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RESEARCH ARTICLE

Analysis and analytical solution of incommensurate fuzzy fractional nabla difference systems in neural networks

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ABSTRACT

Uncertain incommensurate fractional nabla difference systems (IFDSs) in recurrent neural networks (RNNs) are analyzed using fuzzy number theory to address input uncertainties. Fuzzy number theory and its operations are reinvestigated, and the H-differenceable concept is introduced. The existence of a unique H-differenceable solution for incommensurate RNNs is proved. A recursive algorithm is proposed to obtain fuzzy solutions. Illustrative examples with 2-dimensional IFDSs are provided to validate the framework for integrating fractional calculus, fuzzy dynamics and incommensurate RNNs.

1. Introduction

Discrete equations offer notable advantages over continuous-based models in specific scenarios, such as digital signal processing and neural networks. For nonlinear discrete models, the recursive nature of solutions can, under appropriate conditions, eliminate the need for a separate existence analysis. In addition, it introduces recursive methods for obtaining exact or numerical solutions.[1–5]

Recent research has witnessed a growing interest in incommensurate fractional differential systems. However, the body of research on discrete incommensurate fractional difference systems remains relatively undiscovered (see Refs.[6–8] and references therein). Abbes et al.[9] explored an incommensurate fractional discrete macroeconomic system, Al-Taani et al.[10] investigated a discrete memristive model with incommensurate orders, and Shatnaei et al.[11] studied the stability of nonlinear incommensurate fractional order difference systems. Despite these advances, there remains a critical gap in addressing uncertainty in such systems, particularly when fuzzy inputs are involved, a limitation that motivates our work.

In this paper, we focus on a type of dense recurrent neural network (RNN) described by discrete incommensurate nabla fractional difference equations.

Let $x_i : \mathbb{N}_0 \to \mathbb{R}$, $i = 1, \ldots, \nu$, be the inputs of a neural network. These inputs can represent signals or image information. Let $w_{ij} \in \mathbb{R}$ denote the weights and p_i denote the bias terms. We consider an incommensurate fractional nabla

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RNN with ν neurons and activation functions q_i. These RNNs can be expressed as:

$$\begin{aligned}\nabla^{\alpha_1}x_1(t) =& q_1(z_1), \quad \alpha_1 \in (0,1],\\ &\vdots \\ \nabla^{\alpha_\nu}x_\nu(t) =& q_\nu(z_\nu), \quad \alpha_\nu \in (0,1],\end{aligned} \tag{1}$$

where $z_i = w_{i1}x_1(t-1) + \cdots + w_{i\nu}x_i(t-1) + p_i$. Typically, for $t > 0$, $x_i(t)$ represents the output at the time step t. Using vector notation, let $\vec{\alpha} = [\alpha_1, \ldots, \alpha_\nu]^T$, and define $\vec{p}$, $\vec{x}$, and $\vec{f}$ in a similar fashion. We can rewrite Equation (1) in vector form as

$$\nabla^{\vec{\alpha}}\vec{x}(t) = \vec{q}(W\vec{x}(t-1) + \vec{p}) \tag{2}$$

where $W = (w_{ij})$ is a $\nu \times \nu$ matrix and $\nabla^{\vec{\alpha}}$ is a diagonal matrix with the fractional nabla operators ∇^{α_i} on the diagonal, i.e., $\nabla^{\vec{\alpha}} = \text{Diag}[\nabla^{\alpha_1}, \ldots, \nabla^{\alpha_\nu}]$.

Remark 1. *Putting $\vec{\alpha} = [0, \ldots, 0]^T$, we obtain*

$$\vec{x}(t) = \vec{q}(W\vec{x}(t-1) + \vec{p}) \tag{3}$$

which is a layer of classical NN with ν neurons. Putting $\vec{\alpha} = [1, \ldots, 1]^T$, we get

$$\vec{x}(t) = \vec{x}(t-1) + \vec{q}(W\vec{x}(t-1) + \vec{p}) \tag{4}$$

which is equivalent to an RNN. Therefore, for $\alpha_i \in (0,1)$, it is an RNN with memory. Such NNs are widely used in data classification.[12]

If in System Equation (1) all orders are the same, i. e., $\alpha_i = \alpha$ for all $i = 1, \ldots, \nu$, then it is a commensurate system. In this case $\vec{\alpha} = [\alpha, \ldots, \alpha]^T$. Commensurate systems are much easier to investigate than incommensurate systems, since in a commensurate system

$$\nabla^{\vec{\alpha}}W\vec{x}(t) = W\nabla^{\vec{\alpha}}\vec{x}(t),$$

while this equality does not hold for an incommensurate system.

The central problem addressed in this paper is how fuzzy-valued inputs affect the outputs. This represents a problem of uncertainty.

Interval analysis, stochastic analysis, and fuzzy theory have proven to be valuable tools for analyzing uncertainty. In particular, statistical analysis has not yet been employed for uncertainty analysis in incommensurate fractional differential/difference systems. In contrast, fuzzy theory and interval analysis have been utilized in a limited number of studies. For example,[13,14] harnessed fuzzy theory to investigate state uncertainties in incommensurate fractional nabla difference systems (IFDSs). Refs.[15,16] applied interval analysis to address the uncertainties of the parameters. Moreover, Ref.[17] explored a fractional PI observer for IFDSs with parametric uncertainties.

However, for commensurate fractional differential equations, numerous studies[18–20] have established a robust framework for fuzzy theory and uncertainty analysis.

In this paper, we use fuzzy theory to address the uncertainty in IFDSs that arise in the dense RNN described by Equation (2).

We apply the state-of-the-art method of converting a fuzzy number to an interval using r-cuts. In this regard, we carefully define what a fuzzy number is in relation to the r-cut representation. Then, we obtain the corresponding arithmetic operations using Zadeh's extension theory. However, we find that for operations such as the difference, there are inconsistencies. Also, the generalized H-difference suffers from other inconsistencies. To overcome such inconsistencies, we introduce the concept of H-differenceability. Similarly, we extend it to the fuzzy nabla fractional difference operation.

After obtaining clear definitions, we analyze the fuzzy incommensurate neural network. This analysis shows that it has a unique H-differenceable solution. Finally, we introduce an algorithm to compute the fuzzy solution using a recursive formula.

The novelty of this paper is highlighted as follows:

(1) We developed a new RNN model based on fractional nabla operations.
(2) We refine the definition of fuzzy numbers and present the H-differenceable concept.
(3) We prove the existence of a unique H-differenceable solution for incomensurate RNNs with fuzzy input.
(4) We develop a recursive algorithm for calculating fuzzy solutions.

