $p \geqslant 1$;

(ii) *if* $X \in \mathbb{L}_G^p(\Omega_T)$ *for some* $p > \frac{q}{q-1}$*, then we have* $X \in \mathbb{L}_G^{\lambda,1}(\Omega_T)$.

3. *G*-forward performance process

In this section, we propose the concept of G-forward performance process, which is a robust forward performance criterion under model uncertainty. More precisely, we first introduce a financial market where the price process of individual stock is driven by G-Brownian motion, and then we give the definition of G-forward performance process by the self-generation condition under G-expectation.

Let G be given by (2.1) with $\Theta \subset \mathbb{R}^{d\times d}$ being a bounded and closed set of matrices of the form

$$\begin{pmatrix} \gamma^1 & 0 & \cdots & 0 \\ 0 & \gamma^2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \gamma^d \end{pmatrix}, \quad \gamma^i \neq 0,\ i=1,\ldots,d,$$

and let $B=(B^i)_{i=1}^d$ be the d-dimensional G-Brownian motion on the corresponding G-expectation space $(\Omega, \mathbb{L}_G^1(\Omega), \hat{\mathbb{E}})$. According to [39, Corollary 3.5.8], we know that $\langle B\rangle^{ij}=0$ for $i\neq j$, and, for each $j=1,\ldots,d$, there exist $\underline{\sigma}_j, \bar{\sigma}_j>0$ such that

$$\underline{\sigma}_j^2 t \leqslant \langle B\rangle_t^{jj} \leqslant \bar{\sigma}_j^2 t, \quad t\geqslant 0. \tag{3.1}$$

In the following, we write $\langle \vec{B}\rangle = (\langle B\rangle^{jj})_{j=1}^d$ for simplicity.

Consider a financial market consisting of one riskless bond with zero interest rate and n stocks ($n \leqslant d$). The individual stock price process evolves as, for $i=1,\ldots,n$,

$$\frac{\mathrm{d}S_t^i}{S_t^i} = \sum_{j=1}^d \left(\mu_t^{ij}\mathrm{d}\langle B\rangle_t^{jj} + \sigma_t^{ij}\mathrm{d}B_t^j\right) = \mu_t^i \mathrm{d}\langle\vec{B}\rangle_t + \sigma_t^i \mathrm{d}B_t, \quad t\geqslant 0, \tag{3.2}$$

with $S_0^i\in\mathbb{R}$, where $\mu^{ij},\sigma^{ij}\in\mathcal{H}_G^1(0,\infty)$ and σ is of full row rank n.

Let $\tilde{\pi}_t^i$ be the amount of money invested in the stock S_t^i at time t, $i=1,\ldots,n$. We will work with trading strategies rescaled by the volatility, namely, $\pi=(\pi^i)_{i=1}^d$ defined as

$$\pi_t^\top := \tilde{\pi}_t^\top \sigma_t, \quad t\geqslant 0,$$

and denote by $\mathcal{A}$ the admissible set of trading strategies for all time. Assume the self-financing condition holds and, in turn, the wealth process associated with π is

$$\mathrm{d}X_t^\pi = \pi_t^\top\left(\theta_t \mathrm{d}\langle\vec{B}\rangle_t + \mathrm{d}B_t\right), \quad t\geqslant 0, \tag{3.3}$$

Here the market price of risk $\theta_t=(\theta_t^{ij})_{i,j=1}^d$ is defined via $\mu_t=\sigma_t\theta_t$, namely,

$$\theta_t := \sigma_t^\top\left(\sigma_t\sigma_t^\top\right)^{-1}\mu_t, \quad t\geqslant 0.$$

Remark 3.1 *Different from the classical financial market (see, for example, [21, 30]), the dynamics of stocks in (3.2) are driven by G-Brownian motion. We see from (3.1) that the drift uncertainty is reflected in the* $\mathrm{d}\langle\vec{B}\rangle$ *term. Moreover, we note here that the market price of risk* θ *is matrix-valued.*

Definition 3.2 *We call* $U(t,x):[0,\infty)\times\mathbb{R}\times\Omega\to\mathbb{R}$ *a G-forward performance process if it satisfies the following conditions:*

(i) *for each* $x\in\mathbb{R}$, $U(t,x)$ *is progressively measurable;*

(ii) *for each* $(t,\omega)\in[0,\infty)\times\Omega$, $U(t,x)$ *is strictly increasing and strictly concave in* $x\in\mathbb{R}$*;*

(iii) *for any* $\pi\in\mathcal{A}$ *satisfying* $U(t,X_t^\pi)\in\mathbb{L}_G^1(\Omega_t)$ *for all* $t\geqslant 0$, $-U(t,X_t^\pi)$ *is a G-submartingale, and there exists an optimal trading strategy* $\pi^*\in\mathcal{A}$ *such that* $-U(t,X_t^{\pi^*})$ *is a G-martingale.*

Remark 3.3 *Intuitively, G-forward performance process is a robust investment criterion defined in arbitrary time horizon for an ambiguity-averse investor. In fact, according to [43, Proposition 3.4], we have, for any* $0\leqslant t\leqslant s$, *that*

$$\begin{aligned}
-\operatorname*{essinf}_{\pi\in\mathcal{A}_U}\hat{\mathbb{E}}_t\left[-U(s,X_s^\pi)\right] &= -\operatorname*{essinf}_{\pi\in\mathcal{A}_U}\operatorname*{esssup}_{\mathbb{P}'\in\mathcal{P}_G(t,\mathbb{P})}\mathbb{E}^{\mathbb{P}'}\left[-U(s,X_s^\pi)|\,\mathcal{F}_t\right]\\
&= -\operatorname*{essinf}_{\pi\in\mathcal{A}_U}\left(-\operatorname*{essinf}_{\mathbb{P}'\in\mathcal{P}_G(t,\mathbb{P})}\mathbb{E}^{\mathbb{P}'}\left[U(s,X_s^\pi)|\,\mathcal{F}_t\right]\right)\\
&= \operatorname*{esssup}_{\pi\in\mathcal{A}_U}\operatorname*{essinf}_{\mathbb{P}'\in\mathcal{P}_G(t,\mathbb{P})}\mathbb{E}^{\mathbb{P}'}\left[U(s,X_s^\pi)|\,\mathcal{F}_t\right],\ \mathbb{P}\text{-a.s.},
\end{aligned}$$

where $\mathcal{A}_U:=\{\pi\in\mathcal{A},\ U(t,X_t^\pi)\in\mathbb{L}_G^1(\Omega_t),\forall t\geqslant 0\}$ *and* $\mathcal{P}_G(t,\mathbb{P}):=\{\mathbb{P}'\in\mathcal{P}_G,\ \mathbb{P}'=\mathbb{P}\ \text{ on }\ \mathcal{F}_t\}$ *for* $\mathbb{P}\in\mathcal{P}_G$. *Thus, the underlying problem of the G-forward performance process is to find the optimal trading strategy under worst-case scenario in any time horizon, and it generalizes the classical backward utility maximization method from three aspects: infinite investment horizon, risk preference adjustment and model uncertainty, at the same time.*

4. Representations of homothetic *G*-forward performance processes via *G*-BSDEs

This section is devoted to the representations of homothetic (exponential, power, and logarithmic) G-forward performance processes in a single stochastic factor model. Motivated by [30], we apply ergodic and infinite horizon quadratic G-BSDEs to construct them. More precisely, we first consider a type of infinite horizon Markovian G-FBSDEs with quadratic generators and then obtain the solvability of ergodic quadratic G-BSDEs through an approximating sequence of infinite horizon quadratic G-BSDEs with strictly monotonic generator. Finally, with the help of the aforementioned results, we build three types of G-forward performance processes.

