DOI: 10.19884/j.1672-5220.202410003

Distributed Feedback Quadratic Filter for Estimating Moving Target in Time-Varying Non-Gaussian Systems with Limited Sensing Range

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Abstract: This research considers the tracking problem of a moving target in distributed sensor networks with a limited sensing range (LSR) affected by non-Gaussian noise. In such sensor networks, observation loss due to LSR is a prevalent issue that has received insufficient attention. We introduce a time-varying random variable to describe whether the sensor observes a moving target at each moment. When a single sensor node is unable to receive information from other nodes, it cannot update its state estimation of the moving target once the target moves beyond this node's observation range. We propose an information flow topology within distributed sensor networks to facilitate the reception of prior state estimation data transmitted by neighboring nodes. Based on this information, a quadratic distributed estimator is designed for each sensor, and an output injection term is introduced to handle unstable systems. Finally, a numerical example is provided to illustrate the effectiveness of the proposed control scheme.

Keywords: non-Gaussian system; quadratic estimation; moving target; time-varying
CLC number: TP13 **Document code**: A
Article ID: 1672-5220(2025)06-0661-12

Open Science Identity (OSID)

0 Introduction

In recent decades, with the continuous development of wireless sensor networks, filtering and state estimation algorithms that use information interaction between different agents play an important role, and scholars have conducted extensive research on these technologies[1]. Distributed information processing on a network involves nodes performing tasks using both their local data and data from their neighbors[2]. Many systems in our daily life can be classified as one of the distributed networks, including smart community[3], photovoltaic solar cluster panel, smart grid[4], target tracking[5], environmental monitoring[6], and other systems. Due to the constraints imposed by the limited storage capacity of network space in practical engineering, each node of the sensor network needs to collect, calculate, and transmit data all the time, which leads to the limited application range of the distributed network. Based on this, some new distributed network architectures and algorithms were proposed to deal with these problems[7-9]. With the continuous research on distributed network systems, these systems are extensively utilized, and rich research results have yielded substantial research outcomes, especially in the problem of moving target tracking.

In the context of moving target tracking, information loss resulting from unreliable communication channels cannot be overlooked. This loss can result from network-induced issues such as sensor link failure, communication delay, and packet loss. Assuming independent grouped packet loss sequences, Nahi[10] developed an optimal linear state estimator for systems where sensor observations were subject to probabilistic loss. By modeling observation arrivals via a Bernoulli stochastic process, Sinopoli et al.[11] investigated Kalman filtering under intermittent observations. Their analysis identifies a critical arrival rate, beyond which the estimation error covariance fails to converge.

Under some communication transport protocols, such as Round-Robin and Transmission Control Protocol (TCP), the discarded sequence is predetermined, and the sensor knows when to send the packet. However, this assumption is often unrealistic in many scenarios. The sensor network cannot anticipate in advance when the packet will be lost, and the sensor can only know the probability of packet loss[11]. In this case, the packet loss phenomenon in the sensor network is modeled as a random process obeying a Bernoulli distribution. Considering the limited sensing range (LSR), there is also a possibility of packet loss, that is, the sensor's observation of the moving target is lost based on distance. When the moving target is beyond the observation capability of the sensor, the measurement will be lost. For example, Ilić et al.[12] investigated a distributed fault diagnosis system for sensor networks with LSR, where the attenuation of a randomly fluctuating signal observed by a sensor depends on its distance from

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Received date: 2024-10-14
Foundation item: National Natural Science Foundation of China (No. 61803081)
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Citation: SUO J H, ZHU X F. Distributed feedback quadratic filter for estimating moving target in time-varying non-Gaussian systems with limited sensing range[J]. *Journal of Donghua University (English Edition)*, 2025, 42(6): 661-672.

the target. Zhang et al.[13] introduced a novel algorithm for sensor networks with LSR, which used a time-varying random variable to determine whether the sensor could measure the moving target. Based on these studies, Lian et al.[5] developed an information weighted consistency algorithm to solve the problem that moving targets have LSR in distributed networks.

In practical applications, numerous systems are subject to various forms of non-Gaussian noise. Nonetheless, prior research on moving targets with LSR in distributed networks has predominantly concentrated on Gaussian frameworks, neglecting the potential ramifications within non-Gaussian contexts. For the systems with non-Gaussian noise, the conditional expectation of state estimation based on the minimum variance criterion is usually an infinite-dimensional problem[14]. Consequently, it is imperative to explore alternative estimation algorithms to address the state estimation issues inherent in non-Gaussian systems[15-16]. Among them, Masreliez[15] introduced a method to solve the non-Gaussian filtering problem by adjusting the linear predistortion of both past and present observed nonlinear functions. Merwe et al.[16] proposed a recursive Bayesian estimation algorithm for sequential probabilistic reasoning of nonlinear non-Gaussian systems. Currently, polynomial algorithms maintain the advantageous characteristics such as ease of computation and recursion. Quadratic filter using higher-order statistics of non-Gaussian systems has demonstrated enhanced efficiency and improved accuracy[17-18]. The quadratic filter enhances the estimation accuracy of the optimal linear filter. However, it also possesses inherent limitations. This is primarily due to the fact that noise variance is contingent upon the state variance of the original system. Consequently, if the variance of the state is infinite, it follows that the quadratic system noise variance similarly becomes infinite. As a result, the stability of the secondary filter can only be guaranteed within an asymptotically stable system. To address this issue and broaden the range of the system handled by the quadratic filter, the output injection term was introduced to rewrite the system and decompose the system state into deterministic and stochastic components[19-20].

Inspired by the above discussion, this paper designs a feedback quadratic state estimator for moving targets with non-Gaussian noise and distributed time-varying systems with measurement loss to track the trajectory of moving targets. The main contributions are as follows: 1) it is the first attempt to track a moving target in a distributed system with non-Gaussian noise using feedback quadratic filter (FQF); 2) according to the characteristics of the moving target state equation system matrix, the state equation is rewritten by the output injection method to apply to the state estimation of the moving target with non-Gaussian noise; 3) considering the limited observation distance of the sensor, a uniform expression of the estimated gain for both normal and unobserved targets is given. In this paper, it is necessary to transfer the deterministic part of the state, as well as the prior state estimation of the random part.

Notations: $\mathbf{R}^n$ and $\mathbf{R}^{n\times m}$ denote the n-dimensional Euclidean space and the set of all $n \times m$ matrices, respectively. $\mathbf{N}_+$ is the set of positive integers. $\boldsymbol{A}^{\mathrm{T}}$ represents the transpose of $\boldsymbol{A}$. $\mathbf{E}\{\boldsymbol{x}\}$ denotes the expectation of $\boldsymbol{x}$, and the λth-order moment of the random vector $\boldsymbol{x}$ is represented as $e_{\boldsymbol{x}}^{(\lambda)} = \mathbf{E}\{\boldsymbol{x}^{[\lambda]}\}$, $\lambda \in \mathbf{N}_+$. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two vectors or matrices. The Kronecker product of $\boldsymbol{a}$ and $\boldsymbol{b}$ is denoted by $\boldsymbol{a} \otimes \boldsymbol{b}$, and $\boldsymbol{x}^{[\lambda]} = \boldsymbol{x} \otimes \boldsymbol{x}^{[\lambda-1]}$, $\lambda \in \mathbf{N}_+$. Let vec $(\boldsymbol{M})$ represent the vectorization of the $m \times n$ matrix $\boldsymbol{M}$, which is a column vector in the mn dimension obtained by stacking the columns of the matrix $\boldsymbol{M}$ on top of one another. The corresponding inverse transformation of vec$(\boldsymbol{M})$ to $\boldsymbol{M}$ is represented by $\mathrm{sti}_m(\cdot)$, where m represents the number of rows in the $\boldsymbol{M}$ to be obtained. Moreover, $\mathrm{diag}_N\{\boldsymbol{x}_i\}$ represents a block diagonal matrix where block i is $\boldsymbol{x}_i$ and all other entries being zero, $\mathrm{col}_N\{\boldsymbol{x}_i\}$ stands for the column vector $[\boldsymbol{x}_1^{\mathrm{T}}, \boldsymbol{x}_2^{\mathrm{T}}, \cdots, \boldsymbol{x}_N^{\mathrm{T}}]^{\mathrm{T}}$. $\mathrm{tr}\{\boldsymbol{A}\}$ represents the trace of the matrix $\boldsymbol{A}$.

1 Problem Formulation and Preliminaries

Consider the following distributed sensor networks with LSR nodes affected by non-Gaussian noises,

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}_k \boldsymbol{x}_k + \boldsymbol{w}_k,\ k \in \mathbf{N}, \tag{1}$$

$$\boldsymbol{y}_{i,k} = \beta_i(d_{i,k})(\boldsymbol{H}_{i,k}\boldsymbol{x}_k + \boldsymbol{v}_{i,k}), \tag{2}$$

$$\mathbf{z}_{ij,k} = a_{ij}(\hat{\boldsymbol{x}}_{j,k|k-1} + \boldsymbol{\xi}_{ij,k}), \tag{3}$$

where $\boldsymbol{x}_k \in \mathbf{R}^n$ represents the state of the moving target at time k, and $\hat{\boldsymbol{x}}_{j,k|k-1} \in \mathbf{R}^n$ is the prior estimation of it by sensor j; $\boldsymbol{y}_{i,k} \in \mathbf{R}^m$ is the direct observation of the target by node i; $z_{ij,k} \in \mathbf{R}^n$ is the information received by node i from node j at time k; $\boldsymbol{A}_k \in \mathbf{R}^{n\times n}$ is the system matrix and $\boldsymbol{H}_{i,k} \in \mathbf{R}^{m\times n}$ denotes the observation matrix of node i; $\boldsymbol{w}_k \in \mathbf{R}^n$, $\boldsymbol{v}_{i,k} \in \mathbf{R}^m$ and $\boldsymbol{\xi}_{ij,k} \in \mathbf{R}^n$ are non-Gaussian noises at time k.

Here, $\beta_i(d_{i,k})$ is an observation index that is determined by the distance between the sensor and the moving target. We can build the formulation of $\beta_i(d_{i,k})$ as

$$\beta_i(d_{i,k}) = \begin{cases} 1, & \text{when } d_{i,k} \leqslant R_i, \\ 0, & \text{when } d_{i,k} > R_i, \end{cases} \tag{4}$$

where $d_{i,k}$ is the distance between the position of sensor i and the moving target; R_i is the sensing range of sensor i.

Let N_i be a collection of sensors that can transmit information to sensor i. If sensor j can transmit information to i, then $a_{ij} = 1$; otherwise $a_{ij} = 0$. Thus, we can obtain that

$$a_{ij} = \begin{cases} 1, & j \in N_i, \\ 0, & j \notin N_i. \end{cases} \tag{5}$$

We assume that the initial state $\boldsymbol{x}_0$ and the non-Gaussian random sequence $\boldsymbol{w}_k$, $\boldsymbol{v}_{i,k}$ and $\boldsymbol{\xi}_{ij,k}$ satisfy the following conditions for $k \in \mathbf{R}$:

1) $\mathbf{E}\{\boldsymbol{w}_k\} = 0$, $\mathbf{E}\{\boldsymbol{v}_{i,k}\} = 0$, $\mathbf{E}\{\boldsymbol{\xi}_{ij,k}\} = 0$;

2) $\boldsymbol{w}_k$, $\boldsymbol{v}_{i,k}$ and $\boldsymbol{\xi}_{ij,k}$ are mutually uncorrelated white noises;

3) $\boldsymbol{x}_0, \boldsymbol{w}_k, \boldsymbol{v}_{i,k}$ and $\boldsymbol{\xi}_{ij,k}$ are independent from each other;

4) $\boldsymbol{x}_0, \boldsymbol{w}_k, \boldsymbol{v}_{i,k}$ and $\boldsymbol{\xi}_{ij,k}$ have finite fourth moments.

1.1 Output injection

Firstly, by using output Eq. (2), Eq. (1) can be rewritten as

$$\begin{aligned}\boldsymbol{x}_{k+1} &= \boldsymbol{A}_k\boldsymbol{x}_k + \boldsymbol{w}_k + \boldsymbol{L}_{i,k}[\boldsymbol{y}_{i,k} - \beta_i(d_{i,k})(\boldsymbol{H}_{i,k}\boldsymbol{x}_k + \boldsymbol{v}_{i,k})] \\ &= \check{\boldsymbol{A}}_{i,k}\boldsymbol{x}_k + \boldsymbol{L}_{i,k}\boldsymbol{y}_{i,k} + \boldsymbol{h}_{i,k}, \end{aligned} \tag{6}$$

where $\check{\boldsymbol{A}}_{i,k} = \boldsymbol{A}_k - \beta_i(d_{i,k})\boldsymbol{L}_{i,k}\boldsymbol{H}_{i,k}$; $\boldsymbol{L}_{i,k} \in \mathbf{R}^{n \times m}$ is an arbitrarily selected matrix, subject to the condition that it stabilizes the state Eq. (6) after output injection; $\boldsymbol{h}_{i,k} = \boldsymbol{w}_k - \beta_i(d_{i,k})\boldsymbol{L}_{i,k}\boldsymbol{v}_{i,k}$ is the noise of the new equation of state Eq. (6). Based on the conditions for $k \in \mathbf{R}$, we can obtain that $\{\boldsymbol{h}_{i,k}\}$ is a zero-mean white sequence that is independent of the initial state $\boldsymbol{x}_0$ and with known finite second to fourth moment, by definition of $\{\boldsymbol{w}_k\}$ and $\{\boldsymbol{v}_{i,k}\}$. However, unlike convention, $\{\boldsymbol{h}_{i,k}\}$ is correlated with the noise sequence $\{\boldsymbol{w}_k\}$ and $\{\boldsymbol{v}_{i,k}\}$.

