Probability, Uncertainty and Quantitative Risk
Vol. 9, No. 1, 2024, 13–34

https://www.aimsciences.org/PUQR
DOI:10.3934/puqr.2024002

Bi-revealed utilities in a defaultable universe: A new point of view on consumption

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Abstract This paper investigates the inverse problem of bi-revealed utilities in a defaultable universe, defined as a standard universe (represented by a filtration $\mathbb{F}$) perturbed by an exogenous defaultable time τ. We assume that the standard universe does not take into account the possibility of the default, thus τ adds an additional source of risk. The defaultable universe is represented by the filtration $\mathbb{G}$ *up to time* τ (τ included), where $\mathbb{G}$ stands for the progressive enlargement of $\mathbb{F}$ by τ. The basic assumption in force is that τ avoids $\mathbb{F}$-stopping times. The bi-revealed problem consists in recovering a consistent dynamic utility from the observable characteristic of an agent. The general results on bi-revealed utilities, first given in a general and abstract framework, are translated in the defaultable $\mathbb{G}$-universe and then are interpreted in the $\mathbb{F}$-universe. The decomposition of $\mathbb{G}$-adapted processes $X^{\mathbb{G}}$ provides an interpretation of a $\mathbb{G}$-characteristic $X^{\mathbb{G}}_\tau$ stopped at τ as a reserve process. Thanks to the characterization of $\mathbb{G}$-martingales stopped at τ in terms of $\mathbb{F}$-martingales, we establish a correspondence between $\mathbb{G}$-bi-revealed utilities from characteristic and $\mathbb{F}$-bi-revealed pair of utilities from characteristic and reserves. In a financial framework, characteristic can be interpreted as wealth and reserves as consumption. This result sheds a new light on the consumption in utility criterion: the consumption process can be interpreted as a certain quantity of wealth, or reserves, that are accumulated for the financing of losses at the default time.

Keywords Bi-revealed utilities, Defaultable universe, Enlargement of filtration, Preference criteria of wealth and consumption

2020 Mathematics Subject Classification 91B16, 62P05, 91G40

1. Introduction

Bi-revealed utilities in a general framework Decision making under uncertainty is generally

Received 2 January 2023; Accepted 10 February 2024; Early access 12 March 2024
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considered as the selection in an uncertain universe of an optimal sequence of actions, according to a given preference criterion. Nevertheless, as pointed out first by Samuelson in the 40's and reformulated later by Chambers as *"We can never see a utility function, but what we might be able to see are demand observations at a finite list of prices."* This raises the "inverse" problem to recover the criterion from the data, posed by Samuelson in the theory of "revealed preference" [20, 21], and which is more relevant today than ever due to the expansion of computer science. For example the e-commerce is interested in recovering the utility function of a user from her searches and purchases history on the Internet. This revealed utility is then used to target the user with customized proposals of products in accordance with her preferences, see William and al [24].

The preference relations are commonly represented by utility functions (see for example Chambers & Echenique [4] or Arthur [2]) which are concave, continuous, increasing functions, and usually satisfying Inada conditions. Contrary to a standard approach relying on an optimisation context, we formulate and investigate the inverse problem in a very general, abstract and dynamic framework, as in El Karoui and Mrad [13]. The inverse problem starts from the observation of the outcomes $\{X_t(x)\}$ of a "player" from several initial conditions x. This characteristic $X_t(x)$ at date t (starting from x at $t = 0$) is the result of a series of decisions, for example the trajectory followed on a decision tree. For a player, it can represent the amount of rewards collected, the amount of money accumulated or simply the progress or the path taken. For an investor in a financial market or a gambler in a horse race, $\{X_t(x)\}$ is the wealth process. Based on the observation of these outcomes $\{X_t(x)\}$ of the player, we deduce her dynamic utility $\{U(t,z)\}$ by imposing time-consistency constraints. Indeed, in this uncertain and dynamic universe, time transmits a source of risk that players are confronted with, and they expect this effect of time to be mitigated when they act according to the "best" sequence of decisions. In other words, one would like to construct a dynamic utility process $\{U(t,z)\}$ such that along the characteristic process $\{X_t(x)\}$ the criterion is constant in expectation, for any stopping time: $\{X_t(x)\}$ is then said to optimally reveal $\{U(t,z)\}$. To expect a unique solution, we associate the dual problem driven by $\widetilde{U}(t,y)$ the Fenchel conjugate of U ($\widetilde{U}(t,y) := \sup_z\{U(t,z) - z\,y\}$, with equality for $y = U_z(t,z)$), and we require $\widetilde{U}$ to be optimally revealed by some adjoint process $\{Y_t(y)\}$. As in convex analysis, the adjoint processes $\{Y_t(y)\}$ (also known as pricing kernel in economy) are negatively correlated to $\{X_t(x)\}$, that is $\{X_t(x)Y_t(y)\}$ is decreasing in expectation $\forall x, y$ (which means that the economic value of $\{X_t(x)\}$ using the pricing kernel $\{Y_t(y)\}$ is decreasing). The powerful results in [13] translate the above constraints formulated in expectation into a pathwise construction of the bi-revealed dynamic utility given by the *pathwise first order condition* $Y_t(u_z(x)) = U_z(t, X_t(x))$. This highlights the one to one correspondence between the class of stochastic utilities revealed by the characteristic process $\{X_t(x)\}$ and the class of adjoint processes.

Defaultable universe Those algebraic results are applied in a framework where the available information is modeled by a filtration, and we investigate in particular the impact of an unpredictable adverse event arriving at a random time τ. For example, τ could be a default time. We consider a defaultable universe, that consists in a standard universe (represented by a filtration $\mathbb{F}$) perturbed by an exogenous defaultable time τ. We assume that the standard universe does not take into account the possibility of the default, which is not accessible and adds an additional source of risk. We will see that the awareness of such unfavorable event will change the agents' behaviors, the precautionary principle leading them to accumulate some reserves which could be used at the default. In this context, the bi-revealed problem is applied directly in the defaultable universe stopped at τ and then translated in the standard universe: this is done in the spirit of reduction of filtration (and not enlargement of filtration) as in Crépey and Song [7] but in our framework the processes are stopped at time τ (included) and not just before default (τ^-). Under the standard assumption that τ is independent of $\mathbb{F}$, the projection on $\mathbb{F}$ is straightforward. Otherwise, one need to rely on the theory of progressive enlargement of filtration that has been largely studied in the literature, from the seminal papers in the seventies (such as Brémaud and Yor [3], or Jeulin [14]) to more recent works often motivated by credit risk applications (we refer to the survey book of Aksamit and Jeanblanc [1]). To model this additional risk conveyed by τ (that is not an $\mathbb{F}$-stopping time), we consider the defaultable universe as the progressive enlargement of the reference filtration $\mathbb{F}$ by the default time filtration $\sigma(t\wedge\tau)$, denoted as the filtration $\mathbb{G}$. Different assumptions on the dependence between τ and the standard universe are considered in the literature, including the density assumption which is particularly useful when considering processes after τ, such as in El Karoui et al. [11]. We consider here a defaultable universe stopped at time τ, and thus we work under the weak assumption that τ avoids $\mathbb{F}$-stopping times, which is satisfied both in the standard independent case and under the density assumption (with respect to a non-atomic reference measure). Then any $\mathbb{G}$-optional process $H^{\mathbb{G}}$ stopped at τ is characterized by *a pair of* $\mathbb{F}$-*processes* $(H^{\mathrm{bd}}, H^{\mathrm{sd}})$ representing the before default, and stopped at the default, components : $H^{\mathbb{G}}_t = H^{\mathrm{bd}}_t \mathbb{1}_{t<\tau} + H^{\mathrm{sd}}_\tau \mathbb{1}_{\tau\leqslant t}$. This decomposition is key to interpret the characteristic at default $X^{\mathbb{G}}_\tau$ as a reserve process $\{X^{\mathrm{sd}}_t\}$.

Defaultable bi-revealed problem This decomposition also highlights a matching between $\mathbb{G}$-dynamic utility $U^{\mathbb{G}} = (U^{\mathrm{bd}}, U^{\mathrm{sd}})$ and pair of $\mathbb{F}$-dynamic utilities $(U^{\mathbb{F}}, V^{\mathbb{F}})$ determined respectively by U^{bd} and U^{sd}, as well as for the conjugate utilities $\tilde{U}^{\mathbb{G}} = (\tilde{U}^{\mathrm{bd}}, \tilde{U}^{\mathrm{sd}})$ and $(\tilde{U}^{\mathbb{F}}, \tilde{V}^{\mathbb{F}})$. This, together with the characterization of $\mathbb{G}$-martingales stopped at τ in terms of $\mathbb{F}$-martingales, allows us to provide a correspondence between the bi-revealed utilities in the two universes: $U^{\mathbb{G}}$ is bi-revealed by $X^{\mathbb{G}}$ if and only if the pair $(U^{\mathbb{F}}, V^{\mathbb{F}})$ is bi-revealed by (X^{bd}, C^X) (a precise definition will be given in Section 4), where $C^X_t(x) := X^{\mathrm{sd}}_t(x)\frac{Y^{\mathrm{sd}}_t}{Y^{\mathrm{bd}}_t}(u_z(x))$ is the reserve process. In a financial framework, the characteristic can be interpreted as wealth, the reserve as consumption, and the adjoint process as pricing kernel. The previous equality means that the reserve/consumption process is the wealth at the default with a pricing kernel equal to the jump ratio of the $\mathbb{G}$-pricing kernel. This result sheds a new light on the consumption process in preferences and bridges the gap between investment-consumption utility criteria and pure investment utility criteria. *The investment-consumption preference process is indeed equivalent to a pure investment preference process, in a defaultable universe, with a random horizon.*

The paper is organized as follows. Section 2 introduces the setting and states the inverse problem of bi-revealed utilities, in a very general and abstract framework. The main results are recalled, emphasizing on the key properties of the approach. Section 3 introduces the defaultable universe, as the progressive enlargement of a reference filtration $\mathbb{F}$ by a random time τ. Assuming that τ avoids $\mathbb{F}$-stopping times, $\mathbb{G}$-adapted processes are characterized by a pair of $\mathbb{F}$-adapted processes and a characterization of $\mathbb{G}$-martingales stopped at τ, in terms of $\mathbb{F}$-martingales, is provided. Results of the two previous sections are gathered in Section 4 to interpret the characteristic process at default as a reserve process and to establish a correspondence between $\mathbb{G}$-bi-revealed utilities from characteristic and $\mathbb{F}$-bi-revealed pair of utilities from characteristic and reserves. Section 5 further investigates this analogy in the specific framework of a financial market and explores this new interpretation of the consumption as reserves put aside to accumulate a wealth at default.

2. Bi-revealed utilities

2.1 Dynamic utility

Traditionally, the expected utility criterion, first introduced by Neumann and Morgenstern [23] is used to measure the performance of an agent strategy. It is based on a priori specification of a deterministic, concave and increasing (utility) function measuring the terminal performance at a specified time horizon T. The concavity of the criterion expresses the risk aversion of the agent described by the coefficient γ in the popular power utility $u^\gamma(x)=\frac{x^{1-\gamma}}{(1-\gamma)}$. To alleviate this horizon dependence and to allow a dynamic adaptation of agent preferences to be adjusted to the environment evolution, we are concerned with dynamic utilities (see [9, 12, 13]). This concept was introduced for the first time by Musiela and Zariphopoulou [16–18] under the name of "forward utilities" or "performance processes", and it was further studied in many contributions. First concerning the characterization of these random fields, using duality such as in [25], or more recently in the context of consumption and forward utility with no volatility component (decreasing utility) such as in [15]. Other contributions focus on their various applications, either for life insurance contracts [5] or for long-term yield curves [10].

Dynamic utilities provide a flexible setting for dynamic modeling of agents' preferences in a stochastic environment. In Economics, the performance concerns essentially positive quantity (wealth for instance), so the performance criteria are defined on $\mathbb{R}^+$ with non-negative values.

As usual, a " regular " (deterministic) utility is an increasing, concave function $u:\mathbb{R}^+\to\mathbb{R}^+\cup\{\infty\}$ with $u(0)=0$ and whose first order derivative u_z, also called *marginal utility*, is continuous, positive and decreasing. Moreover its asymptotic behavior is specified by the Inada conditions: $u_z(+\infty)=0$ and $u_z(0)=+\infty$. As usual in convex analysis, it is useful to introduce the conjugate utility $\tilde{u}$ defined via the Fenchel-Legendre formula, described by the properties of the Gap function.

$$\text{Fenchel formula}\quad \tilde{u}(y)=u\left(u_z^{-1}(y)\right)-yu_z^{-1}(y)=\max_{z\in\mathbb{Q}^+}(u(z)-zy), \tag{2.1}$$

$$\text{Gap function}\qquad G^u(z,y)=zy-u(z)+\tilde{u}(y)\geqslant 0, \tag{2.2}$$

$$\text{Legendre formula}\quad u_z^{-1}(y)=-\tilde{u}_y(y),\quad G^u\left(z,u_z(z)\right)=0. \tag{2.3}$$

By definition the conjugate $\tilde{u}$ is a decreasing convex function, with derivative $\tilde{u}_y(y)$ taking values in $(0,+\infty)$. In what follows, we will be concerned by the pair of utility function and its conjugate $(u,\tilde{u})$.

The classic notion of " regular " utility and its conjugate can be generalized to universes in which the uncertainty does not concern only the specific risk. The randomness of the universe is modeled via a probability space $(\Omega,\mathcal{H},\mathbb{P})$ equipped with a filtration $\mathbb{H}=(\mathcal{H}_t)$ satisfying the usual conditions of right-continuity and completeness.

A dynamic (or forward) utility U may be interpreted as a collection of random " regular " utility functions $U(t,\omega,.)$ for which the temporal evolution is " updated " over the time in accordance with the new information $(\mathcal{H}_t)$. The dynamic conjugate utility $\widetilde{U}(t,\omega,.)$ is the collection of conjugate functions associated with $U(t,\omega,.)$.

