

# Time-consistent pension policy with minimum guarantee and sustainability constraint

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**Abstract** This paper proposes and investigates an optimal pair investment/pension policy for a pay-as-you-go (PAYG) pension scheme. The social planner can invest in a buffer fund in order to guarantee a minimal pension amount. The model aims at taking into account complex dynamic phenomena such as the demographic risk and its evolution over time, the time and age dependence of agents preferences, and financial risks. The preference criterion of the social planner is modeled by a consistent dynamic utility defined on a stochastic domain, which incorporates the heterogeneity of overlapping generations and its evolution over time. The preference criterion and the optimization problem also incorporate sustainability, adequacy and fairness constraints. The paper designs and solves the social planner's dynamic decision criterion, and computes the optimal investment/pension policy in a general framework. A detailed analysis for the case of dynamic power utilities is provided.

**Keywords** Consistent dynamic utility, PAYG pension policy, Sustainability and actuarial fairness, Demographic and financial risk sharing, Stochastic control

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## 1. Introduction

In many countries, public pensions are financed by a Pay-as-you-go (PAYG) scheme, where current pensions are financed by the redistribution of contributions paid by working participants. In order for the pension system to remain sustainable, these contributions should match approximately the pension liabilities. Intergenerational solidarity is thus one of the main pillars of PAYG pension systems. However, the sustainability of PAYG systems has become a key challenge for policymakers in aging countries, caused by both the decrease in birth rates and an unprecedented increase in the life expectancy (see e.g. [1, 10, 34]). For instance in OECD countries, pension expenditures have increased on average by 1.5% of GDP in nearly two decades, and an

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additional increase of 3.5% of GDP by 2035 is forecasted ([43]). As a consequence, many governments have engaged pension reforms with the aim to reduce pension expenditures. Among them, several countries like Italy or Sweden have redefined the pension amount formula from a Defined Benefit PAYG to a Notional Defined Contribution PAYG, with a pension indexation rate which can depend on demographic and/or economic indicators (see [3, 25]). However, even NDC PAYG pension systems are not guaranteed to be sustainable (see [2, 3]). On the other hand, pension systems should provide adequate benefits for retirees as well as an acceptable level of fairness between generations, as underlined in [2]. As an example, only four OECD countries do not guarantee a minimum pension amount (see [43]).

The aim of this paper is to propose an adaptive decision criterion in order to design an optimal policy that is consistent with both sustainability and adequacy constraints in PAYG framework. Our contribution is twofold.

Firstly, we introduce a general dynamic framework for modeling a PAYG pension system, with a guaranteed minimum annuitized pension amount. Pensions are financed by the workers' contribution, and the social planner has also the flexibility to invest/borrow from a buffer fund to finance the pension system. The buffer fund allows the social planner to invest in the financial market. In addition, the reserves accumulated in the fund allows for demographic (longevity/fertility) shocks to be mitigated over time ([44]). An important challenge is to convey the complexity of the problem, by taking into account key phenomena such as the demographic risk and its evolution over time, the time and age dependence of agents' preferences, or financial risks. To the best of our knowledge, these problems have only been tackled either partially or separately in the literature.

We adopt a dynamic and continuous time approach, which incorporates the heterogeneity of overlapping generations and the non-stationary evolution of the population over time. A similar flexible population dynamics model is studied in [3], however with deterministic age and time dependent birth and mortality rates, while we consider stochastic demographic rates in our setting. This allows us to take into account stylized fact of the population dynamics, such as uncertain longevity or dependency ratio increases. In particular, the population age composition is not assumed to be stationary, as it often the case (see e.g. [14, 11]). Furthermore, we do not restrict ourselves to a two-period models, as in [8] or [15] for instance.

Several papers in the literature study deterministic models of PAYG pension systems, in which the social planner can invest in a buffer fund (see e.g. [3, 25–27, 44]). However, the buffer fund is assumed to have a known return. Besides, when the optimal pension policy is derived from optimizing solvency indicators, a sustainability constraint is only taken into account at an arbitrary terminal time, and there are no adequacy constraints. In this paper, the sustainability of the pension scheme is ensured by imposing a pathwise solvency constraint on the buffer fund (see also [16] for a similar assumption in the case of a fully funded pension system).

In [24], the social planner can invest in a complete market, and a minimum pension amount is considered. However, it is obviously impossible to hedge perfectly the demographic and economic risks through the financial market. Hence, we take into account in this paper the incompleteness of the financial market. Furthermore, the pension benefit in [24] is a lumpsum at retirement. This means that the longevity risk is not taken into account, contrary to this paper, since we consider here an adequacy constraint (minimum pension) directly on the pension amount received by pensioners until their death.

Secondly, this paper provides contributions on the literature on forward dynamic utilities with stochastic constraints and endowments. Indeed, as the representative of past, present and future

generations, the social planner should aggregate preferences of all pensioners. This aggregation is the key in the fairness criterion as this benevolent social planner aims at dealing with successive overlapping generations fairly. Thus, the social planner's decision criterion that appears in the optimization problem's formulation is composed of the buffer fund utility and an aggregated utility which should capture the heterogeneous preferences of different generations. This motivates the use of forward dynamic utilities to deal with such a complex framework ([19]).

Technically, the problem formulation is related to the literature on optimal investment and consumption with labor income, respectively corresponding in our setting to the buffer fund, pensions and contributions. The literature usually states this optimization problem in a backward formulation, see e.g. [12, 22, 28], or more recently [36] for a general setting in incomplete markets. However the backward approach has several drawbacks when considering the framework of PAYG pensions. First it does not incorporate any changes in the agents' preferences, or any uncertain evolution of the environment variables. Furthermore, in the context of pensions, fixing a time-horizon is difficult and can lead to optimal choices that depend on the time horizon, inducing artificial phenomena such as the fund liquidation at terminal time (see e.g. [24, 26]). Finally, the social planner should be able to identify her preference at any intermediate time, in order to ensure consistency across time and generations.

Introduced by Musiela and Zariphopoulou [40, 41], the framework of dynamic utilities (also called forward preference processes) is well suited to solve the issues raised above. Dynamic utilities allow their users to propose long-term, time-coherent policies adjusted to the information flow, in non-stationary and uncertain environment. The framework has been generalized and extended to many classes of dynamic utilities (see e.g. [23, 33, 37, 38]) and to many applications, including (among others) risk measures ([48]), pricing and risk sharing ([4]), yield curve modeling ([21]), or competition in fund management ([5, 17]). In a recent preprint, [42] study the optimal investment of an individual worker in a funded defined contribution pension fund. To the best of our knowledge, our paper is the first attempt to study PAYG pension schemes from a dynamic utility point of view. The setting is close to the forward dynamic utilities of investment and consumption framework in [20], whose results are extended here by considering a given continuous stream of income (contributions) and stochastic pathwise constraints on the buffer fund (wealth) and pensions (consumption).

The population model, PAYG pension system with sustainability and adequacy constraints, and incomplete financial market are introduced in Section 2. The social planner's dynamic decision criterion is formulated in Section 3, by introducing dynamic utilities defined on stochastic domains. In particular, we give sufficient conditions on the local characteristics of the dynamic utility for the utility to be well defined. The aggregation of pensioners' preferences is introduced in Section 3.2. The main results of the paper are presented in Section 4. We first introduce a natural consistency HJB-SPDE derived from the dynamic programming principle, so that the decision criterion is consistent over time. Under this sufficient condition, the optimal constrained investment and pension policy is derived explicitly in Theorem 4.5. The remainder of the section is dedicated to proving this result. Section 5 details two examples in the case of dynamic power utilities.

## 2. The model

All stochastic processes are defined on a standard filtered probability space  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ , where the filtration  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$  is the natural filtration generated a  $n$ -dimensional Brownian motion  $W$ , and that is assumed to be right continuous and complete.

## 2.1 Population dynamics

We consider a population structured continuously in age and time, characterized by the family  $(n(\cdot, a))_{a \geq 0}$  of  $\mathbb{F}$ -adapted processes, where  $n(t, a)$  is the number of individuals of age  $a$  at time  $t$ . For ease of notation, we consider in this paper a single-sex framework. We only assume that the total population size remains finite, i.e.:

$$N_t := \int_0^\infty n(t, a) da < \infty, \quad \forall t \geq 0, \quad \mathbb{P}\text{-a.s.}$$

This general framework allows for the modeling of the stylized facts and *non-stationarity* of an aging population evolution. In particular, the population evolution is usually characterized by age-specific demographic (mortality and birth) rates. In our general framework, they can be modeled for instance by a family of  $\mathbb{F}$ -adapted nonnegative processes  $(d(\cdot, a))_{a \geq 0}$  and  $(b(\cdot, a))_{a \geq 0}$ . Time-dependent stochastic birth and mortality rates can describe the uncertain aging of the population, by taking into account phenomena such as the decrease of mortality rates over time (longevity risk), or birth rates declines. For instance, stochastic mortality models fitted to real data (see e.g. [9, 45]) can be used as inputs for death rates.

The population dynamics can be derived from these rates. Between a small period of time  $[t, t + dt]$ , a (random) proportion  $d(t, a)dt$  of individuals of age  $a$  die, while individuals of age  $a$  give birth to  $b(t, a)n(t, a)$  individuals. In this example, the population dynamics is formally described by a partial differential equation with stochastic coefficients, generalizing the standard McKendrick-Von Foerster equations:

$$(\partial_t + \partial_a)n(t, a) = -d(t, a)n(t, a), \quad (2.1)$$

$$n(t, 0) = \int_0^\infty b(t, a)n(t, a) da. \quad (2.2)$$

A detailed analysis of such equations can be found in [6] (see also e.g. [29, 47]). In particular, such models can be easily extended to include intra-cohort heterogeneity or exogenous population flows.

Workers are assumed to enter the work force at fixed age  $a_e$ , and retires at age  $a_r$ . Thus, the number of pensioners at time  $t$  is

$$N_t^r = \int_{a_r}^\infty n(t, a) da \quad (2.3)$$

social and the number of workers is

$$N_t^w = \int_{a_e}^{a_r} n(t, a) da. \quad (2.4)$$

In the following, all results can be straightforwardly extended when the retirement age  $a_r(t)$  (or  $a_e(t)$ ) depends on time. For ease of notations, the time variable is omitted in the following.

## 2.2 Pension system

We define below the aggregate contribution rate process  $C_t$  payed by the workers at time  $t$ , and the aggregate pension rate process  $P_t$  received by the pensioners at time  $t$ . In what follows, we assume that individuals in a given cohort (of age  $a$  at time  $t$ ) receive the same wage and pension amount. In the case of heterogeneity inside the cohort, the age-dependent wage and pension processes can be interpreted as the average amounts over the cohort.

**Contribution process** The wage process of any worker of age  $a \in [a_e, a_r]$  at time  $t$  is assumed to be an exogenous  $\mathbb{F}$ -adapted process  $(\mathbf{e}_t(a))$ . All workers contribute a fixed proportion  $\alpha_c \in [0, 1]$  of their wage in order to finance pensions of current pensioners. The aggregate contribution process  $(C_t)$  is thus defined by

$$C_t = \alpha_c \int_{a_e}^{a_r} \mathbf{e}_t(a) n(t, a) da, \quad \forall t \geq 0. \quad (2.5)$$

If the wage does not depend on the age, then  $C_t = \alpha_c \mathbf{e}_t N_t^w$ .

**Pension process and adequacy constraint** In order to achieve a minimum adequacy of retirement incomes, we consider the following pension mechanism: Each pensioner of age  $a$  at time  $t$  receives a pension amount  $p_t(a)$  depending on her age, with

$$p_t(a) = p_t^{\min}(a) \rho_t, \quad a \geq a_r, t \geq 0. \quad (2.6)$$

$p_t^{\min}(a)$  corresponds to a minimum pension amount guaranteed to pensioners (see examples below) and  $(\rho_t)$  represents a global adjustment with respect to the benchmark, which is the same for all pensioners living at time  $t$ . The pension process should thus satisfy the **adequacy constraint** (2.7) below:

$$\forall t \geq 0, \forall a \geq a_r, \quad p_t(a) \geq p_t^{\min}(a) \quad \text{a.s.}, \quad \text{or equivalently } \rho_t \geq 1 \quad \text{a.s.} \quad (2.7)$$

Finally, the total pension amount is

$$P_t = \rho_t P_t^{\min}, \quad \text{with } P_t^{\min} = \int_{a_r}^{\infty} p_t^{\min}(a) n(t, a) da. \quad (2.8)$$

In the following, two examples are studied as an application of the general results:

**Example 1** When  $p_t^{\min}(a) \equiv p_t^{\min}$ , all pensioners at time  $t$  receive the same pension amount  $p_t^{\min} \rho_t$  and the minimal aggregate pension amount is  $P_t^{\min} = N_t^r p_t^{\min}$ . For example,  $p_t^{\min}$  can be indexed on current wages, contributions, or any other indicator.

**Example 2** A more realistic choice for  $p^{\min}$  is to take a base pension computed at retirement date, multiplied by an indexation factor:

$$p_t^{\min}(a) = p_{ret}(a_r + t - a) e^{\int_{a_r+t-a}^t \lambda_s ds}. \quad (2.9)$$

•  $p_{ret}(s)$  corresponds to the base pension amount received by an individual retiring (i.e. of age  $a_r$ ) at time  $s$ . Then  $p_{ret}(a_r + t - a)$  is the pension amount received at retirement time by an individual of age  $a$  at time  $t$ . For instance,  $p_{ret}(s)$  can be a proportion  $\alpha_p$  of the average yearly income of an individual retiring at time  $s$ . Then

$$p_{ret}(s) = \alpha_p \frac{c_s(a_r)}{a_r - a_e}, \quad \text{with } c_t(a) = \int_{t-a+a_e}^{t-a+a_r} e^{\int_u^t r_s ds} \mathbf{e}_u(a+u-t) du. \quad (2.10)$$

$c_t(a)$  is the present value of wages earned by an individual's of age  $a$  at  $t$ .

One can also take instead the average income over the last  $h$  years before retirement, then  $p_{ret}(s) = \alpha_p \frac{\int_{s-h}^s e^{\int_u^s r_v dv} \mathbf{e}_u(a+u-t) du}{h}$ .

•  $(\lambda_t)$  is the indexation rate adjusting pension benefits. The indexation rate takes into account changes in prices or wages, using for example the Consumption Price Index. It can be

used to maintain the sustainability of the pension system in case of a demographic shock, such as in [2].

