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Preface for recent advances in forward performance processes

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1. Introduction

Forward performance measurement concept was developed in the context of investment. The idea grew out of desire to propose a framework for investment performance measurement which can run in parallel to another very important idea of modern finance, namely, arbitrage free valuation of derivatives. One may say that the two areas, namely investment and valuation, have little to do with one another. However, in our minds, there were many fundamentally important common ingredients to build on. One of us, the author of this preface, worked before on problems related to arbitrage free valuation. The other, Thaleia Zariphopoulou, worked on questions related to optimization and control. Clearly, between the two of us we had a complementary and relevant set of skills to develop the said idea.

Let me outline now what is common for the two areas of finance from the modelling perspective. Pricing of derivative contracts comes down to the replication of the payoff by trading assets in the market and making sure that there is a simple way to get a value from the replicating strategy. This is done by observing that the price is the expected value of the payoff calculated under the so called arbitrage free measure. This is the *bread and butter* of the so called *sell side* of the finance industry, represented mainly by the investment banks.

There is another side of the finance industry, the *buy side,* represented mainly by the institutional asset managers, insurance companies and hedge funds. What I say here is a huge simplification of reality. However, I do it to explain the origin of the idea. The *sell side* offers to the *buy side* services in form of financial contracts which allow the buyers to control or take certain financial risk.

Investment banks accumulate risk present in the products sold to the *buy side.* They manage this risk at a book level using hedging methodology derived from the arbitrage free valuation. Products are sold at different times and have different maturities. Risk is measured and managed on a daily basis at the book level. Methodology of valuation is additive and adapted to the changing market conditions.

Asset managers behave in a very similar way. They manage money of investors in accordance to their risk profiles. Periodically, some assets are sold and other are bought in a self financing fashion. Of course, this is not always the case in practice. In pension industry, for example, additional funds are added to the pension during accumulation mode and are withdrawn from it in the pension mode. We are not focusing on the impact of the inflow and outflow of capital

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from a fund at this point. We prefer to assume that the sell and buy sides of the industry run self financing strategies. However, their objectives are different. Self financing investment strategy of the buy side aims to *exploit* risk premia. Self financing pricing strategy of the sell side *eliminates* risk premia from the valuation.

The forward performance measurement framework aims to reflect as close as possible mathematical modelling and practice used in both sides of the finance industry. The same mathematical model is used for both sides. Martingale based valuation principle of the sell side is paralleled with martingale property of forward performance process evaluated at the optimal investment strategy. The nonlinear dependence of the forward performance process on wealth generated by an investment strategy aims to capture the risk premia. Strategy is suboptimal in terms of a criteria defined if the investor following it is worse off in some sense. When the market conditions change, the manager may adjust the strategy accordingly. Risk premia the strategy aims to monetize are represented by a stochastic process. Information about this process is revealed as we go. The manager aims to use it at every step of investment process. In fact, his job is to generate a sequence of optimal in some sense investment decisions. This situation is quite different from the more classical formulation of optimization and control problems and hence requires a new approach.

It turns out that such modelling is applicable in many practical situations, way beyond the investment context for which it was originally developed. One of the main benefits is the fact that when model-based decisions are made they depend only on model specification until that moment. Moreover, when conditions change the framework can be adjusted accordingly. The two volumes give various examples of such applications. I will describe some of them in greater detail at the end of this text.

2. Investment context

An investment strategy may be based upon assumptions over a single period returns, like in the case of the Markowitz portfolio, or it may refer to a continuous time model like in the so called Merton optimization problem. When excess returns are zero there is no reason to take any investment risk.

In our model, investment universe is defined by a finite number of risky assets and one riskless asset, the so called savings account. Wealth is denominated in dollars and the investment strategy, exploiting excess returns, generates a portfolio with wealth denominated in dollars and discounted by the savings account. When the risk premia are zero, all wealth is invested in the riskless asset.

In general, wealth can be expressed differently. Also, there is no need to discount risky assets with the savings account. One may choose any asset as a benchmark and use it to discount other assets. One may, also, construct a benchmark/numeraire and use it to measure investment performance.