Definition 2.1 *Let* $(\Omega,(\mathcal{H}_t),\mathbb{P})$ *the reference probability space. A pair of* dynamic utilities $(U,\widetilde{U})$ *is a family of adapted càdlàg random fields* $\{U(t,z),z\in\mathbb{R}^+,\widetilde{U}(t,y),y\in\mathbb{R}^+\}$ *such that* $\mathbb{P}$.a.s.*, for every t, the function* $z\mapsto U(t,z,\omega)$ *is a "regular" utility function (satisfying the Inada* *conditions), and the function* $y \mapsto \widetilde{U}(t,\omega,y)$ *is the associated conjugate random field,* $\tilde{U}(t,y,\omega) = \max_{z\in\mathbb{Q}^+}\big(U(t,\, z,\omega) - z\, y\big)$.

2.2 Inverse problem settings

This paper falls into the theory of "revealed preference" introduced by Samuelson in the 40's, where the problem is to find a dynamic utility maximizing dynamically the observed strategy of a rational agent. More specifically, the problem is as follows: we suppose that we observe a sequence of decisions taken by an agent or her characteristic process X and we want to recover from these observations her preference criterion, that is her utility U at each date.

To achieve this, one needs to specify the constraints and especially the link between the observable (also called characteristic) X and the utility U we are trying to reveal. The first point is that if the agent has decided to make a particular choice, then in view of her criterion this choice must be the best one. Therefore, X is interpreted as an "optimal" (non-negative) process for the revealed criterion. So by analogy with the classical optimization framework where the value function along the characteristic process is a martingale, we look for U verifying the condition $\{U(t,X_t(x))\}$ is a martingale for any $X_0 = x$. In this dynamic point of view, this martingale property expresses the time-stability on the "expected utility" (performance) of the preference criterion in the future. Obviously there are many solutions U, so that as usual in convex analysis, we add dual constraints based on the convex conjugate dynamic utility. A particularly interesting subclass is the class of *bi-revealed utilities*, i.e. the class where revealing U automatically reveals its Fenchel's transform $\widetilde{U}$, i.e., there exists a positive process Y such that $\{\widetilde{U}(t,Y_t(y))\}$ is a martingale for any $y = Y_0$. To summarize, given a data X and an initial utility function $U(0,.) = u(.)$, we are looking for a pair of dynamic utilities $\{U(t,z),\widetilde{U}(t,y)\}$, (linked by the Legendre relation $U(t,z) = \inf_{y\in\mathbb{Q}^+}\{zy - \widetilde{U}(t,y)\}$) such that

- $(U,\widetilde{U})$ are optimally fitted with the observable X and for some positive process Y, that is the processes $\{U(t,X_t(x))\}$ and $\{\widetilde{U}(t,Y_t(y))\}$ are martingales for any $(x,y) > 0$.

- As usual in convex analysis, X and Y are negatively dynamically depending, that is the processes $\{X_t(x)Y_t(y)\}$ are assumed to be supermartingale for any (x,y) (that is Y belongs to the "orthogonal" cone of X).

Remark 2.2 *In El Karoui and Mrad [13], the authors also show that identifying the class of bi-revealed utilities is equivalent to identifying the process Y which we do not assume to be observable.*

The previous features are summarized in the following definition.

Definition 2.3 (Bi-revealed utility). *A pair of dynamic conjugate utility* $\big(U(t,z),\tilde{U}(t,y)\big)$ *is bi-revealed by the system* $\big(u(z),X_t(x),Y_t(y)\big)$ *if* $\forall x,y$

$$U(0,z) = u(z), \tag{2.4}$$

$$\{U(t,X_t(x))\} \quad \textit{and} \quad \{\tilde{U}(t,Y_t(y))\} \quad \textit{are martingales}, \tag{2.5}$$

$$\textit{The product} \quad \{X_t(x)Y_t(y)\} \quad \textit{is a supermartingale}. \tag{2.6}$$

A direct consequence of this definition combined with that of the Fenchel-transform is a direct link between the two processes X and Y and the marginal utility U_z. This is an important result, which can be interpreted as the analog of the maximum principle.

Theorem 2.4 (Necessary condition). *Assume the pair of conjugate utility random fields* $\big(U(t,z),\tilde{U}(t,y)\big)$ *to be bi-revealed by the system* $\big(u(z),\{X_t(x)\},\{Y_t(y)\}\big)$*. Then,*

$$Y_t(u_z(x)) = U_z(t,X_t(x)) \quad \text{a.s. } \forall (t,x), \tag{2.7}$$

$$\textit{The product} \quad \{\, X_t(x)Y_t(u_z(x))\} \quad \textit{is a martingale.} \tag{2.8}$$

If in addition, $\forall t \geqslant 0$, a.s. $x \mapsto X_t(x)$ *is increasing, identity* (2.7) *implies that* Y *is also increasing. Denoting by* $\xi_t(z)$ *the inverse flow of* $x \to X_t(x)$ $\;(X_t(\xi_t(z)) = z)$ *and by* $\zeta_t(v)$ that of $y \mapsto Y_t(y)$ *, the marginal utilities* $(U_z, \widetilde{U}_y)$ *are given by*

$$U_z(t,z) = Y_t(u_z(\xi_t(z))) \quad \textit{and} \quad \widetilde{U}_y(t,v) = -X_t(-\tilde{u}_y(\zeta_t(v))), \quad \text{a.s.} \tag{2.9}$$

Then, since $U(t,0)=0$ *and* $\widetilde{U}(t,\infty)=0$ *, by a change of variable,*

$$U(t,X_t(x)) = \int_0^x Y_t(u_z(z))\mathrm{d}_z X_t(z) \quad \textit{and} \quad \widetilde{U}(t,Y_t(y)) = \int_y^\infty X_t(-\tilde{u}_y(z))\mathrm{d}_z Y_t(z).$$

Note that conditions (2.6) and (2.8) are different: depending on the initial conditions the product XY is a supermartingale or a martingale. These initial conditions play a crucial role in our study and are often the key to the main results established in [13], see Theorem 2.5. As it is shown in Theorem 2.4, the monotonicity with respect to x suffices to characterize these utilities but observe that necessarily $Y_t(u_z(\xi_t(z))$ has to be integrable in a neighborhood of $z=0$ as any marginal utility function.

Proof The key point is that for any pair of utility function $(u,\tilde{u})$ and a pair of dynamic utility $(U,\widetilde{U})$, the Gap formulation of the Fenchel inequality, implies that

$$G^u(z,y) = \tilde{u}(y) - u(z) + zy \geqslant 0 \quad \text{and} \quad G^U(t,X_t(x),Y_t(y)) \geqslant 0, \quad \text{a.s.},$$

where by analogy with (2.2), $G^U(t,X_t(x),Y_t(y)) := \widetilde{U}(t,Y_t(y)) - U(t,X_t(x)) + X_t(x)Y_t(y)$. So, if for any x and y, $\{U(t,X_t(x))\}$ and $\{\tilde{U}(t,Y_t(y))\}$ are martingales, and $\{X_t(x)Y_t(y)\}$ are supermartingales, then taking the expectation in the last inequality, leads to

$$\mathbb{E}\big[G^U(t,X_t(x),Y_t(y))\big] = G^u(x,y) + \mathbb{E}(X_t(x)Y_t(y)) - xy \geqslant 0.$$

For $y = u_z(x)$, $G^u(x,u_z(x)) = 0$, the right-hand side of this inequality is equal to $\mathbb{E}(X_t(x)Y_t(y)) - xy$ which is non negative by the Gap properties and non positive by the supermartingale property. Then, $\mathbb{E}(X_t(x)Y_t(u_z(x))) = xu_z(x)$.

The same relation applied to bounded stopping times is equivalent to the martingale property of $X_t(x)Y_t(u_z(x))$. Moreover the non negative function $G^U(t,X_t(x),Y_t(u_z(x)))$ having a null expectation, for any (x,t), $G^U(t,X_t(x),Y_t(u_z(x))) = 0$, a.s. By monotonicity of $\widetilde{U}(t,.)$, for any x and t $Y_t(u_z(x)) = U_z(t,X_t(x))$, a.s. This means that the condition $y = u_z(x)$ propagates in time and becomes $Y_t(u_z(x)) = U_z(t,X_t(x))$. This achieves (2.7) and (2.8). The end of the proof is straightforward.

As said above, identifying the bi-revealed utilities compatible with the observable characteristic process X and the initial utility u is equivalent to identifying Y satisfying the above requirements. The synthesis of these results is given below (a detailed proof can be found in [13]). □

Theorem 2.5 (Sufficient condition, [13]). *Let* u *be a deterministic utility function and let the characteristic process* X *to be increasing (with respect to* x*). For any positive increasing (with respect to* y*) process* Y *satisfying the conditions:*

$$\forall (x,y) \quad \{X_t(x)Y_t(y)\} \textit{ is a supermartingale}, \quad \{X_t(x)Y_t(u_z(x))\} \textit{ is a martingale}, \qquad (2.10)$$

the pair of random fields $(U,\widetilde{U})$ *is a bi-revealed dynamic utility, where*

$$\left\{U(t,z) := \int_0^z Y_t(u_z(\xi_t(x)))\mathrm{d}x\right\} \quad \textit{and} \quad \left\{\widetilde{U}(t,y) = \int_y^\infty X_t(-\tilde{u}_y(\zeta_t(z)))\mathrm{d}z\right\}.$$

3. Application in a universe with one default

The purpose of this section is to apply the algebraic results of Section 2 in a context where the available information is modeled by a filtration. More precisely, we develop the previous results in a universe (represented by a filtration $\mathbb{G}$) where an exogenous defaultable time τ can perturb the standard universe (represented by a filtration $\mathbb{F}$). As highlighted in Section 2, special attention should be given to the semimartingale decomposition in the new $\mathbb{G}$-universe (here the generic filtration $\mathbb{H}$ in Section 2 will be $\mathbb{G}$) and to the characterization of martingale property relatively to these two filtrations. How to enlarge a standard universe by a random time τ has received a lot of attention since the seventies, under different assumptions on the dependence between τ and the standard universe. Often, papers refer to initial enlargement or progressive enlargement of the standard universe. Applications to finance concern the default problem, when the variable τ is interpreted as a default time. We recall hereafter some results on this enlarged universe in a general setting.

3.1 The general framework of enlargement of filtration

The standard universe is represented by a general filtration $\mathbb{F}$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ satisfying the usual assumption of completeness and right continuity. This standard universe does not take into account the possibility of an unpredictable adverse event, arriving at a random time τ. How the awareness of such unfavorable event will change the agents' behaviors? To answer this question, it is convenient to introduce the extended probability space $(\bar{\Omega}, \bar{\mathcal{A}})$, where $\bar{\Omega} = \Omega \times \mathbb{R}^+$, and $\bar{\mathcal{A}} = \mathcal{A} \otimes \mathcal{B}(\mathbb{R}^+)$. We also denote $\bar{\mathcal{F}}_t = \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+), \bar{\mathcal{O}} = \mathcal{O}^F \otimes \mathcal{B}(\mathbb{R}^+)$, $\bar{\mathcal{P}} = \mathcal{P}^F \otimes \mathcal{B}(\mathbb{R}^+)$, where $\mathcal{O}^F$ (resp $\mathcal{P}^F$) are the $\mathbb{F}$-optional (resp. $\mathbb{F}$-predictable) σ-fields.

Let τ be a **finite** random time, defined as a random horizon for the universe. Often, we make reference to this random horizon as a default time. A priori, τ is not an $\mathbb{F}$-stopping time. The default time τ is considered as a random event that does not perturb the standard universe $\mathbb{F}$ before its arrival, and that has an impact when it arrives and after. Therefore, the new filtration $\mathbb{G}$ of interest is defined as the progressive enlargement of the filtration $\mathbb{F}$ by the random time τ. In the case of a progressive enlargement, the exogenous information is given by the filtration $\mathbb{D}$ generated by the process $(\tau \wedge t)$, where $\mathcal{D}_t = \cap_{s>t}\sigma(\tau \wedge s)$ is the minimal (right continuous) filtration that makes τ a stopping time. A $\mathcal{D}_t$-random variable is a function of $\tau \wedge t$, $f(\tau \wedge t) = f(t)\mathbb{1}_{t<\tau} + f(\tau)\mathbb{1}_{\tau \leqslant t}$. The progressively enlarged filtration $\mathbb{G}$ is the right-continuous and complete version of the filtration generated by the benchmark filtration $\mathbb{F}$ and the filtration $\mathbb{D}$ associated with the process $(\tau \wedge t)$:

$$\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{D}_t.$$

Then, a $\mathcal{G}_t$-random variable is of the form $H_t^G = \bar{H}_t(\tau \wedge t)$, where $\bar{H}_t(\theta)$ is a $\bar{\mathcal{F}}_t$-random variable. More generally, a $\mathbb{G}$-optional (predictable) process is given from a $\bar{H}$-optional (predictable) process by $\bar{H}_t(t \wedge \tau)$. Since τ is a $\mathbb{G}$-stopping time, $\mathcal{G}_\tau = \{\bar{H}_\tau(\tau), \bar{H}_t(\theta) \in \bar{O}^F\}$.

Sometimes, the default time τ is known and has an impact from the beginning, this corresponds to the setting of the initial enlargement filtration $\mathcal{G}_t^I = \mathcal{F}_t \vee \sigma(\tau)$ (considering again the right-continuous and complete version of the filtration $\mathbb{G}^I$). Observe that $\mathcal{F}_t \subset \mathcal{G}_t \subset \mathcal{G}_t^I$. This initially enlarged filtration $\mathbb{G}^I$ is related to the filtration $\bar{\mathbb{F}}$ since any $\mathbb{G}^I$-optional (predictable) process is given from a $\bar{H}$-optional (predictable) process by $\bar{H}_t(\tau)$.

In this paper, we are concerned with processes stopped at time τ, and then to the properties of the σ-algebra $\mathbb{G}_\tau$. Therefore, we only assume the following assumption, that is weaker that the density assumption usually stated in the literature.

