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RESEARCH ARTICLE

Trajectory controllability of integro-differential system of fractional orders in Hilbert spaces

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ABSTRACT

The trajectory controllability for fractional order semilinear integrodifferential systems of order $\nu \in (0,1]$ and $\nu \in (1,2]$ is the subject of this paper. Monotonicity is an important characteristic in many communications applications in which digital-to-analog converter circuits are used. Such applications can function in the presence of nonlinearity, but not in the presence of non-monotonicity. Therefore, it becomes quite interesting to study a problem assuming the monotonicity of the nonlinear function. With the help of fractional calculus, adequate conditions have been developed to verify the trajectory controllability for fractional order semilinear integrodifferential system using the basics of monotone nonlinearity and coercivity. Finally, some examples are presented to demonstrate the viability of the acquired results.

1. Introduction

The principles of fractional calculus and the fractional differential equation have dominated mathematics in recent decades. Some physical problems cannot be solved using differential equations of integer order, but they can be solved using differential equations of fractional order. As a result, numerous academics have recently made significant contributions to the fields of electromagnetics, control theory, signal, porous media, viscoelasticity, biological, engineering difficulties, image processing, fluid flow, diffusion, theology, and other fields. The notation of optimal controls has performed as an important tool in analysis and design of control systems. These kinds of issues can be determined with various signification of fractional derivatives. It has different applications in separate fields, for example, economics, the control of chemical outgrowths, biology, power systems, space technology, engineering, electronics, physics, robotics, transportation, chemistry, and so on; the solution of these types of seeds has become a significant work for young scholars. Controllability research, which was conceived by Kalman, began in earnest at the beginning of the 1960s. Since then, several studies have been conducted utilizing various methodologies in the setting of finite and infinite dimensional deterministic and stochastic systems, one can refer Arora and Sukavanam,[1] Bragdi and Hazi,[2] Davison and

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Kunze,[3] Debnath,[4] George,[5] Klamka,[6–8] Lions,[9] George et al.,[10] Nandakumaran and George,[11] Miller,[12] Cardetti and Gordina[13] Muslim and Kumar,[14] Shukla et al.,[15] Shukla et al.,[16] Wang et al.,[17] Jajarmi and Baleanu,[18] Kumar and Malik,[19] Liu and Li,[20] Muslim and Kumar,[14] Dineshkumar et al.,[21] Mahmudov et al.[22] Mahmudov et al.,[23] Shukla et al.,[24] Vijayakumar.[25–27]

In the setting of fractional systems having order $\nu \in (1, 2]$, Matar,[28] Wang and Zhou,[29] Zhou and Jiao[30] established sufficient criteria for the fractional-order evolution equations in the case of fractional systems of order $\nu \in (0, 1]$. In Kumar and Sukavanam,[31] Sukavanam and Kumar, [32] the authors provided adequate conditions for the approximate controllability of fractional-order semilinear delay control systems utilizing fixed point techniques, based on existence results for mild solutions. In the setting of fractional systems having order $\nu \in (1, 2]$, Kexue et al.[33] discussed the existence and exact controllability of nonlocal fractional systems of order $\nu \in (1, 2]$ in infinite-dimensional spaces by employing Sadovskii's fixed point theorem. In[34] explored mild solutions and approximate controllability of semilinear neutral integro-differential equations in Banach spaces, utilizing resolvent families, optimal control, and sufficient conditions under Lipschitz-type assumptions. In[35] established mild solution existence, uniqueness, and trajectory controllability of conformable Hilfer fractional stochastic systems using semigroup theory, stochastic analysis, and Banach fixed-point theorem. Shukla et al.[36] recently used a sequential technique to establish approximate controllability of fractional system of order $\nu \in (1, 2]$ with indefinite delay. In recent years, a new concept of controllability known as trajectory controllability (T-controllability) has developed as a new area of research, describing the path along which our system goes while under control. There is a scarcity of the literature on T-controllability. We strive to discover a control that drives the system along a predefined trajectory rather than one that steers from a given start state to the final state in T-controllability challenges. Depending on the course chosen, it may reduce some of the costs associated with steering the system.

George introduced the concept of T-controllability for one-dimensional nonlinear systems in 1996. For example, when a rocket is launched into orbit, it may be necessary to have a precise course along with the targeted destination in order to save money and avoid collisions. It could also be used to protect the system. In principle, T-controllability ensures that an appropriate control function can be used to lead a dynamical system from an arbitrary start state to a final state with a predefined trajectory. As a result, T-controllability is a more powerful concept of controllability. The T-controllability of integer order semilinear integrodifferential systems in finite and infinite dimensional settings was established by Chalishajar et al.[37] and Chalishajar et al.[38] In[39] the authors established approximate controllability of Sobolev-type fuzzy Hilfer fractional systems using fractional calculus, Clarke subdifferentials, and Dhage's theorem.[40] This manuscript studied the optimal control problem for coupled semilinear wave systems, proved mild solution existence, optimal pairs, and time optimal control. However, because of their nonlocal nature, fractional differential equations are more efficient in representing real-life processes than classical differential equations. As a result, investigating T-controllability discussion for several types of semilinear fractional dynamical systems is fascinating. The goal of our research is to attain the T-controllability of fractional semilinear integrodifferential symmetries, which is motivated by the previous work. This study explores trajectory controllability (T-controllability) for fractional-order semilinear integro-differential systems, addressing both $\nu \in (0, 1]$ and $\nu \in (1, 2]$. By leveraging Caputo derivatives, monotone nonlinearity, and coercivity conditions, it establishes explicit control functions using Gronwall's inequality and fixed-point methods. Unlike traditional controllability, this approach ensures the system follows a predefined trajectory. The paper extends fractional control theory and provides practical examples, demonstrating its relevance in systems with memory effects and nonlinear dynamics.