### 2.3 Buffer fund

In a “pure” PAYG scheme, the aggregate pension amount  $P_t$  to be paid at time  $t$  to current pensioners is only financed by the contributions of current workers  $C_t$ , inducing the budget constraint:

$$P_t = C_t, \quad \forall t \geq 0. \quad (2.11)$$

The budget constraint (2.11) ensures the sustainability/self-financing of the PAYG system. When the contribution rate  $\alpha_c$  is fixed, the adjustment coefficient  $\rho_t$  is automatically determined by

$$\rho_t = \frac{C_t}{P_t^{\min}} = \alpha_c \frac{\int_{a_e}^{a_r} \mathbf{e}_t(a) n(t, a) da}{\int_{a_r}^{\infty} p_t^{\min}(a) n(t, a) da},$$

and does not necessarily satisfy the adequacy constraint (2.7). The adequacy constraint yields a liquidity issue for the PAYG system, since the minimum pension amount  $P_t^{\min}$  is not covered by the workers contributions when  $P_t^{\min} \geq C_t$ .

In particular, the retirement income can be drastically reduced when there is a decrease in the working population  $N_t^w = \int_{a_e}^{a_r} n(t, a) da$ , resulting in a decrease in the social planner’s stream of income, or an increase of the retired population  $N_t^r = \int_{a_r}^{\infty} n(t, a) da$ , resulting in an increase of the pension amount  $P_t$ . This “demographic risk” can be induced by a fertility shock or an decrease over time of death rates for older ages, as currently observed in most countries (see e.g. [7] on the longevity risk). Pensioners also bear all the economic risk through the wage process ( $\mathbf{e}_t$ ). We thus consider a system where the social planner can invest (or borrow) at each time the amount  $C_t - P_t$  in (from) a buffer fund, hence sharing the demographic risk between the different generations.

**Financial market** As the demographic (and economic) risks are obviously not fully transferable to financial markets, we consider an incomplete Itô market, with short rate ( $r_t$ ) and  $d$  risky assets ( $d \leq n$ ). The price process  $S = (S^i)_{i=1, \dots, d}$  of the risky assets is defined by

$$dS_t^i = S_t^i \left( \mu_t^i dt + \sum_{j=1}^n \sigma_t^{i,j} dW_t^j \right), \quad i = 1, \dots, d. \quad (2.12)$$

The  $d \times n$  volatility matrix ( $\sigma_t$ ) is assumed of full rank (that is  $(\sigma_t^{tr} \sigma_t)$  is invertible, where  $^{tr} \sigma_t$  is the transposed matrix). The  $d$ -dimensional risk premium vector is denoted by  $(\eta_t)$ , where  $\eta_t = ^{tr} \sigma_t \cdot (\sigma_t^{tr} \sigma_t)^{-1} (\mu_t - r_t \mathbf{1}_d)$ .  $\eta_t$  is in the vector subspace  $\mathcal{R}_t = ^{tr} \sigma_t(\mathbb{R}^d)$  of  $\mathbb{R}^n$ . We assume that  $\int_0^T (|r_t| + \|\eta_t\|^2) dt < \infty$ , for any  $T > 0$ ,  $\mathbb{P}$ .a.s.

In what follows, for any  $\mathbb{R}^n$ -valued stochastic process  $(X_t)$ , we denote by  $X^{\mathcal{R}}$  the process such that for all  $t$ ,  $X_t^{\mathcal{R}}$  is the orthogonal projection of  $X_t$  onto  $\mathcal{R}_t$ . Similarly  $X_t^{\perp}$  denotes the orthogonal projection of  $X_t$  onto the orthogonal vector subspace  $\mathcal{R}_t^{\perp}$ .

Note that thanks to the weak assumptions made on the demographic, wage and financial processes, any kind of dependency structure with the financial market can be taken into account in the modeling.

**Investment in the buffer fund** On this (incomplete) market, the social planner manages a buffer fund that aims to absorb demographic and economic shocks. The dynamics of the fund can be considered as the financial wealth of a single agent with a labor income (or endowment)  $C_t = \alpha_c \int_{a_e}^{a_r} c_t(a)n(t,a)da$ , and consumption process  $P_t$ . The amount of money invested in the risky assets at time  $t$  is denoted by the  $d$ -dimensional vector  $\phi_t$ . Then the self-financing dynamics for the wealth  $F$  of the buffer fund with contribution  $C_t$  and pension  $P_t$  is

$$dF_t = F_t r_t dt + (C_t - P_t)dt + \phi_t \cdot \sigma_t \cdot (dW_t + \eta_t dt).$$

In order to ensure the sustainability of the pension system, a maximum amount of debt is imposed to the buffer fund:

$$(\text{Sustainability constraint}) \quad \forall t \geq 0, \quad F_t \geq \mathfrak{K}_t, \quad \text{a.s.}, \quad (2.13)$$

$\mathfrak{K}_t$  is an adapted predictable process, that can be negative, and  $(-\mathfrak{K}_t)$  corresponds to the social planner maximum amount of debt at time  $t$ . For instance,  $(-\mathfrak{K}_t)$  can represent a proportion of the Gross Domestic Product (GDP), while the no borrowing constraint corresponds to  $\mathfrak{K}_t \equiv 0$ . We assume throughout the paper that  $\mathfrak{K}$  is an Itô process:

$$d\mathfrak{K}_t = \mu_t^{\mathfrak{K}} dt + \delta_t^{\mathfrak{K}} \cdot dW_t. \quad (2.14)$$

**Social planner strategy** The strategy of the social planner (pensions and investment) is parametrized by  $(\pi, \rho)$  where  $P_t = \rho_t P_t^{\min}$  and  $\pi_t := {}^{tr} \sigma_t \phi_t \in \mathcal{R}_t$ . The dynamic of the buffer with strategy  $(\pi, \rho)$  is thus given by:

$$dF_t^{\pi, \rho} = F_t^{\pi, \rho} r_t dt + (C_t - \rho_t P_t^{\min})dt + \pi_t \cdot (dW_t + \eta_t dt). \quad (2.15)$$

Observe that the previous equation is not in a multiplicative form since the value of the fund  $F$  can be negative.

**Definition 2.1** (Admissible strategy). *An  $\mathbb{F}$ -adapted strategy  $(\pi, \rho)$  is said to be admissible if and only if*

- $\pi_t \in \mathcal{R}_t, \quad \forall t \geq 0 \quad \mathbb{P}$ -a.s.
- $\int_0^t (|C_s - \rho_s P_s^{\min}| + \|\pi_s\|^2) ds < \infty, \quad \forall t \geq 0 \quad \mathbb{P}$ -a.s.
- $\rho_t \geq 1$  (adequacy) and  $F_t^{\pi, \rho} \geq \mathfrak{K}_t$  (sustainability)  $\forall t \geq 0, \quad \mathbb{P}$ -a.s.

The set of all admissible strategies  $(\pi, \rho)$  is denoted  $\mathcal{A}$ .

Lastly, we introduce the class of the so-called state price density processes (taking into account the discount factor), also called discounted pricing kernels. The discounted pricing kernels  $Y$  are characterized by the property that for any admissible strategy  $(\pi, \rho)$ , the current wealth plus the aggregate pension minus the aggregate consumption, all discounted by  $Y$  (that is  $Y_t F_t^{\pi, \rho} + \int_0^t (\rho_s P_s^{\min} - C_s) Y_s ds$ ) is a local martingale (see [21] for financial and economic viewpoints on the discounted pricing kernels). Discounted pricing kernels are positive Itô-semimartingale with the following dynamics characterized by an orthogonal volatility component  $\nu_t \in \mathcal{R}_t^\perp$ .

**Definition 2.2** (State price density process/discounted pricing kernel). *A positive Itô semimartingale  $Y^\nu$  is called an admissible state price density process (or discounted pricing kernel) if it has the differential decomposition*

$$dY^\nu = Y^\nu[-r_t dt + (\nu_t - \eta_t) \cdot dW_t], \quad \nu_t \in \mathcal{R}_t^\perp, \quad Y_0^\nu = Y_0. \quad (2.16)$$

### 3. The social planner's dynamic decision criterion

The pension allocation and the investment in the buffer fund is decided by a social planner. Her decision are taken upon a preference criterion that should take into account present and future generations. Since the social planner has to aggregate the heterogeneous preferences of the different cohorts, her utility criteria is necessarily complex.

Moreover, due to the long-term characteristics of pension schemes, the decision criteria should be adapted to the non-stationary demographic, economic and financial environment in order to provide a consistent strategy in the long run. For both of these aspects, it has been shown that dynamic utility provides a flexible framework to handle this heterogeneity and to propose long-term policies coherent in time ([19, 20]).

#### 3.1 Definition of dynamic utilities

Dynamic utilities extend the standard notion of deterministic utilities, by allowing the utility criterion to be dynamically adjusted to the available information represented by the filtration  $(\mathcal{F}_t)_{t \geq 0}$ . A dynamic utility  $U$  is a then collection of random utility functions  $\{U(t, \omega, z)\}$  whose temporal evolution is “updated” in accordance with the new information  $(\mathcal{F}_t)$ , starting from an initial utility function  $u(z) = U(0, z)$  (which is deterministic if  $\mathcal{F}_0$  is trivial). Throughout the paper, we adopt the convention of small letters for deterministic utilities and capital letters for stochastic utilities. The specificity of this paper lies in the sustainability and adequacy constraints for the buffer fund and the pension process, which will be taken into account in the definition domain of dynamic utilities. In the following, we give a precise definition of this extension of dynamic utilities on stochastic domains.

**Definition 3.1** (Dynamic utility on stochastic domain).

A dynamic utility defined on a random domain  $U = (U(t, z, \omega))$  is a collection of random utility functions defined on a stochastic domain  $\mathcal{D}_U := \{(t, z, \omega), z \geq X_t(\omega)\}$  such that

- (i)  $(X_t)$  is an  $\mathcal{F}$ -adapted process.
- (ii) For all  $t \geq 0$ , for all  $z \geq X_t$ ,  $U(t, z)$  is  $\mathcal{F}_t$ -adapted.
- (iii) There exists  $N \in \mathcal{F}$  with  $\mathbb{P}(N) = 0$ , such that for any  $\omega \in N^c$ , and for any  $t \geq 0$ , the functions  $z \in [X_t(\omega), \infty[ \rightarrow U(t, z, \omega)$  are nonnegative, strictly concave increasing functions of class  $\mathcal{C}^2$  on  $]X_t(\omega), \infty[$ .
- (iv) Inada conditions:  $\lim_{z \rightarrow X_t(\omega)} U_z(t, z, \omega) = +\infty$  and  $\lim_{z \rightarrow +\infty} U_z(t, z, \omega) = 0$ ,  $\forall t \geq 0$ ,  $\mathbb{P}$ -a.s.

As for a standard utility function, the risk aversion coefficient  $R_A(U)$  is measured by the ratio  $R_A(U)(t, z) = -U_{zz}(t, z)/U_z(t, z)$  and the relative risk aversion by  $R_A^r(U)(t, z) = z R_A(U)(t, z)$ . Note that for dynamic utilities,  $R_A$  and  $R_A^r$  are random coefficients.

**Remark 3.2** Note that the Inada condition in  $X_t$  implies that the absolute risk aversion coefficient explodes in  $X_t$ :

$$\lim_{z \rightarrow X_t(\omega)} R_A(U)(t, z) = \lim_{z \rightarrow X_t(\omega)} -\frac{U_{zz}(t, z)}{U_z(t, z)} = +\infty, \quad \mathbb{P}\text{-a.s.} \quad (3.1)$$

Indeed, denoting  $f(t, z) := -\frac{U_{zz}(t, z)}{U_z(t, z)}$ , we have that  $U_z(t, z) = U_z(t, z_0) \exp(\int_{z_0}^z f(t, u) du)$  for

$z_0 > z$  greater than  $X_t$ . Then, for a fixed  $z_0$ , the Inada condition  $\lim_{z \rightarrow X_t(\omega)} U_z(t, z) = +\infty$  implies  $\lim_{z \rightarrow X_t(\omega)} \int_z^{z_0} f(t, u) du = +\infty$  and therefore  $\lim_{z \rightarrow X_t(\omega)} f(t, z) = +\infty$ .

The Fenchel-Legendre convex conjugate of a dynamic utility  $U$  defined on a stochastic domain  $\mathcal{D}_U := \{(\omega, t, z), z \geq X_t(\omega)\}$  is denoted by  $\tilde{U}$ , where  $\tilde{U}$  satisfies

$$\tilde{U}(t, y) = \sup_{z \geq X_t(\omega)} (U(t, z) - yz), \quad y \in \mathbb{R}^+.$$

In particular,  $\tilde{U}(t, y) \geq U(t, z) - yz$  and the maximum is attained at  $U_z(t, z) = y$ . Note that  $\tilde{U}$  is defined on  $\mathbb{R}^+ \times \mathbb{R}^+$  thanks to Inada conditions on  $U$ .  $\tilde{U}$  is twice continuously  $y$ -differentiable, strictly convex, strictly decreasing. Moreover, the marginal utility  $U_z$  verifies  $U_z^{-1}(t, y) = -\tilde{U}_y(t, y)$ ;  $\tilde{U}(t, y) = U(t, -\tilde{U}(t, y)) + \tilde{U}_y(t, y)y$ , and  $U(t, z) = \tilde{U}(t, U_z(t, z)) + zU_z(t, z)$ .

As this paper has an economic motivation, it is natural to study the particular example of CRRA utility functions, and to consider their dynamic versions.

**Dynamic CRRA utilities** Deterministic Constant Relative Risk Aversion (CRRA) utilities is the standard framework in the economic literature, as explained in [46]. They belong to the class of deterministic Hyperbolic Absolute Risk Aversion (HARA) utility (see e.g. the seminal paper of Merton [35], or Kingston and Thorp [31]) characterized by an hyperbolic absolute risk aversion coefficient:

$$R_A(u)(z) = -\frac{u_{zz}(z)}{u_z(z)} = \frac{1}{az + b}, \quad \text{with } a > 1 \text{ and } b \in \mathbb{R}. \quad (3.2)$$

Integrating (3.2) gives that  $u(z) = \frac{(z+b/a)^{1-\theta}}{1-\theta}$ , defined for  $z > -b/a$ , and where  $\theta = 1/a$ . The case  $b = 0$  corresponds to CRRA utility (also called power utilities) with constant relative risk aversion  $\theta$ . Hereafter, we extend the notion of deterministic CRRA utilities to dynamic ones, still with deterministic  $\theta$ . As it is well known, CRRA dynamic utilities are necessarily time-separable.<sup>1</sup>

**Definition 3.3** *Dynamic CRRA utility  $U^{(\theta)}(t, z)$ , with  $\theta \in ]0, 1[$ , are defined by*

$$U^{(\theta)}(t, z) := Z_t^u \frac{(z - X_t)^{1-\theta}}{1-\theta}, \quad \text{for } z \geq X_t, \quad (3.3)$$

where  $X_t$  is a stochastic process and  $Z_t^u$  is a positive stochastic coefficient reflecting the random evolution of the time preferences.