Relative to a numeraire measurement is very well established in the context of derivative pricing. It turns out that the price is independent on the numeraire/benchmark choice. The same, however, is not true in the context of investment. The benchmark choice and the optimality criteria are linked together. When a criterion is chosen for a given benchmark, in order to obtain analogous consistency one needs to adjust this criterion accordingly. For this reason, we define the criteria in reference to the portfolio wealth discounted with the savings account. When other wealth variables are chosen for portfolio optimization, optimality criteria needs to be adjusted.

One may choose not to require the above mentioned portfolio consistency. In fact this may often be the case in practice. Here we propose a framework in which all potential inconsistencies can be analyzed and their impact on the overall performance measured.

Optimality criteria often refers to a concept of utility. The above mentioned Markowitz and Merton setups can be formulated as utility optimization problems. In the first, the focus is on utility of returns, while in the second is on utility of wealth. In general, wealth variable may refer to many different concepts. It may refer, for example, to a number of shares held in a portfolio or even more generally to a proportion of funds selected from different asset classes.

In the classical Merton problem one maximizes expected utility of terminal wealth over all possible scenarios. There is a single variable present in the utility, namely wealth. If the utility is denoted by U and wealth is denoted by $x \geqslant 0$ the value $U(x)$ represents the utility of wealth x as measured today. In the context of investment, the function U is determined at the beginning of an investment and used to measure utility of wealth X^{π} generated by the investment strategy π and some future time T. The utility optimization consists of maximization of $EU(X_T^{\pi})$ over all possible strategies π. Recall that the wealth process X^{π} of strategy π is discounted with the riskless asset in our analysis. Therefore, the strategy to hold riskless asset with the initial investment x generates constant wealth process also equal to x. Consequently, at the end of the investment horizon T this strategy has utility $U(x)$ which is the same as at the beginning of the investment. This is a direct consequence of the way utility problem was defined. Utility of sure reward $x \geqslant 0$ is the same today as in any future time.

Time plays fundamental role in anything we do in finance. Calculation of the present value of future cash flows is omnipresent in the entire finance industry. Borrowing and lending for term is an important driver of economic activity. The question is: should one also consider utility of time and if yes what should it represent? Clearly, it has to be something different to just calculating present value as explained in the previous paragraph. We focus on this question in the following section.

3. Forward performance process

From now on we consider utility of two variables, namely, of wealth $x \geqslant 0$ and time $t \geqslant 0$. As before, utility is denoted by U but now $U(x,t)$ represents utility of wealth x and time t. We have seen before that wealth does not have to be expressed in dollars. Similarly, time does not have to represent calendar time. For example, it could represent volume time or in any other time concept that is relevant to a particular problem we analyze.

Forward performance process $U(x,t)$, $t \geqslant 0$, is a predictable stochastic process defined on a probability space equipped with filtration $\mathcal{F}_t, t \geqslant 0$. There are k risky assets and one riskless asset, denoted by $S^i, i = 1, ..., k$ and B, respectively. They satisfy

$$\mathrm{d}S^i = S^i \left(\mu^i \mathrm{d}t + \sigma^i \cdot \mathrm{d}W\right), \quad \left|\sigma^i\right| > 0, \quad i = 1, ..., k,$$

$$\mathrm{d}B = rB\mathrm{d}t, \quad B_0 = 1.$$

The process W is a standard d-dimensional Brownian motion.

We analyze performance of self financing investment strategies using a criterion involving the process $U(x,t)$, $t \geqslant 0$. We start with the initial capital $x \geqslant 0$ and invest at any future time t in B and $S^i, i = 1, ..., k$. The discounted amounts invested in the riskless asset B and the risky assets $S^i, i = 1, ..., k$ are denoted, respectively, by

$$\pi^0, \pi = \left(\pi^1, ..., \pi^k\right)^{'}.$$

The strategy π is self financing. The discounted wealth it generates is given by

$$X^\pi = \sum_{i=0}^{k} \pi^i.$$

We assume no arbitrage in our model. The discounted wealth dynamics of a fixed strategy π are

$$\mathrm{d}X^\pi = \sigma\pi \cdot (\lambda \mathrm{d}t + \mathrm{d}W),$$

where

$$\lambda = \left(\sigma^{'}\right)^{+} (\mu - r\mathbf{1}),$$

and

$$\mu = \left(\mu^1, ..., \mu^k\right)^{'}.$$

In the above $'$ stands for transpose and $+$ for the Moore Penrose pseudo inverse matrix.