Assumption 3.1 *τ avoids $\mathbb{F}$-stopping time, that is for any $\mathbb{F}$-stopping time η, $\mathbb{P}(\tau = \eta) = 0$.*

This condition is sometimes called "avoidance assumption", see e.g. Nikeghbali [19], Choulli et al. [6], Corollary 2.5. and in Di Tella et al. [8], or under the name of "thick" random time, see Aksamit and Jeanblanc [1]. A direct consequence of Assumption 3.1 is that any $\mathbb{F}$-adapted càdlàg process Z is τ-continuous, meaning that $Z_\tau = Z_{\tau-}$, a.s. In particular, for any $\mathbb{F}$-stopping time η, the $\mathbb{F}$-optional process $\mathbb{1}_{[0,\eta[}(.)$ stopped at time τ coincides with the $\mathbb{F}$-predictable process $\mathbb{1}_{[0,\eta]}(.)$ stopped at time τ: $\mathbb{1}_{[0,\eta[}(\tau) = \mathbb{1}_{[0,\eta]}(\tau)$ a.s. Since $\mathcal{O}^F$ (resp. $\mathcal{P}^F$) is generated by $\{\mathbb{1}_{[0,\eta[},\ \eta\ \mathbb{F}\text{-stopping time}\}$, (resp. $\{\mathbb{1}_{[0,\eta]},\ \eta\ \mathbb{F}\text{-stopping time}\}$), Assumption 3.1 implies that any $\mathbb{F}$-optional process stopped at τ is indistinguishable to an $\mathbb{F}$-predictable process stopped at τ.

Remark 3.2 (*$\mathcal{F}$-Density assumption). Assumption 3.1 is not standard in the literature, which usually assumes the stronger density assumption with respect to a non-atomic reference measure: there exists a positive σ-finite, non-atomic measure $\mu(\mathrm{d}\theta)$ and for any time $t \geqslant 0$, a non negative $\bar{\mathbb{F}}$-adapted process $\alpha_t(\theta)$ such that,*

$$\mathbb{P}[\tau \in \mathrm{d}\theta | \mathcal{F}_t] = \alpha_t(\theta)\mu(\mathrm{d}\theta), \quad \text{with } \int_0^\infty \alpha_t(\theta)\mu(\mathrm{d}\theta) = 1, \quad \mathbb{P}\text{-a.s.} \tag{3.1}$$

This density assumption implies Hypothesis 2.1. Indeed, let η be a $\mathbb{F}$-stopping time, then the optionalprocess $\mathbb{1}_{\{\eta=\theta\}}$ is μ-negligeablesince μ isnon-atomic: $\mathbb{P}[\tau = \eta] = \mathbb{E}\left[\int \mathbb{1}_{\{\eta=\theta\}}\ \alpha_\eta(\theta)\mu(\mathrm{d}\theta)\right] =$ 0. The second condition in (3.1) expresses that τ is finite a.s. This density assumption is particularly useful when considering processes after τ, such as in El Karoui, Jeanblanc and Jiao [11].

3.2 Before and at the default description

To facilitate the mapping between the $\mathbb{G}$-point of view and the $\mathbb{F}$-point of view, we introduce a pair of $\mathbb{F}$-processes, characterizing the before and at the default behaviors. This decomposition is at the core of the interpretation of the characteristic at default as a reserve process, and thus highlights the precautionary behavior of the agent. Such splitting formula is a priori not immediate and does not hold in all generality for $\mathbb{G}$-optional process, as pointed out by Song [22]. It is usually stated under the density assumption. In our setting, since all processes are stopped at time τ, it is sufficient to assume that τ avoids the $\mathbb{F}$-stopping times: then $\{\tau = t\}$ is negligible and since $\mathbb{F}$ is complete, for $\mathbb{G}$-optional process $H^{\mathbb{G}}$ {stopped at time τ }, the splitting formula holds

$$\big\{H_t^{\mathbb{G}} = H_t^{\mathrm{bd}} \mathbb{1}_{t<\tau} + H_\tau^{\mathrm{sd}} \mathbb{1}_{\tau \leqslant t}\big\}, \tag{3.2}$$

where $\{H_t^{\mathrm{bd}}\}$ is a $\mathbb{F}$-optional process and $\{H_t^{\mathrm{sd}}\}$ is $\mathbb{F}$-predictable process. Remark that to define the "stopped at the default behavior" H_τ^{sd}, one need to define an entire $\mathbb{F}$-predictable process $\{H_t^{\mathrm{sd}}\}$. To refer to this splitting formula we will often use the characterization of a $\mathbb{G}$-optional process $H^{\mathbb{G}}$ stopped at time τ as $H^G = (H^{\mathrm{bd}}, H^{\mathrm{sd}})$. The splitting formula for $\mathbb{G}$-predictable processes stopped at time τ is more straightforward (cf. Jeulin [14] Lemma 4.4): any $\mathbb{G}$-predictable process $H^{\mathbb{G}}$ stopped at τ admits the predictable decomposition

$$\big\{H_t^{\mathbb{G}} = H_t^{\mathrm{bd}}\mathbb{1}_{t\leqslant\tau} + H_\tau^{\mathrm{bd}}\mathbb{1}_{t>\tau}\big\}, \tag{3.3}$$

where $\{H_t^{\mathrm{bd}}\}$ is a $\mathbb{F}$-predictable process.

3.3 Doob-Meyer decompositions of survival processes

The survival process $\mathbb{1}_{\{\tau>t\}}$ is a non-increasing càdlàg $\mathbb{G}$-optional process with finite variation. Its $\mathbb{G}$-dual predictable projection (also called predictable compensator) is denoted $\Lambda^{\mathbb{G}}$: $\Lambda^{\mathbb{G}}$ is an $\mathbb{G}$-increasing (and thus of finite variation) predictable process, constant after τ. Thanks to the predictable splitting formula (3.3), there exists an $\mathbb{F}$-predictable increasing process $\Lambda^{\mathbb{F}}$ such that $\Lambda^{\mathbb{G}}_{.} = \Lambda^{\mathbb{F}}_{.\wedge\tau}$. The avoidance Assumption 3.1 implies that $\Lambda^{\mathbb{G}}$ is continuous (see Lemma 8.13 [19]).

Lemma 3.3 *There exists an $\mathbb{F}$-predictable increasing process $\Lambda^{\mathbb{F}}$ such that*

$$\big\{\tilde{N}_t^\tau = \mathbb{1}_{\{\tau\leqslant t\}} - \Lambda^{\mathbb{F}}_{t\wedge\tau}\big\} \textit{ is a } \mathbb{G}-\textit{finite variation local martingale.} \tag{3.4}$$

Although this additive decomposition is commonly used in the literature, the following (equivalent) multiplicative decomposition is more convenient in our framework.

Proposition 3.4 (i) *In the $\mathbb{G}$-universe, $\{L_t^{\mathbb{G}} = \mathbb{1}_{\{\tau>t\}}\exp(\Lambda_t^{\mathbb{F}})\}$ is an $\mathbb{G}$-finite variation local martingale, associated with the multiplicative decomposition of the $\mathbb{G}$-survival process $\mathbb{1}_{\{\tau>t\}}$.*

(ii) *The $\mathbb{F}$-projection of $\{L_t^{\mathbb{G}}\}$ is the $\mathbb{F}$-local martingale $\{L_t^{\mathbb{F}} = S_t e^{\Lambda_t^{\mathbb{F}}}\}$, where $\{S_t = \mathbb{P}(\tau>t|\mathcal{F}_t)\}$ is an $\mathbb{F}$-supermartingale.*

Proof (i) The equivalence between the additive decomposition given by (3.4) and the multiplicative decomposition is a direct consequence of the dynamics

$$\mathrm{d}L_t^{\mathbb{G}} = -L_{t-}^{\mathbb{G}}\,\mathrm{d}\mathbb{1}_{\{\tau\leqslant t\}} + \mathbb{1}_{\{\tau>t\}}\exp(\Lambda_t^{\mathbb{F}})\mathrm{d}(\Lambda_t^{\mathbb{F}}) = -L_{t-}^{\mathbb{G}}\,\mathrm{d}\tilde{N}_t^\tau.$$

(ii) Since $\exp(\Lambda_t^{\mathbb{F}})$ is $\mathcal{F}_t$-measurable, the $\mathbb{F}$-projection of $L_t^{\mathbb{G}}$ is $L_t^{\mathbb{F}} = \mathbb{P}(\tau>t|\mathcal{F}_t)e^{\Lambda_t^{\mathbb{F}}}$. Then the $\mathbb{F}$-local martingale property of $L^{\mathbb{F}}$ is obtained from the $\mathbb{G}$-local martingale property of $L^{\mathbb{G}}$. Note that $S_t = \mathbb{P}(\tau>t|\mathcal{F}_t)$ is an $\mathbb{F}$-supermartingale by transitivity of the conditional expectation. □

Semimartingales (and more precisely supermartingales) play a key role in the characterization of bi-revealed utility (see Section 2). It is thus useful to specify the decompositions of $\mathbb{F}$ and $\mathbb{G}$-semimartingales, stopped at time τ. Let $\big\{Z_t^{\mathbb{G}} = (Z_t^{\mathrm{bd}}, Z_{t\wedge\tau}^{\mathrm{sd}})\big\}$ be an $\mathbb{G}$-semimartingale stopped at time τ. Hereafter we rather adopt the following multiplicative decomposition, valid for $\mathbb{G}$-semimartingales whose before default process Z^{bd} is positive, on which we focus in the sequel. Then $Z^{\mathbb{G}}$ is equivalently defined by the positive before default $\mathbb{F}$-adapted process $Z_{t\wedge\tau}^{\mathrm{bd}}$ stopped at time τ, and its optional relative jump process $\big\{R_t^Z := \frac{Z_t^{\mathrm{sd}}}{Z_t^{\mathrm{bd}}}\big\}$

$$\big\{Z_t^{\mathbb{G}} = Z_{t\wedge\tau}^{\mathrm{bd}}(\mathbb{1}_{\{\tau>t\}} + R_\tau^Z\mathbb{1}_{\{\tau\leqslant t\}})\big\}.$$

We recall that $Z_\tau^{\mathrm{bd}} = Z_{\tau-}^{\mathrm{bd}}$, a.s. since τ avoids $\mathbb{F}$-stopping time (Assumption 3.1), and thus $Z_{t\wedge\tau}^{\mathrm{bd}}$ is τ-continuous. We will address separately the jump part R_τ^Z at time τ, and $Z_{t\wedge\tau}^{\mathrm{bd}}$.

3.4 Characterization of $\mathbb{G}$-martingales stopped at time τ

3.4.1 $\mathbb{G}$-martingale continuous at time τ

The first step consists in characterizing $\mathbb{G}$-martingale τ-continuous and stopped at τ.

For $\mathbb{G}$-processes which are τ-continuous and stopped at τ, the $\mathbb{G}$-martingale property is equivalent (under Assumption 3.1) to the orthogonality with the finite variation martingale $\{L_t^{\mathbb{G}} = \mathbb{1}_{\{\tau>t\}}\exp(\Lambda_t^{\mathbb{F}}) = \mathbb{1}_{\{\tau>t\}}\frac{L_t^{\mathbb{F}}}{S_t}\}$, whose unique jump is at time τ.

Proposition 3.5 *Let* $\{H_t^{\mathbb{G}} = Z_{t\wedge\tau}^{\mathrm{bd}}\}$ *be an* $\mathbb{G}$*-adapted process stopped and continuous at* τ*. Then, under Assumption 3.1,* $\{H_t^{\mathbb{G}} = Z_{t\wedge\tau}^{\mathrm{bd}}\}$ *is* $\mathbb{G}$*-local martingale if and only if one the following equivalent assertions is satisfied:*

(i) $\{Z_t^{\mathrm{bd}}L_t^{\mathbb{G}}\}$ *is an* $\mathbb{G}$*-local martingale.*

(ii) $\{Z_t^{\mathrm{bd}}L_t^{\mathbb{F}}\}$ *is an* $\mathbb{F}$*-local martingale.*

(iii) $\{Z_t^{\mathrm{bd}} + \int_0^t[\mathrm{d}Z^{\mathrm{bd}}, \frac{\mathrm{d}L^{\mathbb{F}}}{L^{\mathbb{F}}}]_s\}$ *is a* $\mathbb{F}$*-local martingale.*

Proof (i) Let $H_t^{\mathbb{G}} = Z_{t\wedge\tau}^{\mathrm{bd}}$ be an $\mathbb{G}$-adapted process stopped at time τ, with no jump at time τ. Then, since $L_t^{\mathbb{G}}$ is a pure jump finite variation $\mathbb{G}$-local martingale, with one jump at τ only, $H_t^{\mathbb{G}}$ is a $\mathbb{G}$-local martingale if and only if $H_t^{\mathbb{G}}L_t^{\mathbb{G}}$ is a $\mathbb{G}$-local martingale. The first equivalence follows by noticing that $H_t^{\mathbb{G}}L_t^{\mathbb{G}} = Z_t^{\mathrm{bd}}L_t^{\mathbb{G}}$ since $L_t^{\mathbb{G}} = 0$ on $\{\tau \leqslant t\}$.

(ii) Since $\mathcal{G}_t = \mathcal{F}_t$ on $\{t < \tau\}$, the $\mathbb{G}$-local martingale property of $Z_t^{\mathrm{bd}}L_t^{\mathbb{G}}$ is equivalent to the $\mathbb{F}$-local martingale property of its $\mathbb{F}$-projection $Z_t^{\mathrm{bd}}L_t^{\mathbb{F}}$.

(iii) By the rule of stochastic calculus, $\mathrm{d}(Z_t^{\mathrm{bd}}L_t^{\mathbb{F}}) = L_{t-}^{\mathbb{F}}\mathrm{d}Z_t^{\mathrm{bd}} + Z_{t-}^{\mathrm{bd}}\mathrm{d}L_t^{\mathbb{F}} + \mathrm{d}[Z^{\mathrm{bd}}L^{\mathbb{F}}]_t$. Since $L_t^{\mathbb{F}}$ is an $\mathbb{F}$-local martingale, $Z_t^{\mathrm{bd}}L_t^{\mathbb{F}}$ is an $\mathbb{F}$-local martingale if and only if $Z_t^{\mathrm{bd}} + \int_0^t[\mathrm{d}Z^{\mathrm{bd}}, \frac{\mathrm{d}L^{\mathbb{F}}}{L^{\mathbb{F}}}]_s$ is an $\mathbb{F}$-local martingale. □

3.4.2 Pure jump $\mathbb{G}$-martingale

The pure-jump $\mathbb{G}$-martingale $L^{\mathbb{G}}$ is characterized by its initial value $L_0^{\mathbb{G}} = 1$ and its value $L_\tau^{\mathbb{G}} = 0$ at time τ. The following proposition characterizes all positive pure jump $\mathbb{G}$-martingales $J^{M,\mathbb{G}}$ having only one jump at time τ. A positive pure jump $\mathbb{G}$-martingale $J^{M,\mathbb{G}}$ is determined by the positive $\mathbb{F}$-optional process R^J associated with the relative jump : $R_t^J = \frac{J_t^{M,\mathrm{sd}}}{J_t^{M,\mathrm{bd}}}$ (for $L^{\mathbb{G}}$, the relative jump process is zero).