4.1 Markovian infinite horizon quadratic *G*-FBSDEs

We consider the following type of Markovian G-FBSDE on infinite horizon: for any $0\leqslant s\leqslant T<\infty$ and $x\in\mathbb{R}^n$,

$$\left\{\begin{aligned}
X_s^x &= x+\int_0^s b\left(X_r^x\right)\mathrm{d}r+\sum_{i,j=1}^d\int_0^s h^{ij}\left(X_r^x\right)\mathrm{d}\left\langle B\right\rangle_r^{ij}+\int_0^s\kappa\left(X_r^x\right)\mathrm{d}B_r,\\
Y_s^x &= Y_T^x+\int_s^T f\left(X_r^x,Y_r^x,Z_r^x\right)\mathrm{d}r+\sum_{i,j=1}^d\int_s^T g^{ij}\left(X_r^x,Y_r^x,Z_r^x\right)\mathrm{d}\left\langle B\right\rangle_r^{ij}\\
&\quad -\int_s^T Z_r^x\mathrm{d}B_r-\left(K_T^x-K_s^x\right),
\end{aligned}\right. \tag{4.1}$$

where $b, h^{ij} = h^{ji} : \mathbb{R}^n \to \mathbb{R}^n$, $\kappa : \mathbb{R}^n \to \mathbb{R}^{n\times d}$ and $f, g^{ij} = g^{ji} : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ are deterministic continuous functions, and we impose the following assumptions:

Assumption 4.1 (i) *There exist constants* $L_\phi > 0$ 2 $\eta > 0$ 2 $x, x' \in \mathbb{R}^n$,

$$|\phi(x) - \phi(x')| \leqslant L_\phi |x - x'|, \quad for \;\; \phi = b, h^{ij}, \kappa,$$

and

$$G\left((\kappa(x) - \kappa(x'))^\top (\kappa(x) - \kappa(x')) + 2\left((x - x')\left(h^{ij}(x) - h^{ij}(x')\right)\right)_{i,j=1}^{d}\right) \\ + (x - x')(b(x) - b(x')) \leqslant -\eta |x - x'|^2 . \tag{4.2}$$

Moreover, each element of κ *is bounded by a constant* $C_\kappa > 0$, *i.e., for any* $x \in \mathbb{R}^n$,

$$\left|\kappa^{k,j}(x)\right| \leqslant C_\kappa, k = 1, \ldots, n, \quad j = 1, \ldots, d.$$

(ii) *For* $\psi = f, g^{ij}$, *there exist constants* $\alpha_0 > 0$, $L > 0$ *and* $\mu > 0$ *such that, for* $x \in \mathbb{R}^n$,

$$|\psi(x, 0, 0)| \leqslant \alpha_0,$$

and, for any $x, x' \in \mathbb{R}^n$, $y, y' \in \mathbb{R}$ *and* $z, z' \in \mathbb{R}^d$,

$$|\psi(x, y, z) - \psi(x', y', z')| \leqslant L\left[|y - y'| + (1 + |z| + |z'|)(|x - x'| + |z - z'|)\right];$$

and, for any $x \in \mathbb{R}^n$, $z \in \mathbb{R}^d$ *and* $y, y' \in \mathbb{R}$,

$$(y - y')(f(x, y, z) - f(x, y', z)) + 2G\left[\left((y - y')\left(g^{ij}(x, y, z) - g^{ij}(x, y', z)\right)\right)_{i,j=1}^{d}\right] \\ \leqslant -\mu |y - y'|^2 .$$

(iii) *For* $\underline{\sigma}$ *and* $\bar{\sigma}_\Sigma^2$ *given in (2.2) and (2.3), respectively,*

$$\eta - (1 + \bar{\sigma}_\Sigma^2) L \left| \sqrt{L_\kappa} + \sqrt{\frac{2C_\kappa \sqrt{n \bar{\sigma}_\Sigma^2}}{\underline{\sigma}}} \right|^2 \geqslant 0. \tag{4.3}$$

Under Assumption 4.1 (i), according to [39, Section 5.1], we know that there exists a unique solution $X^x \in \mathcal{H}_G^2(0, \infty; \mathbb{R}^n)$ to the G-SDE in (4.1).

Remark 4.2 *Conditions (4.2) and (4.3) are called the strong dissipativity assumptions, which ensure the ergodicity of the process* X^x *(see [4, 13, 30] in the classical probability framework and [18, 19] in the G-expectation framework). In [4, 30], the dissipative constant* η *is assumed to be larger than the Lipschitz constant (in factor) of the ergodic BSDE. Similarly, we impose condition (4.3) while it is more complicated due to the structure of the diffusion term of G-SDE in (4.1) and the G-expectation setting.*

Definition 4.3 *We call the triplet* (Y, Z, K) *a solution to infinite horizon G-BSDE in (4.1) if*

(i) $(Y, Z, K) \in \mathfrak{S}_G^2(0, \infty) := \cap_{T \geqslant 0} \mathfrak{S}_G^2(0, T)$, *where* $\mathfrak{S}_G^2(0, T) := \{(Y, Z, K) : Y \in \mathcal{S}_G^2(0, T), Z \in \mathcal{H}_G^2(0, T; \mathbb{R}^d)$ *and K is a nonincreasing Gmartingale with* K_0=0 *and* $K_T \in \mathbb{L}_G^2(\Omega_T)\}$;

(ii) *it satisfies equation (4.1) for any* $0 \leqslant t \leqslant T < \infty$.

To show the existence of the solution to the G-BSDE in (4.1), the key point is to consider the corresponding truncated G-BSDE with Lipschitz generators and then show the boundedness of the Z-component. This idea comes from [30, Appendix] in the classical probability setting.

Theorem 4.4 *Suppose Assumption 4.1 holds. Then there exists a unique solution* $(X^x, Y^x, Z^x, K^x) \in \mathcal{H}_G^2(0,\infty;\mathbb{R}^n) \times \mathfrak{S}_G^2(0,\infty)$ *to infinite horizon quadratic G-FBSDE (4.1) such that* Y^x *and* Z^x *are bounded.*