The following is the outline for this paper. Under certain necessary conditions, the T-controllability of the fractional semilinear integrodifferential equations having order $\nu \in (0, 1]$ was explored in Section 2. T-controllability of the fractional semilinear integrodifferential equations having order $\nu \in (1, 2]$ was explored in Section 3. We employ the instruments of monotone nonlinearities along with coercivity property to demonstrate this. Certain examples are presented to demonstrate the applicability of the abstract theory in the final section.

2. Fractional differential systems of order $\nu \in (0, 1]$

The primary target of this section is to present the T-controllability of the fractional semilinear integrodifferential equations having order $\nu \in (0, 1]$.

We consider $\mathbb{Y}$, $\mathbb{V}$ be Hilbert spaces with $\mathbb{V}$ is a subspace of $\mathbb{Y}$ and $L_2([0,b],\mathbb{Y})$, $L_2([0,b],\mathbb{V})$ be the corresponding function spaces. We consider the fractional semilinear integrodifferential system of order $\nu \in (0,1]$:

$$\begin{aligned} {}^{c}\mathbf{D}^{\nu}_{\varrho} y(\varrho) =& Ay(\varrho) + B(\varrho, v(\varrho)) \\ &+ G\bigg(\varrho, y(\varrho), \int_0^{\varrho} g(\varrho,\alpha,y(\alpha))d\alpha\bigg), \\ & varrho \in I = [0,b], \nu \in (0,1], \end{aligned} \tag{1}$$

$$y(0) = y_0. \tag{2}$$

In the above system, ${}^{c}\mathbf{D}^{\nu}_{\varrho}$ stands for the Caputo fractional derivative of order ν. The state $y(\varrho)$ assumes its values in Hilbert space $\mathbb{Y}$ and the control $v(\varrho)$ assumes values in Hilbert space $\mathbb{V}$. $A : D(A) \subset \mathbb{Y} \to \mathbb{Y}$ is the infinitesimal generator of a strongly continuous semigroup $\{\mathcal{S}(\varrho),\ \varrho \in \mathbb{R}\}$. The operator $B : J \times \mathbb{V} \to \mathbb{Y}$ is nonlinear. Moreover, $G : J \times \mathbb{Y} \times \mathbb{Y} \to \mathbb{Y}$ and $g : \Delta \times \mathbb{Y} \to \mathbb{Y}$ are appropriate function defined later. $\Delta = \{(\varrho,\alpha) \in I : 0 \le \alpha \le \varrho \le b\}$ and $y_0 \in \mathbb{Y}$.

We assume that $C(I,\mathbb{Y}) : I \to \mathbb{Y}$ be the Banach space of all continuous functions with supremum norm given by

$$\|\psi\| = \sup_{0\le\eta\le b} \|\psi(\eta)\|, \quad \psi \in C(I,\mathbb{Y}).$$

Definition 1. [41] *"The fractional integral of order $\nu \in \mathbb{R}^+$ with the lower limit 0 for a function $f \in L^1(\mathbb{R}^+)$ is defined as*

$$I^{\nu}_t f(\varrho) = \frac{1}{\Gamma(\nu)} \int_0^{\varrho} (\varrho-\alpha)^{\nu-1} f(\alpha)d\alpha, \ \varrho > 0\ , \nu > 0$$

provided the right-hand side is point-wise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is gamma function."

Definition 2. [41] *"The Riemann-Liouville (R-L) derivative of order $\nu \in \mathbb{R}^+$ with lower limit zero for a function $f : [0,\infty) \to \mathbb{R}$ can be written as*

$${}^{L}\mathbf{D}^{\nu}_{\varrho} f(\varrho) = \frac{1}{\Gamma(n-\nu)} \frac{d^n}{dt^n} \int_0^{\varrho} \frac{f(\alpha)}{(\varrho-\alpha)^{\nu+1-n}} d\alpha, \quad \varrho > 0, \quad n-1 < \nu < n."$$

Definition 3. [14] *"The Caputo fractional derivative of order ν for a function $f \in C^{n-1}((0,b);\mathbb{Y}) \cap L^1((0,b);\mathbb{Y})$ can be written as*

$${}^{c}\mathbf{D}^{\nu}_{\varrho} f(\varrho) = \frac{1}{\Gamma(n-\nu)} \int_0^{\varrho} (\varrho-\alpha)^{n-\nu-1} f^n(\alpha)d\alpha$$

where $n-1 < \nu < n$, $n = [\nu]+1$ denotes the integral part of the real number ν. Also

$$I^{\nu}_t({}^{c}\mathbf{D}^{\nu}_{\varrho} f(\varrho)) = f(\varrho) - \sum_{r=0}^{n-1} \frac{t^r}{r!} f^r(0)."$$

where $C^{n-1}((0,b);\mathbb{Y})$ and $L^1((0,b);\mathbb{Y})$ represents the $(n-1)$ time continuously differentiable and space of Y-valued Bochner integrable functions, respectively.