Proposition 3.6 *Any nonnegative pure jump* $\mathbb{G}$*-martingale* $J^{M,\mathbb{G}}$ *such that* $J_0^{M,\mathbb{G}} = 1$*, with only one jump at time* τ *satisfies*

$$\frac{\mathrm{d}J_t^{M,\mathbb{G}}}{J_{t-}^{M,\mathbb{G}}} = (1 - R_t^J)\frac{\mathrm{d}L_t^{\mathbb{G}}}{L_{t-}^{\mathbb{G}}}, \quad \textit{with} \quad R_t^J = \frac{J_t^{M,\mathrm{sd}}}{J_t^{M,\mathrm{bd}}} \tag{3.5}$$

and has the following multiplicative representation

$$\begin{aligned} J_t^{M,\mathbb{G}} &= L_{t-}^{\mathbb{G}}\ (1 + (R_\tau^J - 1)\mathbb{1}_{\{\tau\leqslant t\}})\exp\left(-\int_0^{t\wedge\tau} R_s^J\mathrm{d}\Lambda_s^{\mathbb{F}}\right) \\ &= (1 + (R_\tau^J - 1)\mathbb{1}_{\{\tau\leqslant t\}})\exp\left(-\int_0^{t\wedge\tau}(R_s^J - 1)\mathrm{d}\Lambda_s^{\mathbb{F}}\right). \end{aligned} \tag{3.6}$$

The proof is similar to the one in El Karoui, Jeanblanc and Jiao [11] (stated under the density assumption), where we use that the processes here are stopped at time τ and that τ avoids $\mathbb{F}$-stopping times (Assumption 3.1). Differentiating (3.6) gives

$$\mathrm{d}J_t^{M\mathbb{G}} = -J_t^{M,\mathbb{G}}(R_t^J - 1)\mathrm{d}\Lambda_t^{\mathbb{F}} + J_{t-}^{M,\mathbb{G}}((R_t^J - 1)(\mathrm{d}\tilde{N}_t^\tau + \mathrm{d}\Lambda_t^{\mathbb{F}})) = J_{t-}^{M,\mathbb{G}}(R_t^J - 1)\mathrm{d}\tilde{N}_t^\tau.$$

Therefore $J_t^{M,\mathbb{G}}$ is the exponential martingale of the pure-jump martingale $\int(R_t^J-1)\mathrm{d}\tilde{N}_t^\tau$. This implies (3.5), recalling $\mathrm{d}L_t^{\mathbb{G}}=-L_{t-}^{\mathbb{G}}\mathrm{d}\tilde{N}_t^\tau$.

3.4.3 Characterization of $\mathbb{G}$-martingale stopped at time τ

In this section we are concerned with any $\mathbb{G}$-semimartingale $Z^{\mathbb{G}}$ stopped at time τ, with a positive before-default process Z^{bd}. Let us denote the relative jump of $Z^{\mathbb{G}}$ by $R_t^Z=\frac{Z_t^{\mathrm{sd}}}{Z_t^{\mathrm{bd}}}$, that is $R_t^Z-1=\frac{Z_t^{\mathrm{sd}}-Z_t^{\mathrm{bd}}}{Z_t^{\mathrm{bd}}}$. We have the following result.

Theorem 3.7 *Let* $\{Z_t^{\mathbb{G}}=Z_{t\wedge\tau}^{\mathrm{bd}}\,(1\!\!1_{\{\tau>t\}}+R_\tau^Z 1\!\!1_{\{\tau\leqslant t\}})\}$ *be an* $\mathbb{G}$*-semimartingale stopped at time* τ *, with positive before default process* Z^{bd}*. Then* $Z^{\mathbb{G}}$ *is an* $\mathbb{G}$*-supermartingale (resp. submartingale) if and only if the following equivalent assertions is satisfied:*

(i) $\{Z_t^{\mathrm{bd}}\exp(\int_0^t R_s^Z\mathrm{d}\Lambda_s^{\mathbb{F}})1\!\!1_{\{\tau>t\}}\}$ *is an* $\mathbb{G}$*-supermartingale (resp. submartingale);*

(ii) $\{Z_t^{\mathrm{bd}}\exp(\int_0^t(R_s^Z-1)\mathrm{d}\Lambda_s^{\mathbb{F}})L_t^{\mathbb{F}}\}$ *is an* $\mathbb{F}$*-supermartingale (resp. submartingale).*

Proof Let us consider the multiplicative decomposition of $Z_t^{\mathbb{G}}=Z_{t\wedge\tau}^{\mathrm{bd}}\,(1\!\!1_{\{\tau>t\}}+R_\tau^Z 1\!\!1_{\{\tau\leqslant t\}})$

$$Z_t^{\mathbb{G}}=Z_{t\wedge\tau}^{\mathrm{bd}}\exp\left(\int_0^{t\wedge\tau}(R_s^Z-1)\mathrm{d}\Lambda_s^{\mathbb{F}}\right)(1\!\!1_{\{\tau>t\}}+R_\tau^Z 1\!\!1_{\{\tau\leqslant t\}})\exp\left(-\int_0^{t\wedge\tau}(R_s^Z-1)\mathrm{d}\Lambda_s^{\mathbb{F}}\right)$$

that can be written also in terms of $(Z^{\mathrm{bd}},Z^{\mathrm{sd}})$

$$Z_t^{\mathbb{G}}=Z_{t\wedge\tau}^{\mathrm{bd}}\exp(\int_0^{t\wedge\tau}\frac{Z_s^{\mathrm{sd}}-Z_s^{\mathrm{bd}}}{Z_s^{\mathrm{bd}}}\mathrm{d}\Lambda_s^{\mathbb{F}})\,(1+\frac{Z_\tau^{\mathrm{sd}}-Z_\tau^{\mathrm{bd}}}{Z_\tau^{\mathrm{bd}}}1\!\!1_{\{\tau\leqslant t\}})\exp(-\int_0^{t\wedge\tau}\frac{Z_s^{\mathrm{sd}}-Z_s^{\mathrm{bd}}}{Z_s^{\mathrm{bd}}}\mathrm{d}\Lambda_s^{\mathbb{F}}).$$

It is the product of a pure jump $\mathbb{G}$-martingale $(1\!\!1_{\{\tau>t\}}+R_\tau^Z 1\!\!1_{\{\tau\leqslant t\}})\exp(-\int_0^{t\wedge\tau}(R_s^Z-1)\mathrm{d}\Lambda_s^{\mathbb{F}})$ (see Proposition 3.6) and a process $Z_{t\wedge\tau}^{\mathrm{bd}}\exp(\int_0^{t\wedge\tau}(R_s^Z-1)\mathrm{d}\Lambda_s^{\mathbb{F}})$ stopped at τ and τ-continuous. Therefore, $Z^{\mathbb{G}}$ is an $\mathbb{G}$-supermartingale (resp. submartingale) if and only if the process $Z_{t\wedge\tau}^{\mathrm{bd}}\exp(\int_0^{t\wedge\tau}(R_s^Z-1)\mathrm{d}\Lambda_s^{\mathbb{F}})$ is an $\mathbb{G}$-supermartingale (resp. submartingale) (τ-continuous), that will be necessarily orthogonal to the pure jump martingale. This together with Proposition 3.5 achieves the proof. □

Note that if the process $Z^{\mathbb{G}}$ does not jump at τ, then $R^Z=1$ and $\exp(\int_0^t R_s^Z\mathrm{d}\Lambda_s^{\mathbb{F}})=\exp(\Lambda_t^{\mathbb{F}})$ does not depend on the process $Z^{\mathbb{G}}$ (which is exactly Proposition 3.5). Besides, the compensator $\Lambda^{\mathbb{F}}$ can be explicitly computed under the density assumption (Remark 3.2): $\Lambda_t^{\mathbb{F}}=\alpha_t(t)/S_{t-}$.

We also provide hereafter the additive version of the multiplicative characterization of $\mathbb{G}$-martingales, in terms of $\mathbb{F}$-martingales.

Corollary 3.8 *Let* $Z^{\mathbb{G}}=(Z^{\mathrm{bd}},Z^{\mathrm{sd}})$ *be an* $\mathbb{G}$*-semimartingale stopped at time* τ*. Then* $Z^{\mathbb{G}}$ *is* $\mathbb{G}$*-martingale if and only if* $\{\big(Z_t^{\mathrm{bd}}+\int_0^t(Z_s^{\mathrm{sd}}-Z_s^{\mathrm{bd}})\mathrm{d}\Lambda_s^{\mathbb{F}}\big)L_t^{\mathbb{F}}\}$ *is a* $\mathbb{F}$*-martingale.*

Proof According to Theorem 3.7, $Z^{\mathbb{G}}$ is $\mathbb{G}$-martingale if and only if the process $M_t:=Z_t^{\mathrm{bd}}\exp(\int_0^t(R_s^Z-1)\mathrm{d}\Lambda_s^{\mathbb{F}})L_t^{\mathbb{F}}$ is an $\mathbb{F}$-martingale, with $R_t^Z=\dfrac{Z_t^{\mathrm{sd}}}{Z_t^{\mathrm{bd}}}$. Besides,

$$\begin{aligned}\mathrm{d}M_t=&\exp\left(\int_0^t(R_s^Z-1)\mathrm{d}\Lambda_s^{\mathbb{F}}\right)\left(\mathrm{d}Z_t^{\mathrm{bd}}+(Z_t^{\mathrm{sd}}-Z_t^{\mathrm{bd}})\mathrm{d}\Lambda_t^{\mathbb{F}}\right)L_t^{\mathbb{F}}+Z_t^{\mathrm{bd}}\exp\left(\int_0^t(R_s^Z-1)\mathrm{d}\Lambda_s^{\mathbb{F}}\right)\mathrm{d}L_t^{\mathbb{F}}\\&+\exp\left(\int_0^t(R_s^Z-1)\mathrm{d}\Lambda_s^{\mathbb{F}}\right)[\mathrm{d}Z^{\mathrm{bd}},\mathrm{d}L^{\mathbb{F}}]_t\end{aligned}$$

which implies

$$\exp\Big(-\int_0^t (R^Z_s - 1)\mathrm{d}\Lambda^{\mathbb{F}}_s\Big)\frac{\mathrm{d}M_t}{L^{\mathbb{F}}_t} - Z^{\mathrm{bd}}_t\frac{\mathrm{d}L^{\mathbb{F}}_t}{L^{\mathbb{F}}_t} = \mathrm{d}Z^{\mathrm{bd}}_t + (Z^{\mathrm{sd}}_t - Z^{\mathrm{bd}}_t)\mathrm{d}\Lambda^{\mathbb{F}}_t + [\mathrm{d}Z^{\mathrm{bd}}, \mathrm{d}L^{\mathbb{F}}]_t.$$

Since $L^{\mathbb{F}}$ is an $\mathbb{F}$-martingale, M is an $\mathbb{F}$-martingale if and only if the process $\{\big(Z^{\mathrm{bd}}_t + \int_0^t (Z^{\mathrm{sd}}_s - Z^{\mathrm{bd}}_s)\mathrm{d}\Lambda^{\mathbb{F}}_s\big)L^{\mathbb{F}}_t\}$ is also an $\mathbb{F}$-martingale, which concludes the proof. □

4. Bi-revealed utility in a universe with one default

This section is dedicated to the problem of bi-revealed utility in a universe with one default (the $\mathbb{G}$-universe). Applying results of Sections 2 and 3 provides a nice interpretation in terms of a standard universe (the $\mathbb{F}$-universe) in which the characteristic at default time τ can be interpreted as cumulative reserves kept aside to face this unpredictable event. We then associate with the revealed utility in the $\mathbb{G}$-universe with default a pair of utilities of characteristic and reserve in the $\mathbb{F}$-universe without default.