Note that $\beta_i^p(d_{i,k}) = \beta_i(d_{i,k})$, and $e^{(p)}_{\boldsymbol{h}_{i,k}} = \mathbf{E}\{\boldsymbol{h}^{[p]}_{i,k}\}$ for $p = 2$, 3 and 4 can be computed as

$$e^{(2)}_{\boldsymbol{h}_{i,k}} = e^{(2)}_{\boldsymbol{w}_k} + \beta_i(d_{i,k})\boldsymbol{L}^{[2]}_{i,k}e^{(2)}_{\boldsymbol{v}_{i,k}}, \tag{7}$$

$$e^{(3)}_{\boldsymbol{h}_{i,k}} = e^{(3)}_{\boldsymbol{w}_k} - \beta_i(d_{i,k})\boldsymbol{L}^{[3]}_{i,k}e^{(3)}_{\boldsymbol{v}_{i,k}}, \tag{8}$$

$$\begin{aligned} e^{(4)}_{\boldsymbol{h}_{i,k}} = e^{(4)}_{\boldsymbol{w}_k} &+ \beta_i(d_{i,k})\boldsymbol{M}^4_2(e^{(2)}_{\boldsymbol{w}_k} \otimes \boldsymbol{L}^{[2]}_{i,k}e^{(2)}_{\boldsymbol{v}_{i,k}}) + \\ &\beta_i(d_{i,k})\boldsymbol{L}^{[4]}_{i,k}e^{(4)}_{\boldsymbol{v}_{i,k}}, \end{aligned} \tag{9}$$

where $\boldsymbol{M}^4_2$ is the coefficient matrix of the second-order Kronecker power expansion[21]. Note that $\beta_i(d_{i,k})$ is known at time k, so that $e^{(p)}_{\boldsymbol{h}_{i,k}}$ is known at each time $k \in \mathbf{N}$.

1.2 Deterministic and stochastic parts

Provide the deterministic part $\boldsymbol{x}^{\mathrm{d}}_{i,k}$ and stochastic parts $\boldsymbol{x}^{\mathrm{s}}_{i,k}$ of the system state based on output feedback, as follows:

$$\boldsymbol{x}^{\mathrm{d}}_{i,k+1} = \check{\boldsymbol{A}}_{i,k}\boldsymbol{x}^{\mathrm{d}}_{i,k} + \boldsymbol{L}_{i,k}\boldsymbol{y}_{i,k},\ \boldsymbol{x}^{\mathrm{d}}_{i,0} = \boldsymbol{x}_0, \tag{10}$$

$$\boldsymbol{x}^{\mathrm{s}}_{i,k+1} = \check{\boldsymbol{A}}_{i,k}\boldsymbol{x}^{\mathrm{s}}_{i,k} + \boldsymbol{h}_{i,k},\ \boldsymbol{x}^{\mathrm{s}}_{i,0} = 0. \tag{11}$$

From Eqs. (10) and (11), it follows that $\boldsymbol{x}_k = \boldsymbol{x}^{\mathrm{d}}_{i,k} + \boldsymbol{x}^{\mathrm{s}}_{i,k}$, $\forall k \geqslant 0$. Since $\boldsymbol{L}_{i,k}$ is chosen by oneself, it is not difficult to recursively obtain $\boldsymbol{x}^{\mathrm{d}}_{i,k+1}$ with known $\boldsymbol{x}^{\mathrm{d}}_{i,k}$ at time k. Due to the randomness of the noise term $\boldsymbol{h}_{i,k}$, $\boldsymbol{x}^{\mathrm{s}}_{i,k+1}$ cannot be recursively obtained at time k. Ensure the stability of the system using $\check{\boldsymbol{A}}_{i,k}$ as the state transition matrix by selecting the appropriate matrix $\boldsymbol{L}_{i,k}$, the estimates of $\boldsymbol{x}^{\mathrm{s}}_{i,k}$ will converge.

Now, by using output injection and state decomposition to obtain the stochastic and deterministic parts, the problem of state non-convergence caused by the instability of the original state transition matrix $\boldsymbol{A}_k$ has been solved. The evolution of $\boldsymbol{x}^{\mathrm{d}}_{i,k}$ can be observed by

$$\boldsymbol{x}^{\mathrm{d}}_{i,k} = \Big(\prod_{n=0}^{k-1}\check{\boldsymbol{A}}_{i,n}\Big)\boldsymbol{x}^{\mathrm{d}}_0 + \sum_{n=0}^{k-1}\Big(\prod_{m=n+1}^{k-1}\check{\boldsymbol{A}}_{i,m}\Big)\boldsymbol{L}_{i,n}\boldsymbol{y}_{i,n}. \tag{12}$$

When $n = k-1$, we assume that $\prod_{m=k}^{k-1}\check{\boldsymbol{A}}_{i,m} = \boldsymbol{I}_n$. Here, $\boldsymbol{I}_n$ is an n-dimensional identity matrix.

Similarly, we can obtain the stochastic measurement part,

$$\begin{aligned}\boldsymbol{y}^{\mathrm{s}}_{i,k} = \boldsymbol{y}_{i,k} - \beta_i(d_{i,k})\boldsymbol{H}_{i,k}\boldsymbol{x}^{\mathrm{d}}_{i,k} = \\ \beta_i(d_{i,k})(\boldsymbol{H}_{i,k}\boldsymbol{x}^{\mathrm{s}}_{i,k} + \boldsymbol{v}_{i,k}).\end{aligned} \tag{13}$$

The expressions for prior state estimation and posterior state estimation can be given separately as

$$\hat{\boldsymbol{x}}_{i,k|k-1} = \boldsymbol{x}^{\mathrm{d}}_{i,k} + \hat{\boldsymbol{x}}^{\mathrm{s}}_{i,k|k-1}, \tag{14}$$

and

$$\hat{\boldsymbol{x}}_{i,k|k} = \boldsymbol{x}^{\mathrm{d}}_{i,k} + \hat{\boldsymbol{x}}^{\mathrm{s}}_{i,k|k}. \tag{15}$$

Then, we have

$$\boldsymbol{z}^{\mathrm{s}}_{ij,k} = \vec{\boldsymbol{z}}_{ij,k} - a_{ij}\boldsymbol{x}^{\mathrm{d}}_{j,k} = a_{ij}(\hat{\boldsymbol{x}}^{\mathrm{s}}_{j,k|k-1} + \boldsymbol{\xi}_{ij,k}). \tag{16}$$

From Eqs. (10) to (16), we extract the stochastic parts to ensure the stability of the extended system state and noise, then obtain a recursive expression for the deterministic part.

1.3 Quadratic system for stochastic part

Considering that the random noises $\boldsymbol{w}_k$, $\boldsymbol{v}_{i,k}$ and $\boldsymbol{\xi}_{ij,k}$ are non-Gaussian, second-order statistics are required. To this end, we stack the original vectors together with their second-order Kronecker powers, and an augmented system that captures the stochastic part is derived from Eqs. (11), (13) and (16), which can be given by

$$\begin{cases} \boldsymbol{X}^{\mathrm{s}}_i(k) = \begin{bmatrix}\boldsymbol{x}^{\mathrm{s}}_{i,k} \\ \boldsymbol{x}^{\mathrm{s}[2]}_{i,k}\end{bmatrix},\ \boldsymbol{Y}^{\mathrm{s}}_i(k) = \begin{bmatrix}\boldsymbol{y}^{\mathrm{s}}_{i,k} \\ \boldsymbol{y}^{\mathrm{s}[2]}_{i,k}\end{bmatrix}, \\ \boldsymbol{Z}^{\mathrm{s}}_{ij}(k) = \begin{bmatrix}\boldsymbol{z}^{\mathrm{s}}_{ij,k} \\ \boldsymbol{z}^{\mathrm{s}[2]}_{ij,k}\end{bmatrix},\ \hat{\boldsymbol{X}}^{\mathrm{s}}_i(k \mid k-1) = \begin{bmatrix}\hat{\boldsymbol{x}}^{\mathrm{s}}_{i,k|k-1} \\ \hat{\boldsymbol{x}}^{\mathrm{s}[2]}_{i,k|k-1}\end{bmatrix}, \end{cases} \tag{17}$$

where

$$\boldsymbol{x}^{\mathrm{s}[2]}_{i,k+1} = \check{\boldsymbol{A}}^{[2]}_{i,k}\boldsymbol{x}^{\mathrm{s}[2]}_{i,k} + e^{(2)}_{\boldsymbol{h}_{i,k}} + \boldsymbol{h}^{(2)}_{i,k}, \tag{18}$$

$$\boldsymbol{y}^{\mathrm{s}[2]}_{i,k} = \beta_i(d_{i,k})(\boldsymbol{H}^{[2]}_{i,k}\boldsymbol{x}^{\mathrm{s}[2]}_{i,k} + e^{(2)}_{\boldsymbol{v}_{i,k}} + \boldsymbol{v}^{(2)}_{i,k}), \tag{19}$$

$$\boldsymbol{z}^{\mathrm{s}[2]}_{ij,k} = a_{ij}(\hat{\boldsymbol{x}}^{\mathrm{s}[2]}_{j,k|k-1} + e^{(2)}_{\boldsymbol{\xi}_{ij,k}} + \boldsymbol{\xi}^{(2)}_{ij,k}), \tag{20}$$

here $\boldsymbol{h}^{(2)}_{i,k}$, $\boldsymbol{v}^{(2)}_{i,k}$ and $\boldsymbol{\xi}^{(2)}_{ij,k}$ are zero-mean, temporally uncorrelated sequences and uncorrelated with $\boldsymbol{x}_0$.

$$\boldsymbol{h}^{(2)}_{i,k} = (\boldsymbol{I} + \boldsymbol{T}_{n,n})(\check{\boldsymbol{A}}_{i,k}\boldsymbol{x}^{\mathrm{s}}_{i,k} \otimes \boldsymbol{h}_{i,k}) + \boldsymbol{h}^{[2]}_{i,k} - e^{(2)}_{\boldsymbol{h}_{i,k}}, \tag{21}$$

$$\boldsymbol{v}_{i,k}^{(2)} = (\boldsymbol{I} + \boldsymbol{T}_{m,m})(\boldsymbol{H}_{i,k}\boldsymbol{x}_{i,k}^{\mathrm{s}} \otimes \boldsymbol{v}_{i,k}) + \boldsymbol{v}_{i,k}^{[2]} - e_{\boldsymbol{v}_{i,k}}^{(2)}, \tag{22}$$

$$\boldsymbol{\xi}_{ij,k}^{(2)} = (\boldsymbol{I} + \boldsymbol{T}_{n,n})(\hat{\boldsymbol{x}}_{j,k|k-1}^{\mathrm{s}} \otimes \boldsymbol{\xi}_{ij,k}) + \boldsymbol{\xi}_{ij,k}^{[2]} - e_{\boldsymbol{\xi}_{ij,k}}^{(2)}. \tag{23}$$

However, $\mathbf{E}\{\boldsymbol{h}_{i,k}^{(2)}\boldsymbol{v}_{i,k}^{(2)\mathrm{T}}\}$ is not equal to zero. $\boldsymbol{T}_{n,m}$ is a commutation matrix, which satisfies $\boldsymbol{T}_{n,m}(\boldsymbol{a} \otimes \boldsymbol{b}) = \boldsymbol{b} \otimes \boldsymbol{a}$ for any vector $\boldsymbol{a} \in \mathbf{R}^n$ and $\boldsymbol{b} \in \mathbf{R}^m$.

The augment system can be described as

$$\boldsymbol{X}_i^{\mathrm{s}}(k+1) = \boldsymbol{\mathcal{A}}_i(k)\boldsymbol{X}_i^{\mathrm{s}}(k) + \boldsymbol{f}_i(k) + \boldsymbol{\mathscr{h}}_i(k), \tag{24}$$

$$\boldsymbol{Y}_i^{\mathrm{s}}(k) = \beta_i(d_{i,k})(\boldsymbol{\mathcal{H}}_i(k)\boldsymbol{X}_i^{\mathrm{s}}(k) + \boldsymbol{g}_i(k) + \boldsymbol{\mathscr{v}}_i(k)), \tag{25}$$

$$\boldsymbol{Z}_{ij}^{\mathrm{s}}(k \mid k-1) = a_{ij}(\hat{\boldsymbol{X}}_j^{\mathrm{s}}(k \mid k-1) + \boldsymbol{\mu}_{ij}(k) + \boldsymbol{\omega}_{ij}(k)), \tag{26}$$

where

$$\boldsymbol{\mathcal{A}}_i(k) = \begin{bmatrix} \check{\boldsymbol{A}}_{i,k} & 0 \\ 0 & \check{\boldsymbol{A}}_{i,k}^{[2]} \end{bmatrix};\ \boldsymbol{\mathcal{H}}(k) = \begin{bmatrix} \boldsymbol{H}_{i,k} & 0 \\ 0 & \boldsymbol{H}_{i,k}^{[2]} \end{bmatrix};$$

$$\boldsymbol{f}_i(k) = \begin{bmatrix} 0_{n\times 1} \\ e_{\boldsymbol{h}_{i,k}}^{(2)} \end{bmatrix};\ \boldsymbol{\mathscr{h}}_i(k) = \begin{bmatrix} \boldsymbol{h}_{i,k} \\ \boldsymbol{h}_{i,k}^{(2)} \end{bmatrix};$$

$$\boldsymbol{g}_i(k) = \begin{bmatrix} 0_{m\times 1} \\ e_{\boldsymbol{v}_{i,k}}^{(2)} \end{bmatrix};\ \boldsymbol{\mathscr{v}}_i(k) = \begin{bmatrix} \boldsymbol{v}_{i,k} \\ \boldsymbol{v}_{i,k}^{(2)} \end{bmatrix};$$

$$\boldsymbol{\mu}_{ij}(k) = \begin{bmatrix} 0_{n\times 1} \\ e_{\boldsymbol{\xi}_{ij,k}}^{(2)} \end{bmatrix};\ \boldsymbol{\omega}_{ij}(k) = \begin{bmatrix} \boldsymbol{\xi}_{ij,k} \\ \boldsymbol{\xi}_{ij,k}^{(2)} \end{bmatrix}.$$