As mentioned before, in the classical utility optimization, a single utility function $U(x)$ is chosen. It is often assumed that U is increasing and concave. If the same U is chosen for any investment horizon t, then the classical setup becomes a particular case of a criteria involving the process $U(x,t)$, $t \geqslant 0$, namely,

$$U(x,t) = U(x,0), \quad t \geqslant 0.$$

It follows, as explained previously, that utility of a sure reward x is the same today as in any future time t. This is may not always be the case when performance is measured with respect to process $U(x,t)$, $t \geqslant 0$.

We assume, like in the classical case, that for each t the function $U(x,t)$ is increasing and concave in x. To measure performance of a self financing strategy π generating wealth process X^π we require that

$$E\left(U\left(X_s^\pi, s\right) | \mathcal{F}_t\right) \leqslant U\left(X_t^\pi, t\right), \quad 0 \leqslant t \leqslant s.$$

The above inequality states that the expected at time t utility of future wealth X_s^π, namely $E\left(U\left(X_s^\pi, s\right) | \mathcal{F}_t\right)$, is smaller than the utility of wealth X_t^π calculated at time t, namely $U\left(X_t^\pi, t\right)$. Consequently, when implementing strategy π we loose in expected utility. In mathematical terms this simply means that the process $U\left(X_t^\pi, t\right), t \geqslant 0$, is a supermartingale.

Moreover, we say that the strategy π^* is optimal when the process

$$U\left(X_t^{\pi^*}, t\right), \quad t \geqslant 0$$

is a martingale. This in turn means that when implementing strategy π^* on the average we maintain the same level of utility of wealth across time. So, when starting with wealth x and acting optimally we keep on the average its utility.

Observe that the process $U(x,t)$, $t \geqslant 0$, is a supermartingale. Indeed, x represents wealth process of the strategy consisting of investment of x in the riskless asset B. Note that it is not obvious if the process $U(x,t)$ which satisfies the above requirements exists.

It turns out that if $U(x,t)$ is strictly increasing and concave in x and satisfies the stochastic PDE

$$\mathrm{d}U(x,t)=\frac{1}{2}\frac{\left|U_x(x,t)\lambda_t+\sigma_t\sigma_t^+a_x(x,t)\right|^2}{U_{xx}(x,t)}\mathrm{d}t+a(x,t)\cdot\mathrm{d}W_t, \tag{1}$$

with a suitable volatility $a(x,t)$, $t\geqslant 0$, and initial condition $U(x,0)$, then

$$U\left(X_t^{\pi},t\right),\quad t\geqslant 0$$

is a supermartingale for any π and

$$U\left(X_t^{\pi^*},t\right),\quad t\geqslant 0$$

is a martingale with

$$\pi_t^*=-\sigma_t^+\frac{U_x\left(X_t^{\pi^*},t\right)\lambda_t+a_x\left(X_t^{\pi^*},t\right)}{U_{xx}\left(X_t^{\pi^*},t\right)}, \tag{2}$$

and $X_t^{\pi^*}$ given by

$$\mathrm{d}X_t^{\pi^*}=-\frac{U_x\left(X_t^{\pi^*},t\right)\lambda_t+\sigma_t\sigma_t^+a_x\left(X_t^{\pi^*},t\right)}{U_{xx}\left(X_t^{\pi^*},t\right)}\cdot\left(\lambda_t\mathrm{d}t+\mathrm{d}W_t\right). \tag{3}$$

Remark 1 *General conditions on the process* $a(x,t)$, $t\geqslant 0$, *and the initial condition* $U(x,0)$ *which guarantee existence and uniqueness of solutions to the equation* (1) *are not known. It turns out that even in the simplest case, when the process* $a(x,t)\equiv 0$, *not all increasing and concave functions can serve as initial conditions. This is an important difference with the classical utility problem where arbitrary increasing and concave utility function* U *may be used.*