Proof We first show the existence of the solution. Define a truncation function q by

$$q(z) := \frac{|z| \wedge C_z}{|z|} z \mathbf{1}_{|z|\neq 0} \text{ for } z \in \mathbb{R}^d, \tag{4.4}$$

where

$$C_z := \frac{\eta - \left(1+\bar{\sigma}_\Sigma^2\right) L \left(L_\kappa + \frac{2C_\kappa \sqrt{n\bar{\sigma}_\Sigma^2}}{\underline{\sigma}}\right)}{4\left(1+\bar{\sigma}_\Sigma^2\right) L L_\kappa} \tag{4.5}$$

is a positive constant according to Assumption 4.1 (iii). It is easy to check that

$$|q(z)| \leqslant C_z \text{ and } |q(z) - q(z')| \leqslant |z - z'|.$$

We then consider the following truncated infinite horizon G-BSDE: for any $0 \leqslant s \leqslant T < \infty$,

$$\begin{aligned} Y_s^{q,x} = Y_T^{q,x} &+ \int_s^T f^q\left(X_r^x, Y_r^{q,x}, Z_r^{q,x}\right) \mathrm{d}r + \sum_{i,j=1}^d \int_s^T g^{q,ij}\left(X_r^x, Y_r^{q,x}, Z_r^{q,x}\right) \mathrm{d}\langle B\rangle_r^{ij} \\ &- \int_s^T Z_r^{q,x} \mathrm{d}B_r - \left(K_T^{q,x} - K_s^{q,x}\right), \end{aligned} \tag{4.6}$$

where

$$\psi^q(x,y,z) := \psi(x,y,q(z)), \quad \psi = f, g^{ij}.$$

It is obvious that, for any $x, x' \in \mathbb{R}^n, y, y' \in \mathbb{R}$ and $z, z' \in \mathbb{R}^d$,

$$|\psi^q(x,y,z) - \psi^q(x',y',z')| \leqslant L(1+2C_z)(|x-x'| + |z-z'|) + L|y-y'|,$$

i.e., the truncated generators are uniformly Lipschitz. Then, it follows from [19, Theorem 3.1] that there exists a unique solution $(Y^{q,x}, Z^{q,x}, K^{q,x}) \in \mathfrak{S}_G^2(0,\infty)$ to the truncated G-BSDE (4.6) with $Y^{q,x}$ being bounded.

Now it suffices to prove that $Z^{q,x}$ is indeed bounded by C_z, which implies that $q(Z^{q,x}) = Z^{q,x}$. The truncated G-BSDE (4.6), in turn, coincides with the original G-BSDE in (4.1) and $(X^x, Y^x, Z^x, K^x) := (X^x, Y^{q,x}, Z^{q,x}, K^{q,x})$ satisfies the G-FBSDE (4.1).

Note that $Y^{q,x}$ is a semi-martingale under each $\mathbb{P} \in \mathcal{P}_G$. Then the quadratic variation process of $Y^{q,x}$ is given by

$$\langle Y^{q,x}\rangle_t = \sum_{i,j=1}^d \int_0^t Z_s^{q,x,i} Z_s^{q,x,j} \mathrm{d}\langle B\rangle_s^{ij}, \ \mathbb{P}\text{-a.s.} \tag{4.7}$$

On the other hand, define

$$u^q(x) := Y_0^{q,x} \text{ for } x \in \mathbb{R}^n. \tag{4.8}$$

Following the proof of [19, Lemma 4.2] step by step, one can check that, for any $x, x' \in \mathbb{R}^n$,

$$|u^q(x)| \leqslant \frac{\left(1+\bar{\sigma}_\Sigma^2\right)\alpha_0}{\mu} \text{ and } |u^q(x) - u^q(x')| \leqslant M|x - x'|, \tag{4.9}$$

where

$$M := \frac{\left(1+\bar{\sigma}_\Sigma^2\right) L \left(1+2C_z\right)}{\eta - \left(1+\bar{\sigma}_\Sigma^2\right) LL_\kappa \left(1+2C_z\right)} \tag{4.10}$$

is a positive constant. In fact, from the definition of C_z in (4.5), we obtain that

$$M = \frac{\eta + \left(1+\bar{\sigma}_\Sigma^2\right) LL_\kappa - \frac{2\left(1+\bar{\sigma}_\Sigma^2\right) LC_\kappa \sqrt{n\bar{\sigma}_\Sigma^2}}{\underline{\sigma}}}{\left(\eta - \left(1+\bar{\sigma}_\Sigma^2\right) LL_\kappa + \frac{2\left(1+\bar{\sigma}_\Sigma^2\right) LC_\kappa \sqrt{n\bar{\sigma}_\Sigma^2}}{\underline{\sigma}}\right) L_\kappa},$$

which implies by Assumption 4.1 (iii) that $M > 0$. Furthermore, it follows from [19, Lemma 4.3] that $u^q(X_t^x) = Y_t^{q,x}$. Thus, according to the definition of quadratic variation (see [41, Chap. IV,Proposition 1.18]) and (4.9), we obtain that, for any $0 \leqslant t < s < \infty$ and $\mathbb{P} \in \mathcal{P}_G$,

$$\begin{aligned} \langle Y^{q,x}\rangle_s - \langle Y^{q,x}\rangle_t &= \lim_{N\to\infty} \sum_{j=0}^{N-1} \left| Y^x_{t^N_{j+1}} - Y^x_{t^N_j} \right|^2 \leqslant M^2 \lim_{N\to\infty} \sum_{k=1}^{n} \sum_{j=0}^{N-1} \left| X^{x,k}_{t^N_{j+1}} - X^{x,k}_{t^N_j} \right|^2 \\ &= M^2 \sum_{k=1}^{n} \sum_{i,j=1}^{d} \int_s^t \kappa^{k,i}\left(X_r^x\right) \kappa^{k,j}\left(X_r^x\right) \mathrm{d}\left\langle B \right\rangle_r^{ij}, \ \mathbb{P}\text{-a.s.}, \end{aligned}$$

where $\{t = t_0^N, \ldots, t_N^N = s\}$, $N = 1, 2, \ldots,$ is a sequence of partition of $[t, s]$ such that $\max_j |t_{j+1}^N - t_j^N| \to 0$ as $N \to \infty$. It follows that, for each $\mathbb{P} \in \mathcal{P}_G$,

$$\langle Y^{q,x}\rangle_s - \langle Y^{q,x}\rangle_t \leqslant nM^2 C_\kappa^2 \sum_{i,j=1}^{d} \left(\langle B\rangle_s^{ij} - \langle B\rangle_t^{ij} \right) \leqslant nM^2 C_\kappa^2 \bar{\sigma}_\Sigma^2 \left(s-t\right), \ \mathbb{P}\text{-a.s.},$$

which together with (4.7) and the arbitrariness of $\mathbb{P} \in \mathcal{P}_G$ yields that, for any $0 \leqslant t \leqslant s < \infty$,

$$\sum_{i,j=1}^{d} \int_t^s Z_r^{q,x,i} Z_r^{q,x,j} \mathrm{d}\left\langle B\right\rangle_r^{ij} \leqslant nM^2 C_\kappa^2 \bar{\sigma}_\Sigma^2 \left(s-t\right). \tag{4.11}$$

On the other hand, by the non-degenerate property of G in (2.2) and [39, Corollary 3.5.8], the quadratic variation process $\langle B\rangle_t$ is positive definite and $\mathrm{d}\langle B\rangle_t \geqslant \underline{\sigma}^2 I_d \mathrm{d}t$. Thus, we obtain that

$$\sum_{i,j=1}^{d} \int_t^s Z_r^{q,x,i} Z_r^{q,x,j} \mathrm{d}\left\langle B\right\rangle_r^{ij} \geqslant \underline{\sigma}^2 \int_t^s \left|Z_r^{q,x}\right|^2 \mathrm{d}r. \tag{4.12}$$