For instance  $X_t$  can represent a borrowing constraint of the social planner. Dynamic CRRA utilities satisfy Inada conditions :

$$\lim_{z \rightarrow \infty} U_z^{(\theta)}(t, z) = 0 \text{ and } \lim_{z \rightarrow X_t(\omega)} U_z^{(\theta)}(t, z) = \infty, \quad \forall t, \mathbb{P}\text{-a.s.}$$

CRRA dynamic utilities are also studied in a stochastic factor framework in [33], where the time factor  $Z^u$  is solution of an ergodic BSDE. In the dynamic setting, the time factor  $Z^u$  is more general than a time preference coefficient, in particular its diffusion coefficient will impact the optimal strategies (cf. Section 5). Indeed the dynamics of  $Z^u$  is determined by the consistency condition (in the sense of Definition 3.7).

<sup>1</sup> Indeed, assuming  $-\frac{zU_z^{(\theta)}(t, z)}{U_z^{(\theta)}(t, z)} = \theta \in ]0, 1[$  implies  $\log(U_z^{(\theta)}(t, z)) = -\theta \log(z) + c_t$  for  $c$  an  $\mathbb{F}$ -adapted process, thus  $U^{(\theta)}(t, z) = \frac{1-\theta}{1-\theta} Z_t^u$  with  $Z^u = e^c$ .

An advantage of CRRA dynamic utilities is that they are particularly tractable and lead to explicit solutions. Other classes of dynamic utilities may also be considered, such as exponential (see [40]), time-monotone (see [41]), or mixture of dynamic utilities (see [37]). Nevertheless, more general classes are less tractable and lead to less explicit results (especially concerning the consistency condition), one could then rely on numerical tools to exhibit the optimal solutions.

### 3.2 Buffer fund and pensioners dynamic utilities

The aim of the social planner is to optimize the pension of present and future pensioners, and thus has to take into account an aggregation of cohorts. The buffer fund  $F$  plays the role of a risk sharing mechanism between generations. The greater are the part of the contribution  $C$  invested in the fund, the more reserve are available to pay future pensions and to face unpredictable future demographic and economic risks. On the other hand, an adequate level of pension should be paid to current retirees. This tradeoff is summarized by the social planner's preference process defined as  $U(t, F_t) + \int_0^t V(s, \rho_s) ds$ .  $U$  is the buffer fund dynamic utility (representing future generations), and  $V$  is the aggregate dynamic utility of the pensioners.

The sustainability constraint (1.13) and the adequacy constraint (1.7) are translated into stochastic domains for both  $U$  and  $V$ . The supermartingale property induced by the dynamic programming principle will be translated into condition on local characteristics of  $U$ . This explains why we assume stronger regularity conditions on  $U$  than on  $V$ .

**Definition 3.4** (Buffer fund utility). *The buffer fund utility  $U$  is a dynamic utility with domain  $\mathcal{D}_U = \{(\omega, t, z), z \geq \mathfrak{R}_t\}$ , where  $(\mathfrak{R}_t)$  is the sustainability bound satisfying (2.14) and with  $U(t, \cdot)$  of class  $\mathcal{C}^{3,\delta}$ , that is of class  $\mathcal{C}^3$  with  $U_{zzz}$   $\delta$ -Hölder,  $\delta \in ]0, 1[$ .*

The preference process of a pensioner of age  $a$  at time  $t$ , is defined by  $\bar{v}(t, a, p_t(a))$ , with  $p_t(a)$  then pension amount, and  $\bar{v}$  a dynamic utility depending of the pensioner's age and taking into account uncertain future changes in the pensioners' preferences.

The social planner aggregates preferences of all pensioners to obtain the aggregated pensioners' dynamic utility defined by  $\int_{a_r}^{\infty} \bar{v}(t, a, p_t(a)) \omega_t(a) n(t, a) da$ , with  $\omega_t(a)$  the weight given to a pensioner of age  $a$  at time  $t$ . For instance, the social planner can take into account the actuarial fairness by giving more weight to pensioners who contributed more.

Recalling that  $p_t(a) = p_t^{\min}(a) \rho_t$ , we can define a weighted dynamic utility  $v$  applied to  $\rho_t$ , with

$$v(t, a, \rho) = \omega_t(a) \bar{v}(t, a, p_t^{\min}(a) \rho). \quad (3.4)$$

In the following, we refer to  $v$  as a pensioner dynamics utility from the viewpoint of the social planner.

**Definition 3.5** (Pensioners' utility). *We assume that a pensioner at time  $t$  and of age  $a$  has the dynamic utility  $v(t, a, \cdot)$ , defined as in (3.4) on a domain  $[\underline{\rho}_t, +\infty[$ , with  $\underline{\rho}_t \leq 1$ . The aggregated utility of pensioners is defined by*

$$V(t, \rho) = \int_{a_r}^{\infty} v(t, a, \rho) n(t, a) da. \quad (3.5)$$

**Example 3** We come back to the case where each pensioner receives the same pension amount  $p_t = \rho_t p_t^{\min} \geq p_t^{\min}$  and has the same dynamic utility  $\bar{v}(t, p)$ . In order to take into account actuarial fairness, the social planner can aggregate preferences of all living pensioners by weighting their utility by their past contributions  $\alpha_c c_t(a)$  (in the same spirit of [24]), introduced

in (2.10). Then,

$$v(t, a, \rho) = \alpha_c c_t(a) \bar{v}(t, \rho p_t^{\min}),$$

and the aggregate utility from pensions, taking into account the weight of each pensioners' generation, is

$$V(t, \rho) = \int_{a_r}^{\infty} \bar{v}(t, \rho p_t^{\min}) \alpha_c c_t(a) n(t, a) da = \omega_t^r \bar{v}(t, \rho p_t^{\min}), \quad (3.6)$$

with the total weight of all pensioners living at time  $t$  given by

$$\omega_t^r = \alpha_c \int_{a_r}^{\infty} c_t(a) n(t, a) da. \quad (3.7)$$

**Example 4** In the second example of age-dependent pension,  $p_t(a) = p_t^{\min}(a) \rho_t$  with  $p_t^{\min}(a) = p_{ret}(a_r + t - a) e^{\int_{a_r+t-a}^t \lambda_s ds}$ . We further assume that the pensioners utility  $\bar{v}(t, a, p)$  may depend on their age, so that  $v(s, a, \rho) = \bar{v}(s, a, \rho p_s^{\min}(a))$ . Then the actuarial fairness criteria is taken into account via the choice of initial pension amount  $p_{ret}(s)$ , which increases with the past contribution of the individual retiring at time  $s$ . Thus, the social planner may aggregate the pensioners' utility without using any correcting weight. Then

$$V(t, \rho) = \int_{a_r}^{\infty} \bar{v}(t, a, \rho p_{ret}(a_r + t - a) e^{\int_{a_r+t-a}^t \lambda_u du}) n(t, a) da. \quad (3.8)$$

We introduce the following assumption on the rates of increase of the marginal dual utilities  $\tilde{V}_\rho$  and  $\tilde{U}_z$ , that will play a role in proving the existence of an admissible optimal portfolio in Theorem 4.5.

**Assumption 3.6** We assume the existence of a locally integrable process  $B$  such that

$$|\tilde{V}_\rho(t, z) - \tilde{V}_\rho(t, z')| \leq B_t |\tilde{U}_z(t, z) - \tilde{U}_z(t, z')|, \quad \text{for } t \geq 0, z > 0, z' > 0. \quad (3.9)$$

Note that the relative weight of the buffer fund utility  $U$  with respect to the pensioners' utility  $V$  will play a decisive role. Informally, if the utility of the fund  $U(t, F_t)$  is small compared to the pensioners' aggregate utility  $V(t, \rho_t)$ , the social planner takes more risks regarding the pension system sustainability (see Section 5 for more details in the CRRA framework).

### 3.3 Consistency Property

**Social planner optimization problem** The social planner has to manage a tradeoff between the pension payed to the pensioners and the fund that constitutes reserves for the future generations, among all the admissible strategies satisfying the sustainability and adequacy. In the usual setting, the optimization program is posed backward. It is formulated on a given horizon  $T_H$ , and is written at time  $t = 0$  as follows (given  $F_0 = x$ ):

$$\mathcal{U}(0, x) := \sup_{(\pi, \rho) \in \mathcal{A}} \mathbb{E} \left( u(T_H, F_{T_H}^{\pi, \rho}) + \int_0^{T_H} V(t, \rho_t) dt \right). \quad (3.10)$$

In the backward formulation, the utilities  $u$  of terminal wealth (at  $T_H$ ) and  $V$  of pension rate are given. In our context of intergenerational risk-sharing for pensions, fixing a (long-term) time-horizon  $T_H$  and even more a utility function  $u(T_H, \cdot)$  seems artificial. Extending the optimization program and the optimal strategy to a horizon larger to  $T_H$ , in a time-consistent way, is also a difficult issue. In order to ensure consistency across time and generations, the social

planner should be able to identify which “terminal” criterion  $U(T, \cdot)$  should be considered at any intermediate date  $T \leq T_H$ , while still leading to the same optimal strategy and the same value  $\mathcal{U}(0, x)$ , that is satisfying

$$\text{for any } T \leq T_H, \mathcal{U}(0, x) = \sup_{(\pi, \rho) \in \mathcal{A}} \mathbb{E} \left( U(T, F_T^{\pi, \rho}) + \int_0^T V(t, \rho_t) dt \right).$$

Under regularity assumptions, this criterion is given by the “value function”  $\mathcal{U}(T, z)$  given the wealth  $F_T = z$  at time  $T$

$$\mathcal{U}(T, z) = \text{“sup”} \mathbb{E} \left( u(T_H, F_{T_H}^{\pi, \rho}(T, z)) + \int_T^{T_H} V(s, \rho_s) ds | F_T = z \right), \text{ a.s.} \quad (3.11)$$

This time-consistency translates into a martingale property of the preference process  $\mathcal{U}(t, F_t^{\pi, \rho}) + \int_0^t V(s, \rho_s) ds$  along the optimal strategy. This property, known as the dynamic programming principle, is the main tool in the theory of stochastic control, see Davis [13] or El Karoui [18]. In this backward setting,  $\mathcal{U}(T_H, \cdot) = u(T_H, \cdot)$  is given, and the unknown is the optimal strategy  $(F^*, \rho^*)$  as well as  $\mathcal{U}(t, \cdot)$ , also called “indirect” utility, possibly stochastic. Nevertheless,  $\mathcal{U}$  is difficult to compute (even if  $u(T_H, \cdot)$  is given as a simple deterministic function), and it is even not trivial to prove that  $\mathcal{U}$  defined by (3.11) is indeed concave.

In the forward setting, there is no intrinsic time-horizon  $T_H$  and this is the initial utility  $\mathcal{U}(0, \cdot)$  which is given. This means that forward utilities differ from backward utilities by their boundary conditions, both satisfying a dynamic programming principle, also called consistency given the constraints set  $\mathcal{A}$ .

**Consistency and optimal strategy** The satisfaction provided by an admissible strategy  $(\pi, \rho) \in \mathcal{A}$  is measured by the dynamic criterion  $U(t, F_t^{\pi, \rho}) + \int_0^t V(s, \rho_s) ds$  that is assumed to satisfy a dynamic programming principle.

**Definition 3.7** (Consistent dynamic utility). *Let  $(U, V)$  be a dynamic utility system with admissible strategies set  $\mathcal{A}$ . The utility system  $(U, V)$  is said to be consistent, if*

(i) *For any admissible strategies  $(\pi, \rho) \in \mathcal{A}$ , the preference process  $(U(t, F_t^{\pi, \rho}) + \int_0^t V(s, \rho_s) ds)$  is a non-negative supermartingale.*

(ii) *There exists an optimal strategy  $(\pi^*, \rho^*) \in \mathcal{A}$ , binding the constraints, in the sense that the optimal preference process  $(U(t, F_t^{\pi^*, \rho^*}) + \int_0^t V(s, \rho_s^*) ds)$  is a martingale.*

Under regularity assumptions, the value function  $(\mathcal{U}(t, z), V(t, \rho))$  of the classical optimization problem is an example of consistent utility system, defined from its terminal condition  $\mathcal{U}(T_H, z) = u(z)$  (see [20] and [21] for a general discussion between the forward and the backward viewpoints of utility functions).

### 3.4 Semimartingale dynamic utility

In order to study the preference process  $(U(t, F_t^{\pi, \rho}) + \int_0^t V(s, \rho_s) ds)$ , we assume in the following that the buffer fund dynamic utility  $U$  defined in 3.4 is an Itô random field:

$$dU(t, z) = \beta(t, z) dt + \gamma(t, z) dW_t, \quad z \geq \mathfrak{R}_t, \quad (3.12)$$

with  $\beta(t, z)$  the drift random field and  $\gamma(t, z)$  the multivariate diffusion random field. Since the domain of the buffer fund utility  $U$  is time varying, its dynamics is defined more precisely by

introducing the shifted utility  $\bar{U}$  with fixed domain  $\mathbb{R}^+ \times \mathbb{R}^+$ :

$$\bar{U}(t, z) := U(t, z + \mathfrak{R}_t), \quad \forall (t, z) \in \mathbb{R}^+ \times \mathbb{R}^+, \quad (3.13)$$

and with local characteristics denoted by  $(\bar{\beta}, \bar{\gamma})$ . Obviously  $U$  is a dynamic utility on  $\mathcal{D}_U$  if and only if  $\bar{U}$  is a dynamic utility on  $\mathbb{R}^+ \times \mathbb{R}^+$ . These semimartingale dynamic utilities have been studied in details in [23]. An important part is the connection between the regularity of the dynamic utility  $\bar{U}$  and that of its local characteristics  $(\bar{\beta}, \bar{\gamma})$ . If  $\bar{U}$  is of class  $\mathcal{C}^{2,\delta}$  then its characteristics  $\bar{\beta}$  and  $\bar{\gamma}$  are of class  $\mathcal{C}^{2,\varepsilon}$  for all  $0 < \varepsilon < \delta$ . Conversely if  $\bar{\beta}$  and  $\bar{\gamma}$  are in the class  $\mathcal{C}^{2,\delta}$  then  $U$  is in  $\mathcal{C}^{2,\varepsilon}$  for all  $0 < \varepsilon < \delta$  ([23, 32]). Another part relies on determining conditions on  $(\bar{\beta}, \bar{\gamma})$  to ensure that  $\bar{U}$  is a dynamic utility. Indeed, in the absence of general comparison results for stochastic integrals, it is not straightforward to obtain conditions on the local characteristics  $(\bar{\beta}, \bar{\gamma})$  such that the process  $\bar{U}(t, z) = \bar{U}(t, 0) + \int_0^t \bar{\beta}(t, s) ds + \bar{\gamma}(s, z) \cdot dW_s$  is increasing and concave. The sufficient assumptions recalled below are useful.