3.1 Risk premia

Recall we assume no arbitrage in our model. This is the case when

$$\mu-r\mathbf{1}\in Lin\left(\sigma^{'}\right).$$

We define the excess returns or the risk premia by

$$\lambda=\left(\sigma^{'}\right)^{+}(\mu-r\mathbf{1}).$$

Here $Lin(\sigma^{'})$ stands for the linear space generated by the columns of matrix $\sigma^{'}$. The main aim of investment is to monetize the excess returns. Investment strategies can be profitable if risk premia exist in the market. If there is no risk premia to be exploited one should not hold a risky position in the market and hold only the riskless asset.

Estimation of excess returns is challenging. We are not dealing with this problem here. However, we insure in our model that when there is no risk premia to be exploited it is optimal to hold the riskless asset. In the model terms this means that

$$\lambda=\mathbf{0}\Longrightarrow\ U(x,t)=U(x,0).$$

To achieve this we also assume that the volatility process a satisfies

$$\lambda=\mathbf{0}\Longrightarrow\ a(x,t)=0. \tag{4}$$

Observe that we then have

$$\pi_t^*=\mathbf{0},\quad t\geqslant 0,$$

$$X_t^{\pi^*}=x,\quad t\geqslant 0,$$

and, hence, the entire wealth x is invested in the riskless asset B.

Remark 2 *Here is another difference with the classical utility optimization problem, that is when*

$$U(x,t) = U(x,0), \tag{5}$$

and $U(x,0)$ *is increasing and concave in* x. *Excess returns may not be present and one may still hold positions in risky assets. This is because utility functions and the investment universe can be arbitrary. On the contrary, in our setup we deduce from* (5) *that* $U(x,t)$ *is a (constant) martingale and hence holding the riskless asset is optimal.*

3.2 Decreasing utility process

Assume now that for all x and t

$$a(x,t) = 0.$$

It follows from (1) that U satisfies

$$\mathrm{d}U(x,t) = \frac{1}{2}\frac{\left|U_x(x,t)\lambda_t\right|^2}{U_{xx}(x,t)}\mathrm{d}t. \tag{6}$$

It turns out that all solutions to (6) can be characterized. They are given in terms of solutions to the equation

$$u_t u_{xx} = \frac{1}{2}u_x^2, \tag{7}$$

which turn out to be increasing and concave in x and decreasing in t, and of the aggregate risk premia process given by

$$A_t = \int_0^t |\lambda|^2\,\mathrm{d}s. \tag{8}$$

Indeed, define

$$U(x,t) = u(x,A_t) \tag{9}$$

and verify that for any strategy π,

$$\begin{aligned} \mathrm{d}U(X_t^\pi,t) &= \mathrm{d}u(X_t^\pi,A_t) \\ &= u_x(X_t^\pi,A_t)\,\sigma_t\pi_t\cdot\mathrm{d}W_t + \frac{1}{2}u_{xx}(X_t^\pi,A_t)\left|\sigma_t\pi_t - r(X_t^\pi,A_t)\lambda_t\right|^2\mathrm{d}t, \end{aligned} \tag{10}$$

where the risk tolerance

$$r(x,t) = -\frac{u_x(x,t)}{u_{xx}(x,t)}$$

satisfies

$$r_t + \frac{1}{2}r^2 r_{xx} = 0.$$

From (10), we see that, for any π, the process $U(X_t^\pi,t)$ is a supermartingale and for the strategy

$$\pi_t^* = r\left(X_t^{\pi^*},A_t\right)\sigma_t^+\lambda_t \tag{11}$$

it is a matringale.