For clarity, we provide tables of mathematical notations, notation spaces, and abbreviations in Tables 1-3, respectively.

In Section 2, we review the concept of fuzzy numbers, along with a minor correction to some other available definitions that distinguish fuzzy numbers from fuzzy sets. We present the corresponding transforms with r-cats concerning interval analysis. In Section 3, fuzzy arithmetic is reviewed in connection with shape functions and r-cats, and the nabla fractional difference is extended to fuzzy numbers. In Section 4, we demonstrate that fuzzy IFDSs for RNNs admit a unique solution. In Section 5, we detail the computation of the fuzzy solution for IFDSs. Finally, we provide an illustrative example in Section 6.

Table 1. Table of mathematical notations

Notation	Description
$\nabla^{\alpha}x(t)$	Fractional nabla difference of order α
$\nabla^{-\alpha}x(t)$	Fractional nabla sum of order α
$\vec{\alpha}$	Vector of fractional orders: $[\alpha_1, \ldots, \alpha_\nu]^T$
$\vec{x}(t)$	Vector of variables: $[x_1(t), \ldots, x_\nu(t)]^T$
W	Weight matrix of size $\nu \times \nu$
$\vec{p}$	Bias vector
$\vec{q}$	Vector of activation functions: $[q_1, \ldots, q_\nu]^T$
$\vec{\oplus}$	Element-wise fuzzy sum
$\vec{\ominus}$	Element-wise H-difference
(d, f_1, f_2)	Triple representation of a fuzzy number
$C_\mu(r)$	r-cut of the fuzzy number μ
$\Gamma(\cdot)$	Gamma function
$sgn(\cdot)$	Sign function

Table 2. Table of notations for spaces

Notation	Description
$\mathbb{R}$	Set of real numbers
$\mathbb{N}_a$	Set of natural numbers starting from a
$\mathbb{R}_F$	Set of fuzzy numbers
$\mathbb{U}$ that are m.d.c.	Set of functions $f : [0,1] \to \mathbb{R}^+$ with $f(1) = 0$
$\mathbb{S}$	Set of shape functions with additional conditions on $f(0)$
$\mathbb{S}^{-1}$	Set of inverse shape functions

Table 3. Table of abbreviations

Abbreviation	Description
IFDSs	Incommensurate fuzzy fractional nabla difference systems
RNN	Recurrent Neural Network
NN	Neural Network
H-difference	Hukuhara Difference
GH-difference	Generalized Hukuhara Difference
m.d.c.	Monotonically Decreasing and Continuous
m.i.c.	Monotonically Increasing and Continuous
SGD	Stochastic Gradient Descent

2. What is a fuzzy number?

There is no standardized definition of a fuzzy number. Several authors[21,22] identify two types of definitions relevant to interval analysis: those featuring continuous membership functions and those with semi-continuous membership functions. According to the critiques in Ref.[21], a trapezoidal membership function does not qualify as a fuzzy number. It assigns multiple values without associated uncertainty. Thereby, it represents a set rather than a number.

A key condition for a fuzzy set to qualify as a fuzzy number is that its membership function is normal. That is, there must exist a unique element for which the membership function value is equal to 1. Significantly, this normal element must be unique. This uniqueness specifically prevents the fuzzy number from taking the form of a trapezoidal membership function. A trapezoidal function has multiple elements with constant membership values over an interval, violating the uniqueness of the normal component. As a result, the classical definitions of a fuzzy number are refined in the following manner:

Definition 1. *A membership function $\mu : \mathbb{R} \to [0,1]$ is a fuzzy number ($\mu \in \mathbb{R}_{\mathcal{F}}$), if:*

Normal:: *There exists a unique $d \in \mathbb{R}$ such that $\mu(d) = 1$.*

Compact support:: $\text{Supp}(\mu) = \{w : \mu(w) > 0\}$ *is compact.*

Convexity: *$\mu(x)$ is a convex down function (concave function) that means*

$$\mu(tw_1 + (1-t)w_2) \geq t\mu(w_1) + (1-t)\mu(w_2)$$

for all $w_1, w_2 \in \mathbb{R}$ and $t \in [0,1]$.

Upper continuous: *μ is right continuous on $(-\infty, d)$ and left continuous on (d, ∞).*

The r-cut set of a fuzzy number is defined as

$$C_\mu(r) = \{w | \mu(w) \geq r\}, \ r \in [0,1].$$

The boundaries of these sets are defined by

$$C^*_\mu(r) = \sup C_\mu(r),$$

and

$$C_{\mu*}(r) = \inf C_\mu(r).$$

Theorem 1. *Let μ be a fuzzy number. Then,*

(1) $d \in C_\mu(r)$ and $C_\mu(r) \neq \emptyset$ for all $r \in [0,1]$.
(2) $\exists\ d \in \mathbb{R}$ such that $C_\mu(1) = \{d\}$.
(3) $C_\mu(r_2) \subseteq C_\mu(r_1)$ for $r_1 \leq r_2$.
(4) $\overline{C_\mu(r)}$ is a closed and bounded interval for all $r \in [0,1]$, particularly $\overline{C_\mu(r)} = [C_{\mu*}(r), C^*_\mu(r)]$.
(5) $C_{\mu*} : [0,1] \to \mathbb{R}$ and $C^*_\mu : [0,1] \to \mathbb{R}$ are well-defined functions.

(6) $C_{\mu*}$ is a monotonically increasing function and C^*_μ is a monotonically decreasing function.
(7) $C_{\mu*}$ and C^*_μ are continuous functions.
(8) $C_{\mu*}$ and C^*_μ are bounded and $C_{\mu*}(r) \le C^*_\mu(r)$ for all $r \in [0,1]$.
(9) $C_{\mu*}(1) = C^*_\mu(1) = d$.