In this article, the Kalman-like estimator for quadratic distributed non-Gaussian time-varying systems is constructed as

$$\hat{\boldsymbol{X}}_i^{\mathrm{s}}(k \mid k-1) = \boldsymbol{\mathcal{A}}_i(k-1)\hat{\boldsymbol{X}}_i^{\mathrm{s}}(k-1 \mid k-1) + \boldsymbol{f}_i(k-1), \tag{27}$$

$$\begin{aligned} \hat{\boldsymbol{X}}_i^{\mathrm{s}}(k \mid k) = {} & \hat{\boldsymbol{X}}_i^{\mathrm{s}}(k \mid k-1) + \bar{\boldsymbol{K}}_i(k)(\boldsymbol{Y}_i^{\mathrm{s}}(k) - \hat{Y}_i^{\mathrm{s}}(k \mid k-1)) + \\ & \vec{\boldsymbol{K}}_i(k)\Big(\sum_{j=1}^{N}(\boldsymbol{Z}_{ij}^{\mathrm{s}}(k) + a_{ij}\boldsymbol{x}_j^{\mathrm{d}}(k) - \\ & \hat{\boldsymbol{Z}}_{ij}^{\mathrm{s}}(k \mid k-1) - a_{ij}\boldsymbol{x}_i^{\mathrm{d}}(k))\Big), \end{aligned} \tag{28}$$

where

$$\hat{\boldsymbol{Y}}_i^{\mathrm{s}}(k \mid k-1) = \beta_i(d_{i,k})(\boldsymbol{\mathcal{H}}_i(k)\hat{\boldsymbol{X}}_i^{\mathrm{s}}(k \mid k-1) + \boldsymbol{g}_i(k)), \tag{29}$$

$$\hat{\boldsymbol{Z}}_{ij}^{\mathrm{s}}(k \mid k-1) = a_{ij}(\hat{\boldsymbol{X}}_i^{\mathrm{s}}(k \mid k-1) + \boldsymbol{\mu}_{ij}(k)), \tag{30}$$

$$\boldsymbol{x}_j^{\mathrm{d}}(k) = [\boldsymbol{x}_{j,k}^{\mathrm{d}}, \boldsymbol{0}_{n^2\times 1}]^{\mathrm{T}}, \tag{31}$$

$$\boldsymbol{x}_i^{\mathrm{d}}(k) = [\boldsymbol{x}_{i,k}^{\mathrm{d}}, \boldsymbol{0}_{n^2\times 1}]^{\mathrm{T}}. \tag{32}$$

After giving the expression of the state estimator, we can express the local prior estimation error and the local posterior estimation error as $\tilde{\boldsymbol{X}}_i^{\mathrm{s}}(k \mid k-1) = \boldsymbol{X}_i^{\mathrm{s}}(k) - \hat{\boldsymbol{X}}_i^{\mathrm{s}}(k \mid k-1)$ and $\tilde{\boldsymbol{X}}_i^{\mathrm{s}}(k \mid k) = \boldsymbol{X}_i^{\mathrm{s}}(k) - \boldsymbol{X}_i^{\mathrm{s}}(k \mid k)$.

According to Eqs. (11), (13), (16), (27) and (28), the error can be described by

$$\tilde{\boldsymbol{X}}_i^{\mathrm{s}}(k \mid k-1) = \boldsymbol{\mathcal{A}}_i(k-1)\tilde{\boldsymbol{X}}_i^{\mathrm{s}}(k-1 \mid k-1) + \boldsymbol{\mathscr{h}}_i(k-1), \tag{33}$$

and

$$\begin{aligned} \tilde{\boldsymbol{X}}_i^{\mathrm{s}}(k \mid k) = {} & (\boldsymbol{I}_{n+n^2} - \beta_i(d_{i,k})\bar{\boldsymbol{K}}_i(k)\boldsymbol{\mathcal{H}}_i(k))\tilde{\boldsymbol{X}}_i^{\mathrm{s}}(k \mid k-1) - \\ & \beta_i(d_{i,k})\bar{\boldsymbol{K}}_i(k)\boldsymbol{\mathscr{v}}_i(k) - \\ & \vec{\boldsymbol{K}}_i(k)\sum_{j=1}^{N}a_{ij}(\tilde{\boldsymbol{X}}_i^{\mathrm{s}}(k \mid k-1) + \boldsymbol{x}_j^{\mathrm{d}}(k)) + \\ & \vec{\boldsymbol{K}}_i(k)\sum_{j=1}^{N}a_{ij}(\tilde{\boldsymbol{X}}_j^{\mathrm{s}}(k \mid k-1) + \boldsymbol{x}_i^{\mathrm{d}}(k) - \boldsymbol{\omega}_{ij}(k)). \end{aligned} \tag{34}$$

In addition, we denote that

$$\begin{cases} \tilde{\boldsymbol{X}}^{\mathrm{s}}(k \mid k-1) = \mathrm{col}_N\{\tilde{\boldsymbol{X}}_i^{\mathrm{s}}(k \mid k-1)\}, \\ \tilde{\boldsymbol{X}}^{\mathrm{s}}(k \mid k) = \mathrm{col}_N\{\tilde{\boldsymbol{X}}_i^{\mathrm{s}}(k \mid k)\}. \end{cases}$$

Then, present all sensor errors in a more compact form

$$\tilde{\boldsymbol{X}}^{\mathrm{s}}(k \mid k-1) = \boldsymbol{\mathfrak{A}}(k-1)\tilde{\boldsymbol{X}}^{\mathrm{s}}(k-1 \mid k-1) + \boldsymbol{\mathcal{W}}(k-1), \tag{35}$$

and

$$\tilde{\boldsymbol{X}}^{\mathrm{s}}(k \mid k) = \boldsymbol{M}(k)\tilde{\boldsymbol{X}}^{\mathrm{s}}(k \mid k-1) - \boldsymbol{N}(k), \tag{36}$$

where $\boldsymbol{M}(k) = (\boldsymbol{I}_{N\times(n+n^2)} - \boldsymbol{\beta}(k)\bar{\boldsymbol{K}}(k)\boldsymbol{\mathcal{H}}(k) - \vec{\boldsymbol{K}}(k)\sum_{i=1}^{N}(\boldsymbol{T}_i - \boldsymbol{E}_i\boldsymbol{D}))$; $\boldsymbol{N}(k) = \boldsymbol{\beta}(k)\bar{\boldsymbol{K}}(k)\boldsymbol{\mathcal{V}}(k) + \vec{\boldsymbol{K}}(k)(\sum_{i=1}^{N}\boldsymbol{E}_i\boldsymbol{D}\boldsymbol{\Omega}_i(k)) + \vec{\boldsymbol{K}}(k)(\sum_{i=1}^{N}(\boldsymbol{T}_i - \boldsymbol{E}_i\boldsymbol{D})\boldsymbol{X}_i^{\mathrm{d}}(k))$;

and

$$\begin{aligned} & \boldsymbol{\mathfrak{A}}(k) = \mathrm{diag}_{\mathrm{N}}\{\boldsymbol{\mathcal{A}}_{\mathrm{i}}(k)\};\ \boldsymbol{\mathcal{H}}(k) = \mathrm{diag}_{\mathrm{N}}\{\boldsymbol{\mathcal{H}}_{\mathrm{i}}(k)\}; \\ & \boldsymbol{\mathcal{W}}(\mathrm{k}) = \mathrm{col}_N\{\boldsymbol{\mathscr{h}}_{\mathrm{i}}(k)\};\ \boldsymbol{\mathcal{V}}(k) = \mathrm{col}_N\{\boldsymbol{\mathscr{v}}_i(k)\}; \\ & \bar{\boldsymbol{K}}(k) = \mathrm{diag}_N\{\bar{\boldsymbol{K}}_i(k)\};\ \vec{\boldsymbol{K}}(k) = \mathrm{diag}_N\{\vec{\boldsymbol{K}}_i(k)\}; \\ & \boldsymbol{D} = \{a_{ij}\boldsymbol{I}_{n+n^2}\}_{N\times N}; \\ & \boldsymbol{\Omega}_i(k) = [\boldsymbol{\omega}_{i1}^{\mathrm{T}}(k), \boldsymbol{\omega}_{i2}^{\mathrm{T}}(k), \cdots, \boldsymbol{\omega}_{iN}^{\mathrm{T}}(k)]^{\mathrm{T}}; \\ & \boldsymbol{X}_i^{\mathrm{d}}(k) = [\boldsymbol{x}_1^{\mathrm{d}^{\mathrm{T}}}(k), \boldsymbol{x}_2^{\mathrm{d}^{\mathrm{T}}}(k), \cdots, \boldsymbol{x}_i^{\mathrm{d}^{\mathrm{T}}}(k)]^{\mathrm{T}}; \\ & \boldsymbol{\beta}(k) = \mathrm{diag}_N\{\beta_1(d_{1,k})\boldsymbol{I}_{n+n^2}, \beta_2(d_{2,k})\boldsymbol{I}_{n+n^2}, \cdots, \\ & \qquad \beta_N(d_{N,k})\boldsymbol{I}_{n+n^2}\}; \\ & \boldsymbol{T}_i = \mathrm{diag}_N\{a_{1i}\boldsymbol{I}_{n+n^2}, a_{2i}\boldsymbol{I}_{n+n^2}, \cdots, a_{Ni}\boldsymbol{I}_{n+n^2}\}; \\ & \boldsymbol{E}_i = \mathrm{diag}_N\{0, \underbrace{\cdots}_{i-1}, 0, \boldsymbol{I}_{n+n^2}, 0, \underbrace{\cdots}_{N-i}, 0\}. \end{aligned}$$

The definition of estimation error covariance can be written from the estimation error as

$$\begin{cases} \boldsymbol{P}(k \mid k-1) = \mathbf{E}\{\tilde{\boldsymbol{X}}^{\mathrm{s}}(k \mid k-1)\tilde{\boldsymbol{X}}^{\mathrm{s}\,\mathrm{T}}(k \mid k-1)\}, \\ \boldsymbol{P}(k \mid k) = \mathbf{E}\{\tilde{\boldsymbol{X}}^{\mathrm{s}}(k \mid k)\tilde{\boldsymbol{X}}^{\mathrm{s}\,\mathrm{T}}(k \mid k)\}. \end{cases} \tag{37}$$

2 Distributed Quadratic Estimator Design

In this section, the main task is to design appropriate estimator gains $\bar{\boldsymbol{K}}_i(k)$ and $\vec{\boldsymbol{K}}_i(k)$ to minimize the upper bound of estimation error covariance. Firstly, we introduce several lemmas that will be used to derive the main results.

2.1 Preliminary lemmas

Lemma 1[20] Supposing that there exist three random vectors $\boldsymbol{A} \in \mathbf{R}^p$, $\boldsymbol{B} \in \mathbf{R}^q$ and $\boldsymbol{C} \in \mathbf{R}^r$, where the superscripts p, q and r denote the dimensions of the respective vectors, while $\boldsymbol{A}$ is independent of $\boldsymbol{B}$ and $\boldsymbol{C}$. If $\mathbf{E}\{\boldsymbol{A}\} = 0$, then $\mathbf{E}\{(\boldsymbol{A} \otimes \boldsymbol{B})\boldsymbol{C}^{\mathrm{T}}\} = 0$.

Proof Noting that

$$\mathbf{E}\{(\boldsymbol{A}\otimes\boldsymbol{B})\boldsymbol{C}^{\mathrm{T}}\}=\mathbf{E}\left\{\begin{bmatrix}a_1\boldsymbol{B}\\ \vdots\\ a_p\boldsymbol{B}\end{bmatrix}\boldsymbol{C}^{\mathrm{T}}\right\}=\mathbf{E}\left\{\begin{bmatrix}a_1\mathbf{E}\{\boldsymbol{B}\boldsymbol{C}^{\mathrm{T}}\}\\ \vdots\\ a_p\mathbf{E}\{\boldsymbol{B}\boldsymbol{C}^{\mathrm{T}}\}\end{bmatrix}\right\}=0.$$

The proof of this lemma is complete.

Under the same conditions, it can be obtained that $\mathbf{E}\{\boldsymbol{B}(\boldsymbol{A}\otimes\boldsymbol{B})^{\mathrm{T}}\}=0$ and $\mathbf{E}\{\boldsymbol{A}(\boldsymbol{B}\otimes\boldsymbol{C})^{\mathrm{T}}\}=0$.

Lemma 2[22] Supposing that there exist two random vectors $\boldsymbol{M}$ and $\boldsymbol{N}$, the elementary inequality $(\alpha^{\frac{1}{2}}\boldsymbol{M}-\alpha^{-\frac{1}{2}}\boldsymbol{N})(\alpha^{\frac{1}{2}}\boldsymbol{M}-\alpha^{-\frac{1}{2}}\boldsymbol{N})^{\mathrm{T}}\geqslant 0$ holds, where $\boldsymbol{M}$ and $\boldsymbol{N}$ vectors matrices with compatible dimensions.