Remark 3 *Decreasing in* t, $U(x,t)$ *means that* x *units of wealth is preferred today rather than in later time. This contrasts with the classical utility setup analyzed before. One way to justify such an assumption is to refer to the concept of impatience which assumes preference for earlier rather than later consumption. Preference ordering that exhibit impatience are also described as being myopic or as embodying utility based discounting.*

Remark 4 *Decreasing performance process defined in* (9) *displays very special dependence on the risk premia process* λ_t, $t \geqslant 0$. *Indeed, they are aggregated in the increasing process* A *defined in* (8). *This links to the previous comments that in the process* $U(x,t), t \geqslant 0$ *argument* t *does not have to refer to the calendar time. The methodology is invariant on the random time change. One could work with subordinated processes and filtration instead. This is obviously true in the case of* $a = 0$. *In general, one may need to impose additional assumptions on the volatility process* a *to maintain the same property.*

3.3 Utility functions

In our framework we define utility functions dependent on wealth alone as averages with respect to time t over the process $U(x,t), t \geqslant 0$. They are objects of the general form

$$\int_0^\infty U(x,t)\,\mu(\mathrm{d}t), \quad \mu \geqslant 0, \quad \int_0^\infty \mu(\mathrm{d}t) = 1. \tag{12}$$

The measure μ may be a Dirac delta at T. This choice reduces to the classical utility problem. As usual, utility optimization consists of maximization of expected utility of wealth, i.e.,

$$\sup_\pi E \int_0^\infty U(X_t^\pi, t)\,\mu(\mathrm{d}t).$$

Portfolio $\widetilde{\pi}$ is optimal if

$$\sup_\pi E \int_0^\infty U(X_t^\pi, t)\,\mu(\mathrm{d}t) = E \int_0^\infty U\left(X_t^{\widetilde{\pi}}, t\right)\mu(\mathrm{d}t).$$

Observe that, if π^* is given by (3), then for any π

$$E \int_0^\infty U(X_t^\pi, t)\,\mu(\mathrm{d}t) \leqslant U(x,0) = E \int_0^\infty U\left(X_t^{\pi^*}, t\right)\mu(\mathrm{d}t)$$

because of the suprmartingale property of the process $U(X_t^\pi, t)$ and the martingale property of the process $U\left(X_t^{\pi^*}, t\right)$. Consequently, portfolio π^* is optimal for any utility of form (12).

Remark 5 *This shows again the difference with the classical utility optimization. For the two approaches to coincide, one needs to make sure that the value functions of the classical problem are consistent in some sense with the process* $U(x,t)$. *This, in turn, implies that marginal utility* $U(x,t)$ *at time* t *of the process* $U(x,t), t \geqslant 0$ *cannot be chosen arbitrarily. The process has a structure defined by the equation* (1). *The value functions evaluated at the optimal wealth are martingales and so is the process* U *evaluated at optimal wealth. For the required consistency, consider two classical optimization problems for time horizons* T_1 *and* T_2 *with* $T_1 < T_2$. *Utility* $U(x,T_2)$ *has to be constructed in such a way that the two value functions coincide on the interval* $[0,T_1]$ *for all* T_1 *and* T_2. *If such a structure can be build then, obviously, the optimal strategy with respect to the criteria given by* (1) *is also optimal for all classical problems where* $U(x,T)$ *is the utility and* T *is the investment horizon. Note, however, that* $U(x,T)$ *is a very* *particular utility. Potentially, the only one for which the above mentioned consistency can be guaranteed. In the classical utility setup no such constrains are imposed. Finally, utility process* $U(x,t)$, $t \geqslant 0$ *incorporates information about risk premia defined by the model used. In the classical case there is no imposed link between the utility choice and the opportunity set.*

3.4 Inverse problem

So far we defined an investment universe in which we compared self financing strategies. In order to discriminate between them we defined a dynamic selection criteria, namely, the process $U(x,t), t \geqslant 0$. Finally, under the appropriate conditions we identified optimal strategies.

In practice, however, people use various strategies while investing. The most conservative one is to put all wealth in the riskless asset B. In our framework, this attitude corresponds to the belief that there are no excess returns to be monetized. Another popular strategy is to buy and hold an asset. Yet, another consists of splitting the investment universe into asset classes and depending on the risk profile of an investor divide the initial wealth accordingly. Often 60% of wealth is invested in equity and 40% in fixed income. More conservative portfolios may have more wealth invested in fixed income and less in equity.