Proof. 1. Since μ is normal, there exists a unique $d \in \mathbb{R}$ such that $\mu(d) = 1$. For any $r \in [0,1]$, because $1 \ge r$, we have $\mu(d) \ge r$, so $d \in C_\mu(r)$. Also, since there is at least one element (i. e., d) in $C_\mu(r)$, $C_\mu(r) \ne \emptyset$. 2. By the definition of a fuzzy number, there is a unique element $d \in \mathbb{R}$ for which $\mu(d) = 1$. So, when $r = 1$, $C_\mu(1) = \{x \in \mathbb{R} : \mu(x) \ge 1\} = \{d\}$. 3. Let $r_1 \le r_2$. If $x \in C_\mu(r_2)$, then $\mu(x) \ge r_2$. Since $r_2 \ge r_1$, we have $\mu(x) \ge r_1$, which implies $x \in C_\mu(r_1)$. Thus, $C_\mu(r_2) \subseteq C_\mu(r_1)$. 4. Since μ is a convex function (by the definition of a fuzzy number), for any $r \in [0,1]$, the set $C_\mu(r)$ is convex. In $\mathbb{R}$, convex sets are intervals. The support $\mathrm{Supp}(\mu)$ is compact. Since $C_\mu(r) \subseteq \mathrm{Supp}(\mu)$ for all $r \in [0,1]$, the closure $C_\mu(r)$ is a closed and bounded interval. The endpoints of this interval are precisely $C_{\mu*}(r) = \inf C_\mu(r)$ and $C^*_\mu(r) = \sup C_\mu(r)$, so $C_\mu(r) = [C_{\mu*}(r), C^*_\mu(r)]$. 5. From 4, for each $r \in [0,1]$, the infimum $C_{\mu*}(r)$ and supremum $C^*_\mu(r)$ are well-defined real numbers. So, $C_{\mu*} : [0,1] \to \mathbb{R}$ and $C^*_\mu : [0,1] \to \mathbb{R}$ are well-defined functions. 6. Let $r_1 \le r_2$. From 3, $C_\mu(r_2) \subseteq C_\mu(r_1)$. Then, $\inf C_\mu(r_2) \ge \inf C_\mu(r_1)$ (because a subset cannot have a smaller infimum than the superset), so $C_{\mu*}(r_2) \ge C_{\mu*}(r_1)$, which means that $C_{\mu*}$ increases monotonically. Similarly, $\sup C_\mu(r_2) \le \sup C_\mu(r_1)$, so $C^*_\mu(r_2) \le C^*_\mu(r_1)$, which means that C^*_μ is monotonically decreasing. 7. Let $r_0 \in [0,1]$ and $\epsilon > 0$. Since μ is upper continuous, for a small $\delta > 0$, if $|r - r_0| < \delta$, the change in the set $C_\mu(r)$ is small. Let $r < r_0$. Then $C_\mu(r) \supseteq C_\mu(r_0)$. As $r \to r_0^-$, the infimum $C_{\mu*}(r)$ approaches $C_{\mu*}(r_0)$. Similarly, for $r > r_0$, as $r \to r_0^+$, $C_{\mu*}(r)$ approaches $C_{\mu*}(r_0)$. The proof for the continuity of C^*_μ is similar, using the upper continuity of μ and the properties of the supremum of the r-cuts. 8. From 4, $C_\mu(r) = [C_{\mu*}(r), C^*_\mu(r)]$ is a bounded interval for all $r \in [0,1]$. So, $C_{\mu*}$ and C^*_μ are bounded functions. Also, by the definition of infimum and supremum, $C_{\mu*}(r) \le C^*_\mu(r)$ for all $r \in [0,1]$. 9. From 2, $C_\mu(1) = \{d\}$. Then, $\inf C_\mu(1) = C_{\mu*}(1) = d$ and $\sup C_\mu(1) = C^*_\mu(1) = d$. □

Remark 2. *In comparison, previous definitions do not satisfy properties 2 and 9. This paper presents a minor modification that addresses this shortcoming.*

Equivalently, the boundaries of an r-cut can define a fuzzy number, leading to the following equivalent definition.

Theorem 2. *Let the functions $C_{\mu*} : [0,1] \to \mathbb{R}$ and $C^*_\mu : [0,1] \to \mathbb{R}$ satisfy properties 5–9 of Theorem 1. Then, $\mu : \mathbb{R} \to [0,1]$ defined by*

$$\mu(w) = \begin{cases} \sup\{r : C^*_\mu(r) \ge w\}, & w \ge d, \\ \sup\{r : C_{\mu*}(r) \le w\}, & w \le d, \end{cases} \tag{5}$$

is a fuzzy number.

Proof. It suffices to prove that the function μ satisfies the conditions of Definition Equation (1). The proof is rather straightforward. □

Due to the symmetry of the translated functions $f_1 = d - C_{\mu*}$ and $f_2 = C^*_\mu - d$, we can simplify the fuzzy definition by defining it in a decomposed form, similar to the approach in[21] and in parallel with the interval analysis. To do this, let $d \in \mathbb{R}$. Define the set $\mathbb{U}$ as:

$$\mathbb{U} = \{f : [0,1] \to \mathbb{R}^+ : f \text{ is m. d. c. on } [0,1], \\ f(1) = 0\}.$$

where "m. d. c." abbreviates "monotonically decreasing and continuous".

Theorem 3. *[21] Let C^*_μ and $C_{\mu*}$ be functions from $[0,1]$ to $\mathbb{R}$. Then, $f_1, f_2 \in \mathbb{U}$ if and only if the functions $C^*_\mu = f_2 + d$ and $C_{\mu*} = d - f_1$ satisfy the properties specified in conditions 5–9 of Theorem 1.*

Remark 3. *Based on Theorem 3, a triple (d, f_1, f_2) represents a fuzzy number. To denote the equivalence between this triple and a fuzzy number μ, we use the symbol $\sim$, expressed as:*

$$\mu \sim (d, f_1, f_2).$$

Due to the assumption of compact support, we can impose additional assumptions on the set $\mathbb{U}$. To leverage the results in Ref.[23] for defining arithmetic operations, we impose additional conditions on the set $\mathbb{U}$ and define a new set $\mathbb{S}$ of shape functions as follows:

$$\mathbb{S} = \{f : [0,1] \to \mathbb{R}^+ : f \text{ is m. d. c. on } [0,1], \\ f(1) = 0, f(0) = 1 \text{ or } f \equiv 0\}, \tag{6}$$

where "m. d. c." abbreviates "monotonically decreasing and continuous".