Combining (4.11) and (4.12) implies that

$$\left|Z^{q,x}\right| \leqslant \frac{MC_\kappa \sqrt{n\bar{\sigma}_\Sigma^2}}{\underline{\sigma}}. \tag{4.13}$$

We claim that

$$\frac{MC_\kappa \sqrt{n\bar{\sigma}_\Sigma^2}}{\underline{\sigma}} \leqslant C_z. \tag{4.14}$$

In fact, consider the following quadratic function

$$\varphi\left(x\right) := \left|x\right|^2 - 2C_z x + \frac{C_\kappa \sqrt{n\bar{\sigma}_\Sigma^2}}{2L_\kappa \underline{\sigma}}.$$

One can easily check that the discriminant of quadratic equation $\varphi(x)=0$ is nonnegative according to the definition of C_z in (4.5) and Assumption 4.1 (iii), which implies that $\varphi(C_z)\leqslant 0$. Thus, by the definition of M in (4.10) and the fact that $M>0$, one can easily check that

$$\frac{MC_\kappa\sqrt{n\bar{\sigma}_\Sigma^2}}{\underline{\sigma}}-C_z=\frac{2\left(1+\bar{\sigma}_\Sigma^2\right)LL_\kappa\varphi(C_z)}{\eta-\left(1+\bar{\sigma}_\Sigma^2\right)LL_\kappa\left(1+2C_z\right)}\leqslant 0,$$

which proves (4.14). So we conclude by (4.13) and (4.14) that $|Z^{q,x}|\leqslant C_z$, which gives the desired result by the previous discussion.

To show the uniqueness, since the Z-component is bounded, the generators of G-BSDE in (4.1) can be seen to be Lipschitz continuous in z. So we can directly obtain the uniqueness by [19, Theorem 3.1]. □

4.2 Ergodic quadratic G-BSDE

Now we study the ergodic G-BSDE with quadratic generators: for any $0\leqslant s\leqslant T<\infty$,

$$\begin{aligned}Y_s^x=Y_T^x&+\int_s^T\left(f\left(X_r^x,Z_r^x\right)+\gamma^1\lambda\right)\mathrm{d}r+\sum_{i,j=1}^d\int_s^T\left(g^{ij}\left(X_r^x,Z_r^x\right)+\gamma_{ij}^2\lambda\right)\mathrm{d}\left\langle B\right\rangle_r^{ij}\\&-\int_s^T Z_r^x\mathrm{d}B_r-\left(K_T^x-K_s^x\right),\end{aligned}\tag{4.15}$$

where, for $x\in\mathbb{R}^n$, X^x is the solution to G-SDE in (4.1) under Assumption 4.1 (i). In addition, $\gamma^1\in\mathbb{R}$, $\gamma^2\in\mathbb{S}(d)$ and $f,g^{ij}:\mathbb{R}^n\times\mathbb{R}^d\to\mathbb{R}$ are supposed to satisfy the following conditions:

Assumption 4.5 (i) $\mu_\gamma:=-\gamma^1-2G(\gamma^2)>0$;

(ii) *For* $\psi=f,g^{ij}$, *there exist constants* $\alpha_0>0$ *and* $L>0$ *such that for any* $x,x'\in\mathbb{R}^n$ *and* $z,z'\in\mathbb{R}^d$,

$$|\psi\left(x,0\right)|\leqslant\alpha_0$$

and

$$|\psi\left(x,z\right)-\psi\left(x',z'\right)|\leqslant L\left(1+|z|+|z'|\right)\left(|x-x'|+|z-z'|\right).$$

Following the idea of [19, Section 5], we consider the following Markovian G-BSDE with generators that are monotonic in y: for any $\varepsilon>0$,

$$\begin{aligned}Y_s^{\varepsilon,x}=Y_T^{\varepsilon,x}&+\int_s^T\left(f\left(X_r^x,Z_r^{\varepsilon,x}\right)+\gamma^1\varepsilon Y_r^{\varepsilon,x}\right)\mathrm{d}r+\sum_{i,j=1}^d\int_s^T\left(g^{ij}\left(X_r^x,Z_r^{\varepsilon,x}\right)+\gamma_{ij}^2\varepsilon Y_r^{\varepsilon,x}\right)\mathrm{d}\left\langle B\right\rangle_r^{ij}\\&-\int_s^T Z_r^{\varepsilon,x}\mathrm{d}B_r-\left(K_T^{\varepsilon,x}-K_s^{\varepsilon,x}\right).\end{aligned}\tag{4.16}$$

Note that

$$f^\varepsilon\left(x,y,z\right):=f\left(x,z\right)+\gamma^1\varepsilon y\ \text{ and }\ g^{\varepsilon,ij}\left(x,y,z\right):=g^{ij}\left(x,z\right)+\gamma_{ij}^2\varepsilon y$$

satisfy Assumption 4.1 (ii) with $\mu=\mu_\gamma\varepsilon$. Then it follows from Theorem 4.4 that there exists a unique solution $(Y^{\varepsilon,x},Z^{\varepsilon,x},K^{\varepsilon,x})\in\mathfrak{S}_G^2(0,\infty)$ to G-BSDE (4.16) with

$$|Y^{\varepsilon,x}| \leqslant \frac{\left(1+\bar{\sigma}_{\Sigma}^{2}\right)\alpha_0}{\mu_\gamma \varepsilon} \text{ and } |Z^{\varepsilon,x}| \leqslant C_z,$$

where C_z is given by (4.5).

For each $\varepsilon > 0$, define

$$v^{\varepsilon}(x) := Y_0^{\varepsilon,x},\ x \in \mathbb{R}^n.$$

According to the proof of Theorem 4.4, we know that

$$v^{\varepsilon}(x) = Y_0^{q,\varepsilon,x},$$

where $Y^{q,\varepsilon,x}$ is the first component of the solution to the following truncated G-BSDE: for any $0 \leqslant s \leqslant T < \infty$,

$$\begin{aligned} Y_s^{q,\varepsilon,x} = Y_T^{q,\varepsilon,x} &+ \int_s^T \left(f\left(X_r^x, q\left(Z_r^{q,\varepsilon,x}\right)\right) + \gamma^1 \varepsilon Y_r^{q,\varepsilon,x}\right) \mathrm{d}r \\ &+ \sum_{i,j=1}^{d} \int_s^T \left(g^{ij}\left(X_r^x, q\left(Z_r^{q,\varepsilon,x}\right)\right) + \gamma_{ij}^2 \varepsilon Y_r^{q,\varepsilon,x}\right) \mathrm{d}\left\langle B\right\rangle_r^{ij} - \int_s^T Z_r^{q,\varepsilon,x} \mathrm{d}B_r - \left(K_T^{q,\varepsilon,x} - K_s^{q,\varepsilon,x}\right) \end{aligned}$$

with truncated function q given in (4.4). Thus, by (4.9), we have

$$|v^{\varepsilon}(x)| \leqslant \frac{\left(1+\bar{\sigma}_{\Sigma}^{2}\right)\alpha_0}{\mu_\gamma \varepsilon} \text{ and } |v^{\varepsilon}(x) - v^{\varepsilon}(x')| \leqslant M\,|x - x'|$$

with M in (4.10), and, moreover,

$$v^{\varepsilon}\left(X_t^x\right) = Y_t^{\varepsilon,x}. \tag{4.17}$$

Then, by a similar argument as in [19, Theorems 5.1 and 5.3], we derive the following result.