Lemma 3[18] For $0\leqslant k\leqslant N$, suppose that $\boldsymbol{X}=\boldsymbol{X}^{\mathrm{T}}\geqslant 0$, $\boldsymbol{Y}=\boldsymbol{Y}^{\mathrm{T}}\geqslant 0$ and $\psi_k(\cdot):\mathbf{R}^{nn_x\times nn_x}\to\mathbf{R}^{nn_x\times nn_x}$. If

$$\psi_k(\boldsymbol{X})\leqslant\psi_k(\boldsymbol{Y}),\ \forall\boldsymbol{X}\leqslant\boldsymbol{Y},\tag{38}$$

then the solutions $\boldsymbol{W}_{k+1}$ and $\boldsymbol{M}_{k+1}$ to the following difference equations

$$\boldsymbol{W}_{k+1}=\psi_k(\boldsymbol{W}_k),\ \boldsymbol{M}_{k+1}\leqslant\psi_k(\boldsymbol{M}_k),\ \boldsymbol{M}_0=\boldsymbol{W}_0\tag{39}$$

satisfy

$$\boldsymbol{M}_{k+1}\leqslant\boldsymbol{W}_{k+1}.\tag{40}$$

Lemma 4[20] For matrices $\boldsymbol{M}$, $\boldsymbol{N}$, $\boldsymbol{X}$ and $\boldsymbol{L}$ with compatible dimensions, the following equations are true:

$$\begin{cases}\dfrac{\partial}{\partial\boldsymbol{X}}\mathrm{tr}(\boldsymbol{M}\boldsymbol{X}^{\mathrm{T}})=\boldsymbol{M},\\ \dfrac{\partial}{\partial\boldsymbol{X}}\mathrm{tr}(\boldsymbol{X}\boldsymbol{M})=\boldsymbol{M}^{\mathrm{T}},\\ \dfrac{\partial}{\partial\boldsymbol{X}}\mathrm{tr}(\boldsymbol{M}\boldsymbol{X}\boldsymbol{N})=\boldsymbol{M}^{\mathrm{T}}\boldsymbol{N}^{\mathrm{T}},\\ \dfrac{\partial}{\partial\boldsymbol{X}}\mathrm{tr}(\boldsymbol{M}\boldsymbol{X}^{\mathrm{T}}\boldsymbol{N})=\boldsymbol{N}\boldsymbol{M},\\ \dfrac{\partial}{\partial\boldsymbol{X}}\mathrm{tr}(\boldsymbol{M}\boldsymbol{X}\boldsymbol{N}\boldsymbol{X}^{\mathrm{T}}\boldsymbol{L})=\boldsymbol{M}^{\mathrm{T}}\boldsymbol{L}^{\mathrm{T}}\boldsymbol{X}\boldsymbol{N}^{\mathrm{T}}+\boldsymbol{L}\boldsymbol{M}\boldsymbol{X}\boldsymbol{N}.\end{cases}\tag{41}$$

Furthermore, for any symmetric matrix $\boldsymbol{P}$,

$$\frac{\partial}{\partial\boldsymbol{X}}\mathrm{tr}(\boldsymbol{M}\boldsymbol{X}\boldsymbol{N})\boldsymbol{P}(\boldsymbol{M}\boldsymbol{X}\boldsymbol{N})^{\mathrm{T}}=2\boldsymbol{M}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{X}\boldsymbol{N}\boldsymbol{P}\boldsymbol{N}^{\mathrm{T}}.\tag{42}$$

2.2 Filtering error covariance

By incorporating Eqs. (35) and (36) into Eq. (37), we can obtain the specific expression of Eq. (37) that satisfies the following equations:

$$\begin{aligned}\boldsymbol{P}(k\mid k-1)={}&\mathfrak{A}(k-1)\boldsymbol{P}(k-1\mid k-1)\mathfrak{A}^{\mathrm{T}}(k-1)+\\ &E\{\mathcal{W}(k-1)\mathcal{W}^{\mathrm{T}}(k-1)\}+\mathfrak{I}_k+\mathfrak{I}_k^{\mathrm{T}},\end{aligned}\tag{43}$$

and

$$\begin{aligned}\boldsymbol{P}(k\mid k)={}&\boldsymbol{M}(k)\boldsymbol{P}(k\mid k-1)\boldsymbol{M}^{\mathrm{T}}(k)+\\ &E\{\boldsymbol{N}(k)\boldsymbol{N}^{\mathrm{T}}(k)\}-(\mathfrak{R}_k+\mathfrak{R}_k^{\mathrm{T}}),\end{aligned}\tag{44}$$

where the cross terms expressions are

$$\mathfrak{I}_{\mathrm{k}}=\mathbf{E}\{\mathfrak{A}(k-1)\widetilde{\boldsymbol{X}}^{\mathrm{s}}(k-1\mid k-1)\mathcal{W}^{\mathrm{T}}(k-1)\};$$

$$\mathfrak{R}_k=\mathbf{E}\{\boldsymbol{M}(k)\widetilde{\boldsymbol{X}}^{\mathrm{s}}(k\mid k-1)\boldsymbol{N}^{\mathrm{T}}(k)\}.$$

Remark 1 Noting that $\widetilde{\boldsymbol{X}}^{\mathrm{s}}(k\mid k)$ is independent of $\boldsymbol{w}_k$ and $\boldsymbol{v}_{i,k}$. According to the definition of $\boldsymbol{h}_{i,k}$, it is not difficult to obtain that $\mathbf{E}\{\boldsymbol{X}_i^{\mathrm{s}}(k\mid k)\boldsymbol{h}_{j,k}^{\mathrm{T}}\}=0$. Similarly, we can calculate that $\mathbf{E}\{\widetilde{\boldsymbol{X}}_i^{\mathrm{s}}(k\mid k)\boldsymbol{h}_{j,k}^{(2)\mathrm{T}}\}=0$. Therefore, we can obtain that $\mathbf{E}\{\widetilde{\boldsymbol{X}}_i^{\mathrm{s}}(k\mid k)\boldsymbol{\mathscr{h}}_j^{\mathrm{T}}(k)\}=0$, for any sensor i and sensor j. Similar to the above results, we can clearly see that $\mathbf{E}\{\widetilde{\boldsymbol{X}}^{\mathrm{s}}(k\mid k-1)\mathcal{V}^{T}(\mathrm{k}-1)\}=0$ and $\mathbf{E}\{\boldsymbol{X}^{\mathrm{s}}(k\mid k-1)\boldsymbol{\Omega}_i^{\mathrm{T}}(k)\}=0$.

Therefore, $\mathfrak{I}_{\mathrm{k}}$ can calculate as

$$\mathfrak{I}_k=\mathfrak{A}(k)\{\mathbf{E}\{\widetilde{\boldsymbol{X}}_i^{\mathrm{s}}(k-1\mid k-1)\boldsymbol{\mathscr{h}}_j^{\mathrm{T}}(k-1)\}\}_{N\times N}=0,$$

and from the expression of $\boldsymbol{N}(k)$,

$$\begin{aligned}\mathfrak{R}_{\mathrm{k}}={}&\boldsymbol{M}(k)\mathbf{E}\{\widetilde{\boldsymbol{X}}^{\mathrm{s}}(k\mid k-1)\mathcal{V}^{\mathrm{T}}(k)\}(\boldsymbol{\beta}(k)\overline{\boldsymbol{K}}(k))^{\mathrm{T}}+\\ &\boldsymbol{M}(k)\Big(\sum_{i=1}^{N}\mathbf{E}\{\widetilde{\boldsymbol{X}}^{\mathrm{s}}(k\mid k-1)\boldsymbol{\Omega}_i^{\mathrm{T}}(k)\boldsymbol{D}^{\mathrm{T}}\boldsymbol{E}_i^{\mathrm{T}}\}\Big)\vec{\boldsymbol{K}}^{\mathrm{T}}(k)+\\ &\boldsymbol{M}(k)\Big(\sum_{i=1}^{N}\mathbf{E}\{\widetilde{\boldsymbol{X}}^{\mathrm{s}}(k\mid k-1)\boldsymbol{X}_i^{\mathrm{d}^{\mathrm{T}}}(k)(\boldsymbol{T}_i-\boldsymbol{E}_i\boldsymbol{D})^{\mathrm{T}}\}\Big)\\ &\vec{\boldsymbol{K}}\mathrm{T}(k)=0.\end{aligned}$$

From the content of the Kronecker algebra, we can know that $\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}=\mathrm{sti}_n(\boldsymbol{a}^{[2]})$, $\boldsymbol{a}\boldsymbol{a}^{[2]\mathrm{T}}=\mathrm{sti}_n(\boldsymbol{a}^{[3]})$, and $\boldsymbol{a}^{[2]}\boldsymbol{a}^{[2]\mathrm{T}}=\mathrm{sti}_n(\boldsymbol{a}^{[4]})$ for any vector $\boldsymbol{a}\in\mathbf{R}^n$.

Then, let's calculate the specific expression for $\mathbf{E}\{\mathcal{W}(k-1)\mathcal{W}^{\mathrm{T}}(k-1)\}$, we have

$$\begin{aligned}&\mathbf{E}\{\mathcal{W}(k-1)\mathcal{W}^{\mathrm{T}}(k-1)\}=\\ &\qquad\{\mathbf{E}\{\boldsymbol{\mathscr{h}}_i(k-1)\boldsymbol{\mathscr{h}}_j^{\mathrm{T}}(k-1)\}\}_{N\times N}.\end{aligned}\tag{45}$$

Moreover, we have

$$\mathbf{E}\{\boldsymbol{\mathscr{h}}_i(k)\boldsymbol{\mathscr{h}}_j^{\mathrm{T}}(k)\}=\begin{bmatrix}\mathbf{E}\{\boldsymbol{h}_{i,k}\boldsymbol{h}_{j,k}^{\mathrm{T}}\} & \mathbf{E}\{\boldsymbol{h}_{i,k}\boldsymbol{h}_{j,k}^{(2)\mathrm{T}}\}\\ * & \mathbf{E}\{\boldsymbol{h}_{i,k}^{(2)}\boldsymbol{h}_{j,k}^{(2)\mathrm{T}}\}\end{bmatrix}.\tag{46}$$

Note that $\boldsymbol{x}_0$, $\boldsymbol{w}_k$, $\boldsymbol{v}_{i,k}$ and $\boldsymbol{\xi}_{ij,k}$ are independent from each other. Because of $\boldsymbol{x}_0^{\mathrm{s}}=0$, we can derive that $\boldsymbol{x}_k^{\mathrm{s}}$, $\boldsymbol{w}_k$, $\boldsymbol{v}_{i,k}$ and $\boldsymbol{\xi}_{ij,k}$ are independent from each other, and $\mathbf{E}\{\boldsymbol{x}_k^{\mathrm{s}}\}=0$. It is easy to obtain $\mathbf{E}\{(\check{\boldsymbol{A}}_{i,k}\boldsymbol{x}_k^{\mathrm{s}}\otimes\boldsymbol{w}_k^{\mathrm{T}})\}=0$ and $\mathbf{E}\{(\check{\boldsymbol{A}}_{i,k}\boldsymbol{x}_k^{\mathrm{s}}\otimes\boldsymbol{v}_{i,k}^{\mathrm{T}})\}=0$, so that $\mathbf{E}\{(\check{\boldsymbol{A}}_{i,k}\boldsymbol{x}_k^{\mathrm{s}}\otimes\boldsymbol{h}_{i,k}^{\mathrm{T}})\}=0$. Then, based on Lemma 1, it is not difficult to verify that $\mathbf{E}\{\boldsymbol{h}_{i,k}(\check{\boldsymbol{A}}_{i,k}\boldsymbol{x}_k^{\mathrm{s}}\otimes\boldsymbol{h}_{i,k})^{\mathrm{T}}\}=0$. Then,

$$\begin{aligned}&\mathbf{E}\{\boldsymbol{h}_{i,k}\boldsymbol{h}_{j,k}^{\mathrm{T}}\}=\\ &\mathbf{E}\{(\boldsymbol{w}_k-\beta_i(d_{i,k})\boldsymbol{L}_{i,k}\boldsymbol{v}_{i,k})(\boldsymbol{w}_k-\beta_j(d_{j,k})\boldsymbol{L}_{j,k}\boldsymbol{v}_{j,k})^{\mathrm{T}}\}=\\ &\begin{cases}\mathrm{sti}_n(e_{\boldsymbol{w}_k}^{(2)}), & i\neq j,\\ \mathrm{sti}_n(e_{\boldsymbol{w}_k}^{(2)})+\beta_i(d_{i,k})\boldsymbol{L}_{i,k}\mathrm{sti}_m(e_{\boldsymbol{v}_{i,k}}^{(2)})\boldsymbol{L}_{i,k}^{\mathrm{T}}, & i=j,\end{cases}\end{aligned}\tag{47}$$

and

$$\begin{aligned}&\mathbf{E}\{\boldsymbol{h}_{i,k}\boldsymbol{h}_{j,k}^{(2)\mathrm{T}}\}=\\ &\mathbf{E}\{\boldsymbol{h}_{i,k}((\boldsymbol{I}+\boldsymbol{T}_{n,n})(\check{\boldsymbol{A}}_{j,k}\boldsymbol{x}_{j,k}^{s}\otimes\boldsymbol{h}_{j,k})+\boldsymbol{h}_{j,k}^{[2]}-e_{\boldsymbol{h}_{j,k}}^{(2)})^{\mathrm{T}}\}=\\ &\begin{cases}\mathrm{sti}_n(e_{\boldsymbol{\omega}_k}^{(3)}), & i\neq j,\\ \mathrm{sti}_n(e_{\boldsymbol{\omega}_k}^{(3)})-\beta_i(d_{i,k})\boldsymbol{L}_{i,k}\mathrm{sti}_m(e_{\boldsymbol{v}_{i,k}}^{(3)})\boldsymbol{L}_{i,k}^{[2]^{\mathrm{T}}}, & i=j.\end{cases}\end{aligned}\tag{48}$$

Furthermore, the calculation and expression of $\mathbf{E}\{\boldsymbol{h}_{i,k}^{(2)}\boldsymbol{h}_{j,k}^{(2)\mathrm{T}}\}$ will be more complex, calculated as

$$\mathbf{E}\{\boldsymbol{h}_{i,k}^{(2)}\boldsymbol{h}_{j,k}^{(2)\mathrm{T}}\} = \begin{cases}(\boldsymbol{I}+\boldsymbol{T}_{n,n})((\check{\boldsymbol{A}}_{i,k}\mathbf{E}\{\boldsymbol{x}_{i,k}^{\mathrm{s}}\boldsymbol{x}_{j,k}^{\mathrm{s}^{\mathrm{T}}}\}\check{\boldsymbol{A}}_{j,k}^{\mathrm{T}})\otimes(\mathrm{sti}_n e_{\boldsymbol{w}_k}^{(2)}))(\boldsymbol{I}+\boldsymbol{T}_{n,n})^{\mathrm{T}} - e_{\boldsymbol{w}_k}^{(2)}e_{\boldsymbol{w}_k}^{(2)^{\mathrm{T}}} + \mathrm{sti}_{n^2}(e_{\boldsymbol{w}_k}^{(4)}),\ i\neq j,\\ (\boldsymbol{I}+\boldsymbol{T}_{n,n})((\check{\boldsymbol{A}}_{i,k}\mathbf{E}\{\boldsymbol{x}_{i,k}^{\mathrm{s}}\boldsymbol{x}_{i,k}^{\mathrm{s}^{\mathrm{T}}}\}\check{\boldsymbol{A}}_{i,k}^{\mathrm{T}})\otimes\mathbf{E}\{\boldsymbol{h}_{i,k}\boldsymbol{h}_{i,k}^{\mathrm{T}}\})(\boldsymbol{I}+\boldsymbol{T}_{n,n})^{\mathrm{T}} - e_{\boldsymbol{h}_{i,k}}^{(2)}e_{\boldsymbol{h}_{i,k}}^{(2)^{\mathrm{T}}} + \mathbf{E}\{\boldsymbol{h}_{i,k}^{[2]\|}\boldsymbol{h}_{i,k}^{[2]\|^{\mathrm{T}}}\},\ i=j,\end{cases} \tag{49}$$

where $\mathbf{E}\{\boldsymbol{h}_{i,k}^{[2]}\boldsymbol{h}_{i,k}^{[2]^{\mathrm{T}}}\}$ can be derived as

$$\mathbf{E}\{\boldsymbol{h}_{i,k}^{[2]}h_{i,k}^{[2]^{\mathrm{T}}}\} = \mathrm{sti}_{n^2}(e_{\boldsymbol{w}_k}^{(4)}) + \beta_i(d_{i,k})(\boldsymbol{L}_{i,k}^{[2]}\mathrm{sti}_{m^2}(e_{v_{i,k}}^{(4)})\boldsymbol{L}_{i,k}^{[2]^{\mathrm{T}}}) + \beta_i(d_{i,k})(\boldsymbol{I}+\boldsymbol{T}_{n,n})(\mathrm{sti}_n(e_{w_k}^{(2)})\otimes(\boldsymbol{L}_{i,k}\mathrm{sti}_m(e_{v_{i,k}}^{(2)})\boldsymbol{L}_{i,k}^{\mathrm{T}})) + \beta_i(d_{i,k})(\boldsymbol{I}+\boldsymbol{T}_{n,n})(\boldsymbol{L}_{i,k}^{[2]}e_{v_{i,k}}^{(2)}e_{\boldsymbol{w}_k}^{(2)^{\mathrm{T}}}) + \beta_i(d_{i,k})(\boldsymbol{I}+\boldsymbol{T}_{n,n})(e_{\boldsymbol{w}_k}^{(2)}e_{v_{i,k}}^{(2)^{\mathrm{T}}}\boldsymbol{L}_{i,k}^{[2]^{\mathrm{T}}}).$$

From Eqs. (46) – (49), $\mathbf{E}\{\boldsymbol{\mathcal{W}}(k)\boldsymbol{\mathcal{W}}^{\mathrm{T}}(k)\}$ can be computed, and we denote it as $\mathcal{Q}_w(k)$.