**Assumption 3.8** *Let  $\bar{U}(t, z) = U(t, z + \mathfrak{R}_t)$  the shifted buffer fund utility. We assume that there exists random bounds  $(B_t^1)$  and  $(\zeta_t)$  such that a.s.  $\int_0^T B_t^1 dt < \infty$  and  $\int_0^T \zeta_t^2 dt < \infty$  for any  $T$ , and such that for any  $z > 0$ :*

$$\begin{cases} |\bar{\beta}_z(t, z)| \leq B_t^1 \bar{U}_z(t, z), & |\bar{\beta}_{zz}(t, z)| \leq B_t^1 |\bar{U}_{zz}(t, z)|, \\ \|\bar{\gamma}_z(t, z)\| \leq \zeta_t \bar{U}_z(t, z), & \|\bar{\gamma}_{zz}(t, z)\| \leq \zeta_t |\bar{U}_{zz}(t, z)|. \end{cases} \quad (3.14)$$

$$(3.15)$$

Under Assumption 3.8,  $\bar{U}$  is a well-defined dynamic utility by Corollary 1.3 in [23].

**Itô-Wentzell's formula** The link between the local characteristics  $(\beta, \gamma)$  of  $U$  and  $(\bar{\beta}, \bar{\gamma})$  of  $\bar{U}$  are deduced from the Itô-Wentzell formula, which is a generalization of the Itô formula in the case where the function is itself a random field.

The Itô-Wentzell formula gives the decomposition of a compound random field  $U(t, X_t)$  for  $U(t, z) = u(0, z) + \int_0^t \beta(s, z) ds + \int_0^t \gamma(s, z) \cdot dW_s$  regular enough (of class  $\mathcal{C}^{2,\delta}$ , with  $\delta \in ]0, 1[$ ) and any Itô semimartingale  $X$ . This decomposition is the sum of three terms: the first one is the “differential in  $t$ ” of  $U$ , the second one is the classic Itô's formula (without differentiation in time) and the third one is the infinitesimal covariation between the martingale part of  $U_z$  and the martingale part of  $X$ , all these terms being taken in  $X_t$ .

$$\begin{aligned} dU(t, X_t) &= (\beta(t, X_t) dt + \gamma(t, X_t) dW_t) \\ &+ (U_z(t, X_t) dX_t + \frac{1}{2} U_{zz}(t, X_t) d\langle X, X \rangle_t) + (\langle \gamma_z(t, X_t) dW_t, dX_t \rangle). \end{aligned} \quad (3.16)$$

Applying the Itô-Wentzell formula to  $\bar{U}(t, z - \mathfrak{R}_t)$  yields the following result.

**Proposition 3.9** *Recall that the bound  $\mathfrak{R}_t$  is an Itô process with dynamics  $d\mathfrak{R}_t = \mu_t^{\mathfrak{R}} dt + \delta_t^{\mathfrak{R}} \cdot dW_t$ . Under Assumption 3.8,  $U$  is a dynamic utility on the domain  $\mathcal{D}_U = \{(\omega, t, z), z \geq \mathfrak{R}_t\}$ , with local characteristics*

$$\begin{cases} \beta(t, z) = \bar{\beta}(t, z - \mathfrak{R}_t) - U_z(t, z) \mu_t^{\mathfrak{R}} - \frac{1}{2} U_{zz}(t, z) \|\delta_t^{\mathfrak{R}}\|^2 - \gamma_z(t, z) \cdot \delta_t^{\mathfrak{R}}, \\ \gamma(t, z) = \bar{\gamma}(t, z - \mathfrak{R}_t) - U_z(t, z) \delta_t^{\mathfrak{R}}, \quad \forall z \geq \mathfrak{R}_t. \end{cases} \quad (3.17)$$

$$(3.18)$$

## 4. Optimal PAYG pension policies

Let us recall that a pensioner of age  $a$  at time  $t$  receives pension  $p(t, a) = p_t^{\min}(a) \rho_t \geq p_t^{\min}(a)$ . The buffer fund in which the social planner can borrow/invest has the dynamics (2.14)

given by:

$$dF_t^{\pi, \rho} = F_t^{\pi, \rho} r_t dt + (C_t - \rho_t P_t^{\min}) dt + \pi_t \cdot (dW_t + \eta_t dt),$$

with  $(C_t)$  the contribution process,  $(P_t) = (\rho_t P_t^{\min})$  the pension process introduced in (2.8), and  $(\pi_t)$  the investment strategy in the incomplete market. The aggregated utility of pensioners is given by  $V(t, \rho) = \int_{a_r}^{\infty} v(t, a, \rho) n(t, a) da$  (see (3.5) and subsequent examples).

The first aim is to characterize the buffer fund utility  $U$  of the social planner, such that the preference criteria  $(U, V)$  is consistent, and then to determine the admissible strategy  $(\pi, \rho) \in \mathcal{A}$  optimizing the dynamic criterion  $U(t, F_t^{\pi, \rho}) + \int_0^t V(s, \rho_s) ds$ .

#### 4.1 Consistency SPDE

Itô-Wentzell's formula allows us to transform the supermartingale property implied by the consistency condition into conditions on the differential characteristics of the utility process  $U$ . For standard deterministic utility functions, the infinitesimal counterpart of the dynamic programming principle is a nonlinear Partial Differential Equation (PDE), called dynamic programming equation or Hamilton-Jacobi-Bellman (HJB) equation. In the framework of dynamic utility, the consistency characterization is given in terms of an HJB Stochastic Partial Differential Equation (SPDE), as detailed in [39] and [20, 23]. The presence of pensions to be paid impacts this SPDE in a non-linear way, the non-linear factor involving the utility of pensioners  $V$ . Note that the utility  $U$  and  $V$  are not of the same nature: the consistency is conveyed by  $U$ , which requires then stronger regularity conditions on  $U$  than on  $V$ . This HJB-SPDE provides a constraint on the drift  $\beta$  of  $U$ , as explained below.

**Candidate to be the optimal strategy** Applying Itô-Wentzell's formula to the preference process  $Z_t^{\pi, \rho} := \int_0^t V(s, \rho_s) ds + U(t, F_t^{\pi, \rho})$  with yields

$$\begin{aligned} dZ_t^{\pi, \rho} &= (\beta(t, F_t^{\pi, \rho}) + U_z(t, F_t^{\pi, \rho})(F_t^{\pi, \rho} r_t + C_t)) dt \\ &\quad + (\gamma(t, F_t^{\pi, \rho}) + U_z(t, F_t^{\pi, \rho}) \pi_t) \cdot dW_t \\ &\quad + (\mathcal{P}(t, F_t^{\pi, \rho}, \rho_t) + \mathcal{Q}(t, F_t^{\pi, \rho}, \pi_t)) dt. \end{aligned}$$

with

$$\mathcal{P}(t, z, \rho) := V(t, \rho) - U_z(t, z) P_t^{\min} \rho, \quad (4.1)$$

$$\mathcal{Q}(t, z, \pi) := \frac{1}{2} U_{zz}(t, z) \|\pi_t\|^2 + \pi_t \cdot (\gamma_z(t, z) + U_z(t, z) \eta_t). \quad (4.2)$$

A natural candidate for optimal policy  $(\pi^*, \rho^*)$  are processes which maximize the drift of the preference process  $Z^{\pi, \rho}$ . Thus,  $\pi^*$  should maximize  $\mathcal{Q}(t, z, \cdot)$  and  $\rho^*$  should maximize  $\mathcal{P}(t, z, \cdot)$  on  $[1, +\infty]$ , leading to

$$\pi_t^*(F_t^*) = -\frac{\gamma_z^{\mathcal{R}}(t, F_t^*) + U_z(t, F_t^*) \eta_t}{U_{zz}(t, F_t^*)} \quad \text{and} \quad \rho_t^*(F_t^*) = V_\rho^{-1}(t, P_t^{\min} U_z(t, F_t^*)) \vee 1. \quad (4.3)$$

Note that  $\mathcal{Q}(t, z, \pi_t^*) = -\frac{1}{2} U_{zz}(t, z) \|\pi_t^*\|^2$ .

With a slight abuse of notation, we use interchangeably  $\pi_t^*$  (resp.  $\rho_t^*$ ) or  $\pi_t^*(F_t^*)$  (resp.  $\rho_t^*(F_t^*)$ ).

To alleviate the notations, we introduce the optimal pension process without minimum guarantee, denoted  $\rho^f$  (free constraints).

**Definition 4.1** The maximizer of the operator  $\mathcal{P}(t, z, \rho) := V(t, \rho) - U_z(t, z)P_t^{\min}\rho$  for  $\rho \in \mathbb{R}^+$  is denoted

$$\rho_t^f(z) = V_\rho^{-1}(t, P_t^{\min}U_z(t, z)). \quad (4.4)$$

Remark that  $\mathcal{P}(t, z, \rho_t^f(z)) = \tilde{V}(t, P_t^{\min}U_z(t, z))$ .

**Remark 4.2** Note that if  $U$  satisfies the Inada condition at  $\mathfrak{K}$ , then  $v$  also satisfy the Inada condition at  $\underline{\rho}$ , in order to ensure that the optimal pension is well defined (namely the quantity  $V_\rho^{-1}(t, P_t^{\min}U_z(t, z))$ ). Remark also that Inada conditions for  $U$  and  $v$  at  $+\infty$  can be relaxed into the following condition  $\lim_{\rho \rightarrow +\infty} V_\rho(t, \rho) \leq P_t^{\min} \lim_{z \rightarrow +\infty} U_z(t, z)$ .

**Consistency condition on the drift  $\beta$  of  $U$**  If the candidate  $(\pi^*, \rho^*)$  is indeed the optimal strategy, then to satisfy the time consistency, the drift of the preference process  $Z^{\pi, \rho}$  should be nonpositive for any admissible strategy  $(\pi, \rho)$  and equal to zero for the optimal strategy  $(\pi^*, \rho^*)$ . This leads to the following sufficient condition on the drift of  $U$

$$\begin{aligned} \beta(t, z) &= -U_z(t, z)(zr_t + C_t) - \mathcal{Q}(t, z, \pi^*) - \mathcal{P}(t, z, \rho_t^*(z)) \\ &= -U_z(t, z)(zr_t + C_t - P_t^{\min}\rho^*) + \frac{1}{2}U_{zz}(t, z)\|\pi_t^*(z)\|^2 - V(t, \rho_t^*(z)). \end{aligned} \quad (4.5)$$

## 4.2 Main results

We gather here the main results the paper, proofs and examples being postponed in the next subsections. The first below result shows that under the consistency HJB condition (4.5), the bound  $\mathfrak{K}$  shifting the utility  $U$  is necessarily a buffer fund. This is an interesting new result.

**Theorem 4.3** Let  $U$  be the buffer fund utility introduced in 3.4 and verifying Assumption 3.8, and  $V$  the aggregated pensioners' utility, verifying Assumption 3.6. Assume that the drift  $\beta$  of  $U$  satisfies the HJB constraint

$$\beta(t, z) = -U_z(t, z)(zr_t + C_t - P_t^{\min}\rho_t^*(z)) + \frac{1}{2}U_{zz}(t, z)\|\pi_t^*(z)\|^2 - V(t, \rho_t^*(z)).$$

Then the sustainability bound  $\mathfrak{K}$  is necessarily a buffer fund receiving the contribution  $C$  and paying the minimal pension amount  $P^{\min}$ , that is:

$$d\mathfrak{K}_t = (\mathfrak{K}_t r_t + C_t - P_t^{\min})dt + \delta_t^{\mathfrak{K}} \cdot (dW_t + \eta_t dt), \quad \delta^{\mathfrak{K}} \in \mathcal{R}. \quad (4.6)$$

The proof is postponed to Section 4.3.

**Remark 4.4** Theorem 4.3 states that if the utility system  $(U, V)$  is consistent, then the shift  $\mathfrak{K}_t$  of the utility  $U$  is necessarily a buffer fund with minimal pensions. Nevertheless, in the original problem formulation and Definition 3.4 of  $U$ ,  $\mathfrak{K}_t$  is not assumed to be a buffer fund itself: for instance if  $\mathfrak{K}$  is an index following the GDP, there is no reason why it will follow dynamics (4.6). Therefore, to satisfy the consistency constraint, the dynamic utility should be shifted not with the sustainability constraint  $\mathfrak{K}_t$ , itself but with a buffer fund process  $(X_t)$  that super-replicates (in a pathwise way)  $(\mathfrak{K}_t)$  that is:  $X_t \geq \mathfrak{K}_t$ , for all  $t$ ,  $\mathbb{P}$  a.s. This means that the sustainability constraint is transformed into a stronger one:  $F_t \geq X_t (\geq \mathfrak{K}_t)$ , for all  $t$ ,  $\mathbb{P}$  a.s.. The problem is equivalent to searching for a self-financing portfolio (without contributions and pensions)  $X'$  super-replicating (pathwise) the process  $B_t := \mathfrak{K}_t + \int_0^t (P_s^{\min} - C_s)ds$ . For example,

it may be relevant to choose the “minimal” super-replicating self-financing portfolio  $(X'_t)$  (if it exists) in the following sense:

$$X_0^* := \inf \left\{ X_0 \geq \mathfrak{K}_0 \quad \text{s.t. } \exists \pi' \in \mathcal{R} \quad \text{satisfying} \right. \\ \left. X'_t := X_0 + \int_0^t r_s X'_s ds + \pi'_s \cdot (dW_s + \eta_s ds) \geq B_t, \quad dt \otimes d\mathbb{P} \text{ a.s.} \right\}$$

The existence of a super-replicating self-financing portfolio is not guaranteed, especially in our context of incomplete market, in which demographic risk can not be completely hedged by financial assets. Applying Theorem 5.12 of Karatzas and Kou [30], a sufficient existence condition on  $[0, T]$  is

$$\sup_{\nu \in \mathcal{R}^\perp} \sup_{\tau \in \mathcal{T}_{[0, T]}} \mathbb{E} \left( Y_\tau^\nu (\mathfrak{K}_\tau + \int_0^\tau (P_s^{\min} - C_s) ds)^+ \right) < \infty, \quad \forall T \geq 0, \quad (4.7)$$

where  $\mathcal{T}_{[0, T]}$  is the class of stopping times  $\tau$  with values in the interval  $[0, T]$  and  $Y^\nu$  the state price density process (2.16). Note that this supremum corresponds to  $X_0^*$  which is the super-replicating price of  $(B_t)$ .