Theorem 4.6 *Suppose Assumption 4.1 (i), (iii) and Assumption 4.5 hold. Then, for each* $x \in \mathbb{R}^n$, *there exists a solution* $(Y^x, Z^x, K^x, \lambda) \in \mathfrak{S}_G^2(0,\infty) \times \mathbb{R}$ *to ergodic G-BSDE (4.15) such that* $|Y^x| \leqslant M|X^x|$ *and* $|Z^x| \leqslant C_z$ *with M and* C_z *in (4.10) and (4.5), respectively. Moreover,* λ *is unique.*

4.3 Representations of homothetic G-forward performance processes in a single stochastic factor model

Now we apply the above results to verify the existence of solutions to several specific G-BSDEs and use these solutions to construct three types of homothetic G-forward performance processes in a market with single stock whose dynamics are affected by a single stochastic factor. Moreover, we compare these processes with the classical homothetic forward performance processes obtained in [30] when the G-expectation is a linear expectation with no uncertainty.

Let G be given by

$$G(A) = \frac{1}{2} \sup_{\gamma \in \Theta} \operatorname{tr}\left[\gamma\gamma^{\top} A\right], \quad A \in \mathbb{S}(2), \tag{4.18}$$

where the uncertain set $\Theta \subset \mathbb{R}^{2\times 2}$ is a closed and bounded set including matrices of the form

$$\begin{pmatrix} \gamma^1 & 0 \\ 0 & \gamma^2 \end{pmatrix},\ \gamma^i \neq 0,\ i = 1, 2,$$

and let $B = (B^1, B^2)$ be the 2-dimensional G-Brownian motion on the corresponding G-expectation space $(\Omega, \mathbb{L}_G^1(\Omega), \hat{\mathbb{E}})$.

We deal with an incomplete market including one bond with zero interest rate and one stock S. The stock coefficients are affected by a stochastic factor V, namely, the stock price evolves as

$$\frac{\mathrm{d}S_t}{S_t}=\mu\left(V_t\right)\mathrm{d}\left\langle B\right\rangle_t^{11}+\sigma\left(V_t\right)\mathrm{d}B_t^1,\ t\geqslant 0,$$

where $\mu:\mathbb{R}\rightarrow\mathbb{R}$ and $\sigma:\mathbb{R}\rightarrow\mathbb{R}^+$ are deterministic functions. Here the $\mathbb{R}$-valued factor V solves the G-SDE,

$$\mathrm{d}V_t=b\left(V_t\right)\mathrm{d}t+h^1\left(V_t\right)\mathrm{d}\left\langle B\right\rangle_t^{11}+h^2\left(V_t\right)\mathrm{d}\left\langle B\right\rangle_t^{22}+\kappa\mathrm{d}B_t^1+\sqrt{1-\kappa^2}\mathrm{d}B_t^2,\quad t\geqslant 0,\quad V_0^i\in\mathbb{R}^d, \tag{4.19}$$

where $\kappa\in(0,1)$, and $b,h^i:\mathbb{R}\rightarrow\mathbb{R}$ are also deterministic. The market price of risk with respect to the stochastic factor is

$$\theta(v):=\frac{\mu\left(v\right)}{\sigma\left(v\right)},\ \ v\in\mathbb{R}.$$

We impose following conditions on the market coefficients:

Assumption 4.7 (i) *The function $\theta(v)$ is uniformly bounded and Lipschitz continuous in $v\in\mathbb{R}$;*

(ii) *There exists a $L_\phi>0$ and a large enough $\eta>0$ such that, for any $v,v'\in\mathbb{R}$,*

$$\left|\phi\left(v\right)-\phi\left(v'\right)\right|\leqslant L_\phi\left|v-v'\right|,\phi=b,h^1,h^2,$$

and

$$\begin{aligned}&G\left(2\left(v-v'\right)\begin{pmatrix}h^1(v)-h^1(v') & 0\\ 0 & h^2(v)-h^2(v')\end{pmatrix}\right)+\left(b(v)-b(v')\right)\left(v-v'\right)\\ &\leqslant-\eta\left|v-v'\right|^2.\end{aligned}$$

Under above assumption (ii), G-SDE (4.19) admits a unique solution $V\in\mathcal{H}_G^2(0,\infty)$ by [39, Section 5.1].

Remark 4.8 *We only consider the market with single stock and single stochastic factor herein because it is simple to understand the main idea of the construction via G-BSDEs, and, moreover, we need more structural assumption on the market price of risk for multi-dimensional case in order to obtain the optimal strategy.*

4.3.1 Exponential case

We first investigate the G-forward performance process of the exponential form

$$U\left(t,x\right)=-e^{-x+F_t},\ \left(t,x\right)\in[0,\infty)\times\mathbb{R},$$

where F, related to the stochastic factor, is a process to be chosen.

Let $\tilde{\pi}_t$ be the amount invested in S_t at time $t\geqslant 0$, and the wealth process associated with trading strategy $\pi:=\tilde{\pi}\sigma(V)$ is given by

$$\mathrm{d}X_t^\pi=\pi_t\left(\theta\left(V_t\right)\mathrm{d}\left\langle B\right\rangle_t^{11}+\mathrm{d}B_t^1\right),\quad t\geqslant 0. \tag{4.20}$$

For each $T\geqslant 0$, the set of admissible trading strategies in time horizon $[0,T]$ is defined by

$$\mathcal{A}[0,T]:=\{\pi\in\mathcal{H}_G^2(0,T),\,\pi\text{ is a }G\text{-BMO martingale generator}\},$$

and the set of admissible trading strategies for all $T \geqslant 0$ is defined as $\mathcal{A} := \cup_{T \geqslant 0} \mathcal{A}[0, T]$.

Now we construct two types of exponential G-forward performance processes via ergodic and infinite horizon quadratic G-BSDEs, respectively, with the help of the results in Sections 4.1 and 4.2.

Theorem 4.9 *Define*

$$U(t, x) := -\exp\left(-x + v(V_t) - \rho^1 \lambda \langle B \rangle_t^{11} - \rho^2 \lambda \langle B \rangle_t^{22}\right) \text{ for } (t, x) \in [0, \infty) \times \mathbb{R},$$

where $\rho^1, \rho^2 > 0$ *are constants and* $(v(V_t), Z, K, \lambda) \in \mathfrak{S}_G^2(0, \infty) \times \mathbb{R}$, *with* $v : \mathbb{R} \to \mathbb{R}$ *and* Z *being bounded, is a solution to the following ergodic G-BSDE: for any* $0 \leqslant t \leqslant T < \infty$,

$$\begin{aligned} Y_t = Y_T &+ \int_t^T \left(-\frac{1}{2} |\theta(V_s)|^2 - \theta(V_s) Z_s^1 - \rho^1 \lambda\right) \mathrm{d} \langle B \rangle_s^{11} + \int_t^T \left(\frac{1}{2} \left|Z_s^2\right|^2 - \rho^2 \lambda\right) \mathrm{d} \langle B \rangle_s^{22} \\ &- \int_t^T Z_s \mathrm{d} B_s - (K_T - K_t). \end{aligned} \tag{4.21}$$

Then $U(t, x)$ *is a G-forward performance process with optimal trading strategy* $\pi^* = Z^1 + \theta(V) \in \mathcal{A}$.