We have obtained all representations of $\mathcal{Q}_w(k)$, but there is an unknown term $\mathrm{sti}_n(e_{\boldsymbol{x}_k}^{\mathrm{s}(2)})$ among them. The next proposition is to process it.

Proposition 1 The state covariance $\mathbf{E}\{\boldsymbol{x}_{i,k}^{\mathrm{s}}\boldsymbol{x}_{j,k}^{\mathrm{s}^{\mathrm{T}}}\}$ has the following recursion:

$$\mathbf{E}\{\boldsymbol{x}_{i,k+1}^{\mathrm{s}}\boldsymbol{x}_{j,k+1}^{\mathrm{s}^{\mathrm{T}}}\} = \begin{cases}\check{\boldsymbol{A}}_{i,k}\mathbf{E}\{\boldsymbol{x}_{i,k}^{\mathrm{s}}\boldsymbol{x}_{j,k}^{\mathrm{s}^{\mathrm{T}}}\}\check{\boldsymbol{A}}_{j,k}^{\mathrm{T}} + \mathrm{sti}_n(e_{\boldsymbol{w}_k}^{(2)}), & i\neq j,\\ \check{\boldsymbol{A}}_{i,k}\mathbf{E}\{\boldsymbol{x}_{i,k}^{\mathrm{s}}\boldsymbol{x}_{i,k}^{\mathrm{s}^{\mathrm{T}}}\}\check{\boldsymbol{A}}_{i,k}^{\mathrm{T}} + \mathrm{sti}_n(e_{\boldsymbol{h}_{i,k}}^{(2)}), & i=j.\end{cases} \tag{50}$$

Proof Since $\mathbf{E}\{\boldsymbol{x}_{i,k}^{\mathrm{s}}\boldsymbol{w}_k^{\mathrm{T}}\}=0$ and $\mathbf{E}\{\boldsymbol{x}_{i,k}^{\mathrm{s}}\boldsymbol{v}_{j,k}^{\mathrm{T}}\}=0$, we can derive that $\mathbf{E}\{\boldsymbol{x}_{i,k}^{\mathrm{s}}\boldsymbol{h}_{j,k}^{\mathrm{T}}\}=0$, and it is used for the following calculation:

$$\mathbf{E}\{\boldsymbol{x}_{i,k+1}^{\mathrm{s}}\boldsymbol{x}_{j,k+1}^{\mathrm{s}^{\mathrm{T}}}\} = \mathbf{E}\{(\check{\boldsymbol{A}}_{i,k}\boldsymbol{x}_{i,k}^{\mathrm{s}}+\boldsymbol{h}_{i,k})(\check{\boldsymbol{A}}_{j,k}\boldsymbol{x}_{j,k}^{\mathrm{s}}+\boldsymbol{h}_{j,k})^{\mathrm{T}}\} = \begin{cases}\check{\boldsymbol{A}}_{i,k}\mathbf{E}\{\boldsymbol{x}_{i,k}^{\mathrm{s}}\boldsymbol{x}_{j,k}^{\mathrm{s}^{\mathrm{T}}}\}\check{\boldsymbol{A}}_{j,k}^{\mathrm{T}} + \mathrm{sti}_n(e_{\boldsymbol{w}_k}^{(2)}), & i\neq j,\\ \check{\boldsymbol{A}}_{i,k}\mathbf{E}\{\boldsymbol{x}_{i,k}^{\mathrm{s}}\boldsymbol{x}_{i,k}^{\mathrm{s}^{\mathrm{T}}}\}\check{\boldsymbol{A}}_{i,k}^{\mathrm{T}} + \mathrm{sti}_n(e_{\boldsymbol{h}_{i,k}}^{(2)}), & i=j.\end{cases}$$

Note that $\boldsymbol{x}_{i,0}^{\mathrm{s}}=\boldsymbol{x}_{j,0}^{\mathrm{s}}=0$, for any sensor i and sensor j. The proof of this proposition is complete.

In what follows, we will derive the expressions of $\mathbf{E}\{\boldsymbol{N}(k)\boldsymbol{N}^{\mathrm{T}}(k)\}$.

$$\mathbf{E}\{\boldsymbol{N}(k)\boldsymbol{N}^{\mathrm{T}}(k)\} = \boldsymbol{\beta}(k)\overline{\boldsymbol{K}}(k)\mathbf{E}\{\boldsymbol{\mathcal{V}}(k)\boldsymbol{\mathcal{V}}(k)^{\mathrm{T}}\}(\boldsymbol{\beta}(k)\overline{\boldsymbol{K}}(k))^{\mathrm{T}} + \overrightarrow{\boldsymbol{K}}(k)\aleph_k(\boldsymbol{\beta}(k)\overline{\boldsymbol{K}}(k))^{\mathrm{T}} + \boldsymbol{\beta}(k)\overline{\boldsymbol{K}}(k)\aleph_k^{\mathrm{T}}\overrightarrow{\boldsymbol{K}}^{\mathrm{T}}(k) + \overrightarrow{\boldsymbol{K}}(k)\wp_k\overrightarrow{\boldsymbol{K}}^{\mathrm{T}}(k) + \overrightarrow{\boldsymbol{K}}(k)\left(\sum_{i=1}^{N}(\boldsymbol{T}_i-\boldsymbol{E}_i\boldsymbol{D})\boldsymbol{X}_i^{\mathrm{d}}(k)\right)\left(\sum_{i=1}^{N}(\boldsymbol{T}_i-\boldsymbol{E}_i\boldsymbol{D})\boldsymbol{X}_i^{\mathrm{d}}(k)\right)^{\mathrm{T}}\overrightarrow{\boldsymbol{K}}^{T}(k), \tag{51}$$

where $\aleph_k=\sum_{i=1}^{N}\{\mathbf{E}\{\boldsymbol{E}_i\boldsymbol{D}\boldsymbol{\Omega}_i(k)\boldsymbol{\mathcal{V}}^{\mathrm{T}}(\mathrm{k})\}\}$ and $\wp_k=\sum_{i,j=1}^{N}\mathbf{E}\{(\boldsymbol{E}_i\boldsymbol{D}\boldsymbol{\Omega}_i(k))(\boldsymbol{E}_j\boldsymbol{D}\boldsymbol{\Omega}_j(k))^{\mathrm{T}}\}$.

Considering the term $\mathbf{E}\{\boldsymbol{\mathcal{V}}(k)\boldsymbol{\mathcal{V}}(k)^{\mathrm{T}}\}$, we have

$$\mathbf{E}\{\boldsymbol{\mathcal{V}}(k)\boldsymbol{\mathcal{V}}(k)^{\mathrm{T}}\} = \mathbf{E}\{\boldsymbol{\upsilon}_i(k)\boldsymbol{\upsilon}_j^{\mathrm{T}}(k)\}_{N\times N}. \tag{52}$$

$$\mathbf{E}\{\boldsymbol{\upsilon}_{\mathrm{i}}(\mathrm{k})\boldsymbol{\upsilon}_j^{\mathrm{T}}(k)\} = \begin{bmatrix}\mathbf{E}\{\boldsymbol{v}_{i,k}\boldsymbol{v}_{j,k}^{\mathrm{T}}\} & \mathbf{E}\{\boldsymbol{v}_{i,k}\boldsymbol{v}_{j,k}^{(2)\mathrm{T}}\}\\ * & \mathbf{E}\{\boldsymbol{v}_{i,k}^{(2)}\boldsymbol{v}_{j,k}^{(2)\mathrm{T}}\}\end{bmatrix}. \tag{53}$$

Moreover, one has

$$\mathbf{E}\{\boldsymbol{v}_{i,k}\boldsymbol{v}_{j,k}^{\mathrm{T}}\} = \begin{cases}\mathrm{sti}_m(e_{\boldsymbol{v}_{i,k}}^{(2)}), & i=j,\\ 0, & i\neq j.\end{cases} \tag{54}$$

Moreover, one of the calculations is as

$$\mathbf{E}\{\boldsymbol{v}_{i,k}\boldsymbol{v}_{j,k}^{(2)\mathrm{T}}\} = \mathbf{E}\{\boldsymbol{v}_{i,k}((\boldsymbol{I}+\boldsymbol{T}_{m,m})(\boldsymbol{H}_{j,k}\boldsymbol{x}_k^{\mathrm{s}}\otimes\boldsymbol{v}_{j,k}) + \boldsymbol{v}_{j,k}^{[2]} - e_{\boldsymbol{v}_{j,k}}^{(2)})^{\mathrm{T}}\} = \begin{cases}\mathrm{sti}_m(e_{\boldsymbol{v}_{i,k}}^{(3)}), & i=j,\\ 0, & i\neq j.\end{cases} \tag{55}$$

Furthermore, we can see that

$$\mathbf{E}\{\boldsymbol{v}_{i,k}^{(2)}\boldsymbol{v}_{j,k}^{(2)\mathrm{T}}\} = \mathbf{E}\{((\boldsymbol{I}+\boldsymbol{T}_{m,m})(\boldsymbol{H}_{i,k}\boldsymbol{x}_k^{\mathrm{s}}\otimes\boldsymbol{v}_{i,k}) + \boldsymbol{v}_{i,k}^{[2]} - e_{v_{i,k}}^{(2)})\times((\boldsymbol{I}+\boldsymbol{T}_{m,m})(\boldsymbol{H}_{j,k}\boldsymbol{x}_k^{\mathrm{s}}\otimes\boldsymbol{v}_{j,k}) + \boldsymbol{v}_{j,k}^{[2]} - e_{v_{j,k}}^{(2)})^{\mathrm{T}}\} = \boldsymbol{\ell} + \boldsymbol{\mathcal{J}} + \boldsymbol{\mathcal{K}} + (\boldsymbol{\mathcal{L}}+\boldsymbol{\mathcal{L}}^{T}) - (\boldsymbol{\mathcal{M}}+\boldsymbol{\mathcal{M}}^{\mathrm{T}}) - (\boldsymbol{\mathcal{N}}+\boldsymbol{\mathcal{N}}^{\mathrm{T}}), \tag{56}$$

where

$$\boldsymbol{\ell} = (\boldsymbol{I}+\boldsymbol{T}_{m,m})\mathbf{E}\{(\boldsymbol{H}_{i,k}\boldsymbol{x}_k^{\mathrm{s}}\otimes\boldsymbol{v}_{i,k})(\boldsymbol{H}_{j,k}\boldsymbol{x}_k^{\mathrm{s}}\otimes\boldsymbol{v}_{j,k})^{\mathrm{T}}\}(\boldsymbol{I}+\boldsymbol{T}_{m,m})^{\mathrm{T}};\ \boldsymbol{\mathcal{J}} = \mathbf{E}\{\boldsymbol{v}_{i,k}^{[2]}\boldsymbol{v}_{j,k}^{[2]\mathrm{T}}\};$$

$$\boldsymbol{\mathcal{K}} = \mathbf{E}\{e_{\boldsymbol{v}_{i,k}}^{(2)}(e_{\boldsymbol{v}_{j,k}}^{(2)})^{\mathrm{T}}\};\ \boldsymbol{\mathcal{L}} = \mathbf{E}\{(\boldsymbol{I}+\boldsymbol{T}_{m,m})(\boldsymbol{H}_{i,k}\boldsymbol{x}_k^{\mathrm{s}}\otimes\boldsymbol{v}_{i,k})(\boldsymbol{v}_{j,k}^{(2)})^{\mathrm{T}}\};$$

$$\boldsymbol{\mathcal{M}} = \mathbf{E}\{(\boldsymbol{I}+\boldsymbol{T}_{m,m})(\boldsymbol{H}_{i,k}\boldsymbol{x}_k^{\mathrm{s}}\otimes\boldsymbol{v}_{i,k})(e_{\boldsymbol{v}_{j,k}}^{(2)})^{\mathrm{T}}\};\ \boldsymbol{\mathcal{N}} = \mathbf{E}\{\boldsymbol{v}_{i,k}^{[2]}(e_{\boldsymbol{v}_{j,k}}^{(2)})^{\mathrm{T}}\}.$$

Based on Lemma 1, it is not difficult to verify that $\boldsymbol{\mathcal{L}}=0$ and $\boldsymbol{\mathcal{M}}=0$. Moreover, we have

$$\boldsymbol{\ell} = \begin{cases}(\boldsymbol{I}+\boldsymbol{T}_{m,m})(\boldsymbol{H}_{i,k}\mathrm{sti}_n(e_{\boldsymbol{x}_k}^{\mathrm{s}(2)})\boldsymbol{H}_{i,k}^{\mathrm{T}})\otimes\mathrm{sti}_m(\boldsymbol{v}_{i,k}^{(2)})(\boldsymbol{I}+\boldsymbol{T}_{m,m})^{\mathrm{T}},\ i=j,\\ 0, i\neq j,\end{cases}$$

$$\boldsymbol{\mathcal{J}}=\begin{cases}\mathrm{sti}_{n^2}(e^{(4)}_{\boldsymbol{v}_{i,k}}), & i=j,\\ 0, & i\neq j,\end{cases}\quad \boldsymbol{\mathcal{K}}=\begin{cases}e^{(2)}_{\boldsymbol{v}_{i,k}}(e^{(2)}_{\boldsymbol{v}_{i,k}})^{\mathrm{T}}, & i=j,\\ e^{(2)}_{\boldsymbol{v}_{i,k}}(e^{(2)}_{\boldsymbol{v}_{j,k}})^{\mathrm{T}}, & i\neq j,\end{cases}\quad \boldsymbol{\mathcal{N}}=\begin{cases}\mathrm{e}^{(2)}_{\boldsymbol{v}_{i,k}}(e^{(2)}_{\boldsymbol{v}_{i,k}})^{\mathrm{T}}, & i=j,\\ e^{(2)}_{\boldsymbol{v}_{i,k}}(e^{(2)}_{\boldsymbol{v}_{j,k}})^{\mathrm{T}}, & i\neq j.\end{cases}$$

So far, we can calculate that $\mathcal{Q}_v(k)=\mathbf{E}\{\boldsymbol{\mathcal{V}}(k)\ \boldsymbol{\mathcal{V}}(k)^{\mathrm{T}}\}$.