Definition 4. [30] *A continuous function $x \in C(I,\mathbb{Y})$ is called the mild solution of (1)-(2) along with $u \in L_2(I;\mathbb{V})$ if it fulfills*

$$\begin{aligned} y(\varrho) =& \mathcal{S}_{\nu}(\varrho)y_0 + \int_0^{\varrho} (\varrho-\alpha)^{\nu-1} \hat{\mathcal{S}}_{\nu}(\varrho-\alpha) \\ & \bigg[B(\alpha, v(\alpha)) + G\bigg(\alpha, y(\alpha), \\ & \int_0^{\alpha} g(\alpha,\beta,y(\beta))d\beta\bigg)\bigg] d\alpha, \end{aligned} \tag{3}$$

where

$$\mathcal{S}_{\nu}(\varrho) = \int_0^{\infty} \zeta_{\nu}(\alpha) \mathcal{S}(\varrho^{\nu}\alpha)d\alpha$$

and

$$\hat{\mathcal{S}}_{\nu}(\varrho) = \nu \int_0^{\infty} \alpha\zeta_{\nu}(\alpha) \mathcal{S}(\varrho^{\nu}\alpha)d\alpha.$$

Here

$$\zeta_{\nu}(\alpha) = \frac{1}{\nu} \alpha^{-1-\frac{1}{\nu}} \psi_{\nu}(\alpha^{-\frac{1}{\nu}})$$

is a function defined on $(0,\infty)$ satisfying $\zeta_{\nu}(\alpha) \ge 0$, $\int_0^{\infty} \zeta_{\nu}(\alpha)d\alpha = 1$ and

$$\psi_{\nu}(\alpha) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \alpha^{-\nu n-1} \frac{\Gamma(n\nu+1)}{n!} sin(n\pi\nu), \quad \alpha \in (0,\infty).$$

Lemma 1. *[42] "For any fixed $t \ge 0$, the operators $\mathcal{S}_{\nu}(\varrho)$ and $\hat{\mathcal{S}}_{\nu}(\varrho)$ are linear and bounded, that is, for any $x \in \mathbb{Y}$, $\|\mathcal{S}_{\nu}(\varrho)y\| \le M\|y\|$ and $\|\hat{\mathcal{S}}_{\nu}(\varrho)y\| \le \frac{M\nu}{\Gamma(1+\nu)}\|y\|$, where M is a constant such that $\|\mathcal{S}(\varrho)\| \le M$, for all $\varrho \ge 0$."*

For more details on the fractional systems, refer Bazhlekova,[43] Miller,[AK12] Podlubny[45] and for the detailed study of cosine and sine families, one can relate to Travis and Webb.[46]

Before discussing the T-controllability of fractional integrodifferential system having order $\nu \in (0,1]$, we discuss the relations between T-controllability and complete controllability.

Definition 5. *The system (1)-(2) is called completely controllable provided that for every $y_0, y_1 \in \mathbb{Y}$, and fixed b, $\exists\ v(\cdot) \in L_2(I,\mathbb{V})$ such that $y(\cdot)$ fulfills $y(b) = y_1$.*

Assume that $\mathcal{P}$ be the set of all functions $z \in L_2(I,\mathbb{Y})$ determined on I with $z(0) = y_0$, $z(b) = y_1$ and the fractional derivative ${}^c\mathbf{D}_t^\nu z$ exists almost everywhere for $\nu \in (0,1]$. Noting that $\mathcal{P}$ be the set of all feasible trajectories for (1)-(2).

Definition 6. *The system* (1)-(2) *is called* T*-controllable provided that for every* $z \in \mathcal{P}$*,* $\exists\ u \in L_2(I,\mathbb{V})$ *such that* $y(\varrho)$ *fulfills* $y(\varrho) = z(\varrho)$ *almost everywhere on* I*.*

Clearly, T-controllability ⇒ Complete controllability.

We show that the trajectory controllability of (1)-(2) in the following two cases:

Case-I: If the control appears linearly:

Let us take a look at the linear control system, that is, $B(\varrho, v(\varrho)) = b(\varrho)v(\varrho)$, where $b : I\mathbb{R}$ and $u : I\mathbb{V}$, then the system (1)-(2) becomes

$$
\begin{aligned}
{}^c\mathbf{D}_\varrho^\nu y(\varrho) =& Ay(\varrho) + b(\varrho)v(\varrho)\\
&+ G\Big(\varrho, y(\varrho), \int_0^\varrho g(\varrho,\alpha,y(\alpha))d\alpha\Big),\\
&\varrho \in I, \nu \in (0,1],
\end{aligned}
\tag{4}
$$

$$
y(0) = y_0. \tag{5}
$$

The sufficient conditions to prove the trajectory controllability of (4)-(5) are as follows:

Assumptions [I] [38]

1. A generates a strongly continuous semigroup $\{\mathcal{S}(\varrho) : \varrho \geq 0\}$ on $\mathbb{Y}$.
2. $b(\varrho)$ by no means disappear on I.
3. G is Lipschitz continuous w.r.t. (i) and (ii) argument, that is, $\exists\ \delta_1 > 0$ and $\delta_2 > 0$ such that
$$\|G(\varrho,y_1,y_1) - G(\varrho,x_2,y_2)\| \leq \delta_1\|y_1 - x_2\| + \delta_2\|y_1 - y_2\|,$$
for all $y_1, x_2, y_1, y_2 \in \mathbb{Y}$, $\varrho \in I$.
4. g is L^1-Lipschitz continuous w.r.t. third argument, that is, $\exists\ \gamma > 0$, $\ni$,
$$\int_0^\varrho \|g(\varrho,\alpha,y(\alpha)) - g(\varrho,\alpha,z(\alpha))\|d\alpha \leq \gamma\|y(\varrho) - z(\varrho)\|, y, z \in \mathcal{P}, (\varrho,\alpha) \in \triangle.$$