In the backward framework, El Karoui and Jeanblanc [22], or He and Pagès [28] consider a complete market, which ensures the existence of a super-replicating portfolio. In incomplete market, an analogous assumption as (4.7) is needed in Mostovyi and Sirbu [36] (Assumption 2.5), to ensure the existence of a super-replicating portfolio (see Lemma 3.2 [36]).

The following theorem shows that the policy given by (4.3) is indeed the optimal strategy, and that the corresponding buffer fund satisfies the sustainability constraint (2.13).

**Theorem 4.5** *Let  $U$  be the buffer fund utility verifying Assumptions 3.8, and  $V$  be the aggregated pensioners' utility, verifying Assumption 3.6. Assume  $F_0 > \mathfrak{K}_0$  and that the drift  $\beta$  of  $U$  satisfies the HJB constraint (4.5):*

$$\beta(t, z) = -U_z(t, z)(zr_t + C_t - P_t^{\min} \rho_t^*(z)) + \frac{1}{2} U_{zz}(t, z) \|\pi_t^*(z)\|^2 - V(t, \rho_t^*(z)),$$

$$\text{with } \begin{cases} \pi_t^*(z) = -\frac{\gamma_z^{\mathcal{R}}(t, z) + U_z(t, z)\eta_t}{U_{zz}(t, z)}, & (4.8) \\ \rho_t^*(z) = V_\rho^{-1}(t, P_t^{\min} U_z(t, z)) \vee 1. & (4.9) \end{cases}$$

Then the portfolio/pension plan  $(\pi^*, \rho^*)$  is the optimal strategy. In particular, the buffer fund  $F^*$  following the investment strategy  $\pi_t^* = \pi_t^*(F_t^*)$  and paying the pension amount  $\rho_t^* = \rho_t^*(F_t^*)$  is strictly greater than  $\mathfrak{K}_t$  and satisfies the dynamics

$$\begin{cases} dF_t^* = \mu_t^*(F_t^*) dt + \pi_t^*(F_t^*) dW_t, \\ \mu_t^*(z) := zr_t + C_t - P_t^{\min} \rho_t^*(z) + \pi_t^*(z)\eta_t. \end{cases} \quad (4.10)$$

The proof is postponed to Section 3.4.

Since  $V_\rho^{-1}(t, \cdot)$  and  $U_z(t, \cdot)$  are decreasing functionals, the optimal pension  $p_t^*(a) = \rho_t^* p_t^{\min}(a)$  is increasing in the fund's wealth, which is natural. In particular, if the number of workers and/or the wage amount decreases, the contribution  $C_t$  can become lower to the pension amount  $P_t^*$ , which in turn can lead to a decrease in the optimal pension's value. On the other hand, when minimum pension amount  $P_t^{\min}$  to be paid increases (for instance due to a decrease of death rates at older ages),  $p_t^*(a)$  decreases.

The optimal investment strategy depends on the market risk premium vector  $(\eta_t)$  which can be correlated to wages and demographic rates. The optimal investment strategy also depends on the inverse risk aversion  $-\frac{U_z(t, F_t^*)}{U_{zz}(t, F_t^*)}$  and on the (buffer fund) utility diffusion coefficient through the term  $\gamma_z^{\mathcal{R}}(t, F_t^*)$ . Besides,  $\gamma_z^{\mathcal{R}}(t, F_t^*)$  also depends on the financial, economic and demographic parameters, due to the social planner's preferences and the HJB consistency constraint (4.5). We refer to Section 5 for details on a specific example.

**Corollary 4.6** *Under the assumptions and notations of Theorem 4.5,*

$$\lim_{z \rightarrow \bar{\mathfrak{R}}_t} \pi_t^*(z) = \delta_t^{\bar{\mathfrak{R}}}, \quad \lim_{z \rightarrow \bar{\mathfrak{R}}_t} \rho_t^*(z) = 1, \quad \forall t \geq 0, \quad \mathbb{P}\text{-a.s.} \quad (4.11)$$

*This means that the optimal strategy converges to the “minimal” buffer fund strategy when  $F^*$  tends to the sustainability bound.*

**Proof** First, using the expression of the optimal portfolio (4.8),

$$\begin{aligned} \|\pi_t^*(z) - \delta_t^{\bar{\mathfrak{R}}}\| &= \left\| \frac{\gamma_z^{\mathcal{R}}(t, z) + U_z(t, z)\eta_t}{U_{zz}(t, z)} + \delta_t^{\bar{\mathfrak{R}}}\right\| \\ &\leq \left\| \frac{\gamma_z^{\mathcal{R}}(t, z) + U_{zz}(t, z)\delta_t^{\bar{\mathfrak{R}}}}{U_{zz}(t, z)} \right\| + \|\eta_t\| \frac{|U_z(t, z)|}{|U_{zz}(t, z)|} \\ &\stackrel{(2.15)}{\leq} (\zeta_t + \|\eta_t\|) \frac{|U_z(t, z)|}{|U_{zz}(t, z)|} \xrightarrow{z \rightarrow \bar{\mathfrak{R}}_t} 0, \end{aligned}$$

since  $\lim_{z \rightarrow \bar{\mathfrak{R}}_t} \frac{U_{zz}(t, z)}{U_z(t, z)} = -\infty$  from Remark 3.2. Finally, by the Inada condition on  $U$  and  $V$ ,

$$\lim_{z \rightarrow \bar{\mathfrak{R}}_t} \rho_t^*(z) \vee 1 = \lim_{z \rightarrow \bar{\mathfrak{R}}_t} V_\rho^{-1}(t, P_t^{\min} U_z(t, z)) \vee 1 = \lim_{\rho \rightarrow +\infty} V_\rho^{-1}(t, \rho) \vee 1 = \underline{\rho}_t \vee 1 = 1.$$

□

**Example 1** We come back to the first example, in which the pension and individual pensioners' utility  $\bar{v}$  do not depend on the age of the pensioner. The social planner attributes the global weight  $\omega_t^r = \alpha_c \int_{a_r}^\infty c_t(a) n(t, a) da$  to pensioners living at time  $t$ , based on their past contributions (see (3.7)). In this example, the aggregate utility of pension is given by (3.6):

$$V(t, \rho) = \int_{a_r}^\infty \bar{v}(t, \rho p_t^{\min}) \alpha_c c_t(a) n(t, a) da = \omega_t^r \bar{v}(t, \rho p_t^{\min}).$$

We have  $p_t^{\min} V_\rho^{-1}(t, z) = \bar{v}_\rho^{-1}(t, \frac{z}{p_t^{\min} \omega_t^r})$ . Besides,  $P_t^{\min} = N_t^r p_t^{\min}$  where  $N_t^r = \int_{a_r}^\infty n(t, a) da$  is the number of pensioners at time  $t$ . This implies that the optimal pension for each pensioner is

$$p_t^*(z) = p_t^{\min} \rho_t^*(z) = \bar{v}_\rho^{-1}\left(t, \frac{N_t^r}{\omega_t^r} U_z(t, z)\right) \vee p_t^{\min}. \quad (4.12)$$

The pension amount then increases with the quantity  $\frac{\omega_t^r}{N_t^r}$ , corresponding to the average individual contribution of a pensioner living at time  $t$ .

**Example 2** In the second example of age-dependent pension and utility  $\bar{v}$ , the aggregate utility of pension  $V$  is a complex aggregation between cohorts given by (3.8):

$$V(t, \rho) = \int_{a_r}^\infty \bar{v}(t, a, \rho p_{ret}(a_r + t - a) e^{\int_{a_r+t-a}^t \lambda_u du}) n(t, a) da.$$

In all generality, there is no straightforward formula for  $V_\rho^{-1}$  in terms of the  $\bar{v}_\rho^{-1}$ , except in particular cases of dynamic utilities such as dynamic CRRA utilities (see Section 5).

### 4.3 Proof of Theorem 4.3

Theorem 4.3 states that in order to satisfy the consistency condition (4.5), the sustainability bound  $\mathfrak{K}$  is necessarily a buffer fund receiving the contribution  $C$  and paying the minimal pension amount  $P^{\min}$ . We recall the dynamics of  $\mathfrak{K}$  is given by (2.14):  $d\mathfrak{K}_t = \mu_t^{\mathfrak{K}} dt + \delta_t^{\mathfrak{K}} \cdot dW_t$ . As in Proposition 3.9, we consider hereafter the stochastic utility  $\bar{U}(t, z) = U(t, z + \mathfrak{K}_t)$  whose local characteristics are given, by

$$\begin{cases} \beta(t, z) = \bar{\beta}(t, z - \mathfrak{K}_t) - \bar{U}_z(t, z - \mathfrak{K}_t) \mu_t^{\mathfrak{K}} + \frac{1}{2} \bar{U}_{zz}(t, z - \mathfrak{K}_t) \|\delta_t^{\mathfrak{K}}\|^2 - \bar{\gamma}_z(t, z - \mathfrak{K}_t) \delta_t^{\mathfrak{K}}, \\ \gamma(t, z) = \bar{\gamma}(t, z - \mathfrak{K}_t) - \bar{U}_z(t, z - \mathfrak{K}_t) \delta_t^{\mathfrak{K}}, \end{cases}$$

by the Itô-Wentzell's formula. This combined with the HJB-constraint below,

$$\begin{aligned} \beta(t, z) = & -U_z(t, z)(zr_t + C_t) + \frac{1}{2} U_{zz}(t, z) \left| \frac{\gamma_z^{\mathcal{R}}(t, z) + U_z(t, z) \eta_t}{U_{zz}(t, z)} \right|^2 \\ & - V(t, \rho_t^*(z)) + U_z(t, z) P_t^{\min} \rho_t^*(z), \end{aligned}$$

yields that

$$\begin{aligned} & \bar{\beta}(t, z - \mathfrak{K}_t) - \bar{U}_z(t, z - \mathfrak{K}_t) \mu_t^{\mathfrak{K}} + \frac{1}{2} \bar{U}_{zz}(t, z - \mathfrak{K}_t) \|\delta_t^{\mathfrak{K}}\|^2 - \bar{\gamma}_z(t, z - \mathfrak{K}_t) \cdot \delta_t^{\mathfrak{K}} \\ = & -\bar{U}_z(t, z - \mathfrak{K}_t)(zr_t + C_t) - V(t, \rho_t^*(z)) + \bar{U}_z(t, z - \mathfrak{K}_t) P_t^{\min} \rho_t^*(z) \\ & + \frac{1}{2} \bar{U}_{zz}(t, z - \mathfrak{K}_t) \left\| \frac{\bar{\gamma}_z^{\mathcal{R}}(t, z - \mathfrak{K}_t) - \bar{U}_{zz}(t, z - \mathfrak{K}_t) \delta_t^{\mathfrak{K}, \mathcal{R}} + \bar{U}_z(t, z - \mathfrak{K}_t) \eta_t}{\bar{U}_{zz}(t, z - \mathfrak{K}_t)} \right\|^2. \end{aligned}$$

Reorganizing the terms, we have (omitting the variables  $(t, z - \mathfrak{K}_t)$  to simplify notations):

$$\begin{aligned} \bar{\beta} + V(t, \rho_t^*(z)) = & \bar{U}_z (\mu_t^{\mathfrak{K}} - zr_t - C_t + P_t^{\min} \rho_t^*(z)) \\ & + \frac{1}{2} \frac{1}{\bar{U}_{zz}} (\|\bar{\gamma}_z^{\mathcal{R}} - \bar{U}_{zz} \delta_t^{\mathfrak{K}, \mathcal{R}} + \bar{U}_z \eta_t\|^2 - \|\bar{U}_{zz} \delta_t^{\mathfrak{K}}\|^2) + \bar{\gamma}_z \cdot \delta_t^{\mathfrak{K}}. \end{aligned}$$

Rewriting the last term as follows,

$$\begin{aligned} & \frac{1}{2} \frac{1}{\bar{U}_{zz}} (\|\bar{\gamma}_z^{\mathcal{R}} - \bar{U}_{zz} \delta_t^{\mathfrak{K}, \mathcal{R}} + \bar{U}_z \eta_t\|^2 - \|\bar{U}_{zz} \delta_t^{\mathfrak{K}}\|^2) + \bar{\gamma}_z \cdot \delta_t^{\mathfrak{K}} \\ = & \frac{1}{2} \frac{1}{\bar{U}_{zz}} (\|\bar{\gamma}_z^{\mathcal{R}} + \bar{U}_z \eta_t\|^2 - 2(\bar{\gamma}_z^{\mathcal{R}} + \bar{U}_z \eta_t) \cdot \bar{U}_{zz} \delta_t^{\mathfrak{K}} - \|\bar{U}_{zz} \delta_t^{\mathfrak{K}, \perp}\|^2) + \bar{\gamma}_z \cdot \delta_t^{\mathfrak{K}} \\ = & \frac{1}{2} \frac{1}{\bar{U}_{zz}} (\|\bar{\gamma}_z^{\mathcal{R}} + \bar{U}_z \eta_t\|^2 - \|\bar{U}_{zz} \delta_t^{\mathfrak{K}, \perp}\|^2) + \bar{\gamma}_z^\perp \cdot \delta_t^{\mathfrak{K}} - \bar{U}_z \eta_t \cdot \delta_t^{\mathfrak{K}}, \end{aligned}$$

we get that the consistency condition (4.5) is equivalent to

$$\begin{aligned} & \bar{\beta}(t, z - \mathfrak{K}_t) + V(t, \rho_t^*(z)) \\ = & \bar{U}_z (\mu_t^{\mathfrak{K}} - zr_t - C_t + P_t^{\min} \rho_t^*(z) - \eta_t \cdot \delta_t^{\mathfrak{K}}) + \frac{1}{2} \frac{1}{\bar{U}_{zz}} (\|\bar{\gamma}_z^{\mathcal{R}} + \bar{U}_z \eta_t\|^2 - \|\bar{U}_{zz} \delta_t^{\mathfrak{K}, \perp}\|^2) + \bar{\gamma}_z^\perp \cdot \delta_t^{\mathfrak{K}} \\ = & \bar{U}_z \left[ \mu_t^{\mathfrak{K}} - zr_t - C_t + P_t^{\min} \rho_t^*(z) - \eta_t \cdot \delta_t^{\mathfrak{K}} + \frac{1}{2} \frac{1}{\bar{U}_{zz} \bar{U}_z} (\|\bar{\gamma}_z^{\mathcal{R}} + \bar{U}_z \eta_t\|^2 - \|\bar{U}_{zz} \delta_t^{\mathfrak{K}, \perp}\|^2) + \frac{\bar{\gamma}_z^\perp}{\bar{U}_z} \cdot \delta_t^{\mathfrak{K}} \right]. \end{aligned}$$

Let us analyze the behavior of the left hand size and the right hand size of this identity, when  $z \rightarrow \mathfrak{K}_t$ . We have by 3.4 and 3.5 that  $\lim_{z \rightarrow \mathfrak{K}_t} \rho_t^*(z) = \lim_{z \rightarrow \mathfrak{K}_t} V_p^{-1}(t, P_t^{\min} U_z(t, z)) \vee 1 = \underline{\rho}_t \vee 1 = 1$ .