Theorem 4. *Suppose* $\mathrm{Supp}(\mu) = [a,b]$*. For each $\mu \sim (d, f_1, f_2)$, there exist $\tilde{f}_1, \tilde{f}_2 \in \mathbb{S}$ such that $f_1 = (d-a)\tilde{f}_1$ and $f_2 = (b-d)\tilde{f}_2$.*

Proof. If $d - a \ne 0$, define $\tilde{f}_1 = \frac{f_1}{d-a}$; otherwise, set $\tilde{f}_1 = 0$. If $b - d \ne 0$, define $\tilde{f}_2 = \frac{f_2}{b-d}$; otherwise, set $\tilde{f}_2 = 0$. □

Finally, we recall the shape function from Ref.[23]. Let

$$\mathbb{S}^{-1} = \{L : \mathbb{R}^+ \to [0,1] : L \text{ is upper semi-continous on } \mathbb{R}^+, L(0) = 1, L(1) = 0\}.$$

Any fuzzy number μ_{LR} with $\mathrm{Supp}(\mu_{LR}) = [a, b]$ can be described by L-R functions as

$$\mu_{LR}(x) = \begin{cases} L(\frac{d-w}{d-a}), & w \le d, \\ R(\frac{w-d}{b-d}), & w \ge d, \end{cases} \quad (7)$$

where $R, L \in \mathbb{S}^{-1}$.

For any such representation, there exist $\tilde{f}_1, \tilde{f}_2 \in \mathbb{S}$ such that

$$\tilde{f}_1(L(w)) = \tilde{f}_2(R(w)) = w. \quad (8)$$

Thus,

$$\mu \sim (d, (d-a)\tilde{f}_1, (b-d)\tilde{f}_2).$$

Furthermore, if $r = \mu_{LR}(x)$, then

$$w = d - (d-a)\tilde{f}_1(r), \quad x \le d,$$

and

$$w = d + (b-d)\tilde{f}_2(r), \quad x \ge d.$$

This forms the foundation for converting the result from Ref.[23] to a symmetric representation.

Defining arithmetic operations for fuzzy numbers is complex, relying on Zadeh's extension principle.[24] Each arithmetic operation on two fuzzy numbers requires solving an optimization problem.[25–27] Considerable research has aimed to derive explicit formulas for arithmetic operations. For example, Ref.[27] reviews arithmetic definitions for triangular fuzzy numbers consistent with Zadeh's extension principle. In this paper, we adopt the classical definitions based on the symmetrical representation.[21,22]

2.1. Fuzzy operation

First, we recall the operations between fuzzy numbers and scalars.

Definition 2. *Let $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}_{\mathcal{F}}$. Then,*

- *Scalar Multiplication/Division: $(\lambda\mu)(w) := \mu(w/\lambda)$. Equivalently,*

$$\lambda(d, f_1, f_2) = \begin{cases} (\lambda d, \lambda f_1, \lambda f_2), & \lambda \ge 0, \\ (\lambda d, |\lambda| f_2, |\lambda| f_1), & \lambda < 0. \end{cases} \quad (9)$$

- *Scalar Summation and Difference: $(\lambda \pm \mu)(w) := \mu(w \pm \lambda)$. Equivalently,*

$$\lambda \pm (d, f_1, f_2) = (\lambda \pm d, f_1, f_2).$$

Suppose μ and η are two fuzzy numbers. Then, according to Zadeh's extension principle, the elementary operations of these two fuzzy numbers are defined by

$$(\mu \oplus \ominus \otimes \eta)(w) = \sup_{x+-\times y = w} \min\{\mu(x), \eta(y)\}. \quad (10)$$

Obviously, Definition Equation (10) is not practical. Therefore, we try another approach. We aim to make it consistent with Zadeh's extension principle and previous definitions for as large a class of fuzzy numbers as possible.

Let $\mu \sim (d, f_1, f_2)$ and $\eta \sim (e, g_1, g_2)$. Then,

$$\mu \oplus \eta \sim (d+e, f_1+g_1, f_2+g_2). \quad (11)$$

This definition aligns with Equation (10) for triangular fuzzy numbers. Dubois et al.[23] explicitly defined arithmetic operations for fuzzy numbers of the same type using (L-R) shape functions based on Zadeh's extension theory. They derived

$$\mu_{LR}(w) \oplus \eta_{LR}(w) = \begin{cases} L(\frac{e+d-w}{e-c+d-a}), & w \le e+d, \\ R(\frac{w-e+d}{h-e+b-d}), & w \ge e+d, \end{cases}$$
$$\sim (e+d, (e-c+d-a)\tilde{f}_1, (h-e+b-d)\tilde{f}_2)$$
$$= (d+e, f_1+g_1, f_2+g_2), \quad (12)$$

where $f_1 = (d-a)\tilde{f}_1$, $f_2 = (b-d)\tilde{f}_2$, $g_1 = (e-c)\tilde{f}_1$, $g_2 = (h-e)\tilde{f}_2$, and

$$\eta_{LR}(x) = \begin{cases} L(\frac{e-w}{e-c}), & w \le e, \\ R(\frac{w-e}{h-e}), & w \ge e. \end{cases} \quad (13)$$

In the given mathematical context, the concept of subtraction presents a thorny issue due to inconsistent definitions. Suppose we define the operation $\zeta = \mu \ominus \eta$ as $\zeta = \mu \oplus (-\eta)$. A simple verification shows that $\zeta \oplus \eta \ne \mu$, immediately indicating a deviation from the expected behavior of subtraction.

The complexity escalates with Zadeh's extension theory, which defines

$$\mu \ominus \eta \sim (d-e, f_1+g_2, f_2+g_1).$$

Curiously, this still leads to $\zeta = \mu \oplus (-\eta)$, recreating the same paradox.

In interval theory, Hukuhara and generalized Hukuhara differences (H and GH differences) have been proposed to address this problem.[28] However, a major drawback of these definitions is that they may not be well-defined for arbitrary fuzzy numbers.

We propose using the term "differenceable" when the difference of two fuzzy numbers exists. In the study of discrete dynamic systems, H-differenceable fuzzy numbers appear to play a significant role.[21] Clearly, the H-difference $\mu \ominus \eta$ is differenceable if and only if $f_i - g_i \ge 0$ and thus

$$\zeta = \mu \ominus \eta \sim (d-e, f_1-g_1, f_2-g_2). \quad (14)$$

Theorem 5. *If $\eta \oplus \zeta = \mu$, then the H-differences $\mu \ominus \zeta$ and $\mu \ominus \eta$ are differenceable and equal to η and ζ, respectively.*

Proof. The proof follows directly from Equations (11) and (14). □

Theorem 6. *Let $\mu, \eta, \zeta, \psi \in \mathbb{R}_{\mathcal{F}}$, and assume that $\mu \ominus \eta = \zeta$ is H-differenceable. Then:*

(1) $\zeta \oplus \eta = \eta \oplus \zeta = \mu$;
(2) $\eta \ominus \mu$ is not H-differenceable;
(3) $\mu \ominus \eta \oplus \psi = \mu \oplus \psi \ominus \eta$.