Using these hypotheses, we are able to build a control function explicitly to verify T-controllability of (4)-(5). For showing this, we continue like this:

The existence and uniqueness discussion for (4)-(5) may be easily proved with the help of Lipschitz continuity of functions G and h, for each control $v \in L_2(I,\mathbb{V})$.

Assume that $z(\varrho)$ be the considered trajectory in $\mathcal{P}$. We extract admissible control $v(\varrho)$ from equation (4)-(5) by

$$
v(\varrho) = \frac{{}^c\mathbf{D}_\varrho^\nu z(\varrho) - Az(\varrho) - G\Big(\varrho, z(\varrho), \int_0^\varrho g(\varrho,\alpha,z(\alpha))d\alpha\Big)}{b(\varrho)}.
$$

By using this admissible control function, the state equation (4)-(5) becomes,

$$
\begin{aligned}
{}^c\mathbf{D}_\varrho^\nu y(\varrho) =& Ay(\varrho) + {}^c\mathbf{D}_\varrho^\nu z(\varrho) - Az(\varrho)\\
&- G\Big(\varrho, z(\varrho), \int_0^\varrho g(\varrho,\alpha,z(\alpha))d\alpha\Big)\\
&+ G\Big(\varrho, y(\varrho), \int_0^\varrho g(\varrho,\alpha,y(\alpha))d\alpha\Big)\\
y(0) =& y_0.
\end{aligned}
$$

Let us make the substitution $w(\varrho) = y(\varrho) - z(\varrho)$, Then for $w(\varrho)$, we have the subsequent fractional system

$$
\begin{aligned}
{}^c\mathbf{D}_\varrho^\nu w(\varrho) =& Aw(\varrho) + G\Big(\varrho, y(\varrho), \int_0^\varrho g(\varrho,\alpha,y(\alpha))d\alpha\Big)\\
&- G\Big(\varrho, z(\varrho), \int_0^\varrho g(\varrho,\alpha,z(\alpha))d\alpha\Big),\\
w(0) =& 0.
\end{aligned}
$$

It follows that the solution of the above system with $w(0) = 0$ is described as

$$
\begin{aligned}
w(\varrho) =& \int_0^\varrho (\varrho - \alpha)^{\nu-1}\hat{\mathcal{S}}_\nu(\varrho - \alpha)\\
&\Big[G\Big(\alpha, y(\alpha), \int_0^\alpha g(\alpha,\beta,y(\beta))d\beta\Big)\\
&- G\Big(\alpha, z(\alpha), \int_0^\alpha g(\alpha,\beta,z(\beta))d\beta\Big)\Big]d\alpha.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|w(\varrho)\| \leq& \frac{M\nu}{\Gamma(1+\nu)}\frac{b^\nu}{\nu}\int_0^\varrho \Big(\delta_1\|y(\alpha) - z(\alpha)\|\\
&+ \delta_2\Big\|\int_0^\alpha g(\alpha,\beta,x(\beta))d\beta\\
&- \int_0^\alpha g(\alpha,\beta,z(\beta))d\beta\Big\|\Big)d\alpha\\
\leq& \frac{Mb^\nu}{\Gamma(1+\nu)}\int_0^\varrho (\delta_1\|y(\alpha)\\
&- z(\alpha)\| + \delta_2\gamma\|y(\alpha) - z(\alpha)\|)d\alpha.
\end{aligned}
$$

That is,

$$
\|y(\varrho) - z(\varrho)\| \leq \frac{Mb^\nu}{\Gamma(1+\nu)}(\delta_1 + \delta_2\gamma)\int_0^\varrho \|y(\alpha) - z(\alpha)\|d\alpha.
$$

Hence by 'Gronwall's inequality', we get

$$
\|y(\varrho) - z(\varrho)\| = 0.
$$

Thus, $y(\varrho) = z(\varrho)$, $\forall\ \varrho \in I$ and which concludes the T-controllability of (4)-(5).

Case-II: If the control appears nonlinearly in (1)-(2):

In this case, we need the additional assumptions on B, h, and G to prove the trajectory controllability of (1)-(2) given as:

Assumptions [II].

(1) B, h, and G fulfill Caratheadory conditions, that is, $B(\varrho,\cdot):\mathbb{V}\mapsto\mathbb{Y}$ is continuous for $\varrho\in I$ and $B(\cdot,y):I\mapsto\mathbb{Y}$ is measurable for $y\in\mathbb{V}$ and $g(\varrho,\alpha,\cdot):\mathbb{Y}\mapsto\mathbb{Y}$ is continuous $\forall(\varrho,\alpha)\in\Delta$ and $g(\cdot,\cdot,y):\Delta\mapsto\mathbb{Y}$ is measurable $\forall y\in\mathbb{Y}$ and G fulfills Caratheadory conditions identical to h.