Consider the term $\aleph_k=\sum_{i=1}^{N}\{\mathbf{E}\{\boldsymbol{E}_i\boldsymbol{D}\boldsymbol{\Omega}_i(k)\ \boldsymbol{\mathcal{V}}^{\mathrm{T}}(k)\}\}$, we can obtain that

$$\aleph_k=\sum_{i=1}^{N}\{\mathbf{E}\{\boldsymbol{E}_i\boldsymbol{D}\boldsymbol{\Omega}_i(k)\ \boldsymbol{\mathcal{V}}^{\mathrm{T}}(k)\}\}=\mathbf{E}\left\{\begin{bmatrix}\sum_{j=1}^{N}a_{1j}\boldsymbol{\omega}_{1j}(k)\\ \vdots\\ \sum_{j=1}^{N}a_{Nj}\boldsymbol{\omega}_{Nj}(k)\end{bmatrix}[\boldsymbol{\mathcal{V}}_1^{\mathrm{T}}(k)\cdots\boldsymbol{\mathcal{V}}_N^{\mathrm{T}}(k)]\right\}. \tag{57}$$

We have

$$\mathbf{E}\{a_{ij}\boldsymbol{\omega}_{ij}(k)\boldsymbol{\mathcal{V}}_p^{\mathrm{T}}(k)\}=a_{ij}\mathbf{E}\{[\boldsymbol{\omega}_{ij,k}\boldsymbol{v}^{\mathrm{T}}_{p,k}]\}. \tag{58}$$

Moreover, one has

$$\mathbf{E}\{a_{ij}\boldsymbol{\omega}_{ij}(k)\boldsymbol{\mathcal{V}}_p^{\mathrm{T}}(k)\}=a_{ij}\begin{bmatrix}\mathbf{E}\{\boldsymbol{\xi}_{ij,k}\boldsymbol{v}^{\mathrm{T}}_{p,k}\} & \mathbf{E}\{\boldsymbol{\xi}_{ij,k}\boldsymbol{v}^{(2)\mathrm{T}}_{p,k}\}\\ \mathbf{E}\{\boldsymbol{\xi}^{(2)}_{ij,k}\boldsymbol{v}^{\mathrm{T}}_{p,k}\} & \mathbf{E}\{\boldsymbol{\xi}^{(2)}_{ij,k}\boldsymbol{v}^{(2)\mathrm{T}}_{p,k}\}\end{bmatrix},\ \text{for } i,j,p\in[1,N]. \tag{59}$$

It is not difficult to know that

$$\begin{cases}\mathbf{E}\{\boldsymbol{\xi}_{ij,k}\boldsymbol{v}^{\mathrm{T}}_{p,k}\}=0,\\ \mathbf{E}\{\boldsymbol{\xi}_{ij,k}\boldsymbol{v}^{(2)\mathrm{T}}_{p,k}\}=0,\\ \mathbf{E}\{\boldsymbol{\xi}^{(2)}_{ij,k}\boldsymbol{v}^{\mathrm{T}}_{p,k}\}=0,\end{cases} \tag{60}$$

and

$$\begin{aligned}\mathbf{E}\{\boldsymbol{\xi}^{(2)}_{ij,k}\boldsymbol{v}^{(2)\mathrm{T}}_{p,k}\}&=\mathbf{E}\{((\boldsymbol{I}+\boldsymbol{T}_{n,n})(\widehat{\boldsymbol{x}}^{s}_{j,k|k-1}\otimes\boldsymbol{\xi}_{ij,k})+\boldsymbol{\xi}^{[2]}_{ij,k}-e^{(2)}_{\xi_{ij,k}})((\boldsymbol{I}+\boldsymbol{T}_{m,m})(\boldsymbol{H}_{p,k}\boldsymbol{x}^{\mathrm{s}}_k\otimes\boldsymbol{v}_{p,k})+\boldsymbol{v}^{[2]}_{p,k}-e^{(2)}_{v_{p,k}})\}=\\ &\quad e^{(2)}_{\xi_{ij,k}}e^{(2)\mathrm{T}}_{v_{p,k}}+e^{(2)}_{\xi_{ij,k}}e^{(2)\mathrm{T}}_{v_{p,k}}-e^{(2)}_{\xi_{ij,k}}e^{(2)\mathrm{T}}_{v_{p,k}}-e^{(2)}_{\xi_{ij,k}}e^{(2)\mathrm{T}}_{v_{p,k}}=0.\end{aligned} \tag{61}$$

From Eqs. (57)–(61), we can directly calculate $\aleph_k=0$.

Considering the term $\wp_k=\sum_{i,j=1}^{N}\mathbf{E}\{(\boldsymbol{E}_i\boldsymbol{T}\boldsymbol{\Omega}_i(k))(\boldsymbol{E}_j\boldsymbol{D}\boldsymbol{\Omega}_j(k))^{\mathrm{T}}\}$, we can obtain that

$$\wp_k=\sum_{i,j=1}^{N}\mathbf{E}\{(\boldsymbol{E}_i\boldsymbol{D}\boldsymbol{\Omega}_i(k))(\boldsymbol{E}_j\boldsymbol{D}\boldsymbol{\Omega}_j(k))^{\mathrm{T}}\}=\sum_{i,j=1}^{N}\boldsymbol{E}_i\boldsymbol{D}\mathbf{E}\{\boldsymbol{\Omega}_i(k)\boldsymbol{\Omega}_j(k)^{\mathrm{T}}\}\boldsymbol{D}^{\mathrm{T}}\boldsymbol{E}_j^{\mathrm{T}}, \tag{62}$$

$$\mathbf{E}\{\boldsymbol{\Omega}_i(k)\boldsymbol{\Omega}_j(k)^{\mathrm{T}}\}=\mathbf{E}\left\{\begin{bmatrix}\boldsymbol{\omega}_{i1}(k)\\ \vdots\\ \boldsymbol{\omega}_{iN}(k)\end{bmatrix}[\boldsymbol{\omega}_{j1}^{\mathrm{T}}(k)\cdots\boldsymbol{\omega}_{jN}^{\mathrm{T}}(k)]\right\}. \tag{63}$$

Moreover, we have

$$\mathbf{E}\{\boldsymbol{\omega}_{ip}(k)\boldsymbol{\omega}_{jq}(k)^{\mathrm{T}}\}=\begin{bmatrix}\mathbf{E}\{\boldsymbol{\xi}_{ip,k}\boldsymbol{\xi}^{\mathrm{T}}_{jq,k}\} & \mathbf{E}\{\boldsymbol{\xi}_{ip,k}\boldsymbol{\xi}^{(2)\mathrm{T}}_{jq,k}\}\\ \mathbf{E}\{\boldsymbol{\xi}^{(2)}_{ip,k}\boldsymbol{\xi}^{\mathrm{T}}_{jq,k}\} & \mathbf{E}\{\boldsymbol{\xi}^{(2)}_{ip,k}\boldsymbol{\xi}^{(2)\mathrm{T}}_{jq,k}\}\end{bmatrix},\ \text{for } i,j,p\in[1,N]. \tag{64}$$

Similar to the previous proof, we can obtain that

$$\mathbf{E}\{\boldsymbol{\xi}_{ip,k}\boldsymbol{\xi}^{\mathrm{T}}_{jq,k}\}=\begin{cases}\mathrm{sti}_n(e^{(2)}_{\xi_{ip,k}}), & i=j,\ p=q,\\ 0, & \text{else},\end{cases} \tag{65}$$

$$\mathbf{E}\{\boldsymbol{\xi}_{ip,k}\boldsymbol{\xi}^{(2)\mathrm{T}}_{jq,k}\}=\begin{cases}\mathrm{sti}_n(e^{(3)}_{\xi_{ip,k}}), & i=j,\ p=q,\\ 0, & \text{else},\end{cases} \tag{66}$$

$$\mathbf{E}\{\boldsymbol{\xi}^{(2)\mathrm{T}}_{ip,k}\boldsymbol{\xi}^{(2)\mathrm{T}}_{jq,k}\}=\begin{cases}(\boldsymbol{I}+\boldsymbol{T}_{n,n})((\mathbf{E}\{\hat{\boldsymbol{x}}^{\mathrm{s}}_{p,k\,|\,k-1}\hat{\boldsymbol{x}}^{s\,\mathrm{T}}_{p,k|k-1}\})\otimes\mathrm{sti}_n(e^{(2)}_{\xi_{ip,k}}))(\boldsymbol{I}+\boldsymbol{T}_{n,n})^{\mathrm{T}}+\\ \qquad \mathrm{sti}_{n^2}(e^{(4)}_{\xi_{ip,k}})-e^{(2)}_{\xi_{ip,k}}(e^{(2)}_{\xi_{ip,k}})^{\mathrm{T}},\ i=j,\ p=q,\\ 0,\ \text{else}.\end{cases} \tag{67}$$

So far, we have obtained the complete expression of $\mathbf{E}\{\boldsymbol{N}(k)\boldsymbol{N}^{\mathrm{T}}(k)\}$, up to the unknown term $\mathbf{E}\{\hat{\boldsymbol{x}}^{\mathrm{s}}_{p,k|k-1}\hat{\boldsymbol{x}}^{\mathrm{s}^{\mathrm{T}}}_{p,k|k-1}\}$. In the following, a proposition is presented to handle this term.

Proposition 2 Based on Lemma 2, there is a positive scalar ε, satisfies the following expression:

$$\mathbf{E}\{\hat{\boldsymbol{x}}^{\mathrm{s}}_{i,k|k-1}\hat{\boldsymbol{x}}^{\mathrm{s}^{\mathrm{T}}}_{i,k|k-1}\} \leqslant (1+\varepsilon)\mathrm{sti}_n(e^{(2)}_{\boldsymbol{x}^{\mathrm{s}}_k}) + (1+\varepsilon^{-1})\boldsymbol{P}_{i,k|k-1}, \tag{68}$$

where $\mathrm{sti}_n(e^{(2)}_{\boldsymbol{x}^{\mathrm{s}}_k})$ can be obtained in Proposition 1, and $\boldsymbol{P}_{i,k|k-1}$ is the prior estimation error covariance of sensor i.

Proof From the definition of $\hat{\boldsymbol{x}}^{\mathrm{s}}_{i,k}$, it can be seen that

$$\begin{aligned}\mathbf{E}\{\hat{\boldsymbol{x}}^{\mathrm{s}}_{i,k|k-1}\hat{\boldsymbol{x}}^{\mathrm{s}^{\mathrm{T}}}_{i,k|k-1}\} &= \mathbf{E}\{(\boldsymbol{x}^{\mathrm{s}}_k - \tilde{\boldsymbol{x}}^{\mathrm{s}}_{i,k|k-1})(\boldsymbol{x}^{\mathrm{s}}_k - \tilde{\boldsymbol{x}}^{\mathrm{s}}_{i,k|k-1})^{\mathrm{T}}\} = \mathbf{E}\{\boldsymbol{x}^{\mathrm{s}}_k\boldsymbol{x}^{\mathrm{s}^{\mathrm{T}}}_k\} + \mathbf{E}\{\tilde{\boldsymbol{x}}^{\mathrm{s}}_{i,k|k-1}(\tilde{\boldsymbol{x}}^{\mathrm{s}}_{i,k|k-1})^{\mathrm{T}}\} + \\ &\mathbf{E}\{\boldsymbol{x}^{\mathrm{s}}_k(-\tilde{\boldsymbol{x}}^{\mathrm{s}}_{i,k|k-1})^{\mathrm{T}} + (-\tilde{\boldsymbol{x}}^{\mathrm{s}}_{i,k|k-1})x^{\mathrm{s}^{\mathrm{T}}}_k\} \leqslant (1+\varepsilon)\mathrm{sti}_n(e^{(2)}_{\boldsymbol{x}^{\mathrm{s}}_k}) + (1+\varepsilon^{-1})\boldsymbol{P}_{i,k|k-1},\end{aligned}$$

The proof of this proposition is complete.

We define the positive semidefinite matrix $\boldsymbol{\sigma}_k$ to meet the following condition:

$$\mathbf{E}\{\hat{\boldsymbol{x}}^{\mathrm{s}}_{i,k}\hat{\boldsymbol{x}}^{\mathrm{s}^{\mathrm{T}}}_{i,k}\} + \boldsymbol{\sigma}_k = (1+\varepsilon)\mathrm{sti}_n(e^{(2)}_{\boldsymbol{x}^{\mathrm{s}}_k}) + (1+\varepsilon^{-1})\boldsymbol{P}_{i,k|k-1}. \tag{69}$$

So far, all terms of the estimation error covariance of the system dynamic equation have been discussed. Next, we will provide the following theorem.