Proof. The proof follows directly from Equations (11), (14), and the definition of $\mathbb{R}_{\mathcal{F}}$. □

The multiplication is defined as:

$$\mu_{LR}(w) \otimes \eta_{LR}(w) = \begin{cases} L\left(\frac{ed-w}{d(e-c)+e(d-a)}\right), & w \leq e+d, \\ R\left(\frac{w-ed}{d(h-e)+e(b-d)}\right), & w \geq e+d, \end{cases}$$
$$\sim (d(e-c)+e(d-a))\tilde{f}_1, (ed, (d(h-e)+e(b-d))\tilde{f}_2)$$
$$\sim (de, ef_1+dg_1, ef_2+dg_2). \tag{15}$$

Let $g : \mathbb{R} \to \mathbb{R}$ be a function. It can be extended to a fuzzy function $g : \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ by the extension theorem:

$$(g(\mu))(w) = \sup_{x:g(x)=w} \{\mu(x), 0\}. \tag{16}$$

Fortunately, Equation (16) can be simplified for one-to-one functions. In this case, $(q(\mu))(w)$ is equal to

$$\begin{cases} \mu(q^{-1}(w)), & \text{if } q^{-1}(w) \text{exists \& } q^{-1}(w) \in \operatorname{Supp}(\mu), \\ 0, & \text{oth.} \end{cases} \tag{17}$$

It can be easy to verify that for $f(x) = \lambda x$, the Definition Equations (9) and (17) are consistent.

Most activation functions in neural networks (NN) are m. i. c. functions such as the Sigmoid function, rectified linear unit and hyperbolic tangent functions. To find a symmetric form of Equation (17), m. i. c. functions play an important role in achieving some interesting simplifications. Activation functions change the shape of the input fuzzy number according to the following theorems.

Theorem 7. *Let q be an invertible m. i. c. function and $\mu \sim (d, (b-a)f_1, (d-a)f_2)$ where $f_1, f_2 \in \mathbb{S}$, $a \leq d \leq b$ such that $[a, b] = \operatorname{Supp}(\mu)$. Then,*

$$q(\mu) \sim (q(d), (q(d)-q(a))\hat{f}_1, (q(b)-q(d))\hat{f}_2), \tag{18}$$

where

$$\hat{f}_1(r) = \frac{q(d) - q(d-(d-a)f_1(r))}{q(d)-q(a)}, \tag{19}$$

and

$$\hat{f}_2(r) = \frac{q(d+(b-d)f_2(r)) - q(d)}{q(b)-q(d)}. \tag{20}$$

Proof. Since q is a m. i. c. function, from Equation (17), we have

$$r := q(\mu(w)) = \begin{cases} L\left(\frac{d-q^{-1}(w)}{d-a}\right), & w \leq q(d), \\ R\left(\frac{q^{-1}(w)-d}{b-d}\right), & w \geq q(d), \end{cases} \tag{21}$$

and $\operatorname{Supp}(q(\mu)) = [q(a), q(b)]$.

Then, from Equation (8), we get:

$$f_2(r) = f_2\left(R\left(\frac{q^{-1}(w)-d}{b-d}\right)\right) = \frac{q^{-1}(w)-d}{b-d},$$

for $w \geq d$ and

$$f_1(r) = f_1\left(L\left(\frac{d-q^{-1}(w)}{d-a}\right)\right) = \frac{d-q^{-1}(w)}{d-a},$$

for $w \leq d$. By rearranging the equations from the previous step, we can rewrite them equivalently as:

$$w = q(d+(b-d)f_2(r)), \quad q(w) \geq q(d),$$

and

$$w = q(d-(d-a)f_1(r)), \quad q(w) \leq q(d).$$

On the other hand, by the definition of $\hat{f}_1$ and $\hat{f}_2$, we should have:

$$w = q(d) + (q(b)-q(d))\hat{f}_2(r), \quad q(w) \geq q(d),$$

and

$$w = q(d) - (q(d)-q(a))\hat{f}_1(r), \quad q(w) \leq q(d).$$

By comparing these equations, we can conclude that:

$$q(d) + (q(b)-q(d))\hat{f}_2(r) = q(d+(b-d)f_2(r)),$$

and

$$q(d) - (q(d)-q(a))\hat{f}_1(r) = q(d-(d-a)f_1(r)).$$

□

Corollary 1. *Let q be invertible m. i. c. function and $\mu \sim (d, f_1, f_2)$ where $f_1, f_2 \in \mathbb{U}$. Also, assume $a \leq d \leq b$ such that $[a, b] = \operatorname{Supp}(\mu)$. Then,*

$$q(\mu) \sim (q(d), \hat{f}_1, \hat{f}_2) \tag{22}$$

where

$$\hat{f}_1(r) = q(d) - q(d - f_1(r)), \tag{23}$$

and

$$\hat{f}_2(r) = q(d + f_2(r)) - q(d). \tag{24}$$

2.2. Nabla fractional differences and sum

There are different definitions for nabla fractional differences and sums. We follow the definitions in.[21]

Definition 3. *Let $x : \mathbb{N}_a \to \mathbb{R}$, $\alpha > 0$ and $N = \operatorname{ceil}(\alpha)$. Then, the fractional nabla sum is defined as*

$$\nabla^{-\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t} \frac{\Gamma(t-s+\alpha)}{\Gamma(t-s+1)} x(s),$$

and the fractional nabla difference is defined as

$$\nabla^{\alpha}x(t) = \nabla^{N}\nabla^{-(N-\alpha)}x(t), \quad t \in \mathbb{N}_{a+N},$$

where $\nabla x(t) = x(t) - x(t-1)$, *and* $\mathbb{N}_a = \{a, a+1, \cdots\}$.