Proof It is easy to check that the generators satisfy Assumption 4.5 since θ is uniformly bounded and Lipschitz continuous. So, applying Theorem 4.6 yields that there is a solution $(Y, Z, K, \lambda) \in \mathfrak{S}_G^2(0, \infty) \times \mathbb{R}$ to ergodic G-BSDE (4.21) such that Z is uniformly bounded and $Y = v(V)$ for a function $v : \mathbb{R} \to \mathbb{R}$.

Now we verify that $U(t, x)$ is a G-forward performance process. (i) and (ii) in Definition 3.2 are obvious, so we only need to verify the self-generation condition. Combining (4.20) and (4.21), one can easily obtain that, for any $0 \leqslant t \leqslant T < \infty$ and $\pi \in \mathcal{A}[0, T]$,

$$\begin{aligned} & - X_T^\pi + Y_T - \rho^1 \lambda \langle B \rangle_T^{11} - \rho^2 \lambda \langle B \rangle_T^{22} - \left(-X_t^\pi + Y_t - \rho^1 \lambda \langle B \rangle_t^{11} - \rho^2 \lambda \langle B \rangle_t^{22}\right) \\ = & \int_t^T \frac{1}{2} \left|\pi_s - Z_s^1 - \theta(V_s)\right|^2 \mathrm{d} \langle B \rangle_s^{11} + (K_T - K_t) \\ & + \int_t^T \left(Z_s^1 - \pi_s\right) \mathrm{d} B_s^1 - \frac{1}{2} \int_t^T \left|Z_s^1 - \pi_s\right|^2 \mathrm{d} \langle B \rangle_s^{11} + \int_t^T Z_s^2 \mathrm{d} B_s^2 - \frac{1}{2} \int_t^T \left|Z_s^2\right|^2 \mathrm{d} \langle B \rangle_s^{22}. \end{aligned}$$

It follows that

$$-U(T, X_T^\pi) = -U(t, X_t^\pi) \mathcal{E}_T^t \left(\int \Lambda^\pi \mathrm{d} B\right) \exp\left(\int_t^T \frac{1}{2} \left|\pi_s - Z_s^1 - \theta(V_s)\right|^2 \mathrm{d} \langle B \rangle_s^{11} + (K_T - K_t)\right), \tag{4.22}$$

where $\Lambda^\pi := (Z^1 - \pi, Z^2)^\top$ is a 2-dimensional G-BMO martingale generator. According to G-Girsanov transformation introduced in Section 2, define

$$\hat{\mathbb{E}}_t^{\Lambda^\pi}[X] := \hat{\mathbb{E}}_t \left[\mathcal{E}_T^t \left(\int \Lambda^\pi \mathrm{d} B\right) X\right] \text{ for } X \in \mathbb{L}_G^{\Lambda^\pi, 1}(\Omega_T).$$

Since K is a nonincreasing G-martingale and $K_0 = 0$, we have $K \leqslant 0$ and, by the characterization of $\mathbb{L}_G^p$ in Theorem 2.1, we obtain that $e^{K_t} \in \mathbb{L}_G^p(\Omega_t)$ for each $p \geqslant 1$. Thus, by Remark 2.7, $\hat{\mathbb{E}}_t^{\Lambda^\pi}[e^{K_T - K_t}]$ is well-defined. Thus, for any $\pi \in \mathcal{A}$ satisfying $U(t, X_t^\pi) \in \mathbb{L}_G^1(\Omega_t), \forall t \geqslant 0$, taking conditional G-expectation on both sides of (4.22) yields that

$$\begin{aligned}\hat{\mathbb{E}}_t\left[-U\left(T,X_T^\pi\right)\right]&=-U\left(t,X_t^\pi\right)\hat{\mathbb{E}}_t\left[\mathcal{E}_T^t\left(\int\Lambda^\pi\mathrm{d}B\right)\exp\left(\int_t^T\frac{1}{2}\left|\pi_s-Z_s^1-\theta\left(V_s\right)\right|^2\mathrm{d}\left\langle B\right\rangle_s^{11}\right)e^{K_T-K_t}\right]\\&\geqslant-U\left(t,X_t^\pi\right)\hat{\mathbb{E}}_t^{\Lambda^\pi}\left[e^{K_T-K_t}\right].\end{aligned}\tag{4.23}$$

It is obvious that $e^{K_t}\geqslant\hat{\mathbb{E}}_t^{\Lambda^\pi}[e^{K_T}]$ because K is nonincreasing, and, on the other hand, we obtain by G-Jensen's inequality (see [39, Section 4.3]) that $e^{K_t}\leqslant\hat{\mathbb{E}}_t^{\Lambda^\pi}[e^{K_T}]$. Thus, we have $\hat{\mathbb{E}}_t^{\Lambda^\pi}[e^{K_T-K_t}]=1$ and hence $\hat{\mathbb{E}}_t[-U(T,X_T^\pi)]\geqslant-U(t,X_t^\pi)$.

Set $\pi^*:=Z^1+\theta\left(V\right)$, which belongs to $\mathcal{A}$ since Z^1 and $\theta(V)$ are bounded. By (4.22), (2.4) and the fact that $e^{K_t}\in\mathbb{L}_G^p(\Omega_t)$ for each $p\geqslant1$, applying Theorem 2.1 yields that, for each $t\geqslant0$,

$$U\left(t,X_t^{\pi^*}\right)=U\left(0,x\right)\mathcal{E}_t\left(\int\Lambda^{\pi^*}\mathrm{d}B\right)e^{K_t}\in\mathbb{L}_G^1(\Omega_t).$$

Thus, by the definition of π^*, we obtain by (4.23) that

$$\hat{\mathbb{E}}_t\left[-U\left(T,X_T^{\pi^*}\right)\right]=-U\left(t,X_t^{\pi^*}\right)\hat{\mathbb{E}}_t^{\Lambda^{\pi^*}}\left[e^{K_T-K_t}\right]=-U\left(t,X_t^{\pi^*}\right),$$

which proves (iii) in Definition 3.2. So we conclude that $U(t,x)$ is a G-forward performance process. □

Similarly, we also have the following representation of exponential G-forward performance process via infinite horizon G-BSDE through Theorem 4.4.

Theorem 4.10 *Define*

$$U\left(t,x\right):=-\exp\left(-x+u\left(V_t\right)-\int_0^t\rho^1u\left(V_s\right)\mathrm{d}\left\langle B\right\rangle_s^{11}-\int_0^t\rho^2u\left(V_s\right)\mathrm{d}\left\langle B\right\rangle_s^{22}\right)\ \text{for}\ (t,x)\in[0,\infty)\times\mathbb{R},$$

where $\rho^1,\rho^2>0$ are constants and $\left(u\left(V_t\right),Z,K\right)$, with $u:\mathbb{R}\to\mathbb{R}$ and Z being bounded, is the unique solution to the following infinite horizon quadratic G-BSDE: for any $0\leqslant t\leqslant T<\infty$,

$$\begin{aligned}Y_t=Y_T&+\int_t^T\left(-\frac{1}{2}\left|\theta\left(V_s\right)\right|^2-\theta\left(V_s\right)Z_s^1-\rho^1Y_s\right)\mathrm{d}\left\langle B\right\rangle_s^{11}+\int_t^T\left(\frac{1}{2}\left|Z_s^2\right|^2-\rho^2Y_s\right)\mathrm{d}\left\langle B\right\rangle_s^{22}\\&-\int_t^TZ_s\mathrm{d}B_s-\left(K_T-K_t\right).\end{aligned}\tag{4.24}$$

Then $U(t,x)$ is a G-forward performance process with optimal trading strategy $\pi^*=Z^1+\theta(V)\in\mathcal{A}$.