(2) B, g, and G satisfy subsequent characteristics:

$$\|B(\varrho,v)\|_{\mathbb{Y}}\le l_0(\varrho)+l_1\|v\|_{\mathbb{V}},\ \forall\ v\in\mathbb{V},\ \varrho\in I,$$
$$\|g(\varrho,\alpha,y)\|\le m_0(\varrho)+m_1\|y\|_{\mathbb{Y}}\quad\forall\ \varrho\in I, y\in\mathbb{Y},$$
$$\|G(\varrho,y,z)\|_{\mathbb{Y}}\le n_0(\varrho)+n_1\|y\|_{\mathbb{Y}}+n_2\|z\|_{\mathbb{Y}}.$$

(3) B fulfills monotonicity and coercivity conditions, i.e.,

$$\langle B(\varrho,v)-B(\varrho,w),v-w\rangle\ge 0\quad\forall\quad v,w\in\mathbb{V},\ \varrho\in I$$

and

$$\lim_{\|u\|\to\infty}\frac{\langle B(\varrho,v),v\rangle}{\|v\|}=\infty.$$

Theorem 1. *Under assumptions [I] ((i),(iii),(iv)) and [II], the nonlinear system* (1)-(2) *is T-controllable.*

Proof. The existence and uniqueness of the nonlinear system (1)-(2) can be proved by employing Lipschitz continuity of G and g for each fixed v and the solution fulfills

$$\begin{aligned}y(\varrho)=&\mathcal{S}_\nu(\varrho)y_0+\int_0^\varrho(\varrho-\alpha)^{\nu-1}\hat{\mathcal{S}}_\nu(\varrho-\alpha)\\&B(\alpha,v(\alpha))d\alpha\\&+\int_0^\varrho(\varrho-\alpha)^{\nu-1}\hat{\mathcal{S}}_\nu(\varrho-\alpha)\\&G\Big(\alpha,y(\alpha),\int_0^\alpha g(\alpha,\beta,y(\beta))d\beta\Big)d\alpha.\end{aligned}$$

Assume that $z\in\mathcal{P}$ is the given trajectory with $z(0)=y_0$. Our duty is to check the control v fulfilling

$$\begin{aligned}z(\varrho)=&\mathcal{S}_\nu(\varrho)y_0+\int_0^\varrho(\varrho-\alpha)^{\nu-1}\hat{\mathcal{S}}_\nu(\varrho-\alpha)\\&B(\alpha,v(\alpha))d\alpha\\&+\int_0^\varrho(\varrho-\alpha)^{\nu-1}\hat{\mathcal{S}}_\nu(\varrho-\alpha)\\&G\Big(\alpha,z(\alpha),\int_0^\alpha g(\alpha,\beta,z(\beta))d\beta\Big)d\alpha.\end{aligned}$$

To show this, we put $B_1(\varrho)=B(\varrho,v(\varrho))$ in (1)-(2), we get

$$\begin{aligned}{}^c\mathbf{D}^\nu_\varrho y(\varrho)&=Ay(\varrho)+B_1(\varrho)\\&+G\Big(\varrho,y(\varrho),\int_0^\varrho g(\varrho,\alpha,y(\alpha))d\alpha\Big),\quad(6)\\&\varrho\in I,\ \nu\in(0,1]\end{aligned}$$

$$y(0)=y_0.\tag{7}$$

We define $B_1(\varrho)$ by

$$\begin{aligned}B_1(\varrho)=&{}^c\mathbf{D}^\nu_\varrho z(\varrho)-Az(\varrho)\\&-G\Big(\varrho,z(\varrho),\int_0^\varrho g(\varrho,\alpha,z(\alpha))d\alpha\Big)\end{aligned}$$

for the given trajectory $z(\varrho)\in\mathcal{P}$.

Along with this control, (6) can be wriiten as,

$$\begin{aligned}{}^c\mathbf{D}^\nu_\varrho y(\varrho)=&Ay(\varrho)+{}^c\mathbf{D}^\nu_\varrho z(\varrho)-Az(\varrho)\\&-G\Big(\varrho,z(\varrho),\int_0^\varrho g(\varrho,\alpha,z(\alpha))d\alpha\Big)\\&+G\Big(\varrho,y(\varrho),\int_0^\varrho g(\varrho,\alpha,y(\alpha))d\alpha\Big),\\y(0)=&y_0.\end{aligned}$$

Setting $w(\varrho)=y(\varrho)-z(\varrho)$, one can get

$$\begin{aligned}{}^c\mathbf{D}^\nu_\varrho w(\varrho)=&Aw(\varrho)+G\Big(\varrho,y(\varrho),\int_0^\varrho g(\varrho,\alpha,y(\alpha))d\alpha\Big)\\&-G\Big(\varrho,z(\varrho),\int_0^\varrho g(\varrho,\alpha,z(\alpha))d\alpha\Big),\end{aligned}\tag{8}$$

$$w(0)=0.\tag{9}$$

From semigroup theory, the solution of (8)-(9) given by

$$\begin{aligned}w(\varrho)=&\int_0^\varrho(\varrho-\alpha)^{\nu-1}\hat{\mathcal{S}}_\nu(\varrho-\alpha)\\&\Big[G\Big(\alpha,y(\alpha),\int_0^\alpha g(\alpha,\beta,y(\beta))d\beta\Big)\\&-G\Big(\alpha,z(\alpha),\int_0^\alpha g(\alpha,\beta,z(\beta))d\beta\Big)\Big]d\alpha.\end{aligned}$$