Definition 4. *Let* $\alpha \in (0,1]$ *and* $x : \mathbb{N}_a \to \mathbb{R}_{\mathcal{F}}$. *Then, the fractional nabla H-difference is defined as*

$$\begin{aligned}\nabla^{\alpha}x(t) =&\nabla\nabla^{-(1-\alpha)}x(t)\\ =&\nabla^{-(1-\alpha)}x(t) \ominus \nabla^{-(1-\alpha)}x(t-1),\end{aligned} \tag{25}$$

for $t \in \mathbb{N}_{a+1}$. *If Equation* (25) *is feasible, we say it is fractional nabla H-differenceable.*

It is useful to decompose $\nabla^{-(1-\alpha)}x(t)$ into peak and tail operators.

$$\begin{aligned}\nabla^{-(1-\alpha)}x(t) =& \frac{1}{\Gamma(1-\alpha)}\sum_{s=0}^{t}\frac{\Gamma(t-s+(1-\alpha))}{\Gamma(t-s+1)}x(s)\\ =&x(t) \oplus \frac{1}{\Gamma(1-\alpha)}\sum_{s=0}^{t-1}\frac{\Gamma(t-s+(1-\alpha))}{\Gamma(t-s+1)}x(s).\end{aligned} \tag{26}$$

Thus, $x(t)$ is the peak operator and

$$T^{-(1-\alpha)}x(t-1) := \sum_{s=0}^{t-1}\frac{\Gamma(t-s+(1-\alpha))}{\Gamma(1-\alpha)\Gamma(t-s+1)}x(s),$$

is the tail operator. We can rewrite Equation (26) as

$$\nabla^{-(1-\alpha)}x(t) = x(t) \oplus T^{-(1-\alpha)}x(t-1). \tag{27}$$

3. Analysis of solution

The first question is whether there exists an H-differenceable solution for System Equation (1) if $x_i(0) \in \mathbb{R}_{\mathcal{F}}$. We build a constructive method to obtain the fuzzy solution of Equation (2) based on the H-difference operator. Equation (2) for fuzzy inputs is

$$\nabla^{\vec{\alpha}}\vec{x}(t) = \vec{q}(W\vec{x}(t-1)\vec{\oplus}\vec{p}). \tag{28}$$

Using Equation (25), we obtain

$$\nabla^{-(1-\vec{\alpha})}\vec{x}(t)\vec{\ominus}\nabla^{-(1-\vec{\alpha})}\vec{x}(t-1) = \vec{q}(W\vec{x}(t-1)\vec{\oplus}\vec{p}). \tag{29}$$

Since the fuzzy difference is Hukuhara type, we can rewrite

$$\nabla^{-(1-\vec{\alpha})}\vec{x}(t) = \nabla^{-(1-\vec{\alpha})}\vec{x}(t-1)\vec{\oplus}\vec{q}(W\vec{x}(t-1)\vec{\oplus}\vec{p}) \tag{30}$$

where $\vec{\oplus}$ and $\vec{\ominus}$ stand for the element-wise fuzzy sum and H-difference, respectively. Using the decomposition in Equation (27), we get

$$\begin{aligned}&\vec{x}(t)\vec{\oplus}T^{-(1-\vec{\alpha})}\vec{x}(t-1) =\\ &\quad\nabla^{-(1-\vec{\alpha})}\vec{x}(t-1)\vec{\oplus}\vec{q}(W\vec{x}(t-1)\vec{\oplus}\vec{p}).\end{aligned} \tag{31}$$

Theorem 8. *Suppose* $x_i \in \mathbb{R}_{\mathcal{F}}$, $i = 1, \ldots, \nu$. *Then, H-difference*

$$\mathfrak{O}^{-(1-\vec{\alpha})}\vec{x}(t-1) = \nabla^{-(1-\vec{\alpha})}\vec{x}(t-1)\ominus T^{-(1-\vec{\alpha})}\vec{x}(t-1)$$

is feasible and

$$\begin{aligned}&\mathfrak{O}^{-(1-\vec{\alpha})}\vec{x}(t-1) =\\ &\frac{\vec{\alpha}}{\Gamma(1-\vec{\alpha})}\sum_{s=0}^{t-1}\left(\frac{\Gamma(t-s-\vec{\alpha})}{\Gamma(t-s+1)}\right)\vec{x}(s).\end{aligned} \tag{32}$$

Proof. First, we observe that for $s = 0, \cdots, t-1$,

$$\frac{(t-s-\alpha_i)\Gamma(t-s-\alpha_i)}{(t-s)\Gamma(t-s)} < \frac{\Gamma(t-s-\alpha_i)}{\Gamma(t-s)},$$

and thus

$$0 \leq \frac{\Gamma(t+1-s-\alpha_i)}{\Gamma(t+1-s)} < \frac{\Gamma(t-s-\alpha_i)}{\Gamma(t-s)}.$$

Therefore, the H-differences

$$\frac{1}{\Gamma(1-\alpha_i)}\frac{\Gamma(t-s-\alpha_i)}{\Gamma(t-s)}x_i(s)\ominus \frac{1}{\Gamma(1-\alpha_i)}\frac{\Gamma(t+1-s-\alpha_i)x_i(s)}{\Gamma(t+1-s)}$$

are feasible for $s = 0, \cdots, t-1$. Especially, their sum is also nabla H-differenceable and

$$\begin{aligned}&\mathfrak{O}^{-(1-\vec{\alpha})}x_i(t-1)\\ =&\frac{1}{\Gamma(1-\alpha_i)}\sum_{s=0}^{t-1}\left(\frac{\Gamma(t-s-\alpha_i)}{\Gamma(t-s)}\right.\\ &\left.-\frac{\Gamma(t+1-s-\alpha_i)}{\Gamma(t+1-s)}\right)x_i(s)\\ =&\frac{1}{\Gamma(1-\alpha_i)}\sum_{s=0}^{t-1}\left(\frac{\Gamma(t-s-\alpha_i)}{\Gamma(t-s)}\right.\\ &\left.(1-\frac{(t-s-\alpha_i)}{(t-s)})\right)x_i(s)\\ =&\frac{\alpha_i}{\Gamma(1-\alpha_i)}\sum_{s=0}^{t-1}\frac{\Gamma(t-s-\alpha_i)}{\Gamma(t-s+1)}x_i(s).\end{aligned} \tag{33}$$

□

Taking into account Theorem 8, System Equation (31) can be written as

$$\vec{x}(t) =\mathfrak{O}^{-(1-\vec{\alpha})}\vec{x}(t-1)\vec{\oplus}\vec{q}(W\vec{x}(t-1)\vec{\oplus}\vec{p}). \tag{34}$$

Since Equation (34) expresses $\vec{x}(t)$ in terms of $\vec{x}(t-1)$, $\vec{x}(t)$ can be obtained recursively.