4.1 Semimartingale interpretation

In the $\mathbb{G}$-universe, the characteristic process $X^{\mathbb{G}}$ and the adjoint process $Y^{\mathbb{G}}$ are semimartingales and have the following decomposition

$$\begin{cases} X^{\mathbb{G}}_{t\wedge\tau}(x) = X^{\mathrm{bd}}_{t\wedge\tau}(x)\big(\mathbb{1}_{\{\tau>t\}} + R^X_\tau(x)\mathbb{1}_{\{\tau\leqslant t\}}\big), \quad R^X_t(x) := \dfrac{X^{\mathrm{sd}}_t(x)}{X^{\mathrm{bd}}_t(x)}, \\ Y^{\mathbb{G}}_{t\wedge\tau}(y) = Y^{\mathrm{bd}}_{t\wedge\tau}(y)\big(\mathbb{1}_{\{\tau>t\}} + R^Y_\tau(y)\mathbb{1}_{\{\tau\leqslant t\}}\big), \quad R^Y_t(y) := \dfrac{Y^{\mathrm{sd}}_t(y)}{Y^{\mathrm{bd}}_t(y)}. \end{cases}$$

Let $U^{\mathbb{G}}$ be a bi-revealed utility by the system $\big(u(.), X^{\mathbb{G}}, Y^{\mathbb{G}}\big)$. By definition and by Theorem 2.4, we have necessarily that

$$\Big\{X^{\mathbb{G}}(x)Y^{\mathbb{G}}(y) \text{ is a } \mathbb{G}\text{-supermartingale}, \quad \forall x, y \quad \text{and} \quad \text{a } \mathbb{G}\text{-martingale if } y = u_z(x)\Big\}.$$

Applying Theorem 3.7 and denoting $R^{XY}_t(x,y) := \frac{X^{\mathrm{sd}}_t(x)Y^{\mathrm{sd}}_t(y)}{X^{\mathrm{bd}}_t(x)Y^{\mathrm{bd}}_t(y)}$, this is equivalent to

$$\Big\{X^{\mathrm{b\,d}}_t(x)Y^{\mathrm{b\,d}}_t(y)e^{\int_0^t\big(R^{XY}_s(x,y)-1\big)\mathrm{d}\Lambda^{\mathbb{F}}_s}L^{\mathbb{F}}_t\Big\} \quad \begin{cases} \text{is an } \mathbb{F}\text{-supermartingale}, \forall x, y \\ \text{and } \mathbb{F}\text{-local martingale if } y = u_z(x). \end{cases} \tag{4.1}$$

Relation (4.1) seems first to indicate that the appropriate probability measure to be considered on the small filtration $\mathbb{F}$ is $\mathbb{Q}^L$, defined as $\mathrm{d}\mathbb{Q}^L := L^{\mathbb{F}}\mathrm{d}\mathbb{P}$. $\mathbb{Q}^L$ is a probability measure as soon as $L^{\mathbb{F}}$ is an uniformly integrable martingale, which will be assumed hereafter. This change of probability only depends on τ and thus is conveying the impact of the "pure default" risk. Then, the $(\mathbb{F}, \mathbb{Q}^L)$(super)-martingale property of $\Big\{X^{\mathrm{bd}}_tY^{\mathrm{bd}}_te^{\int_0^t(R^{XY}_s-1)\mathrm{d}\Lambda^{\mathbb{F}}_s} = X^{\mathrm{bd}}_te^{\int_0^t R^{XY}_s\mathrm{d}\Lambda^{\mathbb{F}}_s}Y^{\mathrm{bd}}_te^{-\Lambda^{\mathbb{F}}_t}\Big\}$ can be related to a similar property in a universe without default with a reserve rate R^{XY} (that is the rate of the characteristic that is put aside), and with $Y^{\mathrm{bd}}e^{-\Lambda^{\mathbb{F}}}$ as the adjoint process. The analogy with a financial market with consumption as in [9] is developed in Section 5.

4.2 Correspondence between $\mathbb{G}$- and $\mathbb{F}$-bi-revealed utilities

The following Theorem 4.1 specifies the mapping between an $\mathbb{G}$-utility and an $\mathbb{F}$-pair of utilities of characteristic and reserve. Since the reserve is integrated with respect to $\mathrm{d}\Lambda^{\mathbb{F}}_t$ in (4.1), we use in the following notations the upper script Λ to remind this point.

Theorem 4.1 *Let $U^{\mathbb{G}}$ be an $\mathbb{G}$-dynamic utility bi-revealed by the triplet $(u, X^{\mathbb{G}}, Y^{\mathbb{G}})$. From $U^{\mathbb{G}} = (U^{\mathrm{bd}}, U^{\mathrm{sd}})$, we define the pair $(U^{\mathbb{F}}, V^{\mathbb{F},\Lambda})$ of $\mathbb{F}$-dynamic utilities as follows*

$$\Big\{U^{\mathbb{F}}(t,z) := e^{-\Lambda^{\mathbb{F}}_t} U^{\rm bd}(t,z)\Big\} \quad and \quad \Big\{V^{\mathbb{F},\Lambda}(t,c) := e^{-\Lambda^{\mathbb{F}}_t} U^{\rm sd}(t, c\frac{Y^{\rm bd}_t}{Y^{\rm sd}_t})\Big\}. \tag{4.2}$$

Let $\mathbb{Q}^L$ be the equivalent probability measure defined from the $\mathbb{P}$-martingale $L^{\mathbb{F}}$ by $d\mathbb{Q}^L = L^{\mathbb{F}} d\mathbb{P}$ with $\{L^{\mathbb{F}}_t = S_t e^{\Lambda^{\mathbb{F}}_t}\}$ and $\{S_t = \mathbb{P}(\tau > t|\mathcal{F}_t)\}$. We denote by C^Λ the reserve process $\Big\{C^X_t(x) := X^{\rm sd}_t(x)\frac{Y^{\rm sd}_t(u_z(x))}{Y^{\rm bd}_t(u_z(x))}\Big\}$. Then the following martingale properties are equivalent

$$\Big\{U^{\mathbb{G}}(t, X^{\mathbb{G}}_t(x)) = U^{\rm bd}(t, X^{\rm bd}_t(x))\mathbb{1}_{t<\tau} + U^{\rm sd}(t, X^{\rm sd}_t(x))\mathbb{1}_{\tau\leqslant t}\Big\} \text{ is an } (\mathbb{G},\mathbb{P})\text{-martingale}, \quad \forall x. \tag{4.3}$$

$$\Big\{U^{\mathbb{F}}(t, X^{\rm bd}_t(x)) + \int_0^t V^{\mathbb{F},\Lambda}(s, C^X_s(x)) d\Lambda^{\mathbb{F}}_s\Big\} \text{ is an } (\mathbb{F},\mathbb{Q}^L)\text{-martingale}, \quad \forall x. \tag{4.4}$$

By construction, the random fields $U^{\mathbb{F}}$ and $V^{\mathbb{F}}$ inherit the property of monotonicity and concavity of U^G and they are $\mathbb{F}$-progressive processes: therefore $U^{\mathbb{F}}$ and $V^{\mathbb{F}}$ are $\mathbb{F}$-dynamic utilities. The conjugate utilities in the $\mathbb{F}$-universe are characterized from $U^{\mathbb{F}}$ and $V^{\mathbb{F},\Lambda}$ using the Legendre relation:

$$\Big\{\tilde U^{\mathbb{F}}(t,z) := e^{-\Lambda^{\mathbb{F}}_t}\tilde U^{\rm bd}(t, z e^{\Lambda^{\mathbb{F}}_t})\Big\} \quad \text{and} \quad \Big\{\tilde V^{\mathbb{F},\Lambda}(t,c) := e^{-\Lambda^{\mathbb{F}}_t}\tilde U^{\rm sd}(t, c e^{\Lambda^{\mathbb{F}}_t}\frac{Y^{\rm sd}_t}{Y^{\rm bd}_t})\Big\}. \tag{4.5}$$

In what follows we will concentrate on the dynamic utilities $U^{\mathbb{F}}$ and $V^{\mathbb{F},\Lambda}$, equivalent results for the dynamic conjugate utilities $\tilde U^{\mathbb{F}}$ and $\tilde V^{\mathbb{F},\Lambda}$ can be directly deduced from (4.5).

A direct consequence of Theorem 4.1 is the following result.

Corollary 4.2 *A $\mathbb{G}$-dynamic utility $U^{\mathbb{G}} = (U^{\rm bd}, U^{\rm sd})$ and its conjugate $\tilde U^{\mathbb{G}}$ is bi-revealed by the triplet $(u, X^{\mathbb{G}}, Y^{\mathbb{G}})$ if and only if the pair $(U^{\mathbb{F}}, V^{\mathbb{F},\Lambda})$ of $\mathbb{F}$-dynamic utilities and their conjugate $(\tilde U^{\mathbb{F}}, \tilde V^{\mathbb{F},\Lambda})$ given in* (4.2), (4.5) *is bi-revealed by the triplet $(u, (X^{\rm bd}, C^X), Y^{\rm bd}e^{-\Lambda^{\mathbb{F}}})$.*

Remark 4.3 *The results of Theorem 4.1 and Corollary 4.2 remain valid if the characteristic processes $(X^{\rm bd}, Y^{\rm bd})$ are changed into $(X^{\rm bd}e^{\Psi}, Y^{\rm bd}e^{-\Psi})$ for any $\mathbb{F}$-adapted process Ψ, since the martingale property of $\{X^{\rm bd}_t e^{\int_0^t R^{XY}_s d\Lambda_s} Y^{\rm bd}_t e^{-\Lambda^{\mathbb{F}}_t}\}$ is preserved by such a transformation.*

Proof of Theorem 4.1 Let $U^{\mathbb{G}} = (U^{\rm bd}, U^{\rm sd})$ be a bi-revealed utility by the system $\big(u(.), X^{\mathbb{G}}, Y^{\mathbb{G}}\big)$. Adopting the additive form for the performance process $\{U^{\mathbb{G}}(t, X^{\mathbb{G}}_t(x))\}$ and for its jump size $\{J^{U\circ X}_t(x) := U^{\rm sd}(t, X^{\rm sd}_t(x)) - U^{\rm bd}(t, X^{\rm bd}_t(x))\}$, the martingale property (4.3) of $\{U^{\mathbb{G}}(t, X^{\mathbb{G}}_t)\}$ can be rewritten as

$$\Big\{U^{\mathbb{G}}(t, X^{\mathbb{G}}_t(x)) = U^{\rm bd}(t\wedge\tau, X^{\rm bd}_{t\wedge\tau}(x)) + J^{U\circ X}_\tau(x)\mathbb{1}_{\tau\leqslant t}\Big\} \text{ is an } (\mathbb{G},\mathbb{P})\text{-martingale}, \quad \forall x.$$

According to Corollary 3.8 this property is equivalent to

$$\Big\{\big(U^{\rm bd}(t, X^{\rm bd}_t(x)) + \int_0^t J^{U\circ X}_s(x) d\Lambda^{\mathbb{F}}_s\big)L^{\mathbb{F}}_t\Big\} \text{ is an } (\mathbb{F},\mathbb{P})\text{-martingale}, \quad \forall x,$$

or, under the equivalent probability measure $\mathbb{Q}^L$, to

$$\Big\{U^{\rm bd}(t, X^{\rm bd}_t(x)) - \int_0^t \big(U^{\rm bd}(s, X^{\rm bd}_s(x)) - U^{\rm sd}(s, X^{\rm sd}_s(x))\big) d\Lambda^{\mathbb{F}}_s\Big\} \text{ is an } (\mathbb{F},\mathbb{Q}^L)\text{-martingale}, \quad \forall x. \tag{4.6}$$

Using the notation

$$U^{\mathbb{F}}(t,z) := e^{-\Lambda^{\mathbb{F}}_t}U^{\rm bd}(t,z),\ V^{\mathbb{F},\Lambda}(t,c) := e^{-\Lambda^{\mathbb{F}}_t}U^{\rm sd}\left(t, c\frac{Y^{\rm bd}_t}{Y^{\rm sd}_t}\right)$$
$$\text{and } \ C^\Lambda_t(x) := X^{\rm sd}_t(x)\frac{Y^{\rm sd}_t(u_z(x))}{Y^{\rm bd}_t(u_z(x))},$$

(4.6) is equivalent to

$$\Big\{U^{\mathbb{F}}(t,X_t^{\rm bd}(x))+\int_0^t V^{\mathbb{F},\Lambda}(s,C_s^{\Lambda}(x))\mathrm{d}\Lambda_s^{\mathbb{F}}\Big\}\text{ is an }(\mathbb{F},\mathbb{Q}^L)\text{-martingale},\quad\forall x$$

since

$$\mathrm{d}\Big(U^{\mathbb{F}}(t,X_t^{\rm bd})+\int_0^t V^{\mathbb{F},\Lambda}(s,C_s^{\Lambda})\mathrm{d}\Lambda_s^{\mathbb{F}}\Big)=\mathrm{d}\Big(e^{-\Lambda_t^{\mathbb{F}}}U^{\rm bd}(t,X_t^{\rm bd})+\int_0^t e^{-\Lambda_s^{\mathbb{F}}}U^{\rm sd}(s,X_s^{\rm sd})\mathrm{d}\Lambda_s^{\mathbb{F}}\Big)$$
$$=e^{-\Lambda_t^{\mathbb{F}}}\Big[\mathrm{d}U^{\rm bd}(t,X_t^{\rm bd})+\big(U^{\rm sd}(t,X_t^{\rm sd})-U^{\rm bd}(t,X_t^{\rm bd})\big)\mathrm{d}\Lambda_t^{\mathbb{F}}\Big]$$

which achieves the proof. □

Interpretation and comments The analogy between the $\mathbb{G}$ and $\mathbb{F}$-universes in terms of utility processes and characteristic processes are gathered below:

(i) The before-default component of the $\mathbb{G}$-dual characteristic process $Y^{\mathbb{G}}=(Y^{\rm bd},Y^{\rm sd})$ determines the $\mathbb{F}$-dual characteristic process $Y^{\mathbb{F}}=Y^{\rm bd}e^{-\Lambda^{\mathbb{F}}}$.

(ii) The $\mathbb{G}$-characteristic process $X^{\mathbb{G}}=(X^{\rm bd},X^{\rm sd})$ is analog to the $\mathbb{F}$-characteristic process $X^{\mathbb{F}}:=X^{\rm bd}$ with accumulation of reserves $C^{\Lambda}=X^{\rm sd}R^{Y}$. This means that the $\mathbb{F}$-characteristic coincides with the $\mathbb{G}$-characteristic before default, and the reserve process $C^{\Lambda}=X^{\rm sd}R^{Y}$ is the $\mathbb{G}$-characteristic at the default $X^{\rm sd}$ multiplied by the jump ratio R^{Y} of the $\mathbb{G}$-dual process.

(iii) The before-default component of the $\mathbb{G}$-utility process $U^{\mathbb{G}}=(U^{\rm bd},U^{\rm sd})$ determines the $\mathbb{F}$-utility from wealth $U^{\mathbb{F}}=e^{-\Lambda^{\mathbb{F}}}U^{\rm bd}$ and its at the-default component determines the $\mathbb{F}$-utility $V^{\mathbb{F},\Lambda}=e^{-\Lambda^{\mathbb{F}}}U^{\rm sd}$ of reserve.

In a financial framework, the reserve process $C^{\Lambda}=X^{\rm sd}R^{Y}$ can be interpreted as the wealth at the default multiplied by the jump ratio of the pricing kernel $Y^{\mathbb{G}}$. This is equivalent to the identity $C^{\Lambda}Y^{\rm bd}=X^{\rm sd}Y^{\rm sd}$ where the quantities C^{Λ} and $X^{\rm sd}$ are evaluated with the corresponding pricing kernels before default $Y^{\rm bd}$ and at the default $Y^{\rm sd}$.