4.3.2 Power case

Now we aim to construct a power G-forward performance process of the form

$$U\left(t,x\right)=\frac{1}{\delta}x^\delta e^{F_t},\ \left(t,x\right)\in[0,\infty)\times\mathbb{R}^+,$$

with constant $\delta\in(0,1)$ and F to be specified. In this case, we will work with the trading strategies $\alpha_t:=\tilde{\alpha}_t\sigma\left(V_t\right)$, with $\tilde{\alpha}_t$ being the proportion of the total wealth invested into S_t at time $t\geqslant0$, and we still consider the admissible set $\mathcal{A}$ for α. Then the wealth process associated with α solves the linear G-SDE

$$\mathrm{d}X_t^\alpha=X_t^\alpha\alpha_t\left(\theta\left(V_t\right)\mathrm{d}\left\langle B\right\rangle_t^{11}+\mathrm{d}B_t^1\right),\quad t\geqslant0.\tag{4.25}$$

One can easily check that the solution to the above G-SDE satisfies

$$X_T^\alpha = X_t^\alpha \exp\left(\int_t^T \left(\theta\left(V_s\right)\alpha_s - \frac{1}{2}\left|\alpha_s\right|^2\right) \mathrm{d}\left\langle B\right\rangle_s^{11} + \int_t^T \alpha_s \mathrm{d}B_s^1\right), \quad 0 \leqslant t \leqslant T < \infty. \tag{4.26}$$

Theorem 4.11 *Define*

$$U\left(t,x\right) := \frac{1}{\delta}x^\delta \exp\left(v\left(V_t\right) - \rho^1\lambda\left\langle B\right\rangle_t^{11} - \rho^2\lambda\left\langle B\right\rangle_t^{22}\right) \ \textit{for}\ (t,x) \in [0,\infty)\times\mathbb{R}, \tag{4.27}$$

where $\rho^1, \rho^2 > 0$ are constant and $(v(V_t), Z, K, \lambda)$, *with* $v : \mathbb{R} \to \mathbb{R}$ *and Z being bounded, is a solution to the following ergodic G-BSDE: for any* $0 \leqslant t \leqslant T < \infty$,

$$Y_t = Y_T + \int_t^T \left(\frac{\delta}{2\left(1-\delta\right)}\left|Z_s^1 + \theta\left(V_s\right)\right|^2 + \frac{1}{2}\left|Z_s^1\right|^2 - \rho^1\lambda\right)\mathrm{d}\left\langle B\right\rangle_s^{11} + \int_t^T \left(\frac{1}{2}\left|Z_s^2\right|^2 - \rho^2\lambda\right)\mathrm{d}\left\langle B\right\rangle_s^{22} - \int_t^T Z_s \mathrm{d}B_s - \left(K_T - K_t\right). \tag{4.28}$$

Then $U(t,x)$ is a G-forward performance process with optimal trading strategy $\alpha^* := \frac{Z^1+\theta(V)}{1-\delta} \in \mathcal{A}$.

Proof One can easily verify that all assumptions in Theorem 4.6 hold. Thus, there exists a solution $(Y, Z, K, \lambda) \in \mathfrak{S}_G^2(0,\infty)\times\mathbb{R}$ to ergodic G-BSDE (4.28) such that Z is bounded and Y can be written as $Y = v(V)$ for a deterministic function $v : \mathbb{R} \to \mathbb{R}$.

By (4.26) and the ergodic G-BSDE (4.28), we easily deduce by (4.27) that, for any $0 \leqslant t \leqslant T < \infty$ and $\alpha \in \mathcal{A}$,

$$-U\left(T, X_T^\alpha\right) = -U\left(t, X_t^\alpha\right)\mathcal{E}_T^t\left(\int \Lambda^\alpha \mathrm{d}B\right)\exp\left(-\frac{\delta\left(1-\delta\right)}{2}\int_t^T \left|\alpha_s - \frac{Z_s^1 + \theta\left(V_s\right)}{1-\delta}\right|^2 \mathrm{d}\left\langle B\right\rangle_s^{11}\right)e^{K_T - K_t},$$

where $\Lambda^\alpha := (Z^1 + \delta\alpha, Z^2)^\top$ is a G-BMO martingale generator. By a similar argument as in the proof of the exponential case, we have that, for $\alpha \in \mathcal{A}$ satisfying $U(t, X_t^\alpha) \in \mathbb{L}_G^1(\Omega_t), \forall t \geqslant 0$, $-U\left(t, X_t^\alpha\right)$ is a G-submartingale, and $-U\left(t, X_t^{\alpha^*}\right)$ is a G-martingale with

$$\alpha^* := \frac{Z^1 + \theta\left(V\right)}{1-\delta} \in \mathcal{A}.$$

Thus, $U(t,x)$ defined in (4.27) is a G-forward performance process. □

We also have the representation of power G-forward performance process via infinite horizon G-BSDE.

Theorem 4.12 *Define*

$$U\left(t,x\right) := \frac{1}{\delta}x^\delta \exp\left(u\left(V_t\right) - \int_0^t \rho^1 u\left(V_s\right)\mathrm{d}\left\langle B\right\rangle_s^{11} - \int_0^t \rho^2 u\left(V_s\right)\mathrm{d}\left\langle B\right\rangle_s^{22}\right) \ \textit{for}\ (t,x) \in [0,\infty)\times\mathbb{R},$$

where $\rho^1, \rho^2 > 0$ *are constants and* $(u(V_t), Z, K) \in \mathfrak{S}_G^2(0,\infty)$, *with* $u : \mathbb{R} \to \mathbb{R}$ *and Z being bounded, is the unique solution to the following infinite horizon G-BSDE: for any* $0 \leqslant t \leqslant T < \infty$,

$$Y_t = Y_T + \int_t^T \left(\frac{\delta}{2\left(1-\delta\right)}\left|Z_s^1 + \theta\left(V_s\right)\right|^2 + \frac{1}{2}\left|Z_s^1\right|^2 - \rho^1 Y_s\right)\mathrm{d}\left\langle B\right\rangle_s^{11} + \int_t^T \left(\frac{1}{2}\left|Z_s^2\right|^2 - \rho^2 Y_s\right)\mathrm{d}\left\langle B\right\rangle_s^{22} - \int_t^T Z_s \mathrm{d}B_s - \left(K_T - K_t\right). \tag{4.29}$$

Then $U(t,x)$ is a G-forward performance process with optimal trading strategy $\alpha^* := \frac{Z^1+\theta(V)}{1-\delta} \in \mathcal{A}$.