A natural question is if such strategies may be optimal. If the answer is yes, what does our model say about the risk premia in the market and the risk appetite of the investor? While topics covered so far were either published in academic journals or at least presented at conferences, the inverse problem discussed in this part of the preface is new to the best of my knowledge. For this reason I prefer to concentrate here on examples rather than on a comprehensive analysis of the general case which, in my opinion, should be presented in a paper submitted to a peer reviewed journal.

Reduce the general model to the case when we have one risky asset S, driven by one dimensional Brownian motion W and the riskless asset B. Consider the case when all wealth x is invested in the risky asset. The strategy, say π, consists of buying x/S_0 of shares of the risky asset and keeping it. Clearly, such a strategy is self financing and its discounted dollar value at time t, denoted by X^π, is

$$X_t^\pi = \frac{x}{S_0}\frac{S_t}{B_t}, \quad t \geqslant 0.$$

In this case we have

$$\pi_t = X_t^\pi, \quad t \geqslant 0 \tag{13}$$

because the entire initial wealth x is invested in S and nothing in B. Now, assume that π is optimal. Then it satisfies (2) and hence it must be that

$$\pi_t = -\frac{1}{\sigma_t}\frac{U_x(\pi_t,t)\,\lambda_t + a_x(\pi_t,t)}{U_{xx}(\pi_t,t)}, \tag{14}$$

for all $t \geqslant 0$. It follows from (3) that

$$\mathrm{d}\pi_t = \sigma_t\pi_t\left(\lambda_t \mathrm{d}t + \mathrm{d}W_t\right) = \pi_t\left(\left(\mu_t - r_t\right)\mathrm{d}t + \sigma_t W_t\right), \quad \pi_0 = x,$$

and, hence, the optimal wealth coincides with the wealth generated by the strategy π of holding risky asset S.

Condition (14) imposes constraint on the volatility process a. Indeed, replace π_t with a variable x and observe that the above condition (14) holds for an arbitrary but fixed time t.

To simplify the notation, skip the t argument and write (14) as follows

$$\sigma x U_{xx} = -\left(U_x \lambda + a_x\right).$$

Also assume that $\lambda_t = \lambda > 0$ and $\sigma_t = \sigma > 0$. Then, equation (1) reduces to

$$\mathrm{d}U = \frac{1}{2}\sigma^2 x^2 U_{xx}\mathrm{d}t + \left(\left(\sigma - \lambda\right)U - \sigma x U_x\right)\mathrm{d}W.$$

We are going to look for solutions to this equation in the general class of processes defined by

$$U\left(x,t\right) = u\left(\frac{x}{Y_t}, A_t\right) Z_t, \tag{15}$$

where u satisfies (7), while Y, Z and A are given by

$$\mathrm{d}Y_t = Y_t \delta\left(\lambda \mathrm{d}t + \mathrm{d}W_t\right), \quad Y_0 = 1,$$

$$\mathrm{d}Z_t = Z_t \phi \mathrm{d}W_t, \quad Z_0 = 1,$$

and

$$A_t = \left(\lambda + \phi - \delta\right)^2 t.$$

It turns out that, for any strategy π, we have

$$\begin{aligned} \mathrm{d}U\left(X_t^\pi, t\right) = & \left(U_x\left(X_t^\pi, t\right)\left(\sigma \pi_t - X_t^\pi \delta\right) + U\left(X_t^\pi, t\right)\phi\right)\mathrm{d}W_t \\ & + \frac{1}{2}u_{xx}\left(\frac{X_t^\pi}{Y_t}, A_t\right) Z_t \left|\frac{1}{Y_t}\sigma\pi_t - \left(\frac{X_t^\pi}{Y_t}\delta + r\left(\frac{X_t^\pi}{Y_t}, A_t\right)\left(\lambda + \phi - \delta\right)\right)\right|^2 \mathrm{d}t. \end{aligned} \tag{16}$$