Furthermore,  $\bar{U}(t, 0)$  is well defined. Hence, by continuity the limit of  $\bar{\beta}(t, z - \mathfrak{R}_t)$  when  $z - \mathfrak{R}_t \rightarrow 0$  exists and is equal to  $\bar{\beta}(t, 0)$ . Thus, the left-hand side of the previous equation tends to a constant  $l_t < \infty$ .

For the right-hand side, since by the Inada condition  $\bar{U}_z(t, z - \mathfrak{R}_t) \rightarrow \infty$  when  $z \rightarrow \mathfrak{R}_t$ , the bracketed term can therefore only tend towards zero, that is

$$\lim_{z \rightarrow \mathfrak{R}_t} \left[ \mu_t^{\mathfrak{R}} - zr_t - C_t + P_t^{\min} \rho_t^*(z) - \eta_t \cdot \delta_t^{\mathfrak{R}} + \frac{1}{2} \frac{1}{\bar{U}_{zz} \bar{U}_z} (\|\bar{\gamma}_z^{\mathcal{R}} + \bar{U}_z \eta_t\|^2 - \|\bar{U}_{zz} \delta_t^{\mathfrak{R}, \perp}\|^2) + \frac{\bar{\gamma}_z^{\perp}}{\bar{U}_z} \cdot \delta_t^{\mathfrak{R}} \right] = 0,$$

which we rewrite

$$\lim_{z \rightarrow \mathfrak{R}_t} \left[ \mu_t^{\mathfrak{R}} - zr_t - C_t + P_t^{\min} \rho_t^*(z) - \eta_t \cdot \delta_t^{\mathfrak{R}} + \frac{\bar{\gamma}_z^{\perp}}{\bar{U}_z} \cdot \delta_t^{\mathfrak{R}} + \frac{1}{2} \left( \frac{\|\bar{\gamma}_z^{\mathcal{R}}\|^2}{\bar{U}_{zz} \bar{U}_z} + 2 \frac{\bar{\gamma}_z^{\mathcal{R}} \cdot \eta_t}{\bar{U}_{zz}} + \frac{\bar{U}_z}{\bar{U}_{zz}} \|\eta_t\|^2 - \frac{\bar{U}_{zz}}{\bar{U}_z} \|\delta_t^{\mathfrak{R}, \perp}\|^2 \right) \right] = 0,$$

or equivalently,

$$\lim_{z \rightarrow \mathfrak{R}_t} \left[ A_t(z) + \frac{1}{2} (B_t(z) - \frac{\bar{U}_{zz}}{\bar{U}_z} \|\delta_t^{\mathfrak{R}, \perp}\|^2) \right] = 0, \quad (4.13)$$

where we have used the notations

$$\begin{cases} A_t(z) = \mu_t^{\mathfrak{R}} - zr_t - C_t + P_t^{\min} \rho_t^*(z) - \eta_t \cdot \delta_t^{\mathfrak{R}} + \frac{\bar{\gamma}_z^{\perp}}{\bar{U}_z} \cdot \delta_t^{\mathfrak{R}}, \\ B_t(z) = \frac{\|\bar{\gamma}_z^{\mathcal{R}}\|^2}{\bar{U}_{zz} \bar{U}_z} + 2 \frac{\bar{\gamma}_z^{\mathcal{R}} \cdot \eta_t}{\bar{U}_{zz}} + \frac{\bar{U}_z}{\bar{U}_{zz}} \|\eta_t\|^2. \end{cases}$$

We shall now study the limits of  $A_t(z)$  and  $B_t(z)$  when  $z$  tends to  $\mathfrak{R}_t$ . For this, we recall according to Remark 3.2, that if the Inada condition holds, then

$$\lim_{z \rightarrow \mathfrak{R}_t} \frac{\bar{U}_{zz}(t, z - \mathfrak{R}_t)}{\bar{U}_z(t, z - \mathfrak{R}_t)} = \lim_{z \rightarrow \mathfrak{R}_t} \frac{U_{zz}(t, z)}{U_z(t, z)} = +\infty.$$

Also, under Assumption 3.8, there exist a random bound  $\zeta_t$  satisfying a.s.  $\int_0^T \zeta_t^2 dt < \infty$  for any  $T$  such that

$$\lim_{z \rightarrow \mathfrak{R}_t} \frac{\|\bar{\gamma}_z(t, z - \mathfrak{R}_t)\|}{\bar{U}_z(t, z - \mathfrak{R}_t)} \leq \zeta_t.$$

Thus,  $|B_t(z)| = \left| \frac{\|\bar{\gamma}_z^{\mathcal{R}}\|^2}{\bar{U}_{zz} \bar{U}_z} + 2 \frac{\bar{\gamma}_z^{\mathcal{R}} \cdot \eta_t}{\bar{U}_{zz}} + \frac{\bar{U}_z}{\bar{U}_{zz}} \|\eta_t\|^2 \right| \leq \frac{\bar{U}_z}{\bar{U}_{zz}} (\zeta_t + 3 \|\eta_t\|^2) \rightarrow 0$  when  $(z - \mathfrak{R}_t) \rightarrow 0$ .

Furthermore,

$$\begin{aligned} |A_t(z)| &= \left| (\mu_t^{\mathfrak{R}} - zr_t - C_t + P_t^{\min} \rho_t^*(z) - \eta_t \cdot \delta_t^{\mathfrak{R}}) + \frac{\bar{\gamma}_z^{\perp}}{\bar{U}_z} \cdot \delta_t^{\mathfrak{R}} \right| \\ &\leq |(\mu_t^{\mathfrak{R}} - zr_t - C_t + P_t^{\min} \rho_t^*(z) - \eta_t \cdot \delta_t^{\mathfrak{R}})| + \zeta_t \|\delta_t^{\mathfrak{R}, \perp}\|, \end{aligned}$$

consequently  $A_t(z)$  has a finite limit when  $(z - \mathfrak{R}_t) \rightarrow 0$ . This combined with  $|B_t(z)| \rightarrow 0$  when  $(z - \mathfrak{R}_t) \rightarrow 0$ , (4.13) and the fact that  $\lim_{z \rightarrow \mathfrak{R}_t} \frac{\bar{U}_{zz}(t, z - \mathfrak{R}_t)}{\bar{U}_z(t, z - \mathfrak{R}_t)} = +\infty$ , implies that necessarily

$$\delta_t^{\mathfrak{R}, \perp} = 0. \quad (4.14)$$

It follows from (4.13) that

$$\lim_{z \rightarrow \mathfrak{R}_t} (\mu_t^{\mathfrak{R}} - zr_t - C_t + P_t^{\min} \rho_t^*(z) - \eta_t \cdot \delta_t^{\mathfrak{R}}) = \mu_t^{\mathfrak{R}} - \mathfrak{R}_t r_t - C_t + P_t^{\min} - \eta_t \cdot \delta_t^{\mathfrak{R}} = 0, \quad (4.15)$$

where we have used, according to (4.11),  $\lim_{z \rightarrow \mathfrak{R}_t} \rho_t^*(z) = 1$ . This concludes the proof of Theorem 4.3.

#### 4.4 Proof of Theorem 4.5

In order to show that the sustainability condition ( $F_t^* \geq \mathfrak{R}_t$ ,  $dt \otimes d\mathbb{P}$  a.s.) is satisfied under

the assumptions of Theorem 4.5, we study the intermediate process  $(U_z(t, F_t^*))$ . This process is actually the optimal state price density process  $(Y_t^*)$ . We refer to [20] for more details.

Combining the bounds (2.15) on the derivatives of the diffusion coefficient  $\bar{\gamma}(t, z)$  of  $\bar{U}$  and Theorem 4.3, we first show that  $U_z(t, F_t^*)$  is the unique strong (non-explosive) solution of an SDE with Lipschitz coefficients. This property, combined with Inada conditions, ensures that the sustainability condition is necessarily satisfied. This preliminary result is stated in the following Proposition 4.7.

**Proposition 4.7** *Under the assumptions and notations of Theorem 4.5,*

(i) *The dynamics of the marginal utility  $U_z$  is given by*

$$\begin{aligned} dU_z(t, z) = & \left( -U_{zz}(t, z)\mu_t^*(z) - U_z(t, z)r_t - \frac{1}{2}U_{zzz}(t, z)\|\pi_t^*(z)\|^2 - \gamma_{zz}^{\mathcal{R}} \cdot \pi_t^*(z) \right) dt \\ & + \gamma_z(t, z) \cdot dW_t \end{aligned}$$

(ii) *Let  $\tau^F = \inf\{t \geq 0, F_t^* = \mathfrak{R}_t\}$ .  $U_z(t, F_t^*)$  is the solution of the following SDE on  $[0, \tau^F[$  of*

$$dU_z(t, F_t^*) = -U_z(t, F_t^*)r_t dt + (\gamma_z^\perp(t, F_t^*) - U_z(t, F_t^*)\eta_t) \cdot dW_t. \quad (4.16)$$

The proof of Theorem 4.5 is derived from Theorem 4.3 and Proposition 4.7, by following the same steps than in the proof of Theorem 4.8 in [20].

**Proof of Theorem 4.5** Inspired by (4.16), let us consider the following SDE

$$dY_t = -Y_t r_t dt + (\gamma_z^\perp(t, U_z^{-1}(t, Y_t)) - Y_t \eta_t) \cdot dW_t.$$

By Theorem 4.3,  $\delta^{\mathfrak{R}, \perp} = 0$  a.s. Hence, using the relation (3.18) between  $\gamma$  and  $\bar{\gamma}$ , the condition (3.15) can be rewritten as  $\|\gamma_z^\perp(t, z)\| \leq \zeta_t U_z(t, z)$  for some non negative process  $\zeta$  such that a.s.  $\int_0^T \zeta_t^2 dt < \infty$  for any  $T$ . This implies that the coefficients of this SDE are Lipschitz, yielding that this SDE admits a unique strong (non explosive) solution. Furthermore, by Proposition 4.7,  $U_z(t, F_t^*)$  is solution of this SDE, which yields that  $\tau^F = +\infty$  i.e.  $F_t^* > \mathfrak{R}_t$ ,  $\forall t > 0$ ,  $\mathbb{P}$  a.s. Besides, as a consequence of Assumptions 3.6 and 3.8,  $(\pi_t^*(F_t^*), \rho_t^*(F_t^*))$  verify the required integrability condition, which concludes the proof.

The last step consists in proving Proposition 4.7. □

**Proof of Proposition 4.7**

(i) Since by assumption  $U$  satisfies the HJB constraint (4.5), its dynamics is

$$\begin{aligned} dU(t, z) = & \left( -U_z(t, z)(zr_t + C_t - P_t^{\min} \rho_t^*(z)) + \frac{1}{2}U_{zz}(t, z)\|\pi_t^*(z)\|^2 - V(t, \rho_t^*(z)) \right) dt \\ & + \gamma(t, z) \cdot dW_t. \end{aligned} \quad (4.17)$$

In addition, since  $U$  is of class  $\mathcal{C}^{3, \delta}$  and by Assumption 3.8, this implies by Corollary 2.3 in [23], that  $\beta_z$  and  $\gamma_z$  are the local characteristics of the space derivative  $U_z$ .

On  $\{1 < \rho_t^f(z)\}$ ,  $\rho_t^*(z) = \rho_t^f(z) \in \mathcal{C}^1$  with  $U_z(t, z)P_t^{\min} = V_\rho(t, \rho_t^f(z))$  by (4.4). Thus  $U_z(t, z)P_t^{\min} \partial_z \rho_t^*(z) = V_\rho(t, \rho_t^f(z)) \partial_z \rho_t^f(z)$  on  $\{1 < \rho_t^f(z)\}$ .

This last inequality also holds on  $\{\rho_t^f(z) \leq 1\}$  since on this set,  $\rho_t^*(z) \equiv 1$  implying  $\partial_z \rho_t^*(z) = 0$ . Using those simplifications, the derivative with respect to  $z$  of the dynamics (4.17) of  $U$  yields by [23, Theorem 2.2],

$$\begin{aligned} dU_z(t, z) = & \left( -U_{zz}(t, z)(zr_t + C_t - P_t^{\min}\rho_t^*(z)) - U_z(t, z)r_t \right. \\ & \left. + \frac{1}{2}U_{zzz}(t, z)\|\pi_t^*(z)\|^2 + U_{zz}(t, z)\partial_z\pi_t^*(z)\cdot\pi_t^*(z) \right) dt + \gamma_z(t, z)dW_t. \end{aligned} \quad (4.18)$$

Now, using the definition (4.8) of  $\pi^*$ ,

$$\partial_z\pi_t^*(z) = -\frac{U_{zzz}(t, z)}{U_{zz}(t, z)}\pi_t^*(z) - \frac{\gamma_{zz}^{\mathcal{R}}(t, F_t^*) + U_{zz}(t, F_t^*)\eta_t}{U_{zz}(t, F_t^*)}, \quad (4.19)$$

which implies that

$$\begin{aligned} & \frac{1}{2}U_{zzz}(t, z)\|\pi_t^*(z)\|^2 + U_{zz}(t, z)\partial_z\pi_t^*(z)\pi_t^*(z) \\ &= -\frac{1}{2}U_{zzz}(t, z)\|\pi_t^*(z)\|^2 - \gamma_{zz}^{\mathcal{R}}\pi_t^*(z) - U_{zz}(t, z)\pi_t^*(z)\eta_t. \end{aligned} \quad (4.20)$$

Using the notation  $\mu_t^*(z) = zr_t + C_t - P_t^{\min}\rho_t^*(z) + \pi_t^*(z)\eta_t$  for the drift of  $F^*$  and injecting (4.19) and (4.20) in the dynamics (4.18), yields

$$\begin{aligned} dU_z(t, z) = & \left( -U_{zz}(t, z)\mu_t^*(z) - U_z(t, z)r_t - \frac{1}{2}U_{zzz}(t, z)\|\pi_t^*(z)\|^2 - \gamma_{zz}^{\mathcal{R}}\pi_t^*(z) \right) dt \\ & + \gamma_z(t, z)dW_t. \end{aligned}$$

This establishes the first statement of the theorem.