2.3 Estimator gain design

Theorem 1 Based on Lemma 3, assume that there exist two matrices $\varXi(k\mid k) > 0$, $\varXi(k\mid k-1) > 0$, satisfying the following difference equations:

$$\varXi(k\mid k-1) = \mathfrak{A}(k-1)\varXi(k-1\mid k-1)\mathfrak{A}(k-1)^{\mathrm{T}} + \mathcal{Q}_w(k-1), \tag{70}$$

$$\varXi(k\mid k) = \boldsymbol{M}(k)\varXi(k\mid k-1)\boldsymbol{M}^{\mathrm{T}}(k) + \boldsymbol{\beta}(k)\bar{\boldsymbol{K}}(k)\mathcal{Q}_v(k)(\boldsymbol{\beta}(k)\bar{\boldsymbol{K}}(k))^{\mathrm{T}} + \vec{\boldsymbol{K}}(k)\wp_k\vec{\boldsymbol{K}}(k) + \boldsymbol{\sigma}_k, \tag{71}$$

with the initial condition $\varXi(0\mid 0) = \boldsymbol{P}(0\mid 0) = 0$. Then, we can obtain that $\varXi(k\mid k)$ and $\varXi(k\mid k-1)$ are the upper bounds of the $\boldsymbol{P}(k\mid k)$ and $\boldsymbol{P}(k\mid k-1)$, respectively.

Proof Firstly, at time $k=0$, we know that $0 \leqslant \boldsymbol{P}(0\mid 0) \leqslant \varXi(0\mid 0)$. Then, from Eqs. (43)–(45), (50)–(51), (57), (62), and (68), it can be concluded that

$$\boldsymbol{P}(k\mid k-1) = \mathfrak{A}(k-1)\boldsymbol{P}(k-1\mid k-1)\mathfrak{A}(k-1)^{\mathrm{T}} + \mathcal{Q}_{\mathrm{w}}(\mathrm{k}-1), \tag{72}$$

$$\boldsymbol{P}(k\mid k) \leqslant \boldsymbol{M}(k)\boldsymbol{P}(k\mid k-1)\boldsymbol{M}^{\mathrm{T}}(k) + \boldsymbol{\beta}(k)\bar{\boldsymbol{K}}(k)\mathcal{Q}_v(k)(\boldsymbol{\beta}(k)\bar{\boldsymbol{K}}(k))^{\mathrm{T}} + \vec{\boldsymbol{K}}(k)\wp_k\vec{\boldsymbol{K}}(k) + \boldsymbol{\sigma}_k. \tag{73}$$

Therefore, based on Lemma 3 and Eqs. (70)–(73), we can obtain that $\varXi(k\mid k) \geqslant \boldsymbol{P}(k\mid k)$, $\forall k \geqslant 0$. The proof of this theorem is complete.

What we need to do later is to minimize the upper bound of the estimation error covariance $\varXi(k\mid k)$ by designing appropriate estimator gains.

Definition $\mathcal{T} = \sum_{i=1}^{N}(\boldsymbol{T}_i - \boldsymbol{E}_i\boldsymbol{D})$. Based on Lemma 4, we can calculate the trace of the upper bound of the estimation error covariance $\mathrm{tr}\{\varXi(k\mid k)\}$, and we can get the partial derivative of the trace pair $\bar{\boldsymbol{K}}(k)$ and $\vec{\boldsymbol{K}}(k)$. We can obtain that

$$\begin{aligned}\frac{\partial}{\partial\bar{\boldsymbol{K}}(k)}\mathrm{tr}\{\varXi(k\mid k)\} = 2\boldsymbol{\beta}(k)(&\bar{\boldsymbol{K}}(k)\boldsymbol{\mathcal{H}}(k)\varXi(k\mid k-1)\boldsymbol{\mathcal{H}}(k)^{\mathrm{T}} + \bar{\boldsymbol{K}}(k)\boldsymbol{Q}_v(k) + \\ &\vec{\boldsymbol{K}}(k)T\varXi(k\mid k-1)\boldsymbol{\mathcal{H}}(K)^{\mathrm{T}} - \varXi(k\mid k-1)\boldsymbol{\mathcal{H}}(k)^{\mathrm{T}}),\end{aligned} \tag{74}$$

and

$$\begin{aligned}\frac{\partial}{\partial\vec{\boldsymbol{K}}(k)}\mathrm{tr}\{\varXi(k\mid k)\} = {}&2\vec{\boldsymbol{K}}(k)\sum_{i=1}^{N}(\boldsymbol{T}_i - \boldsymbol{E}_i\boldsymbol{D})\boldsymbol{X}^d_i(k)\left(\sum_{i=1}^{N}(\boldsymbol{T}_i - \boldsymbol{E}_i\boldsymbol{D})\boldsymbol{X}^d_i(k)\right)^{\mathrm{T}} + 2\vec{\boldsymbol{K}}(k)\boldsymbol{\mathcal{T}}\varXi(k\mid k-1)\boldsymbol{\mathcal{T}}^{\mathrm{T}} - \\ &2\varXi(k\mid k-1)\boldsymbol{\mathcal{T}}^{\mathrm{T}} + 2\vec{\boldsymbol{K}}(k)\sum_{i=1}^{N}(\boldsymbol{E}_i\boldsymbol{D}(\wp_k + \boldsymbol{\sigma}_k)(\boldsymbol{E}_i\boldsymbol{D})^{\mathrm{T}}) + 2\boldsymbol{\beta}(k)\bar{\boldsymbol{K}}(k)\boldsymbol{\hbar}(k)\varXi(k\mid k-1)\boldsymbol{\mathcal{T}}^{\mathrm{T}}.\end{aligned} \tag{75}$$

Let us solve Eqs. (76) and (77),

$$\frac{\partial}{\partial\bar{\boldsymbol{K}}(k)}\mathrm{tr}\{\varXi(k\mid k)\} = 0, \tag{76}$$

$$\frac{\partial}{\partial\vec{\boldsymbol{K}}(k)}\mathrm{tr}\{\varXi(k\mid k)\} = 0. \tag{77}$$

Definition $\boldsymbol{\Xi}(k \mid k-1) = \mathrm{col}_N\{\boldsymbol{\Xi}_i(k \mid k-1)\}$, $\overline{\boldsymbol{K}}(k) = \mathrm{col}_N\{\overline{\boldsymbol{K}}_i(k)\}$ and $\vec{\boldsymbol{K}}(k) = \mathrm{col}_N\{\vec{\boldsymbol{K}}_i(k)\}$. For convenience, define the following formulas:

$$\begin{cases}\boldsymbol{\Delta}_1 = \boldsymbol{\mathcal{H}}(k)\boldsymbol{\Xi}(k \mid k-1)\boldsymbol{\mathcal{H}}^{\mathrm{T}}(k) + \boldsymbol{\mathcal{Q}}_v(k), \\ \boldsymbol{\Delta}_2 = \boldsymbol{\mathcal{T}}\,\boldsymbol{\Xi}(k \mid k-1)\boldsymbol{\mathcal{H}}^{\mathrm{T}}(k), \\ \boldsymbol{\Delta}_3 = \boldsymbol{\Xi}_i(k \mid k-1)\boldsymbol{\mathcal{H}}^{\mathrm{T}}(k), \\ \boldsymbol{\Delta}_4 = \boldsymbol{\mathcal{H}}(k)\boldsymbol{\Xi}(k \mid k-1)\boldsymbol{\mathcal{T}}^{\mathrm{T}}, \\ \boldsymbol{\Delta}_5 = \boldsymbol{\mathcal{T}}\,\boldsymbol{\Xi}(k \mid k-1)\boldsymbol{\mathcal{T}}^{\mathrm{T}} + \sum_{i=1}^{N}(\boldsymbol{E}_i\boldsymbol{D}(\wp_k + \boldsymbol{\sigma}_k)(\boldsymbol{E}_i\boldsymbol{D})^{\mathrm{T}}) + \\ \qquad \sum_{i=1}^{N}(\boldsymbol{T}_i - \boldsymbol{E}_i\boldsymbol{D})\boldsymbol{X}_i^d(k)\left(\sum_{i=1}^{N}(\boldsymbol{T}_i - \boldsymbol{E}_i\boldsymbol{D})\boldsymbol{X}_i^d(k)\right)^{\mathrm{T}}, \\ \boldsymbol{\Delta}_6 = \boldsymbol{\Xi}_i(k \mid k-1)\boldsymbol{\mathcal{T}}^{\mathrm{T}}.\end{cases} \tag{78}$$

To obtain the estimated gains, the value of $\beta_i(d_{i,k})$ need to be discussed. Due to the random movement of the moving target and LSR, the sensor may not be able to directly observe the target at certain times, resulting in $\beta_i(d_{i,k})$ with a value of 0 or 1.

1) If $\beta_i(d_{i,k}) = 0$, sensor i cannot receive direct measurement information, and

$$\overline{\boldsymbol{K}}_i(k) = 0, \tag{79}$$

$$\vec{\boldsymbol{K}}_i(k) = \boldsymbol{\Delta}_6(\boldsymbol{\Delta}_5)^{-1}. \tag{80}$$

2) If $\beta_i(d_{i,k}) \neq 0$, the filter updates

$$\overline{\boldsymbol{K}}_i(k) = (\boldsymbol{\Delta}_3 - \vec{\boldsymbol{K}}_i(k)\boldsymbol{\Delta}_2)(\boldsymbol{\Delta}_1)^{-1}, \tag{81}$$

$$\vec{\boldsymbol{K}}_i(k) = (\boldsymbol{\Delta}_6 - \boldsymbol{\Delta}_3(\boldsymbol{\Delta}_1)^{-1}\boldsymbol{\Delta}_4)(\boldsymbol{\Delta}_5 - \boldsymbol{\Delta}_2(\boldsymbol{\Delta}_1)^{-1}\boldsymbol{\Delta}_4)^{-1}. \tag{82}$$

Based on the two scenarios above, merging them to obtain the final estimator $\overline{\boldsymbol{K}}_i(k)$ and $\vec{\boldsymbol{K}}_i(k)$,

$$\overline{\boldsymbol{K}}_i(k) = \beta_i(d_{i,k})(\boldsymbol{\Delta}_3 - \vec{\boldsymbol{K}}_i(k)\boldsymbol{\Delta}_2)(\boldsymbol{\Delta}_1)^{-1}, \tag{83}$$

$$\begin{aligned}\vec{\boldsymbol{K}}_i(k) = {} & (\boldsymbol{\Delta}_6 - \beta_i(d_{i,k})\boldsymbol{\Delta}_3(\boldsymbol{\Delta}_1)^{-1}\boldsymbol{\Delta}_4) \\ & (\boldsymbol{\Delta}_5 - \beta_i(d_{i,k})\boldsymbol{\Delta}_2(\boldsymbol{\Delta}_1)^{-1}\boldsymbol{\Delta}_4)^{-1}.\end{aligned} \tag{84}$$

Remark 2 There are two forms of the estimator gain $\overline{\boldsymbol{K}}_i(k)$ and $\vec{\boldsymbol{K}}_i(k)$ calculated above, and the sensor node needs to know which form to choose for state estimation at the current moment. Due to the LSR of the sensor, $\beta_i(d_{i,k}) = 0$ when the target moves outside the sensor range, according to $\boldsymbol{y}_{i,k}$ expression, $\boldsymbol{y}_{i,k} = 0$ when $\beta_i(d_{i,k}) = 0$. Therefore, the sensor does not need to determine the specific location of the moving target at every moment, but only needs to judge the value of the directly observed $\boldsymbol{y}_{i,k}$ to determine the value of $\beta_i(d_{i,k})$.

3 Numerical Example

In this section, a simulation example is given to illustrate the effectiveness of using FQF to estimate the state of a moving target in distributed sensor networks with LSR. Considering a sensor network with six sensor nodes, the communication topology is given in Fig. 1.

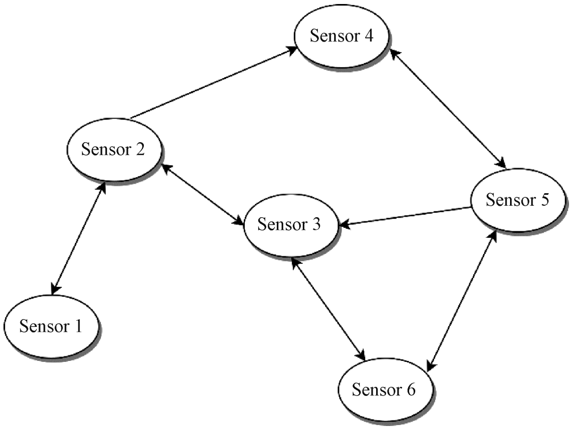


Fig. 1 Communication topology of sensor network

We assume that there is a target moving in this sensor network. Considering a linear time-varying dynamic model, the state transition matrix of the dynamic model (Ⅰ) is shown as

$$\boldsymbol{A}_k = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1.06\cos(0.1\pi k) & -1.06\sin(0.1\pi k) \\ 0 & 0 & -1.06\sin(0.1\pi k) & 1.06\cos(0.1\pi k) \end{bmatrix}.$$

Considering the measurement model (Ⅱ) with the following coefficient matrix:

$$\boldsymbol{H}_{i,k} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

The state is $\boldsymbol{x}_k = [x_{k,1}, x_{k,2}, x_{k,3}, x_{k,4}]^{\mathrm{T}}$, where $x_{k,1}$, $x_{k,2}$, $x_{k,3}$ and $x_{k,4}$ respectively represent the transverse position, longitudinal position, transverse velocity, and longitudinal velocity of the moving target. Moreover, the initial state is given as $\boldsymbol{x}_0 = [1 \quad 1 \quad 0 \quad 0]^{\mathrm{T}}$. In addition, set the positive scalar ε mentioned in Proposition 2 to 0.5.