In this example, we use $Y = S/S_0$ and hence $\delta = \sigma > 0$. Moreover, $\pi_t = X_t^\pi$, as stated in (13), and, thus, the expression reduces further to

$$\begin{aligned} \mathrm{d}U\left(X_t^\pi, t\right) = & U\left(X_t^\pi, t\right)\phi_t \mathrm{d}W_t \\ & + \frac{1}{2}u_{xx}\left(\frac{X_t^\pi}{Y_t}, A_t\right) Z_t \left|r\left(\frac{X_t^\pi}{Y_t}, A_t\right)\left(\lambda + \phi - \sigma\right)\right|^2 \mathrm{d}t. \end{aligned}$$

Now choose

$$\phi = \sigma - \lambda$$

and note that

$$A = 0.$$

Consequently, $U\left(X_t^\pi, t\right) = U\left(\pi_t, t\right)$ is a martingale and the buy and hold policy is optimal with the process

$$U\left(x,t\right) = u\left(\frac{x}{S_0}, 0\right) Z_t, \tag{17}$$

where

$$\mathrm{d}Z_t = Z_t\left(\sigma - \lambda\right)\mathrm{d}W_t,$$

and the function $u\left(\cdot, 0\right)$ is such that equation (7) has a solution.

Remark 6 *The above conclusion is intuitively obvious. Indeed, in our reduced model there are only two assets, one riskless and one risky called, respectively,* B *and* S*. It is straightforward to verify that* B *expressed in units of* S *satisfies*

$$\mathrm{d}\frac{B_t}{S_t} = \frac{B_t}{S_t}\sigma\left(\left(\sigma-\lambda\right)\mathrm{d}t - \mathrm{d}W_t\right).$$

Consequently the risk premia are

$$\sigma-\lambda.$$

Observe that now wealth is expressed in the number of shares of S *and not in the discounted with* B *value of* S *as analyzed initially. The process* Z *represents the view of the investor. Indeed, we see that*

$$\mathrm{d}\frac{B_t}{S_t} = -\frac{B_t}{S_t}\sigma\left(\mathrm{d}W_t - \left(\lambda-\sigma\right)\mathrm{d}t\right) = \frac{B_t}{S_t}\sigma\mathrm{d}\widetilde{W}_t$$

and the process

$$\widetilde{W}_t = W_t - \left(\lambda-\sigma\right)t$$

is a Brownian motion under the measure $\widetilde{\mathbb{P}}$ *defined by*

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \exp\left(\left(\lambda-\sigma\right)W_T - \frac{1}{2}\left(\lambda-\sigma\right)^2 T\right).$$

Clearly under $\widetilde{\mathbb{P}}$ *there are no risk premia to monetize and, hence, it is optimal to hold* S*. However, under* $\mathbb{P}$ *one may try to outperform the benchmark* S *by using the optimal strategy. Finally, observe that the first variable of* u *in* (17) *refers to wealth expressed in the number of shares bought at time* 0 *while in* U *wealth is measured in dollars at time* 0.

Next, consider the strategy, say π, which consists of holding a fixed proportion of wealth, say ω, in the risky asset S with the reminder invested in B. In this case we have, for all t,

$$\omega = \frac{\pi_t}{X_t^\pi}.$$

It then follows trivially from (16) that with $\delta=\sigma\omega$ and $\phi=\sigma\omega-\lambda$ the process $U\left(X_t^\pi,t\right)$, $t\geqslant 0$ is a martingale and, hence, π is optimal in such a configuration of the input variables.

Remark 7 *The above examples demonstrate that with the appropriate choice of* ϕ *in the definition of the process* Z *one can eliminate the risk premia and prove that a given strategy is optimal.*

To eliminate this argument we make sure that the model parameters remain unchanged, or alternatively, that we take no view on the market. Therefore, we assume that $\phi=0$ and hence $Z=1$. Consequently, the general class reduces to

$$U\left(x,t\right) = u\left(\frac{x}{Y_t},A_t\right),$$

with u satisfying (7) and Y and A given by

$$\mathrm{d}Y_t = Y_t\delta\left(\lambda\mathrm{d}t+\mathrm{d}W_t\right), \quad Y_0=1,$$

$$A_t = \left(\lambda-\delta\right)^2 t.$$

It follows trivially from (16) that the strategy π of holding proportion ω in S and the rest in B is optimal if

$$\frac{X^\pi}{Y}\left(\sigma\omega-\delta\right) = r\left(\frac{X^\pi}{Y},A\right)\left(\lambda-\delta\right). \tag{18}$$