Theorem 9. *System Equation* (28) *subject to input* $\vec{x}(0) = \vec{x}_0$ *has a unique H-difference solution.*

Proof. We start with $t = 1$ to initiate the induction process. Then,

$$\vec{x}(1) = \nabla^{-(1-\vec{\alpha})}\vec{x}(0)\vec{\ominus}T^{-(1-\vec{\alpha})}\vec{x}(0)\vec{\oplus}\vec{q}(W\vec{x}(0)\vec{\oplus}\vec{p}). \tag{35}$$

From Theorem 8,

$$\nabla^{-(1-\vec{\alpha})}\vec{x}(0)\vec{\ominus}T^{-(1-\vec{\alpha})}\vec{x}(0) = \vec{\alpha}\vec{x}_0$$

(Here, all vector operations of the same type are element-wise). Thus, $\vec{x}(1)$ exists and satisfies

$$\vec{x}(1) = \vec{\alpha}\vec{x}_0 \vec{\oplus} \vec{q}(W\vec{x}_0 \vec{\oplus} \vec{p}). \tag{36}$$

Therefore, $\vec{x}(1)$ is unique because $\vec{x}_0$ is unique. We complete the proof by induction. Suppose there exists a unique $\vec{x}(i)$ for $i < N$, satisfying Equation (2). We show $\vec{x}(N)$ exists and is unique. From Equation (34)

$$\vec{x}(N) = \left(\frac{\vec{\alpha}}{\Gamma(1-\vec{\alpha})}\sum_{s=0}^{N-1}\left(\frac{\Gamma(N-s-\vec{\alpha})}{\Gamma(N-s+1)}\right)\vec{x}(s)\right) \vec{\oplus} \vec{q}(W\vec{x}(N-1)\vec{\oplus}\vec{p}). \tag{37}$$

The right hand side of Equation (37) is a fuzzy combination of $\vec{x}(0), \vec{x}(1), \ldots, \vec{x}(N-1)$. Since they are well-defined fuzzy numbers by the induction hypothesis, $\vec{x}(N)$ is well-defined.

Furthermore, suppose $\vec{y}$ is another solution of Equation (2). Then, it satisfies

$$\vec{y}(N) = \left(\frac{\vec{\alpha}}{\Gamma(1-\vec{\alpha})}\sum_{s=0}^{N-1}\left(\frac{\Gamma(N-s-\vec{\alpha})}{\Gamma(N-s+1)}\right)\vec{y}(s)\right) \vec{\oplus} \vec{q}(W\vec{y}(N-1)\vec{\oplus}\vec{p}). \tag{38}$$

But $y(s) = x(s)$ for $s \leq N-1$ by the uniqueness assumption of induction, which leads to

$$\vec{y}(N) = \left(\frac{\vec{\alpha}}{\Gamma(1-\vec{\alpha})}\sum_{s=0}^{N-1}\left(\frac{\Gamma(N-s-\vec{\alpha})}{\Gamma(N-s+1)}\right)\vec{x}(s)\right) \vec{\oplus} \vec{q}(W\vec{x}(N-1)\vec{\oplus}\vec{p}) = \vec{x}(N), \tag{39}$$

and it completes the proof. □

4. Algorithm for finding fuzzy solution

A base for constructing an algorithm to compute a fuzzy solution is Equation (37) and the triple representation. Suppose $\vec{x}(0) = [x_1(0), \ldots, x_\nu(0)]^T$, is a given fuzzy input. Let $\mathrm{Supp}(x_i(0)) = [a_i(0), b_i(0)]$, with $x_i(0) \sim (d_i(0), f_i^{[0]}, g_i^{[0]})$ where $f_i^{[0]}, g_i^{[0]} \in \mathbb{U}$, for $i = 1, \ldots, \nu$. Then,

$$\begin{aligned} z_i(0) =& w_{i1}x_1(0) \oplus \cdots \oplus w_{i\nu}x_\nu(0) + p_i \\ &\sim (\tilde{d}(0), \tilde{f}_i^{[0]}, \tilde{g}_i^{[0]}) \end{aligned} \tag{40}$$

where

$$\begin{aligned} \tilde{d}_i(0) =& \sum_{j=1}^{\nu} w_{ij}d_j(0) + p_i, \\ \tilde{f}_i^{[0]} =& \sum_{j=1}^{\nu} |w_{ij}| k(f_j^{[0]}, g_j^{[0]}, \mathrm{sgn}(w_{ij})), \\ \tilde{g}_i^{[0]} =& \sum_{j=1}^{\nu} |w_{ij}| k(g_j^{[0]}, f_j^{[0]}, \mathrm{sgn}(w_{ij})), \end{aligned} \tag{41}$$

and

$$k(u, v, s) = \frac{1}{2}(u+v) + \frac{1}{2}(u-v)s \tag{42}$$

is used to interchange $f_i^{[t]}$ and $g_i^{[t]}$ if coefficients are negative. We note that $k(u, v, 1) = u$ and $k(u, v, -1) = v$.

Now, applying q_i to $z_i(0) \sim (\tilde{d}_i(0), \tilde{f}_i^{[0]}, \tilde{g}_i^{[0]})$ gives the right-hand side of the NN. Corollary 1 plays an important role, yielding

$$\nabla^{\alpha_i} x_i(1) = q_i(z_i(0)) \sim (\tilde{\tilde{d}}_i(0), \tilde{\tilde{f}}_i^{[0]}, \tilde{\tilde{g}}_i^{[0]}),$$

where

$$\begin{aligned} \tilde{\tilde{d}}_i(0) =& q_i(\tilde{d}_i(0)), \\ \tilde{\tilde{f}}_i^{[0]}(r) =& \tilde{\tilde{d}}_i(0) - q_i(\tilde{d}_i(0) - \tilde{f}_i^{[0]}(r)), \\ \tilde{\tilde{g}}_i^{[0]}(r) =& q_i(\tilde{d}_i(0) + \tilde{g}_i^{[0]}(r)) - \tilde{\tilde{d}}_i(0). \end{aligned} \tag{43}$$

Since $\alpha_i \geq 0$, it follows from Equation (36) that

$$x_i(1) \sim (d_i(1), f_i^{[1]}, g_i^{[1]}), \tag{44}$$

where

$$\begin{aligned} d_i(1) =& \alpha_i d_i(0) + \tilde{\tilde{d}}_i(0), \\ f_i^{[1]}(r) =& \alpha_i f_i^{[0]}(r) + \tilde{\tilde{f}}_i^{[0]}(r), \\ g_i^{[1]}(r) =& \alpha_i g_i^{[0]}(r) + \tilde{\tilde{g}}_i^{[0]}(r). \end{aligned} \tag{45}$$

Thus, $\vec{x}(1) = [x_1(1), \ldots, x_\nu(1)]^T$.