This result sheds a new light on the consumption process in preferences and bridges the gap between investment-consumption utility criteria and pure investment utility criteria. Indeed, depending on the topics, utility optimization focuses either on the consumption or on the wealth. In economics, especially in studies concerning long term investments, the consumption plays a major role. But in finance, putting the emphasis on consumption is not common. Denoting (U,V) the stochastic utilities from wealth and consumption respectively, the investment-consumption preference process is written as $\int_0^T V(s,c_s)\mathrm{d}s+U(T,X_T)$. Observe the different natures of the two parts in the preference process. On the one hand, $\int_0^T V(s,c_s)\mathrm{d}s$ is an integral of the utility of a rate (per unit of time) of consumption. On the other hand, $U(t,X_T)$ is the utility of the "aggregate" wealth (which is itself an integral of a rate of wealth). In these two terms, the integral operators and utility functions are inverted. Despite these difference, very often the same type of functions, with eventually different risk aversion parameters, are used. This apparent discrepancy is solved thanks to this new viewpoint on the consumption process. The consumption is interpreted as precautionary reserves to accumulate wealth that will be available at default time. The next section further investigates this analogy in the framework of financial market and explores this new interpretation of the consumption.

5. Defaultable financial market and consumption

In an incomplete financial framework, the observable (characteristic) processes X and its dual Y are respectively the *optimal* wealth process and *optimal* discounted pricing kernel, *for a given utility criterion.* For this criterion, X is optimal among the set of admissible self-financing strategies, and Y is optimal among the set of pricing kernels (also called state price density processes), that are processes orthogonal to the financial assets tradable in the market. We define below a generic financial market, in the $\mathbb{G}$ and $\mathbb{F}$-universes. A financial market is characterized by the tradable assets and the set of admissible self-financing strategies. The No Arbitrage condition is stated in terms of the existence of at least one pricing kernel. In the $\mathbb{G}$-market, to hedge against the default, investors may invest in a defaultable asset, in addition to non-defaultable assets. Relying on previous results in Section 3 and Section 4, the link with an $\mathbb{F}$-market, in which investors are allowed to consume, is provided. This highlights the analogy between consumption and wealth at the default.

5.1 The defaultable $\mathbb{G}$-market

We define below the defaultable market and the set of admissible wealth processes. This set contains the observable wealth X, that is optimal for some criterion U. To avoid confusion, an admissible wealth is denoted by W, while the notation X is kept for the optimal one.

5.1.1 The tradable assets in the $\mathbb{G}$-market

In the defaultable market (also called $\mathbb{G}$-market), agents can invest in non-defaultable $\mathbb{F}$-adapted risky assets $\{S^i,\ i=1,\cdots,d\}$ and a risk free asset S^0 characterized by the short interest rate $\{r_t\}$. In addition, in the situation of a random horizon, investors may want to protect themselves against this additional source of risk induced by this uncertain time τ. For example τ is the time of switching to a new regime of the economy, or the time of an ecological catastrophe, or the time of a sovereign default, or the death of the investor. Therefore in the $\mathbb{G}$-market, a defaultable ($\mathbb{G}$-adapted) asset is tradable to propose a hedge against the risk induced by the default.

The defaultable asset Typically, this defaultable asset is a perpetual Credit Default Swap (CDS), characterized by continuous payments (the CDS "fee" or "spread") and, in exchange, the payoff of a compensation in the event of default. As usually for a swap, its value at time 0 is zero. The cash flow of such defaultable contract has one jump (of size 1) at time τ, and is modelled (in the additive form) by

$$SW_t = \mathbf{1}_{\tau\leqslant t} - \int_0^{t\wedge\tau} \varphi^{\mathrm{sw}}_s \mathrm{d}s, \tag{5.1}$$

where φ^{sw} is the CDS spread. If the CDS spread $\varphi^{\mathrm{sw}}_t\mathrm{d}t$ coincides with the default compensator $\mathrm{d}\Lambda_t$ then the CDS price SW is an $\mathbb{G}$-martingale. Remark that the defaultable asset jumps at τ, on the contrary to the non-defaultable assets S^i, which are $\mathbb{F}$-adapted, and therefore are continuous at τ (as a consequence of Assumption 3.1). The following quantities will be extensively used in the sequel:

$$\Phi^{\mathrm{sw}}_t := \int_0^t \varphi^{\mathrm{sw}}_s \mathrm{d}s \ \text{ and } \ \Psi_t := \int_0^t \psi_s \mathrm{d}s \text{ for } \psi_t := \varphi^{\mathrm{sw}}_t - \Phi^{\mathrm{sw}}_t r_t. \tag{5.2}$$

In the $\mathbb{G}$-market, all investments in these tradable assets are stopped at time τ and the strategies are self-financing. We first make precise Remark 4.3 in this framework of $\mathbb{G}$-financial market, by characterizing the set of adjoint processes $Y^{\mathbb{G}}$ in this setting.

5.1.2 Pricing kernels in the $\mathbb{G}$-universe

The No Arbitrage condition implies the existence of at least one pricing kernel (state price density process). A pricing kernel is defined as a non-negative adapted process $Y^{\mathbb{G}}$ which is orthogonal to the tradable assets in the market, that is $(S^i Y^{\mathbb{G}})$ (for $i=0,\cdots,d$) and $(SW\,Y^{\mathbb{G}})$ are $(\mathbb{G},\mathbb{P})$-local-martingales. Thanks to the self financing dynamics of the admissible wealth processes, pricing kernels are also orthogonal to any admissible wealth process, whatever the initial wealth. In this financial context, pricing kernels are the adjoint processes, with a stronger condition of orthogonality compared to the abstract framework of Section 2 ($S{\cdot}Y^{\mathbb{G}}$ are here local martingales instead of only supermartingales). The orthogonality of $Y^{\mathbb{G}}=(Y^{\rm bd},Y^{\rm sd})$ with the non-defaultable assets S^i determines the before default part $Y^{\rm bd}$ of $Y^{\mathbb{G}}$, while the orthogonality with respect to the defaultable asset determines the jump relative jump $R^Y=\frac{Y^{\rm bd}}{Y^{\rm sd}}$ as explained in the following theorem.

Theorem 5.1 *We recall that* $\Phi^{\rm sw}_t=\int_0^t\varphi^{\rm sw}_s{\rm d}s$ *and* $\Psi_t=\int_0^t\psi_s{\rm d}s$ for $\psi_t=\varphi^{\rm sw}_t-\Phi^{\rm sw}_t r_t$. *Then* $Y^{\mathbb{G}}=(Y^{\rm bd},Y^{\rm sd})$ *is an* $(\mathbb{G},\mathbb{P})$*-pricing kernel if and only if*

$$Y^{\rm bd}e^{\Psi-\Lambda^{\mathbb{F}}}\ \textit{is an } (\mathbb{F},\mathbb{Q}^L)\textit{-pricing kernel},\quad \textit{and}\quad \frac{Y^{\rm sd}_t}{Y^{\rm bd}_t}{\rm d}\Lambda^{\mathbb{F}}_t=(\varphi^{\rm sw}_t-\Phi^{\rm sw}_t r_t){\rm d}t=\psi_t{\rm d}t.$$

The relative jump $R^Y=\frac{Y^{\rm sd}}{Y^{\rm bd}}$ *is independent on the initial condition* y *and is determined by the CDS-spread* $\varphi^{\rm sw}$, *the spot-rate* r *and the cumulative compensator* $\Lambda^{\mathbb{F}}$.

No Arbitrage condition implies the positivity of ψ, so that $Y^{\rm sd}$ remains positive. Indeed $\psi_t=(\varphi^{\rm sw}_t-\Phi^{\rm sw}_t r_t)>0$ for all t is equivalent to $\forall\ \ 0<t_0<t,\ \Phi^{\rm sw}_t>\Phi^{\rm sw}_{t_0}\exp(\int_{t_0}^t r_s{\rm d}s)$ that is the CDS' return is greater than the risk free rate : on $\{t<\tau\}$, $\frac{SW_t}{SW_{t_0}}>\frac{S^0_t}{S^0_{t_0}}$. If this condition is not fulfilled, arbitrage opportunities (through a short position in the CDS and a long position in the risk free asset) can be realized.

Proof $Y^{\mathbb{G}}=(Y^{\rm bd},Y^{\rm sd})$ is an $\mathbb{G}$-pricing kernel if and only if it is orthogonal to the non-defaultable assets S^i (for $i=0,\cdots,d$) and to the defaultable asset SW.

(i) $Y^{\mathbb{G}}$ is orthogonal to SW if and only if $\left\{Y^{\mathbb{G}}_t SW_t=-\Phi^{\rm sw}_t Y^{\rm bd}_t\mathbb{1}_{t<\tau}+Y^{\rm sd}_t(1-\Phi^{\rm sw}_\tau)\mathbb{1}_{\tau\leqslant t}\right\}$ is an $(\mathbb{G},\mathbb{P})$-local-martingale. By Theorem 3.7, and recalling the equivalent probability measure ${\rm d}\mathbb{Q}^L:=L^{\mathbb{F}}{\rm d}\mathbb{P}$, this is equivalent to

$$\left\{\Phi^{\rm sw}_t Y^{\rm bd}_t\exp\left(\int_0^t(R^Y_s\frac{\Phi^{\rm sw}_s-1}{\Phi^{\rm sw}_s}-1){\rm d}\Lambda^{\mathbb{F}}_s\right)\right\}\ \text{is an } (\mathbb{F},\mathbb{Q}^L)\text{-local-martingale.}$$

Denoting by $\mu^{\rm bd}_t$ the finite variation part of the $(\mathbb{F},\mathbb{Q}^L)$-semi-martingale $Y^{\rm bd}$, this is equivalent to $e^{-\int_0^t\left(\frac{R^Y_s}{\Phi^{\rm sw}_s}+1\right){\rm d}\Lambda^{\mathbb{F}}_s}\left(\Phi^{\rm sw}_t{\rm d}\mu^{\rm bd}_t+Y^{\rm bd}_t\varphi^{\rm sw}_t{\rm d}t+(Y^{\rm sd}_t(\Phi^{\rm sw}_t-1)-\Phi^{\rm sw}_t Y^{\rm bd}_t){\rm d}\Lambda^{\mathbb{F}}_t\right)=0$, a.s., $\forall t$, that is

$${\rm d}\mu^{\rm bd}_t=-Y^{\rm bd}_t\frac{\varphi^{\rm sw}_t}{\Phi^{\rm sw}_t}{\rm d}t-\left(Y^{\rm sd}_t\frac{\Phi^{\rm sw}_t-1}{\Phi^{\rm sw}_t}-Y^{\rm bd}_t\right){\rm d}\Lambda^{\mathbb{F}}_t,\quad \text{a.s. }\forall t. \tag{5.3}$$

(ii) Using again Theorem 3.7, the $(\mathbb{G},\mathbb{P})$-local-martingale property of $\{Y^{\mathbb{G}}_t S^0_t\}$ is equivalent to (since S^0 is continuous at τ) $\{Y^{\rm bd}_t S^0_t\exp(\int_0^t(R^Y_s-1){\rm d}\Lambda^{\mathbb{F}}_s)\}$ is an $(\mathbb{F},\mathbb{Q}^L)$-local-martingale, that is

$$d\mu_t^{\text{bd}} = -Y_t^{\text{bd}} r_t dt - (Y_t^{\text{sd}} - Y_t^{\text{bd}}) d\Lambda_t^{\mathbb{F}}, \quad \text{a.s. } \forall t. \tag{5.4}$$

Combining (5.3) and (5.4) determines the relative jump $R^Y = \frac{Y^{\text{sd}}}{Y^{\text{bd}}}$ of $Y^{\mathbb{G}}$

$$\frac{Y_t^{\text{sd}}}{Y_t^{\text{bd}}} d\Lambda_t^{\mathbb{F}} = (\varphi_t^{\text{sw}} - \Phi_t^{\text{sw}} r_t) dt = d\Psi_t. \tag{5.5}$$

(iii) $Y^{\mathbb{G}}$ is orthogonal to the non-defaultable $\mathbb{F}$-adapted assets S^i ($i = 1, \cdots, d$) (S^i has no jump at τ) iff $\{Y_t^{\text{bd}} S_t^i \exp(\int_0^t (R_s^Y - 1) d\Lambda_s^{\mathbb{F}})\}$ is an $(\mathbb{F}, \mathbb{Q}^L)$-local-martingale, that is iff $\{Y_t^{\text{bd}} S_t^i e^{\Psi_t} e^{-\Lambda_t^{\mathbb{F}}}\}$ is an $(\mathbb{F}, \mathbb{Q}^L)$-local martingale, since $R_t^Y d\Lambda_t^{\mathbb{F}} = (\varphi_t^{\text{sw}} - \Phi_t^{\text{sw}} r_t) dt = d\Psi_t$. This means that $Y^{\text{bd}} e^{\Psi - \Lambda^{\mathbb{F}}}$ is $(\mathbb{F}, \mathbb{Q}^L)$-pricing kernel.