Now taking norm on both sides and using Gronwall's inequality just in the previous case, one can attain

$$\|y(\varrho)-z(\varrho)\|=0.$$

Therefore, $y(\varrho)=z(\varrho),\ \forall\ \varrho\in I$. Therefore, the mild solution of the given system equals the prescribed trajectory $z(\varrho)$, when the control u is given by $B_1(\varrho)=B(\varrho,v(\varrho))$. Hence to prove the trajectory controllability, it is enough to extract $v(\varrho)$ from $B_1(\varrho)$. For extracting $v(\varrho)$, we define $N:L_2(I,\mathbb{V})\to L_2(I,\mathbb{Y})$ by

$$(Nv)(\varrho)=B(\varrho,v(\varrho)).\tag{10}$$

By referring to the assumption [II(i)-(ii)], we observe that the operator N is well-defined, continuous, and bounded. Using the assumption[II(iii)], we can say that N is monotone and coercive. A hemi-continuous monotone map is of the type (M) [44]. Hence, by employing Theorem 3.6.9 of Joshi and Bose,[44] we can say that the nonlinear map N is onto. Therefore, there exists a control u fulfilling (10). Also, since $u\in L_2(I,\mathbb{V})$, therefore u is measurable. So, we can extract $v(\varrho)$ from $B(\varrho, v(\varrho))$ so that for this control, the mild solution of the considered system equals to the prescribed trajectory $z(\varrho)$, which concludes T-controllability of (1)-(2).

3. Fractional differential systems of order $\nu \in (1,2]$

We are primarily focusing on the trajectory controllability of the subsequent fractional semilinear integrodifferential system of order $\nu \in (1,2]$

$$\begin{aligned} {}^{c}\mathbf{D}^{\nu}_{\varrho} y(\varrho) =& Ay(\varrho) + B(\varrho, v(\varrho)) \\ &+ G\left(\varrho, y(\varrho), \int_0^{\varrho} g(\varrho, \alpha, y(\alpha))d\alpha\right), \\ &\varrho \in I, \nu \in (1,2], \end{aligned} \tag{11}$$

$$y(0) = y_0, y'(0) = z_0. \tag{12}$$

Here, A is the infinitesimal generator of the strongly continuous cosine family (CF) $\{C_\nu(\varrho), t \geq 0\}$ and $y_0, z_0 \in \mathbb{Y}$.

To define the mild solution of (11)-(12), we assume the subsequent system

$${}^{c}D^{\nu}_{\varrho} y(\varrho) = Ay(\varrho),\ y(0) = \eta\ y'(0) = 0. \tag{13}$$

In the above, $\nu \in (1,2]$; $A : D(A) \subseteq \mathbb{Y} \to \mathbb{Y}$ is a closed and densely defined operator in $\mathbb{Y}$. Taking Riemann fractional integral of order ν on both sides of (13), one can attain

$$y(\varrho) = \eta + \frac{1}{\Gamma(\nu)} \int_0^{\varrho} (\varrho - \alpha)^{\nu - 1} Ax(\alpha) d\alpha \tag{14}$$

Definition 7. [43] *"Let $\nu \in (1,2]$. A family $\{C_\nu(\varrho)\}_{t\geq 0} \subset \mathbb{L}(\mathbb{Y})$ is called a solution operator (or a strongly continuous ν-order fractional CF) for* (13) *if the following conditions are satisfied:*

(1) $C_\nu(\varrho)$ is strongly continuous for $t \geq 0$ and $C_\nu(0) = I$;
(2) $C_\nu(\varrho)D(A) \subset D(A)$ and $AC_\nu(\varrho)\eta = C_\nu(\varrho)A\eta$ for all $\eta \in D(A)$, $t \geq 0$;
(3) $C_\nu(\varrho)\eta$ is a solution of (13)*,$\forall\ \eta \in D(A)$, $\varrho \geq 0$.*

A is called infinitesimal generator of $C_\nu(\varrho)$. The strongly continuous ν-order fractional CF is also called ν-order CF."

Definition 8. *The sine family (SF) $S_\nu : [0,\infty) \to \mathbb{L}(\mathbb{Y})$ connected with C_ν is presented as*

$$S_\nu(\varrho) = \int_0^{\varrho} C_\nu(\alpha) d\alpha, \quad \varrho \geq 0.$$

Definition 9. *The fractional R-L family $P_\nu : [0,\infty) \to \mathbb{L}(\mathbb{Y})$ connected with C_ν is presented as*

$$P_\nu(\varrho) = J^{\nu-1}_{\varrho} C_\nu(\varrho).$$

Definition 10. [43] *"A function $x(\cdot) \in C(I;\mathbb{Y})$ is said to be the mild solution of* (11)-(12) *if it satisfies*

$$\begin{aligned} y(\varrho) =& C_\nu(\varrho)y_0 + S_\nu(\varrho)z_0 + \int_0^{\varrho} P_\nu(\varrho - \alpha)B(\alpha, v(\alpha))d\alpha \\ &+ \int_0^{\varrho} P_\nu(\varrho - \alpha) G\left(\alpha, x(s), \int_0^{\varrho} g(\varrho, \alpha, y(\alpha))d\alpha\right)\}ds \end{aligned} \tag{15}$$

where $S_\nu(\varrho)$ and $P_\nu(\varrho)$ are the fractional SF and fractional R-L family, respectively, associated with strongly continuous CF $C_\nu(\varrho)$ as defined above."