(ii) Applying Itô-Wentzell's formula to  $U_z(t, F_t^*)$  on  $\{t < \tau^F\}$  gives (recalling the dynamics  $dF_t^* = \mu_t^*(F_t^*)dt + \pi_t^*(F_t^*)dW_t$ ):

$$dU_z(t, F_t^*) = -U_z(t, F_t^*)r_t dt + (\gamma_z(t, F_t^*) + U_{zz}(t, F_t^*)\pi_t^*(F_t^*))dW_t.$$

Using the identity  $\gamma_z(t, F_t^*) + U_{zz}(t, F_t^*)\pi_t^*(F_t^*) = \gamma_z^\perp(t, F_t^*) - U_z(t, F_t^*)\eta_t$ , we conclude that  $U_z(t, F_t^*)$  is solution of the SDE (4.16). □

## 5. Application to dynamic CRRA utilities

We provide the resolution in the important example of dynamic Constant Relative Risk Aversion (CRRA) utilities, also called power dynamic utilities, introduced in 3.1. We now assume that the aggregated utility  $V$  of pensioners is given by an aggregation of individual power dynamic utilities  $\bar{v}$ , with pensioners having the same relative risk aversion  $\theta$ .

The social planner needs to infer a dynamic utility  $U$  of the fund such that her preference criterion  $(U, V)$  is consistent. It is then natural to search the utility  $U$  in the same class.<sup>2</sup> Therefore the goal is to find a consistent CRRA utility  $U$  (if it exists) satisfying assumptions of Theorem 4.5:

$$U(t, z) = Z_t^u \frac{(z - X_t)^{1-\theta}}{1-\theta}, \quad (5.1)$$

where  $Z^u$  is assumed to be a positive process with dynamics

$$dZ_t^u = Z_t^u(b_t dt + \delta_t \cdot dW_t), \quad \delta_t \in \mathbb{R}^n. \quad (5.2)$$

As discussed in Remark 4.4, to take into account the sustainability constraint,  $X$  should be taken as a buffer fund with pension rate  $P^{\min}$  and contribution rate  $C$ , and that super-replicates (pathwise) the sustainability bound  $\mathfrak{R}$ . The dynamics of the shift  $X$  is, according to

<sup>2</sup>If  $U$  is power then necessarily  $V$  is power with the same risk aversion coefficient, to ensure the consistency of the criterion  $(U, V)$  (see [20]).

Theorem 4.3,

$$dX_t = (X_t r_t + C_t - P_t^{\min})dt + \delta_t^X \cdot (dW_t + \eta_t dt), \quad \delta^X \in \mathcal{R}. \quad (5.3)$$

The dynamics of the utility process  $U$  is deduced easily from that of  $Z^u$  and  $X$ .

**Lemma 5.1** *Denoting  $dX_t = \mu_t^X dt + \delta_t^X \cdot dW_t$ , the dynamics of the shift  $X$ , the dynamics of the CRRA shifted utility  $U(t, z) = Z_t^u \frac{(z - X_t)^{1-\theta}}{1-\theta}$  is  $dU(t, z) = \beta(t, z)dt + \gamma(t, z) \cdot dW_t$ , with local characteristics*

$$\begin{cases} \beta(t, z) = -U_z(t, z)(\mu_t^X + \delta_t \cdot \delta_t^X) + \frac{1}{2}U_{zz}(t, z)\|\delta_t^X\|^2 + U(t, z)b_t, \\ \gamma(t, z) = -U_z(t, z)\delta_t^X + U(t, z)\delta_t. \end{cases}$$

*If the shift  $X$  is a buffer fund (with pension rate  $P^{\min}$  and contribution rate  $C$ ) then  $\mu_t^X = X_t r_t + C_t - P_t^{\min} + \delta_t^X \cdot \eta_t$  and  $\delta^X \in \mathcal{R}$ .*

**Proof** The proof is a straightforward application of the Itô Lemma.  $\square$

We also provide some useful relations for power utilities and their derivatives of first and second order and their conjugate:

$$\begin{cases} (z - X_t)U_z(t, z) = Z_t^u (z - X_t)^{1-\theta} = (1 - \theta)U(t, z), \\ (z - X_t)^2 U_{zz}(t, z) = -\theta(z - X_t)U_z(t, z) = -\theta(1 - \theta)U(t, z). \end{cases} \quad (5.4)$$

The first term  $\frac{\theta}{1-\theta}(Z_t^u)^{(1/\theta)}y^{\frac{\theta-1}{\theta}}$  of the dual  $\tilde{U}$  is standard, the second term  $X_t y$  (linear in  $y$ ) corresponds to the support function of the convex domain  $[X_t, +\infty[$ .

## 5.1 Age-independent pensions

We start with Example 1. Each pensioner receives the same pension amount  $p_t = \rho_t p_t^{\min}$  (with  $\rho \geq 1$ ), and has the same dynamic power utility  $\bar{v}$ ,

$$\bar{v}(t, p) = Z_t \frac{(p - p_t^{\min})^{1-\theta}}{1 - \theta},$$

where  $Z$  is a positive adapted process. The social planner aggregates preferences of pensioners by weighting their utility by their past contributions  $\alpha_c c_t(a)$ , thus

$$V(t, \rho) = \omega_t^r \bar{v}(s, \rho p_t^{\min}) = \omega_t^r Z_t (p_t^{\min})^{1-\theta} \frac{(\rho - 1)^{1-\theta}}{1 - \theta}, \quad (5.5)$$

where  $\omega_t^r = \alpha_c \int_{a_r}^{\infty} c_t(a)n(t, a)da$  is the total weight of all pensioners living at time  $t$ .

### 5.1.1 Optimal policy

**Proposition 5.2** *Assume that the shifted power utility  $U$  defined by (5.1) is a consistent utility verifying assumptions of Theorem 4.5. Then the optimal strategy is given by*

$$\begin{cases} \pi_t^* = \delta_t^X + \frac{1}{\theta}(F_t^* - X_t)(\delta_t^{\mathcal{R}} + \eta_t), \\ p_t^* = p_t^{\min} + (F_t^* - X_t) \left( \frac{Z_t \omega_t^r}{Z_t^u N_t^r} \right)^{\frac{1}{\theta}}, \end{cases} \quad (5.6)$$

$$\quad (5.7)$$

and the optimal fund  $F^*$  is solution of the dynamics

$$dF_t^* = dX_t + (F_t^* - X_t) \left( \left( r_t - \left( \frac{Z_t \omega_t^r}{Z_t^u N_t^r} \right)^{\frac{1}{\theta}} \right) dt + \frac{1}{\theta}(\delta_t^{\mathcal{R}} + \eta_t) \cdot (dW_t + \eta_t dt) \right). \quad (5.8)$$

Note that due to the form of the pension utility  $\bar{v}$ ,  $\rho_t^f$  defined by (4.4) is always greater than the bound 1 and thus  $\rho_t^* = \rho_t^f$  a.s.

The optimal strategy has a particular additive form. The optimal pension is the minimal pension  $p_t^{\min}$  plus an additional term that is proportional to the ‘‘cushion’’ ( $F_t^* - X_t$ ), where we recall that  $X$  is a buffer fund receiving the contribution rate  $C$  and paying the minimal pension amount  $P^{\min}$ , which (sur)replicates the sustainability bound  $\mathfrak{R}$ . The proportionality factor  $\left(\frac{Z_t \omega_t^r}{Z_t^u N_t^r}\right)^{\frac{1}{\theta}}$  is decreasing in the risk aversion parameter  $\theta$ . The ratio  $\frac{\omega_t^r}{N_t^r} = \frac{\int_{a_r}^{\infty} \alpha_c c_t(a) n(t,a) da}{\int_{a_r}^{\infty} n(t,a) da}$  represents the average past contributions of one pensioner at time  $t$ . With this choice of weight  $w^r$ , the more pensioners contributed during their working period, the higher their pension will be. This can be interpreted as some kind of actuarial fairness.

The ratio  $\frac{Z_t}{Z_t^u}$  represents the relative importance of the pensioners utility with respect to the buffer fund utility, which represents future generations. This can be interpreted as an ‘‘intergenerational risk sharing’’ coefficient. If  $\frac{Z_t}{Z_t^u} \gg 1$ , current pensioners will receive higher pensions, while the fund return rate will decrease, potentially putting the pension system’s sustainability at risk (see Theorem 5.5). On the other hand, if  $\frac{Z_t}{Z_t^u} \ll 1$ , then future pensioners will benefit from a higher fund value.

The optimal portfolio is the portfolio  $\delta^X$  of buffer fund shift  $X$  plus an additional term proportional to the ‘‘cushion’’ ( $F_t^* - X_t$ ). Again the proportionality factor  $\frac{(\delta_t^{\mathcal{R}} + \eta_t)}{\theta}$  is decreasing in the risk aversion parameter  $\theta$ . Note that the optimal strategy also depends on the volatility of buffer fund utility.

**Proof** By a simple derivation with respect to  $z$  and  $\rho$ , we have  $U_z(t, z) = Z_t^u (z - X_t)^{-\theta}$  and  $V_\rho(t, \rho) = Z_t \omega_s^r p_t^{\min} (p_t^{\min} (\rho - 1))^{-\theta}$  so that  $p_t^{\min} V_\rho^{-1}(t, y) = p_t^{\min} + \left(\frac{y}{Z_t \omega_s^r p_t^{\min}}\right)^{-\frac{1}{\theta}}$ . Thus, applying Theorem 4.5

$$\begin{aligned} p_t^* &= p_t^{\min} V_\rho^{-1}(t, P_t^{\min} U_z(t, F_t^*)) \\ &= p_t^{\min} V_\rho^{-1}(t, P_t^{\min} Z_t^u (F_t^* - X_t)^{-\theta}) \\ &= p_t^{\min} + \left(\frac{Z_t^u (F_t^* - X_t)^{-\theta} P_t^{\min}}{Z_t \omega_t^r p_t^{\min}}\right)^{-\frac{1}{\theta}} \\ &= p_t^{\min} + \left(\frac{Z_t \omega_t^r}{Z_t^u N_t^r}\right)^{\frac{1}{\theta}} (F_t^* - X_t), \end{aligned}$$

where we used  $P_t^{\min} = p_t^{\min} N_t^r$  for the last equality. In addition, the optimal investment strategy is given by

$$\begin{aligned} \pi_t^* &= -\frac{\gamma_x^{\mathcal{R}}(t, F_t^*) + U_z(t, F_t^*) \eta_t}{U_{zz}(t, F_t^*)} \\ &= -\frac{-U_{zz}(t, F_t^*) \delta_t^X + U_z(t, F_t^*) (\delta_t^{\mathcal{R}} + \eta_t)}{U_{zz}(t, F_t^*)} \\ &= \delta_t^X + \frac{1}{\theta} (F_t^* - X_t) (\delta_t^{\mathcal{R}} + \eta_t), \end{aligned}$$

where we used Lemma 4.1. Plugging this strategy into the buffer fund dynamics (2.15) yields

$$\begin{aligned} dF_t^* &= \left( F_t^* r_t + C_t - N_t^r p_t^{\min} \left( 1 + \left( \frac{Z_t \omega_t^r}{Z_t^u N_t^r} \right)^{\frac{1}{\theta}} (F_t^* - X_t) \right) \right) dt \\ &\quad + \left( \frac{1}{\theta} (F_t^* - X_t) (\delta_t^{\mathcal{R}} + \eta_t) + \delta_t^X \right) \cdot (dW_t + \eta_t dt). \end{aligned}$$

which implies (5.8) by (5.3).  $\square$

### 5.1.2 Consistency

The next step consists in verifying the assumptions of Theorem 4.5 and in particular the consistency condition, that translates into condition on  $Z^u$ .

**Proposition 5.3** *Let  $V$  be the aggregated pensioners utility (5.5) and  $U(t, z) = Z_t^u \frac{(z - X_t)^{1-\theta}}{1-\theta}$ . If  $(U, V)$  is consistent then  $Z^u$  must be a well-defined solution to the SDE:*

$$-dZ_t^u = Z_t^u \left( \left( (1-\theta)r_t + \frac{(1-\theta)}{2\theta} \|\delta_t^{\mathcal{R}} + \eta_t\|^2 + \theta (Z_t \omega_t^r)^{\frac{1}{\theta}} (N_t^r)^{\frac{\theta-1}{\theta}} (Z_t^u)^{-\frac{1}{\theta}} \right) dt - \delta_t \cdot dW_t \right). \quad (5.9)$$

**Proof** One the one hand, by application of Lemma 5.1, the buffer fund dynamic utility  $U$  has the following dynamics

$$\begin{cases} dU(t, z) = \beta(t, z)dt + \gamma(t, z)dW_t, \\ \beta(t, z) = -U_z(t, z)(X_t r_t + C_t - P_t^{\min} + \delta_t^X \cdot (\eta_t + \delta_t)) + \frac{1}{2} U_{zz}(t, z) \|\delta_t^X\|^2 + U(t, z) b_t, \\ \gamma(t, z) = -U_z(t, z) \delta_t^X + U(t, z) \delta_t. \end{cases} \quad (5.10)$$

On the other hand, if  $(U, V)$  is consistent the consistency constraint (4.5) should be satisfied:

$$\beta(t, z) = -U_z(t, z)(z r_t + C_t - P_t^{\min} \rho_t^*(z)) + \frac{1}{2} U_{zz}(t, z) \|\pi_t^*(z)\|^2 - V(t, \rho_t^*(z)).$$

By the expression of optimal controls given in Proposition 5.2, the elementary identities (5.4) and using (5.5) and (5.6) we can rewrite the consistency constraint (4.5) as

$$\begin{aligned} \beta(t, z) &= -U_z(t, z)(z r_t + C_t - P_t^{\min} + \delta_t^X \cdot (\eta_t + \delta_t)) + \frac{1}{2} U_{zz}(t, z) \|\delta_t^X\|^2 \\ &\quad + U(t, z) \left( (1-\theta) N_t^r \left( \frac{Z_t \omega_t^r}{Z_t^u N_t^r} \right)^{\frac{1}{\theta}} - \frac{(1-\theta)}{2\theta} \|\delta_t^{\mathcal{R}} + \eta_t\|^2 - \omega_t^r \frac{Z_t}{Z_t^u} \left( \frac{Z_t \omega_t^r}{Z_t^u N_t^r} \right)^{\frac{1}{\theta}-1} \right). \end{aligned}$$

Identifying this with (5.10), we obtain

$$\begin{aligned} &-U_z(t, z) X_t r_t + U(t, z) b_t \\ &= -U_z(t, z) z r_t + U(t, z) \left( (1-\theta) N_t^r \left( \frac{Z_t \omega_t^r}{Z_t^u N_t^r} \right)^{\frac{1}{\theta}} - \frac{(1-\theta)}{2\theta} \|\delta_t^{\mathcal{R}} + \eta_t\|^2 - \omega_t^r \frac{Z_t}{Z_t^u} \left( \frac{Z_t \omega_t^r}{Z_t^u N_t^r} \right)^{\frac{1}{\theta}-1} \right), \end{aligned}$$

which implies that the drift  $b_t$  of  $Z^u$  must satisfy

$$b_t = -(1-\theta)r_t - \frac{(1-\theta)}{2\theta} \|\delta_t^{\mathcal{R}} + \eta_t\|^2 - \theta (N_t^r)^{1-\frac{1}{\theta}} \left( \frac{\omega_t^r Z_t}{Z_t^u} \right)^{\frac{1}{\theta}}.$$

$\square$

SDE (5.9) is only defined on  $Z^u > 0$ , due to the term  $(Z_t^u)^{-\frac{1}{\theta}}$  in the drift.