Denoting by μ_t^i the finite variation part of the $(\mathbb{F}, \mathbb{Q}^L)$-semi-martingale S^i, this is equivalent to $d\mu_t^{\text{bd}} + Y_t^{\text{bd}} d\mu_t^i + (Y_t^{\text{sd}} - Y_t^{\text{bd}}) d\Lambda_t^{\mathbb{F}} + [dY^{\text{bd}}, \frac{dS^i}{S^i}]_t = 0$, a.s. $\forall t$, which becomes, using (5.4), $d\mu_t^i - [\frac{dY_t^{\text{bd}}}{Y^{\text{bd}}}, \frac{dS^i}{S^i}]_t = r_t dt$, a.s. $\forall t$. □

5.1.3 Admissible wealth processes in the $\mathbb{G}$-market

The agent invests in this financial market a fraction π of her positive wealth $W^{\mathbb{G}}$ in the risky assets S, the quantity α in the defaultable asset SW and the rest in the cash[1]:

$$W_t^{\mathbb{G}} = \left(W_t^{\mathbb{G}} \pi_t\right).S_t + \alpha_t SW_t + \left(\frac{W_t^{\mathbb{G}} - W_t^{\mathbb{G}} \pi_t.S_t - \alpha_t SW_t}{S_t^0}\right) S_t^0. \tag{5.6}$$

The class of admissible strategies (π, α) reflects the incompleteness of the market by restrictions on the risky portfolios π that are constrained to live in a given adapted subspace. All investments are stopped at time τ. The portfolio evolves according to the *self-financing* dynamics, with positive initial wealth $W_0^{\mathbb{G}} \in \mathbb{R}_+^*$:

$$dW_t^{\mathbb{G}} = \mathbf{1}_{t \leqslant \tau} \left(W_t^{\mathbb{G}} r_t dt + W_t^{\mathbb{G}} \pi_t.(dS_t - r_t S_t dt) + \alpha_t (d\, SW_t - r_t\, SW_t\, dt)\right). \tag{5.7}$$

The wealth process $W^{\mathbb{G}}$ can be written as $W_t^{\mathbb{G}} = W_{t\wedge\tau}^{\text{bd}} + (W_\tau^{\text{sd}} - W_\tau^{\text{bd}}) \mathbf{1}_{\tau \leqslant t}$.

According to (5.6), since $SW_t = \mathbf{1}_{\tau \leqslant t} - \mathbf{1}_{t<\tau} \int_0^{t\wedge\tau} \varphi_s^{\text{sw}} ds = \mathbf{1}_{\tau \leqslant t} - \Phi_t^{\text{sw}} \mathbf{1}_{t<\tau}$

$$W_t^{\text{bd}} = \left(W_t^{\text{bd}} \pi_t\right).S_t - \alpha_t \Phi_t^{\text{sw}} + \left(\frac{W_t^{\text{bd}} - W_t^{\text{bd}} \pi_t.S_t + \alpha_t \Phi_t^{\text{sw}}}{S_t^0}\right) S_t^0, \tag{5.8}$$

$$W_\tau^{\text{sd}} - W_\tau^{\text{bd}} = W_\tau^{\mathbb{G}} - W_{\tau^-}^{\mathbb{G}} = \alpha_\tau (SW_\tau - SW_{\tau^-}) = \alpha_\tau. \tag{5.9}$$

The $\mathbb{G}$-wealth process $W^{\mathbb{G}}$ has one jump at time τ, of size α_τ. If the agent wants to obtain the wealth $W_\tau^{\text{sd}} = W_\tau^{\mathbb{G}}$ at the default time, she has to invest $\alpha_t = (W_t^{\text{sd}} - W_t^{\text{bd}})$ in the defaultable asset at any time t. Therefore one can equivalently parametrize the $\mathbb{G}$-wealth process by (π, α) or by (π, W^{sd}).

5.1.4 From the class of admissible wealth processes to the observable process

Given a utility criterion, the investor aims to optimize her preferences among the set of admissible wealths described above. This leads to a stochastic control problem, usually formulated backward in the literature, and for which the value function is an example of dynamic consistent utility (under technical regularity assumptions to ensure - among other properties - its concavity, see [9]). In the backward problem setting, authors give little importance to the role of the initial condition. Indeed, in the optimization program, they consider a large class of controls (so-called Open-Loop controls) and at the optimum they select

[1] Hereafter the dot denotes the scalar product.

a Markovian control. But it should be noted that these optimal strategies depend in a complex way on the state $X_t(x)$ at each date t, except in very particular cases (such as power utilities). Therefore the optimal wealth, which is a controlled process belonging to the general class of admissible wealths, depends on x in a non-linear way: it is a random field depending on x and t, such as the controlled strategies $\pi(t,x)$, $\alpha(t,x)$. Similarly, the optimal adjoint process $Y_t(y)$ also depends on its initial condition y in a not necessarily linear form, see [10] for more details on this dependency. In our inverse problem, where we assume to observe the optimal process, the role of the initial condition cannot be neglected. In a sense, assuming that we have data for several initial conditions is enough to fill the lack of information about the universe and its uncertainties. As recalled in Theorems 2.4 and 2.5, the monotonicity of X and Y is a key property to reconstruct the utility process from the observable. This monotonicity can be proved in an Itô framework by considering the Stochastic Differential Equations satisfied by X and Y and checking that these SDEs have sufficiently regular coefficients.

Example of an Itô financial market We provide here the classic example of an Itô financial market, in which the filtration $\mathbb{F}$ is driven by a n-standard Brownian motion B. The risk free asset is characterized by the short rate $\{r_t\}$, the non-defaultable $\mathbb{F}$-assets $\{S^i_t,\ i=1,\cdots,d\}$ by the n-dimensional risk premium vector $\{\eta_t\}$ and the $d\times n$ volatility matrix $\{\sigma_t\}$. The existence of a multivariate risk premium η is a weak form of absence of arbitrage opportunity in this $\mathbb{F}$-market (without default). The incompleteness of the $\mathbb{F}$-market is expressed by restrictions on the risky portfolios $\sigma_t\pi_t$ that are constrained to live in a given progressive vector space $\mathcal{R}_t$.

In this Itô framework, the dynamics of an $\mathbb{F}$-pricing kernel $Y^{\mathbb{F},\nu}$ is characterized by an orthogonal volatility $\nu\in\mathcal{R}^\perp$ as follows[2] (we refer to [9] for more details)

$$\begin{aligned} \mathrm{d}Y^{\mathbb{F},\nu}_t(y) &= Y^{\mathbb{F},\nu}_t(y)[-r_t\mathrm{d}t + (\nu_t(Y^{\mathbb{F},\nu}_t(y)) - \eta_t).\mathrm{d}B_t],\\ \nu_t(Y^{\mathbb{F},\nu}_t(y)) &\in \mathcal{R}^\perp_t, \quad Y^{\mathbb{F},\nu}_0(y) = y>0. \end{aligned} \tag{5.10}$$

In the $\mathbb{G}$-market, the defaultable asset is still given by the CDS (5.1) and the characteristic $X^{\mathbb{G}}$ is an $\mathbb{G}$-admissible wealth satisfying the self-financing dynamics (5.7),

$$\mathrm{d}X^{\mathbb{G}}_t(x) = \mathbf{1}_{t\leqslant\tau}\left(X^{\mathbb{G}}_t(x)(r_t\mathrm{d}t + (\sigma_t\pi_t(X^{\mathbb{G}}_t(x)).(\mathrm{d}B_t+\eta_t\mathrm{d}t)) + \alpha_t(X^{\mathbb{G}}_t(x)(\mathrm{d}\,SW_t - r_t\,SW_t\,\mathrm{d}t)\right).$$

The dynamics of $(\mathbb{G},\mathbb{P})$-pricing kernels follow from Theorem 5.1: $Y^{\mathbb{G}}=(Y^{\mathrm{bd}},Y^{\mathrm{sd}})$ is an $\mathbb{G}$-pricing kernel if and only if $Y^{\mathrm{sd}}_t\mathrm{d}\Lambda^{\mathbb{F}}_t = Y^{\mathrm{bd},\nu}_t\mathrm{d}\Psi_t$ and $Y^{\mathrm{bd}}=Y^{\mathrm{bd},\nu}$ ($\nu\in\mathcal{R}^\perp$) solution of

$$\mathrm{d}Y^{\mathrm{bd},\nu}_t(y) = Y^{\mathrm{bd},\nu}_t(y)\big[-r_t\mathrm{d}t - \mathrm{d}\Psi_t + \mathrm{d}\Lambda^{\mathbb{F}}_t + (\nu_t(Y^{\mathrm{bd},\nu}_t(y)) - \eta_t).\mathrm{d}B_t\big].$$

5.2 The defaultable $\mathbb{G}$-market as a $\mathbb{F}$-market with consumption

This section investigates the link between the $\mathbb{G}$-market with the defaultable asset and an $\mathbb{F}$-market with consumption, providing a new point of view for the consumption. From equations (5.8) and (5.9), a self-financing portfolio, stopped at time τ, is decomposed as follows

$$W^{\mathbb{G}}_t = W^{\mathrm{bd}}_t\mathbf{1}_{t<\tau} + \big(W^{\mathrm{bd}}_\tau + \alpha_\tau(1+\Phi^{\mathrm{sw}}_\tau)\big)\mathbf{1}_{\tau\leqslant t},$$

with $W^{\mathrm{bd}}_t = [W^{\mathrm{bd}}_t\pi_t].S_t - [\alpha_t]\Phi^{\mathrm{sw}}_t + \left[\dfrac{W^{\mathrm{bd}}_t - W^{\mathrm{bd}}_t\pi_t.S_t + \alpha_t\Phi^{\mathrm{sw}}_t}{S^0_t}\right]S^0_t$. The dynamics of the before default wealth W^{sd}_τ satisfies

[2] For $\mathcal{R}$ a vector subspace of $\mathbb{R}^n$, $\mathcal{R}^\perp$ denotes its orthogonal space. For any $x\in\mathbb{R}^n$, $x^{\mathcal{R}}$ is the orthogonal projection of the vector x onto $\mathcal{R}$.

$$\begin{aligned}
\mathrm{d}W_t^{\mathrm{bd}} &= W_t^{\mathrm{bd}}\pi_t \mathrm{d}S_t - \alpha_t \mathrm{d}\Phi_t^{\mathrm{sw}} + \big(W_t^{\mathrm{bd}} - W_t^{\mathrm{bd}}\pi_t.S_t + \alpha_t\Phi_t^{\mathrm{sw}}\big) r_t \mathrm{d}t. \\
&= W_t^{\mathrm{bd}} r_t \mathrm{d}t + W_t^{\mathrm{bd}}\pi_t.(\mathrm{d}S_t - r_t S_t \mathrm{d}t) + (W_t^{\mathrm{sd}} - W_t^{\mathrm{bd}})\mathrm{d}t \\
&= W_t^{\mathrm{bd}}\Big(r_t \mathrm{d}t - (R_t^W - 1)\psi_t \mathrm{d}t + \pi_t.(\mathrm{d}S_t - r_t S_t \mathrm{d}t)\Big)
\end{aligned} \tag{5.11}$$

where we used the parametrization $\alpha_t = W_t^{\mathrm{sd}} - W_t^{\mathrm{bd}}$ for the second equality, and the previous notations $\psi_t = \varphi_t^{\mathrm{sw}} - \Phi_t^{\mathrm{sw}} r_t$ and $R^W = \frac{W^{\mathrm{sd}}}{W^{\mathrm{bd}}}$ for the third equality. The multiplicative form of the dynamics (5.11) ensures the positivity of the wealth. (5.11) is similar to the dynamics of an admissible wealth in an $\mathbb{F}$-market with consumption. In the $\mathbb{F}$-market, the agent is allowed to invest only in the non-defaultable ($\mathbb{F}$-adapted) assets $\{S^i,\ i = 1, \cdots, d\}$ and S^0. She can not invest in the defaultable asset, but she is allowed to consume a part of her non-negative wealth at the progressive rate $c_t = \rho_t W_t^{\mathbb{F},\pi,\rho} \geqslant 0$, where ρ is the fraction of wealth that is consumed. Indeed, by denoting as before by π_t the fraction of her wealth $W_t^{\mathbb{F},\pi,\rho}$ invested in the risky assets, the dynamics of a positive wealth process in the $\mathbb{F}$ market with admissible risky portfolio π and relative consumption rate $\rho \geqslant 0$, starting from the positive initial wealth $W_0^{\mathbb{F},\pi,\rho}$, is given by

$$\mathrm{d}W_t^{\mathbb{F},\pi,\rho} = W_t^{\mathbb{F},\pi,\rho}\Big((r_t - \rho_t)\,\mathrm{d}t + \pi_t.(\mathrm{d}S_t - r_t S_t \mathrm{d}t)\Big). \tag{5.12}$$

In depth discussion on the backward/forward approaches for utilities of investment and consumption is provided in [9].

5.2.1 The consumption process as reserves to face a default event

We recall that the appropriate probability measure to consider on the small filtration $\mathbb{F}$ is $\mathbb{Q}^L$ defined as $\mathrm{d}\mathbb{Q}^L = L^{\mathbb{F}}\mathrm{d}\mathbb{P}$, and that $\Psi_t = \int_0^t (\varphi_s^{\mathrm{sw}} - \Phi_s^{\mathrm{sw}} r_s)\mathrm{d}s = \int_0^t \psi_s \mathrm{d}s$. Then, according to (5.11) and Theorem 5.1, we have the following correspondence between $\mathbb{F}$-market processes and $\mathbb{G}$-market processes

$$\begin{cases} W_t^{\mathrm{bd},\pi} = W_t^{\mathbb{F},\pi,\rho} e^{\Psi_t}, \\ Y_t^{\mathrm{bd}} = Y_t^{\mathbb{F}} e^{\Lambda_t^{\mathbb{F}} - \Psi_t}, \end{cases}$$

where $W^{\mathbb{F},\pi,\rho}$ is an $\mathbb{F}$-wealth process with strategy π and consumption rate $\rho = R^W\psi$, and $Y^{\mathbb{F}}$ is an $\mathbb{F}$-pricing kernel.

Proposition 5.2 *We have the equivalence between $\mathbb{G}$-self-financing strategies and $\mathbb{F}$-self-financing strategies with consumption, for $t \leqslant \tau$:*

- $\mathbb{G}$-*strategies*: $\{\pi_t\}$ *on the non-defaultable risky assets,* $\{\alpha_t = W_t^{\mathrm{sd}} - W_t^{\mathrm{bd},\pi}\}$ *on the defaultable asset, leading to the wealth* $\{W_t^{\mathbb{G},\pi}\}$ *on* $[0,\tau]$;
- $\mathbb{F}$-*strategies*: $\{\pi_t\}$ *on the non-defaultable risky assets, consumption rate* $\{\rho_t = \psi_t R_t^W\}$, *leading to the wealth* $\{W_t^{\mathbb{F},\pi,\rho} = e^{-\Psi_t} W_t^{\mathrm{bd},\pi}\}$;
- $\mathbb{G}$-*pricing kernels* $\{Y_t^{\mathbb{G}}\}$ *and* $\mathbb{F}$-*pricing kernels* $\{Y_t^{\mathbb{F}}\}$ *are linked by* $\{Y_t^{\mathrm{bd}} = Y_t^{\mathbb{F}} e^{\Lambda_t^{\mathbb{F}} - \Psi_t}\}$.