Assume that $\mathcal{P}$ be the set of all functions defined on I with $z(0) = y_0$, $z'(0) = z_0$, $z(b) = y_1$ and the fractional derivative ${}^{c}\mathbf{D}^{\nu}_{\varrho} z$ exists almost everywhere for $\nu \in (1,2]$. Note that $\mathcal{P}$ be the set of all feasible trajectories for (11)-(12).

We prove the trajectory controllability of the system (11)-(12) in two cases.

Case-I: If the control appears linearly:

Consider the system (11)-(12) with the control appears linearly, i.e. , $B(\varrho, v(\varrho)) = b(\varrho)v(\varrho)$, where $b : I\mathbb{R}$ and $u : I\mathbb{V}$, then the system (11)-(12) becomes

$$\begin{aligned} {}^{c}\mathbf{D}^{\nu}_{\varrho} y(\varrho) =& Ay(\varrho) + b(\varrho)v(\varrho) \\ &+ G\left(\varrho, y(\varrho), \int_0^{\varrho} g(\varrho, \alpha, y(\alpha))d\alpha\right), \\ &\varrho \in I, \nu \in (1,2], \end{aligned} \tag{16}$$

$$y(0) = y_0 \text{ and } y'(0) = z_0. \tag{17}$$

Assumptions [III].

(1) A generates a strongly continuous CF $\{C_\nu(\varrho) : t \geq 0\}$ on $\mathbb{Y}$.
(2) $b(\varrho)$ does not disappear on I.
(3) G and h are Lipschitz continuous in a similar manner as in Assumptions [I] of Section 2.

By referring these two assumptions, we may create the control explicitly to verify T-controllability of (16)-(17). For verifying this, we continue in the following way:

For every $v \in L_2(I,\mathbb{V})$, the existence and uniqueness of (16)-(17) follows from Assumptions [III] by employing the Lipschitz continuity of functions G and h.

Assume that $z(\varrho)$ be the given trajectory in $\mathcal{P}$. We present $v(\varrho)$ as

$$v(\varrho) = \frac{{}^{c}\mathbf{D}^{\nu}_{\varrho} z(\varrho) - Az(\varrho) - G\left(\varrho, z(\varrho), \int_0^{\varrho} g(\varrho, \alpha, z(\alpha))d\alpha\right)}{b(\varrho)}$$

By using this control, (16)-(17) becomes,

$$
\begin{aligned}
{}^c\mathbf{D}^\nu_\varrho y(\varrho) =& Ay(\varrho) + {}^c\mathbf{D}^\nu_\varrho z(\varrho) - Az(\varrho)\\
& - G\Big(\varrho, z(\varrho), \int_0^\varrho g(\varrho,\alpha,z(\alpha))d\alpha\Big)\\
& + G\Big(\varrho, y(\varrho), \int_0^\varrho g(\varrho,\alpha,y(\alpha))d\alpha\Big),\\
& \nu \in (1,2]\\
y(0) =& y_0 \text{ and } y'(0) = z_0.
\end{aligned}
$$

Fixing $w(\varrho) = y(\varrho) - z(\varrho)$, one can get

$$
\begin{aligned}
{}^c\mathbf{D}^\nu_\varrho w(\varrho) =& Aw(\varrho)\\
& + G\Big(\varrho, y(\varrho), \int_0^\varrho g(\varrho,\alpha,y(\alpha))d\alpha\Big)\\
& - G\Big(\varrho, z(\varrho), \int_0^\varrho g(\varrho,\alpha,z(\alpha))d\alpha\Big),\\
& \nu \in (1,2],\\
w(0) =& 0 \text{ and } w'(0) = 0.
\end{aligned}
$$

By using the semigroup theory, the solution of the above equation may be given by

$$
\begin{aligned}
w(\varrho) = \int_0^\varrho P_\nu(\varrho-\alpha)\Big[G\Big(&\alpha, y(\alpha),\\
\int_0^\alpha g(\alpha,\beta,y(\beta))d\beta\Big) &- G\Big(\alpha, z(\alpha),\\
\int_0^\alpha g(\alpha,\beta,z(\beta))d\beta\Big)\Big]d\alpha&
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|w(\varrho)\| \le& \frac{Mb^{\nu-1}}{\Gamma(\nu)}\int_0^\varrho \Big(\delta_1\|y(\alpha)-z(\alpha)\|\\
& + \delta_2\Big\|\int_0^\alpha g(\alpha,\beta,x(\beta))d\beta\\
& - \int_0^\alpha g(\alpha,\beta,z(\beta))d\beta\Big\|\Big)d\alpha\\
\le& \frac{Mb^{\nu-1}}{\Gamma(\nu)}\int_0^\varrho (\delta_1\|x(s)-z(s)\| + \delta_2\gamma\|x(\alpha)\\
& - z(\alpha)\|)d\alpha.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|y(\varrho)-z(\varrho)\| \le& \frac{Mb^{\nu-1}}{\Gamma(\nu)}(\delta_1+\delta_2\gamma)\\
& \int_0^\varrho \|y(\alpha)-z(\alpha)\|d\alpha.
\end{aligned}
$$

Using "Grownwall's inequality," we get

$$
\|y(\varrho)-z(\varrho)\| = 0.
$$

Therefore, $y(\varrho) = z(\varrho)$ for all $\varrho \in I$. This proves T-controllability of (16)-(17).