4.3.3 Logarithmic case

We end this section with the construction of logarithmic G-forward performance process of the form

$$U(t,x)=\ln x+F_t,\quad (t,x)\in[0,\infty)\times\mathbb{R}^+,$$

with a unknown process F, and we still work with the same trading strategies as in the power case.

Theorem 4.13 *Define*

$$U(t,x):=\ln x+v(V_t)-\rho^1\lambda\langle B\rangle_t^{11}-\rho^2\lambda\langle B\rangle_t^{22}\ \text{for } (t,x)\in[0,\infty)\times\mathbb{R}, \tag{4.30}$$

where $\rho^1,\rho^2>0$ *are constants and* $(v(V_t),Z,K,\lambda)$, $v:\mathbb{R}\to\mathbb{R}$, *is a solution to the following ergodic G-BSDE: for any* $0\leqslant t\leqslant T<\infty$,

$$Y_t=Y_T+\int_t^T\left(\frac{1}{2}|\theta(V_s)|^2-\rho^1\lambda\right)\mathrm{d}\langle B\rangle_s^{11}-\rho^2\lambda\left(\langle B\rangle_T^{22}-\langle B\rangle_t^{22}\right)-\int_t^T Z_s\mathrm{d}B_s-(K_T-K_t),$$

Then $U(t,x)$ *is a G-forward performance process with optimal trading strategy* $\alpha^*:=\theta(V)\in\mathcal{A}$.

Proof Note that, for any $0\leqslant t\leqslant T<\infty$ and $\alpha\in\mathcal{A}$,

$$\ln X_T^\alpha=\ln X_t^\alpha+\int_t^T\left(\theta(V_s)\alpha_s-\frac{1}{2}|\alpha_s|^2\right)\mathrm{d}\langle B\rangle_s^{11}+\int_t^T\alpha_s\mathrm{d}B_s^1,$$

which together with the ergodic G-BSDE implies that

$$\begin{aligned}-U(T,X_T^\alpha)=&-U(t,X_t^\alpha)+\int_t^T\frac{1}{2}|\alpha_s-\theta(V_s)|^2\mathrm{d}\langle B\rangle_s^{11}\\&-\int_t^T\left(\alpha_s+Z_s^1\right)\mathrm{d}B_s^1-\int_t^T Z_s^2\mathrm{d}B_s^2-(K_T-K_t).\end{aligned}$$

Since K is a G-martingale under G-expectation $\hat{\mathbb{E}}$, taking conditional G-expectation on both sides yields that, for all $\alpha\in\mathcal{A}$ satisfying $U(t,X_t^\alpha)\in\mathbb{L}_G^1(\Omega_t),\forall t\geqslant 0$,

$$\hat{\mathbb{E}}_t\left[-U(T,X_T^\alpha)\right]=-U(t,X_t^\alpha)+\hat{\mathbb{E}}_t\left[\int_t^T\frac{1}{2}|\alpha_s-\theta_s|^2\mathrm{d}\langle B\rangle_s^{11}-(K_T-K_t)\right]\geqslant-U(t,X_t^\alpha).$$

Choose $\alpha^*:=\theta(V)\in\mathcal{A}$ and we obtain that $-U\left(t,X_t^{\alpha^*}\right)$ is a G-martingale, which yields that $U(t,x)$ is a G-forward performance process. □

Furthermore, the construction of logarithmic G-forward performance process via infinite horizon G-BSDE follows easily.

Theorem 4.14 *Define*

$$U(t,x):=\ln x+u(V_t)-\int_0^t\rho^1u(V_s)\mathrm{d}\langle B\rangle_s^{11}-\int_0^t\rho^2u(V_s)\mathrm{d}\langle B\rangle_s^{22}\ \text{for } (t,x)\in[0,\infty)\times\mathbb{R}, \tag{4.31}$$

where $\rho^1,\rho^2>0$ *are constants and* $(u(V_t),Z,K)\in\mathfrak{S}_G^2(0,\infty)$, $u:\mathbb{R}\to\mathbb{R}$, *is the unique solution to the following infinite horizon G-BSDE: for any* $0\leqslant t\leqslant T<\infty$,

$$Y_t=Y_T+\int_t^T\left(\frac{1}{2}|\theta(V_s)|^2-\rho^1Y_s\right)\mathrm{d}\langle B\rangle_s^{11}-\int_t^T\rho^2Y_s\mathrm{d}\langle B\rangle_s^{22}-\int_t^T Z_s\mathrm{d}B_s-(K_T-K_t).$$

Then $U(t,x)$ *is a* G*-forward performance process with optimal trading strategy* $\alpha^* := \theta(V) \in \mathcal{A}$.

4.3.4 Connection with classical homothetic forward performance processes

Now we consider the case when the uncertainty set of G-expectation is reduced to a singleton. Let the set Θ in (4.18) be given by

$$\Theta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

and then

$$G\left(A\right) = \frac{1}{2}\left(a^{11} + a^{22}\right), \quad A = \begin{pmatrix} a^{11} & a^{12} \\ a^{12} & a^{22} \end{pmatrix} \in \mathbb{S}\left(2\right). \tag{4.32}$$

In this case, the canonical process $B = (B^1, B^2)$ is the classical Brownian motion and the corresponding sublinear G-expectation is a linear expectation. Moreover, the quadratic variation process of B satisfies

$$\langle B \rangle_t^{11} = \langle B \rangle_t^{22} = t.$$

For the exponential case, if we choose ρ^i, $i = 1, 2$, to satisfy $\rho^1 + \rho^2 = 1$, the ergodic G-BSDE (4.21) becomes

$$\begin{aligned} Y_t = Y_T &+ \int_t^T \left(-\frac{1}{2}\left|\theta\left(V_s\right)\right|^2 - \theta\left(V_s\right) Z_s^1 + \frac{1}{2}\left|Z_s^2\right|^2 - \lambda \right) \mathrm{d}s \\ &- \int_t^T Z_s \mathrm{d}B_s - \left(K_T - K_t\right). \end{aligned} \tag{4.33}$$

Since K is a nonincreasing martingale under linear expectation with $K_0 = 0$, we have $K \equiv 0$. So BSDE (4.33) coincides with (42) in [30] (see also [4, Section 4]). According to Theorem 4.9,

$$U\left(t, x\right) := -\exp\left(-x + v\left(V_t\right) - \lambda t\right) \text{ for } \left(t, x\right) \in [0, \infty) \times \mathbb{R},$$

is a G-forward performance process with G in (4.32), which aligns with the representation of exponential forward performance process under linear expectation in [30, Theorem 4.2]. One can similarly check that the representations for power and logarithmic G-forward performance processes also align with the classical ones obtained in [30].

Intuitively, the classical forward performance process is a special case of G-forward performance process when the G-expectation is a linear one, which can be observed by Definition 3.2 and the definition of classical forward performance process (see, for example, [30, Definition 2.1]).

5. Conclusion

In this paper, we give the definition of the G-forward performance processes in the market model driven by G-Brownian motion. Then we establish the well-posedness of Markovian infinite horizon and ergodic G-BSDEs whose generators are quadratic in z, based on the idea of truncation. According to these abstract results, we construct some examples of homothetic G-forward performance processes in a single stochastic factor model driven by 2-dimensional G-Brownian motion.

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