Note that the quantities in the $\mathbb{F}$-market are recapitalized by $e^{-\Psi_t}$, the recapitalization rate depending on the CDS-spread φ^{sw} and the market spot rate r.

5.2.2 $\mathbb{F}$ and $\mathbb{G}$-bi-revealed utilities in financial markets

In the framework of a financial market, the characteristic processes (X, Y) one would like to consider are optimal wealth process for X and optimal pricing kernel for Y. In the light of Proposition 5.2, for the $\mathbb{F}$-market, the characteristic processes should be $e^{-\Psi} X^{\mathrm{bd},\pi}$ (instead of

$X^{\mathrm{bd},\pi}$) and $Y^{\mathrm{bd}} = Y^{\mathbb{F}} e^{\Lambda^{\mathbb{F}}-\Psi}$ (instead of $Y^{\mathrm{bd}} e^{\Lambda^{\mathbb{F}}}$). This is coherent with Remark 4.3 since

$$X_t^{\mathrm{bd}}(x) Y_t^{\mathrm{bd}}(y) e^{\int_0^t (R_s^{XY}(x)-1)\mathrm{d}\Lambda_s^{\mathbb{F}}} = \Big(X_t^{\mathrm{bd}}(x) e^{\int_0^t R_s^{XY}(x)\mathrm{d}\Lambda_s^{\mathbb{F}}+\Psi_t}\Big)\Big(Y_t^{\mathrm{bd}}(y) e^{-\Lambda_t^{\mathbb{F}}-\Psi_t}\Big).$$

Ψ is interpreted as a parameter of adjustment specific to the market-model. Note that the coefficient $R^{XY} = R^X R^Y$ depends only on x and not on y, as underlined in Theorem 5.1. Thus the consumption C^X and the $\mathbb{F}$-utilities defined below do not depend on y, which is crucial to ensure the monotonicity of the characteristic process and the concavity of $V^{\mathbb{F}}$. Theorem 4.1 can be rewritten in the following slightly different form, taking into account the adjustment parameter Ψ.

Corollary 5.3 *Let u a deterministic utility, $X^{\mathbb{G}}$ an $\mathbb{G}$-self-financing optimal wealth and $Y^{\mathbb{G}}$ an $\mathbb{G}$-optimal pricing kernel, to which we associate the $\mathbb{F}$-self-financing optimal wealth $X^{\mathbb{F}}(x) := X^{\mathrm{bd}}(x)e^{-\Psi}$ with consumption $C_t^{\mathbb{F}}(x) := X_t^{\mathrm{sd}}(x)\psi_t$ and the $\mathbb{G}$-optimal pricing kernel $Y^{\mathbb{F}}(y) := Y^{\mathrm{bd}}(y)e^{\Psi-\Lambda^{\mathbb{F}}}$ as in Proposition 5.2. Then an $\mathbb{G}$-dynamic utility $U^{\mathbb{G}} = (U^{\mathrm{bd}}, U^{\mathrm{sd}})$ and its conjugate $\tilde{U}^{\mathbb{G}}$ is bi-revealed by the triplet $(u, X^{\mathbb{G}}, Y^{\mathbb{G}})$ if and only if the pair of utility of wealth and consumption $(U^{\mathbb{F}}, V^{\mathbb{F}})$ and their conjugate $(\tilde{U}^{\mathbb{F}}, \tilde{V}^{\mathbb{F}})$*

$$\begin{cases} \Big\{U^{\mathbb{F}}(t,z) := e^{-\Lambda_t^{\mathbb{F}}} U^{\mathrm{bd}}(t, z e^{\Psi_t})\Big\} \text{ and } \Big\{V^{\mathbb{F}}(t,c) := e^{-\Lambda_t^{\mathbb{F}}} U^{\mathrm{sd}}\Big(t, \dfrac{c}{\psi_t}\Big)\dfrac{\psi_t}{R_t^Y}\Big\}, \\ \Big\{\tilde{U}^{\mathbb{F}}(t,y) := e^{-\Lambda_t^{\mathbb{F}}} \tilde{U}^{\mathrm{bd}}(t, y e^{\Lambda_t^{\mathbb{F}}-\Psi_t})\Big\} \text{ and } \Big\{\tilde{V}^{\mathbb{F}}(t,c) := e^{-\Lambda_t^{\mathbb{F}}} \tilde{U}^{\mathrm{sd}}(t, c e^{\Lambda_t^{\mathbb{F}}} R_t^Y)\dfrac{\psi_t}{R_t^Y}\Big\}. \end{cases}$$

is bi-revealed by the triplet $(u, (X^{\mathbb{F}}, C^{\mathbb{F}}), Y^{\mathbb{F}})$.

Using the following identities of $\rho_t(x) = R_t^X(x)\psi_t$ (Proposition 5.2) and of the $\mathbb{G}$-pricing kernel jump $R_t^Y(y)\mathrm{d}\Lambda_t^{\mathbb{F}} = \psi_t \mathrm{d}t$ (Theorem 5.1), we remark that the consumption $C_t^{\mathbb{F}}(x) = X_t^{\mathrm{bd}}(x)\rho_t(x)$ can be written as $C_t^{\mathbb{F}}(x)\mathrm{d}t = X_t^{\mathrm{sd}}(x)\psi_t \mathrm{d}t = X_t^{\mathrm{sd}}(x) R_t^Y \mathrm{d}\Lambda_t^{\mathbb{F}} = C_t^{\Lambda}(x)\mathrm{d}\Lambda_t^{\mathbb{F}}$ using the notations of Section 4. Then Corollary 5.3 is an immediate consequence of Theorem 4.1, by observing the equivalence of the following martingale properties:

$$\begin{aligned} &\Big\{U^{\mathbb{G}}(t, X_t^{\mathbb{G}}(x)) = U^{\mathrm{bd}}(t, X_t^{\mathrm{bd}}(x))\mathbb{1}_{t<\tau} + U^{\mathrm{sd}}(t, X_t^{\mathrm{sd}}(x))\mathbb{1}_{\tau\leqslant t}\Big\} \text{ is an } (\mathbb{G},\mathbb{P})\text{-martingale}, \quad \forall x; \\ &\Big\{U^{\mathbb{F}}(t, X_t^{\mathbb{F}}(x)) + \int_0^t V^{\mathbb{F},\Lambda}(s, C_s^{\Lambda}(x))\mathrm{d}\Lambda_s^{\mathbb{F}}\Big\} \text{ is an } (\mathbb{F},\mathbb{Q}^L)\text{-martingale}, \quad \forall x; \\ &\Big\{U^{\mathbb{F}}(t, X_t^{\mathbb{F}}(x)) + \int_0^t V^{\mathbb{F}}(s, C_s^{\mathbb{F}}(x))\mathrm{d}s\Big\} \text{ is an } (\mathbb{F},\mathbb{Q}^L)\text{-martingale}, \quad \forall x. \end{aligned}$$

5.2.3 Example of CRRA utilities

To illustrate the previous result, we detail the example of dynamic power-type utilities. An $\mathbb{G}$-dynamic utility $U^{\mathbb{G}}$ with relative risk aversion $\alpha \in]0,1[$ is necessarily of the following time-separable form: $U^{\mathbb{G}}(t,z) = Z_t^{\mathbb{G}} \frac{z^{1-\alpha}}{1-\alpha}$ where $Z^{\mathbb{G}} = (Z^{\mathrm{bd}}, Z^{\mathrm{sd}})$ is an $\mathbb{G}$-optional positive process. If $U^{\mathbb{G}}$ is revealed by the wealth $X^{\mathbb{G}}$, then

$$\begin{aligned} &\Big\{U^{\mathbb{G}}(t, X_t^{\mathbb{G}}) = \frac{Z_t^{\mathbb{G}}(X_t^{\mathbb{G}})^{1-\alpha}}{1-\alpha}\Big\}_{t\geqslant 0} \text{ is a } \mathbb{G}\text{-martingale, and} \\ &\Big\{U_x^{\mathbb{G}}(t, X_t^{\mathbb{G}}) = Z_t^{\mathbb{G}}(X_t^{\mathbb{G}})^{-\alpha}\Big\}_{t\geqslant 0} \text{ is a pricing kernel (starting from } x^{-\alpha}) \text{ denoted } Y_t^{\mathbb{G}}(x^{-\alpha}). \end{aligned}$$

In this $\mathbb{G}$-universe, we prove that $X^{\mathbb{G}}$ and $Y^{\mathbb{G}}$ are linear with respect to their initial conditions, which is a well known result for CRRA utilities in a non-defaultable market, see [9]. Indeed, using the natural factorization[3] $Z^{\mathbb{G}} = Z^{\mathbb{G},\sigma} Z^{\mathbb{G},\perp}$, one deduces that $Z_t^{\mathbb{G},\sigma}(X_t^{\mathbb{G}}(x))^{-\alpha} = x^{-\alpha} Y_t^0$, where Y_t^0 is the minimal pricing kernel which is intrinsic to the market and thus does not depend on the investor's initial wealth x. This means that $X_t^{\mathbb{G}}(x) = xX_t^{\mathbb{G}}(1)$ where $X_t^{\mathbb{G}}(1)$, also denoted $\bar{X}_t^{\mathbb{G}}$, is the optimal portfolio starting from $x = 1$. Similarly, $Z^{\mathbb{G},\perp} = Y^{\mathbb{G},\perp}$ which implies that $Y_t^{\mathbb{G}}(y) = yY_t^0 Z_t^{\mathbb{G},\perp} = y\bar{Y}_t^{\mathbb{G}}$. In addition the utility $U^{\mathbb{G}}(t,x)$ and its conjugate $\widetilde{U}^{\mathbb{G}}(t,y)$ can be written in terms of the characteristics $X^{\mathbb{G}}$ and $Y^{\mathbb{G}}$ as follows:

$$U^{\mathbb{G}}(t,x) = \frac{1}{1-\alpha}\bar{Y}_t^{\mathbb{G}}\bar{X}_t^{\mathbb{G}}\left(\frac{x}{\bar{X}_t^{\mathbb{G}}}\right)^{1-\alpha}, \quad \widetilde{U}^{\mathbb{G}}(t,y) = -\frac{\bar{Y}_t^{\mathbb{G}}\bar{X}_t^{\mathbb{G}}}{1-\frac{1}{\alpha}}\left(\frac{y}{\bar{Y}_t^{\mathbb{G}}}\right)^{1-\frac{1}{\alpha}}.$$

According to Corollary 5.3, $U^{\mathbb{G}}(t,x)$ is related to the $\mathbb{F}$-dynamic utility system $(U^{\mathbb{F}}, V^{\mathbb{F}})$ of wealth and consumption given by

$$\left\{U^{\mathbb{F}}(t,z) := e^{-\Lambda_t^{\mathbb{F}}+(1-\alpha)\Psi_t} Z_t^{\mathrm{bd}}\frac{z^{1-\alpha}}{1-\alpha}\right\} \quad \text{and} \quad \left\{V^{\mathbb{F}}(t,c) := e^{-\Lambda_t^{\mathbb{F}}}\frac{\psi_t^{\alpha}}{R_t^Y} Z_t^{\mathrm{sd}}\frac{c^{1-\alpha}}{1-\alpha}\right\}.$$

Note that the components $(Z^{\mathrm{bd}}, Z^{\mathrm{sd}})$ are linked. Indeed, $U^{\mathbb{G}}(t, X_t^{\mathbb{G}}(x)) = \frac{x^{1-\alpha}}{1-\alpha}Z_t^{\mathbb{G}}(\bar{X}_t^{\mathbb{G}})^{1-\alpha}$ is a $(\mathbb{G},\mathbb{P})$-martingale, or equivalently, by Theorem 3.7,

$$M_t^{\mathbb{Q}} := Z_t^{\mathrm{bd}}(\bar{X}_t^{\mathrm{bd}})^{1-\alpha}\exp\left(\int_0^t \left(\frac{Z_s^{\mathrm{sd}}(\bar{X}_s^{\mathrm{sd}})^{1-\alpha}}{Z_s^{\mathrm{bd}}(\bar{X}_s^{\mathrm{bd}})^{1-\alpha}} - 1\right)\mathrm{d}s\right) \text{ is a } (\mathbb{F},\mathbb{Q}^L)\text{-martingale.}$$

Therefore the drift of the dynamics of Z^{bd} depends on Z^{sd} as follows:

$$\mathrm{d}Z_t^{\mathrm{bd}} = -\left(\left(\frac{\bar{X}_t^{\mathrm{sd}}}{\bar{X}_t^{\mathrm{bd}}}\right)^{1-\alpha} Z_t^{\mathrm{sd}} - Z_t^{\mathrm{bd}}\right)\mathrm{d}t - Z_t^{\mathrm{bd}}\frac{d\left[(\bar{X}_t^{\mathrm{bd}})^{1-\alpha}\right]}{\left[(\bar{X}_t^{\mathrm{bd}})^{1-\alpha}\right]} + Z_t^{\mathrm{bd}}\mathrm{d}M_t^{\mathbb{Q}}.$$

This link between the time-component process of CRRA utility system $(U^{\mathbb{F}}, V^{\mathbb{F}})$ of wealth and consumption has already been highlighted in [9, Proposition 4.5].

Conclusion By adopting the general abstract viewpoint on bi-revealed utilities and applying it to a defaultable universe, this paper proves the equivalence between a utility criterion from aggregate wealth and consumption with a utility criterion from aggregate wealth with random horizon. The consumption process is interpreted as an accumulation of reserves that are kept aside in order to face an unpredictable event arriving at the random date τ. Given the interpretation of τ, many models are possible. In the case of an ecological risk, τ is an exponential time independent to the basic financial assets of the market. In the case of a sovereign default, the natural hypothesis is, as in the credit literature, that the default admits a stochastic intensity. All these models are included in the general setting of defaultable universe considered in this paper.

Acknowledgements

This work is with the financial support of the "Chaire Risque Financier" of the "Fondation du Risque", the Labex MME-DII. The authors's research is part of the ANR project DREAMeS (ANR-21-CE46-0002).

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