**Proposition 5.4** *Let  $\xi$  be the process defined by*

$$\xi_t = \exp \left( \int_0^t \left( (1-\theta)r_s + \frac{(1-\theta)}{2\theta} \|\delta_s^{\mathcal{R}} + \eta_s\|^2 + \frac{\|\delta_s\|^2}{2} \right) ds - \int_0^t \delta_s \cdot dW_s \right).$$

There exists a unique solution of the SDE (5.9), defined by

$$Z_0^u = \xi_t^{-1} \left( (Z_0^u)^{\frac{1}{\theta}} - \int_0^t (N_s^r)^{1-\frac{1}{\theta}} (Z_s \omega_s^r \xi_s)^{\frac{1}{\theta}} ds \right)^\theta, \quad \forall t \in [0, \tau^Z[, \quad (5.11)$$

where  $\tau^Z$  the first hitting time of 0 satisfies

$$\tau^Z = \inf \left\{ t; \int_0^t (N_s^r)^{1-\frac{1}{\theta}} \left( \frac{Z_s \omega_s^r \xi_s}{Z_0^u} \right)^{\frac{1}{\theta}} ds \geq 1 \right\}. \quad (5.12)$$

**Proof** Let  $Z^u$  be a solution of (5.9) on  $[0, \tau[$ , with  $\tau = \inf\{t \geq 0; Z_t^u = 0\}$ , and denote

$$\tilde{r}_t := (1 - \theta)r_t + \frac{(1 - \theta)}{2\theta} \|\delta_t^{\mathcal{R}} + \eta_t\|^2, \quad \text{and} \quad \tilde{\beta}_t := (N_t^r)^{1-\frac{1}{\theta}} (Z_t \omega_t^r)^{\frac{1}{\theta}},$$

so that

$$dZ_t^u = -(Z_t^u \tilde{r}_t + \theta \tilde{\beta}_t (Z_t^u)^{1-\frac{1}{\theta}}) dt + Z_t^u \delta_t \cdot dW_t.$$

This equation may be simplified by the following change of variables:

$$\bar{Z}_t := Z_t^u \exp \left( \int_0^t \left( \tilde{r}_s + \frac{\|\delta_s\|^2}{2} \right) ds - \int_0^t \delta_s \cdot dW_s \right) = Z_t^u \xi_t.$$

Then,

$$\frac{1}{\theta} (\bar{Z}_t)^{\frac{1}{\theta}-1} d\bar{Z}_t = -\tilde{\beta}_t \xi_t^{\frac{1}{\theta}} dt,$$

i.e.

$$(\bar{Z}_t)^{\frac{1}{\theta}} - (\bar{Z}_0)^{\frac{1}{\theta}} = - \int_0^t \tilde{\beta}_s \xi_s^{\frac{1}{\theta}} ds,$$

and thus

$$Z_t^u = \xi_t^{-1} \left( (Z_0^u)^{\frac{1}{\theta}} - \int_0^t (N_s^r)^{1-\frac{1}{\theta}} (Z_s \omega_s^r \xi_s)^{\frac{1}{\theta}} ds \right)^\theta,$$

and  $\tau^Z$  verifies (5.12). □

The necessary condition for  $Z^u$  given in Proposition 5.3 is actually a sufficient condition.

### 5.1.3 Existence of the optimal investment/pension policy

**Theorem 5.5** *Let  $Z^u$  be defined as in Proposition 5.4. Then the shifted power dynamic utilities  $(U, V)$  given by (5.1)–(5.5) are consistent on  $[0, \tau^Z[$ , and verify the assumption of Theorem 4.5 on this interval. Therefore the optimal strategy is given on  $[0, \tau^Z[$  by*

$$\begin{cases} \pi_t^* = \delta_t^X + \frac{1}{\theta} (F_t^* - X_t) (\delta_t^{\mathcal{R}} + \eta_t), \\ p_t^* = p_t^{\min} + (F_t^* - X_t) \left( \frac{Z_t \omega_t^r}{Z_t^u N_t^r} \right)^{\frac{1}{\theta}}. \end{cases}$$

Furthermore,  $\tau^Z = \inf\{t \geq 0; F_t^* = X_t\}$ .

Observe that  $p^*$  depends on the population dynamics only through the ratio  $\omega_t^r/N_t^r$ . Since all

pensioners receive the same pension amount  $p_t^*$ , there is also an intergenerational risk-sharing among pensioners.

For  $t \geq \tau^Z$ ,  $Z_t^u = 0$  and thus the buffer fund utility  $U(t, \cdot) \equiv 0$ . At this stopping time, the buffer fund hits the boundary  $X$ . In order to stay sustainable, the pension system parameters have to be updated, for instance by decreasing the sustainability bound  $\mathfrak{K}_t$ , the minimum pension amount  $p_t^{\min}(a)$  or the importance given to the pensioners' preferences. Observe that thanks to (5.12), the distribution  $\tau^Z$  can be approximated numerically. In particular,  $\tau^Z$  decreases with the number of pensioners and the weight of pensioner's preferences attributed by the social planner. However,  $\tau^Z$  does not depend on the initial buffer fund amount  $F_0$ , due to the choice of dynamic power utilities.

**Proof of Theorem 5.5** The consistency SDE is verified by Proposition 5.3 and 5.4. It is straightforward to check Assumption 3.8, since  $\bar{U}(t, z) = U(t, z - X_t) = \frac{z^{1-\theta}}{1-\theta}$ .

Furthermore,  $\tilde{V}(t, z) = z + \frac{\theta}{1-\theta} (\omega_t^r Z_t (p_t^{\min})^{1-\theta})^{\frac{1}{\theta}} z^{1-\frac{1}{\theta}} = z + \frac{\theta}{1-\theta} (\omega_t^r Z_t)^{\frac{1}{\theta}} \left( \frac{z}{p_t^{\min}} \right)^{1-\frac{1}{\theta}}$ , and it is straightforward to check that Assumption 3.6 is verified. The optimal strategy is then obtained from Proposition 5.2. By Proposition 5.2, and using the short notation  $\mathcal{E}(\int_0^t g_s \cdot dW_s)$  for the Doléans-Dade exponential martingale<sup>3</sup>

$$(F_t^* - X_t) = (F_0^* - X_0) \exp\left(\int_0^t (r_s - (N_s^r)^{1-\frac{1}{\theta}} \left(\frac{Z_s \omega_s^r}{Z_s^u}\right)^{\frac{1}{\theta}} + \frac{1}{\theta} (\delta_s^{\mathcal{R}} + \eta_s) \cdot \eta_s) ds\right) \mathcal{E}\left(\int_0^t \frac{1}{\theta} (\delta_s^{\mathcal{R}} + \eta_s) \cdot dW_s\right),$$

and by Proposition 5.3,

$$Z_t^u = Z_0^u \exp\left(-\int_0^t \left((1-\theta)r_s + \frac{(1-\theta)}{2\theta} \|\delta_s^{\mathcal{R}} + \eta_s\|^2 + \theta(N_s^r)^{1-\frac{1}{\theta}} \left(\frac{Z_s \omega_s^r}{Z_s^u}\right)^{\frac{1}{\theta}}\right) ds\right) \mathcal{E}\left(\int_0^t \delta_s \cdot dW_s\right).$$

Then,

$$(F_t^* - X_t)(Z_t^u)^{-\frac{1}{\theta}} = (F_0^* - X_0)(Z_0^u)^{-\frac{1}{\theta}} \exp\left(\frac{1}{\theta} \int_0^t (r_s + \frac{1}{2} \|\delta_s^{\perp} - \eta_s\|^2) ds - \int_0^t \frac{1}{\theta} (\delta_s^{\perp} - \eta_s) \cdot dW_s\right).$$

The previous equation can be rewritten as

$$(F_t^* - X_t)(Z_t^u)^{-\frac{1}{\theta}} = (F_0^* - X_0)(Z_0^u)^{-\frac{1}{\theta}} (Y_t)^{-\frac{1}{\theta}},$$

where we recognize the state price density process  $Y$

$$Y_t = \exp\left(-\int_0^t r_s ds\right) \mathcal{E}\left(\int_0^t (\delta_s^{\perp} - \eta_s) \cdot dW_s\right).$$

In particular  $\tau^Z = \inf\{t \geq 0; F_t^* = X_t\}$ .

□

Section 5.2 explains how to extend the results obtained for Example 1 to Example 2.

## 5.2 Age-dependent pensions

In Example 2 of age-dependent pension and utility  $\bar{v}$ , the aggregate utility of pension  $V$  is a complex aggregation between cohorts given by (3.8)

$$V(t, \rho) = \int_{a_r}^{\infty} \bar{v}(t, a, \rho) p_{ret}(a_r + t - a) e^{\int_{a_r+t-a}^t \lambda_u du} n(t, a) da.$$

<sup>3</sup>  $\mathcal{E}(\int_0^t g_s \cdot dW_s) := \exp(\int_0^t g_s \cdot dW_s - \frac{1}{2} \int_0^t \|g_s\|^2 ds)$ .

We assume that pensioner utility depends on the age  $a$  through the shift  $p^{\min}(t, a)$  in the shifted power utility

$$\bar{v}(t, a, p) = Z_t \frac{(p - p_t^{\min}(a))^{1-\theta}}{1-\theta} \quad \text{and} \quad v(t, a, \rho) = (p_t^{\min}(a))^{1-\theta} Z_t \frac{(\rho - 1)^{1-\theta}}{1-\theta}.$$

Therefore in the case of power utility, the aggregate utility of pension has the following multiplicative form

$$V(t, \rho) = \tilde{\omega}_t^r Z_t \frac{(\rho - 1)^{1-\theta}}{1-\theta}, \quad \text{with} \quad \tilde{\omega}_t^r = \int_{a_r}^{\infty} (p_t^{\min}(a))^{1-\theta} n(t, a) da.$$

Observe that the problem is formulated similarly than in Example 1, with a different weight  $\tilde{\omega}_t^r$  and in this case  $P_t^{\min} = \int_{a_r}^{\infty} p^{\min}(t, a) n(t, a) da$ . Thus similar computations as in Section 5.1 yield

$$\begin{aligned} \rho_t^* &= V_\rho^{-1}(t, P_t^{\min} U_z(t, F_t^*)) \\ &= V_\rho^{-1}(t, P_t^{\min} Z_t^u (F_t^* - X_t)^{-\theta}) \\ &= 1 + \left( \frac{Z_t^u (F_t^* - X_t)^{-\theta} P_t^{\min}}{Z_t \tilde{\omega}_t^r} \right)^{-\frac{1}{\theta}} \\ &= 1 + (F_t^* - X_t) \left( \frac{Z_t \int_{a_r}^{\infty} (p_t^{\min}(y))^{1-\theta} n(t, y) dy}{Z_t^u \int_{a_r}^{\infty} p_t^{\min}(y) n(t, y) dy} \right)^{\frac{1}{\theta}}. \end{aligned}$$

Therefore the optimal strategy in this Example 2 with CRRA utility is (on  $[0, \tau^Z]$ )

$$\left\{ \begin{array}{l} \pi_t^* = \delta_t^X + \frac{1}{\theta} (F_t^* - X_t) (\delta_t^R + \eta_t), \\ p_t^*(a) = p_t^{\min}(a) \left( 1 + (F_t^* - X_t) \left( \frac{Z_t \int_{a_r}^{\infty} (p_t^{\min}(y))^{1-\theta} n(t, y) dy}{Z_t^u \int_{a_r}^{\infty} p_t^{\min}(y) n(t, y) dy} \right)^{\frac{1}{\theta}} \right). \end{array} \right.$$

Observe that in Examples 1 and 2, the portfolio are the same. What differs are the pensions. Nevertheless if  $p_t^{\min}(a) = p_t^{\min}$  does not depend on the age, then the optimal pension  $p_t^*(a)$  simplifies in  $p_t^*(a) = p_t^* = p_t^{\min} + (F_t^* - X_t) \left( \frac{Z_t}{Z_t^u} \right)^{\frac{1}{\theta}}$ . It is coherent with Example 1 since it corresponds to the particular case of Example 1 in which each pensioner has weight 1, that is  $\omega_t^r = N_t^r$ .

The increase of the pension amount with age depends on the indexation rate  $\lambda_t$  in  $p_t^{\min}(a)$  and on the optimal adjustment  $\rho_t^* = 1 + (F_t^* - X_t) \left( \frac{Z_t \int_{a_r}^{\infty} (p_t^{\min}(y))^{1-\theta} n(t, y) dy}{Z_t^u \int_{a_r}^{\infty} p_t^{\min}(y) n(t, y) dy} \right)^{\frac{1}{\theta}}$ . As in Example 1,  $\rho^*$  still depends on the importance attributed to the pensioners' preferences with respect to  $Z^u$ . The pension can also be written as follows, to make appear the relative weight of cohort  $a$ , captured through  $p_t^{\min}(a)$ , with respect to the other cohorts:

$$p_t^*(a) = p_t^{\min}(a) + (F_t^* - X_t) \left( \frac{Z_t \int_{a_r}^{\infty} \left( \frac{p_t^{\min}(y)}{p_t^{\min}(a)} \right)^{1-\theta} n(t, y) dy}{Z_t^u \int_{a_r}^{\infty} \left( \frac{p_t^{\min}(y)}{p_t^{\min}(a)} \right) n(t, y) dy} \right)^{\frac{1}{\theta}}.$$

**Conclusion** This paper designs a social planner’s dynamic decisions criterion under sustainability, adequacy and fairness constraints. The optimal investment/pension policy is derived when the social planner can invest in/borrow from a buffer fund, with the aim to provide a better demographic and financial risk-sharing across generations. This flexible modeling can be easily extended to heterogenous cohorts or open populations, thus considering also intra generational risk sharing.

The explicit computations for the optimal policies allows these theoretical results to be applied to an empirical setting with real data. For instance, an interesting application will consist in computing and analyzing actuarial fairness criteria for each generation, and to test the impact of demographic shocks (such as the “baby boom”) on optimal contribution/benefit plans.

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