The non-Gaussian noise sequences $\{\boldsymbol{w}_k\}$, $\{\boldsymbol{v}_{i,k}\}$ and $\{\boldsymbol{\xi}_{ij,k}\}$ are defined as

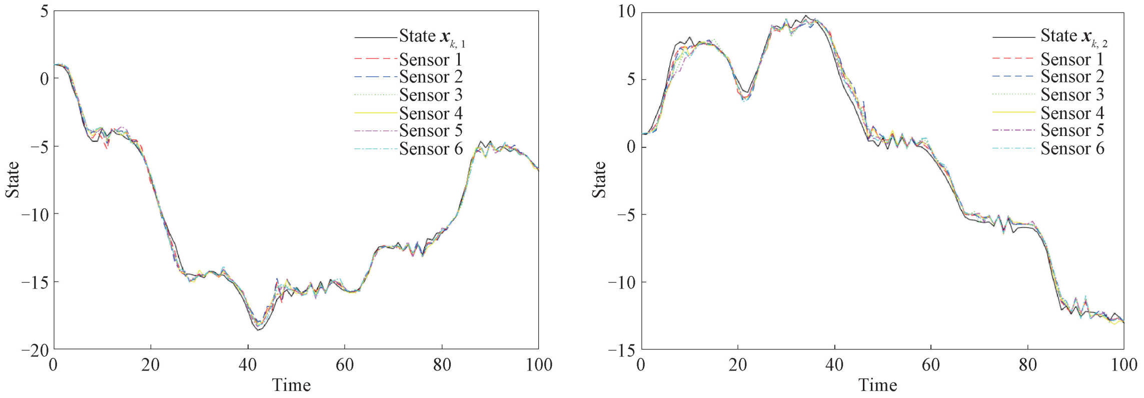
$$\begin{bmatrix} \boldsymbol{w}_k \\ \boldsymbol{v}_{i,k} \\ \boldsymbol{\xi}_{ij,k} \end{bmatrix} = \begin{bmatrix} -1/18 & 1/6 & 1/2 \\ 1/18 & -1/18 & -1/2 \\ 1/18 & -5/18 & -5/18 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix},$$

where γ_1, γ_2 and γ_3 are mutually exclusive and independent Bernoulli random variables that satisfy the following probability distribution:

$$\mathbb{P}\{\gamma_1 = 1\} = 5/6, \mathbb{P}\{\gamma_2 = 1\} = 1/9, \mathbb{P}\{\gamma_3 = 1\} = 1/18.$$

Therefore, we conducted a set of simulation experiments. By using the stochastic model mentioned above, different observation probabilities of different sensors in the sensor network can be obtained to analyze the performance of the state estimator designed for the moving target. Based on the observation range of the sensor, the observation probability can range from 0 to 1.0.

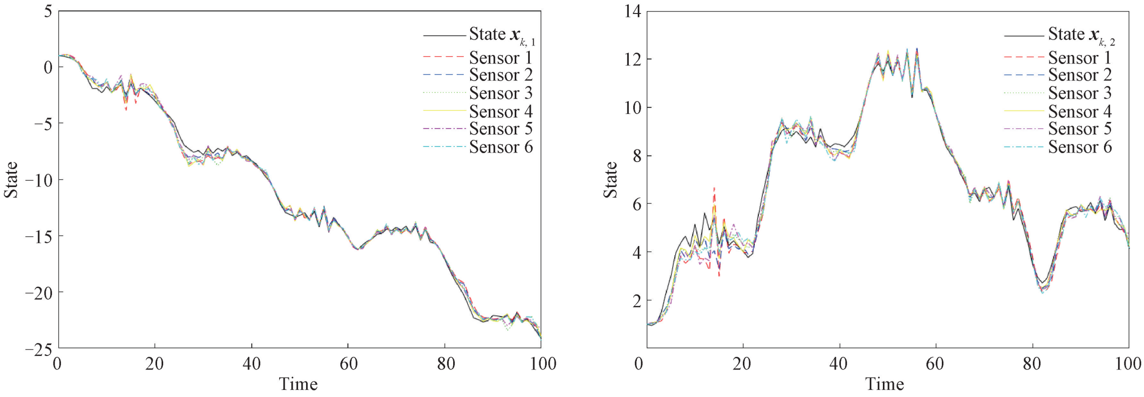
First, the first observation probability scheme of the sensor network is set to: $p_1 = 0.8$, $p_2 = 0.8$, $p_3 = 0.8$, $p_4 = 0.8$, $p_5 = 0.8$ and $p_6 = 0.8$, for all the sensors. Figure 2 shows the true state component $\boldsymbol{x}_{k,j}$ ($j = 1, 2$) of the moving target and the state estimate $[\hat{x}_{i,k|k}]_j$ ($i = 1, 2, 3, 4, 5, 6$) under this observation probability strategy (strategy 1). It is proven that the proposed filter scheme can perform state estimation well under the condition of loss of measurement value caused by LSR.



(a)
(b)

Fig. 2 State estimates with strategy 1: (a) state $\boldsymbol{x}_{k,1}$; (b) state $\boldsymbol{x}_{k,2}$

Then, the second observation probability strategy (strategy 2) is set to: $p_1 = 0$, $p_2 = 0.9$, $p_3 = 0.8$, $p_4 = 0.7$, $p_5 = 0.6$ and $p_6 = 0.5$. The observation probability of sensor node 1 is 0, which is called a naive node. It can be seen from Fig. 3 that when there are naive nodes, the estimator can still perform a good estimation of the state of the moving target due to the information passed between sensors.



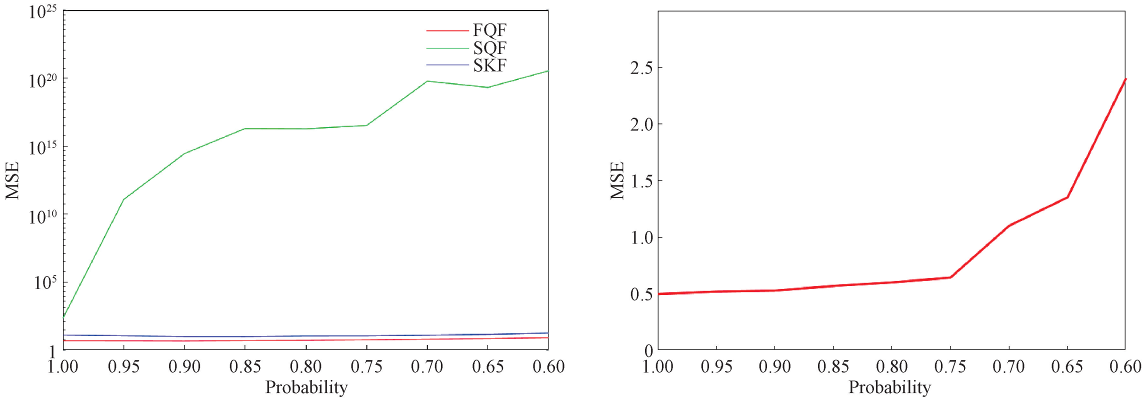
(a)
(b)

Fig. 3 State estimates with strategy 2: (a) state $\boldsymbol{x}_{k,1}$; (b) state $\boldsymbol{x}_{k,2}$

To quantify the estimation accuracy and show the estimation effect of the proposed filter, we define the mean square estimation error (MSE) as

$$\mathcal{M} = (1/N)(1/T)(1/M_c)\sum_{m_c=1}^{M_c}\sum_{i=1}^{N}\sum_{t=1}^{T}(\boldsymbol{x}_k - \hat{\boldsymbol{x}}_{i,k})^{\mathrm{T}}(\boldsymbol{x}_k - \hat{\boldsymbol{x}}_{i,k}),$$

where i, t, and m_c denote the sensor index, discrete-time step, and Monte-Carlo run index, respectively. Based on $M_c = 100$ independent Monte-Carlo runs with N sensors and T discrete-time steps per run, we can get the MSE changing with the sensor observation probability, as shown in Fig. 4. In Fig. 4, we compare the MSEs of FQF, standard Kalman filter (SKF), and standard quadratic filter (SQF). The MSE of FQF is also shown in Fig. 4. Additionally, the vertical axis has been locally enlarged to highlight the details. It can be seen that the MSE of FQF is lower than that of SKF and SQF, which illustrates the superiority of the proposed estimator.



(a)

(b)

Fig. 4 MSE with different probabilities: (a) FQF, SKF and SQF; (b) FQF

4 Conclusions

In this paper, we have been concerned with the tracking problem of a moving target in a distributed sensor network with LSR affected by non-Gaussian noise. Since the modeling of a moving target is often an unstable system, the method of output injection has been used, and the state has then been decomposed into deterministic and stochastic parts. A quadratic filter has been used to estimate the state of a system with non-Gaussian noise. The upper bound of the estimation error covariance has been obtained, and two estimator gains have been designed to minimize this upper bound. Finally, simulation results have shown that the proposed filter has a good estimation performance. In the future, our work can focus on the state estimation of multiple moving targets in distributed sensor networks and the selection of appropriate communication protocols for information transmission, with consideration of some network-induced phenomena.

References

[1] YICK J, MUKHERJEE B, GHOSAL D. Wireless sensor network survey [J]. *Computer Networks*, 2008, 52(12): 2292-2330.

[2] LI D, WONG K D, HU Y H, et al. Detection, classification, and tracking of targets [J]. *IEEE Signal Processing Magazine*, 2002, 19(2): 17-29.

[3] BLEDA A L, FERNÁNDEZ-LUQUE F J, ROSA A, et al. Smart sensory furniture based on WSN for ambient assisted living [J]. *IEEE Sensors Journal*, 2017, 17(17): 5626-5636.

[4] KANNA S, DINI D H, XIA Y, et al. Distributed widely linear frequency estimation in unbalanced three phase power systems [J]. *Computer Science*, 2014, 61(1): 74-83.

[5] LIAN B S, WAN Y, ZHANG Y, et al. Distributed Kalman consensus filter for estimation with moving targets [J]. *IEEE Transactions on Cybernetics*, 2022, 52(6): 5242-5254.

[6] MOIS G, FOLEA S, SANISLAV T. Analysis of three IoT-based wireless sensors for environmental monitoring [J]. *IEEE Transactions on Instrumentation and Measurement*, 2017, 66(8): 2056-2064.

[7] XU Y, LÜ W J, LIN W Q, et al. On extended state estimation for nonlinear uncertain systems with round-robin protocol [J]. *Automatica*, 2022, 138: 110154.

[8] HE X K, XUE W C, ZHANG X C, et al. Distributed filtering for uncertain systems under switching sensor networks and quantized communications [J]. *Automatica*, 2020, 114: 108842.

[9] DAS S, MOURA J M F. Distributed Kalman filtering with dynamic observations consensus [J]. *IEEE Transactions on Signal Processing*, 2015, 63(17): 4458-4473.

[10] NAHI N. Optimal recursive estimation with uncertain observation [J]. *IEEE Transactions on Information Theory*, 1969, 15(4): 457-462.

[11] SINOPOLI B, SCHENATO L, FRANCESCHETTI M, et al. Kalman filtering with intermittent observations [J]. *IEEE Transactions on Automatic Control*, 2004, 49(9): 1453-1464.

[12] ILIĆ N, STANKOVIĆ M S, STANKOVIĆ S S. Adaptive consensus-based distributed target tracking in sensor networks with limited sensing range [J]. *IEEE Transactions on Control Systems Technology*, 2014, 22(2): 778-785.

[13] ZHANG Y, SUN L C, HU G Q. Distributed consensus-based multitarget filtering and its application in formation-containment control [J]. *IEEE Transactions on Control of Network Systems*, 2020, 7(1): 503-515.

[14] ZAKAI M. On the optimal filtering of diffusion processes [J]. *Zeitschrift Für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 1969, 11(3): 230-243.

[15] MASRELIEZ C. Approximate non-Gaussian filtering with linear state and observation relations［J］. *IEEE Transactions on Automatic Control*，1975，20(1)：107-110.

[16] MERWE R V D，WAN E. Gaussian mixture sigma-point particle filters for sequential probabilistic inference in dynamic state-space models［C］. Proceedings of 2003 IEEE International Conference on Acoustics，Speech，and Signal Processing. NewYork：IEEE，2003：701-704.

[17] CACACE F，FASANO A，GERMANI A. Quadratic filtering of non-Gaussian systems with intermittent observations［C］. Proceedings of 52nd IEEE Conference on Decision and Control. NewYork：IEEE，2013：4024-4029.

[18] LIU Q Y，WANG Z D，HAN Q L，et al. Quadratic estimation for discrete time-varying non-Gaussian systems with multiplicative noises and quantization effects［J］. *Automatica*，2020，113：108714.

[19] CACACE F，CONTE F，GERMANI A，et al. Feedback quadratic filtering［J］. *Automatica*，2017，82：158-164.

[20] LIU Q Y，ZHENG S Y，WANG Z D，et al. Distributed estimation for multi-agent systems with relative measurements and quantized communication：a feedback quadratic framework［J］. *International Journal of Robust and Nonlinear Control*，2023，33(5)：3164-3184.

[21] CARRAVETTA F，GERMANI A，RAIMONDI M. Polynomial filtering for linear discrete time non-Gaussian systems［J］. *SIAM Journal on Control and Optimization*，1996，34(5)：1666-1690.

[22] LIU Q Y，WANG Z D，HE X，et al. Event-based $H\infty$ consensus control of multi-agent systems with relative output feedback：the finite-horizon case［J］. *IEEE Transactions on Automatic Control*，2015，60(9)：2553-2558.

有限感知范围下时变非高斯系统的移动目标分布式反馈二次滤波器

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摘　要： 探讨了受非高斯噪声影响的有限感测范围（limited sensing range，LSR）分布式传感器网络中运动目标的跟踪问题。在这种传感器网络中，LSR 引起的观测损失是一个普遍存在的问题，但没有得到足够的重视。为此，引入了一个时变随机变量来描述传感器在每个时刻是否观察到运动目标。当单个传感器节点无法从其他节点接收信息时，一旦目标移动到该节点的观测范围之外，它就无法更新其对移动目标的状态估计。该文提出了一种分布式传感器网络内的信息流拓扑，以方便接收相邻节点传输的先验状态估计数据。基于这些信息，为每个传感器设计了一个二次分布估计器，并引入了一个输出注入项来处理不稳定系统。最后，通过一个数值算例说明所提出的控制方案的有效性。

关键词： 非高斯系统；二次估计；移动目标；时变