Case-II: If the control appears nonlinearly in (11)-(12):

In this case, trajectory controllability of the system (11)-(12) can be verified by the same approach discussed in Theorem 1 using the Lipschitz continuity of G and h and some additional assumptions on B, g, and f given in Assumptions [II] of Section 2.

4. Examples

In this section, we present numerical examples to support and illustrate the theoretical results.

Example 1. Assume that the following nonlinear integrodifferential system with control$b(\varrho, v) = v|v|$.

$$
\begin{aligned}
{}^c\mathbf{D}^\nu_\varrho y(\varrho) =& a(\varrho)y(\varrho) + B(\varrho, v(\varrho))\\
& + \sin\Big(y(\varrho) + 3\int_0^\varrho x(\alpha)d\alpha\Big), 0 < \nu \le 1,
\end{aligned}
\tag{18}
$$

$$
y(0) = y_0. \tag{19}
$$

The control $B(\varrho, v)$ is continuous and coercive. It is easy to show that G and h fulfill all the requirements of the Theorem 1. Therefore, the system (18) is T-controllable.

Example 2. Assume that $\Omega = (0,1)$ be the bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Assume that the subsequent fractional system

$$
\begin{aligned}
{}^c\mathbf{D}^\nu_\varrho y(\varrho) =& \frac{\partial^2 y}{\partial\varrho^2} + v(y,\varrho) + \frac{1}{2}[\sin^2 y(\varrho)\\
& + \sin y(\varrho)] \text{ in } \Omega\times(0,b), 0<\nu\le 1,
\end{aligned}
\tag{20}
$$

$$
z(y,0) = 0 \text{ in } \Omega, \tag{21}
$$

$$
z(y,\varrho) = 0 \text{ in } \partial\Omega\times(0,b). \tag{22}
$$

Let us define $A : L^2(0,1) \to L^2(0,1)$ by $Aw = w''$, where

$D(A) = \{w \in \mathbb{Y} : w, w'$ are absolutely continuous ,

$$
w(0) = w(1) = 0\}
$$

and

$$
Aw = \sum_{k=1}^{+\infty} k^2 < w, w_k > w_k, \quad w \in D(A),
$$

where $w_k(\alpha) = (2/\pi)^{1/2}\sin k\alpha$, $\quad k = 1,2,3\cdots$ is the orthogonal set of eigenfunctions of A. Here, $< w, w_k >$ stands for the L^2 inner product. Additionally, A generates a strongly continuous semigroup $\{\mathcal{S}(\varrho), \varrho \ge 0\}$ in $\mathbb{Y}$ presented as

$$
\mathcal{S}(\varrho)w = \sum_{k=1}^{+\infty} \exp(-k^2\varrho) < w, w_k > w_k, \quad w \in \mathbb{Y},
$$

and $\mathbb{Y} = \mathbb{V} = L^2(\Omega)$. The control term $B(\varrho, v(\varrho)) = v(\varrho)$ is linear. Also,

$$F(\varrho, y(\varrho), z(\varrho)) = \frac{1}{2}[\sin^2 y(\varrho) + \sin z(\varrho)]$$

and $G(\varrho, \alpha, y(\alpha)) = \frac{1}{2}[\cos y(\alpha)]$, both are Lipschitz continuous and satisfy all the assumptions. Therefore, using the results of Case-I of Section 2, we get that (20) is T-controllable.

Example 3. Consider the nonlinear fractional integro-differential system represented in the sense of a Caputo fractional derivative

$$\begin{aligned} {}^c\mathbf{D}_\varrho^\nu y(\varrho) =& 4y(\varrho) + v^2(\varrho) + 10\cos y(\varrho) \\ &+ 9\int_0^\varrho (\varrho + \alpha^2 + \sin y(\varrho))d\alpha, \\ &\varrho \in [0,1], \\ &y(0) = y_0, \ \ y(0) = \eta_0. \end{aligned} \tag{23}$$

where $\nu \in (1,2]$. The control function $Bv(\varrho) = v^2(\varrho)$ is continuous on $[0,1] \times \mathbb{R}$ and coercive. Further, the nonlinear functions $G\left(\varrho, y(\varrho), \int_0^\varrho g(\varrho, \alpha, y(\alpha))d\alpha\right) = 10\cos y(\varrho) + 9\int_0^\varrho (\varrho + \alpha^2 + \sin y(\varrho))d\alpha$ is Lipschitz with Lipschitz constants δ_1 and δ_2. All the hypotheses of Theorem 1 are satisfied. Therefore, the nonlinear fractional integro-differential system (3.1)-(3.2) is T-controllable at any value of $\nu \in (1,2]$ and y_0, $\eta_0 \in \mathbb{R}$.

5. Conclusion

Future research can explore T-controllability for fractional-order systems with delays, stochastic influences, or impulsive effects, extending the current framework to more complex real-world applications. Investigating optimal control strategies (see,[47][48]) for minimizing energy or cost during trajectory tracking could enhance practical implementation. Additionally, applying machine learning or data-driven techniques to improve control accuracy in fractional systems is a promising direction. Further studies on nonlinear fractional systems in Banach spaces and their applications in robotics, biomedical engineering, and financial modeling could provide valuable advancements in the field.

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