Assume now that

$$r\left(x,t\right)=\gamma x,\quad \gamma>1.$$

This corresponds to

$$u\left(x,t\right)=\frac{\gamma}{\gamma-1}x^{\frac{\gamma-1}{\gamma}}e^{-\frac{1}{2}\left(\gamma-1\right)t}.$$

Obviously (18) holds true when

$$\sigma\omega+\left(\gamma-1\right)\delta=\gamma\lambda. \tag{19}$$

Recall that ω is the proportion of wealth invested in S, λ represents the excess return of S per unit of risk σ, which is the volatility of S, δ is the benchmark Y volatility , and finally γ represents the growth rate of the risk aversion function $r\left(x,t\right)=\gamma x$. From (19) we see that the investor has the flexibility to choose the proportion ω, the benchmark Y, and hence its volatility σ, and the growth rate γ of his risk aversion, while keeping specification of the model unchanged. When $\delta=0$, condition (19) imposes constrains on λ and σ for the strategy of investing proportion ω of wealth in the risky asset S to remain optimal. This is because we must have $\gamma>1$ and $0\leqslant\omega\leqslant1$.

Example 8 *We finish with analysis of an example of parameter values and the balanced portfolio where 60% of wealth is invested in equity and 40% in fixed income. Assume* $\mu=0.05, r=0.03$ *and* $\sigma=0.2$, *then* $\lambda=0.1$ *Then,* (19) *becomes*

$$2\omega+10\left(\gamma-1\right)\delta=\gamma.$$

Obviously, with $\omega=0.6$ *and* $\delta=0$, *we get* $\gamma=1.2$, *and for an arbitrary* δ *we get the condition on* ω *and* γ *under which the portfolio is optimal.*

4. Conclusions

We have introduced a general arbitrage free model of the market, and defined a new criteria measuring performance of self financing strategies, the so called forward performance process. Under certain conditions we have identified optimal strategies.

In a reduced model we analyzed the inverse problem. Namely, for a given strategy we constructed the forward performance process for which the said strategy is optimal. It turned out that the balanced portfolio is optimal for the forward performance process for which risk tolerance is a linear function of wealth.

The new performance criteria gave rise to a new stochastic PDE for which general conditions for existence and uniqueness of solutions are still unknown. It is not yet clear what is the practical role that volatility process $a(x,t)$ should play. Analysis of the inverse problem seems to indicate that one may want to control the market view variable in the forward performance process by restricting the volatility $a(x,t)$ appropriately.

Beyond serving as performance measuring mechanisms for investment strategies, forward performance processes opened an entirely new way in formulating optimization criteria as they offer substantial flexibility in terms of model choice and dynamic revisions, horizon specification and dynamic evolution of risk preferences. The last fifteen years or so, forward performance criteria have been used to produce the so-called forward indifference prices and, for the exponential class, the forward entropic measures. Their definition was also extended to encompass several features like intermediate consumption, model ambiguity, defaultable claims, long-term investments, and actuarial risks. More recently, they were extended to games of finite and infinite (mean field limit) number of players.

Besides their modeling contributions, forward performance processes gave rise to new problems in stochastic analysis and stochastic optimization. These are primarily *ill-posed* problems in fully non-linear degenerate stochastic PDE that have not been studied before. For the class of homothetic forward processes (power, logarithmic and exponential), direct connections with infinite-horizon and ergodic BSDE emerged that resulted in new developments in this direction. Furthermore, forward performance processes in finite-dimensions introduced a new class of ill-posed Hamilton-Jacobi-Bellman and quasilinear equations.

The two Volumes on "Recent advances in Forward Performance Processes offer new insights to the theory and applicability of this new class of performance criteria. Topics like model ambiguity, forward robust control, forward mean field games and forward G-expectations are included, together with theoretical and numerical developments. The two Volumes also contain works on new applications of forward performance criteria in pension fund management, optimal liquidation, indifference valuation, and in risk preferences under default.