We obtain $\vec{x}(T)$ for $T > 1$ recursively. Suppose $x_i(t) \sim (d_i(t), f_i^{[t]}, g_i^{[t]})$ where $f_i^{[0]}, g_i^{[0]} \in \mathbb{U}$, for $t = 1, \ldots, T$ and $i = 1, \ldots, \nu$. We derive a recursive formula for each component of the triple. Define

$$\vec{z}(t-1) = W\vec{x}(t-1)\vec{\oplus}\vec{p}.$$

Then,

$$\begin{aligned} z_i(t-1) =& w_{i1}x_1(t-1) \oplus \cdots \oplus w_{i\nu}x_\nu(t-1) \oplus p_i \\ &\sim (\tilde{d}(t-1), \tilde{f}_i^{[t-1]}, \tilde{g}_i^{[t-1]}), \end{aligned} \tag{46}$$

where

$$\begin{aligned}\tilde{d}_i(t-1) =& \sum_{j=1}^{\nu} w_{ij} d_j(t-1) + p_i,\\ \tilde{f}_i^{[t-1]} =& \sum_{j=1}^{\nu} |w_{ij}| k(f_j^{[t-1]}, g_j^{[t-1]}, \mathrm{sgn}(w_{ij})),\\ \tilde{g}_i^{[t-1]} =& \sum_{j=1}^{\nu} |w_{ij}| k(g_j^{[t-1]}, f_j^{[t-1]}, \mathrm{sgn}(w_{ij})),\end{aligned} \tag{47}$$

for $t = 1, \ldots, T-1$.

From Corollary 1, we obtain

$$\nabla^{\alpha_i} x_i(t) = q_i(z_i(t-1)) \sim (\tilde{\tilde{d}}_i(t-1), \tilde{\tilde{f}}_i^{[t-1]}, \tilde{\tilde{g}}_i^{[t-1]}), \tag{48}$$

where

$$\begin{aligned}&\tilde{\tilde{d}}_i(t-1) = q_i(\tilde{d}_i(t-1)),\\ &\tilde{\tilde{f}}_i^{[t-1]}(r) = \tilde{\tilde{d}}_i(t-1) - q_i(\tilde{d}_i(t-1) - \tilde{f}_i^{[t-1]}(r)),\\ &\tilde{\tilde{g}}_i^{[t-1]}(r) = q_i(\tilde{d}_i(t-1) + \tilde{g}_i^{[t-1]}(r)) - \tilde{\tilde{d}}_i(t-1).\end{aligned} \tag{49}$$

It follows from Equations (34) and (32) that

$$\vec{x}(t) = \left(\frac{\vec{\alpha}}{\Gamma(1-\vec{\alpha})} \sum_{s=0}^{t-1} \left(\frac{\Gamma(t-s-\vec{\alpha})}{\Gamma(t-s+1)} \right) \vec{x}(s) \right) \vec{\oplus} \vec{q}(\vec{z}). \tag{50}$$

Finally, since $\alpha_i \geq 0$ and $t > 1$, the coefficients

$$\frac{\alpha_i}{\Gamma(1-\alpha_i)} \frac{\Gamma(t-s-\alpha_i)}{\Gamma(t-s+1)}$$

are positive for $s = 0, \ldots, t-1$. Thus,

$$\begin{aligned}d_i(t) =& \frac{\alpha_i}{\Gamma(1-\alpha_i)} \sum_{s=0}^{t-1} \left(\frac{\Gamma(t-s-\alpha_i)}{\Gamma(t-s+1)} \right) d_i(s)\\ &+ \tilde{\tilde{d}}_i(t-1),\\ f_i^{[t]}(r) =& \frac{\alpha_i}{\Gamma(1-\alpha_i)} \sum_{s=0}^{t-1} \left(\frac{\Gamma(t-s-\alpha_i)}{\Gamma(t-s+1)} \right) f_i^{[s]}(r)\\ &+ \tilde{\tilde{f}}_i^{[t-1]}(r),\\ g_i^{[t]}(r) =& \frac{\alpha_i}{\Gamma(1-\alpha_i)} \sum_{s=0}^{t-1} \left(\frac{\Gamma(t-s-\alpha_i)}{\Gamma(t-s+1)} \right) g_i^{[s]}(r)\\ &+ \tilde{\tilde{g}}_i^{[t-1]}(r),\end{aligned} \tag{51}$$

and $\vec{x}(t) = [x_1(t), \ldots, x_\nu(t)]^T$.

We summarized the method in Algorithm 1.

Remark 4. *The computational complexities of Equations* (47), (49), *and* (51) *are* $O(\nu)$, $O(\nu)$, *and* $O(\nu t)$ *for each* t. *Consequently, the complexity of Algorithm 1 for computing* $\vec{x}(t)$ *is* $O(\nu t(t-1))$, *which reflects its scalability with respect to parameters* ν *and* t.

(1) **Input data**: $x_i(0) \sim (d_i(0), f_i^{[0]}, g_i^{[0]})$ for $i = 1, \ldots, \nu$.
(2) **Input parameters**: T, $\vec{p}$, W, $\vec{q}$.
(3) **Output**: $x_i(T) \sim (d_i(T), f_i^{[T]}, g_i^{[T]})$ for $i = 1, \ldots, \nu$.
(4) **For** $t = 1$ to T :
(5) **For** $i = 1$ to ν :
 (a) **Calculate** $z_i(t-1) \sim (\tilde{d}_i(t-1), \tilde{f}_i^{[t-1]}, \tilde{g}_i^{[t-1]})$ using Equation (47).
 (b) **Calculate** $(\tilde{\tilde{d}}_i(t-1), \tilde{\tilde{f}}_i^{[t-1]}, \tilde{\tilde{g}}_i^{[t-1]})$ using Equation (49).
 (c) **Calculate** $x_i(t) \sim (d_i(t), f_i^{[t]}, g_i^{[t]})$ using Equation (51).
(6) **Return** $\vec{x}(T) = [x_1(T), \ldots, x_\nu(T)]^T$.

Algorithm 1. Fuzzy solution of System (2)

5. Illustrations and examples

In this section, we present simpler examples to ensure understanding of the analysis and notation used throughout this paper. The first example details the implementation of Algorithm 1 for a 2D IFDS. The second example demonstrates an application of the proposed incommensurate RNN for local prediction of time series. These examples include the necessary computations to train the incommensurate RNN and to obtain the incommensurate order $\vec{\alpha}$.

5.1. Illustrative example

We consider a 2D NN. Let us consider

$$x_1(0)(w) = \begin{cases} 1-(w-d_1)^2/\epsilon^2, & w \in [-\epsilon+d_1, d_1+\epsilon],\\ 0, & \text{oth.,}\end{cases} \tag{52}$$

and

$$x_2(0)(w) = \begin{cases} 1+(w-d_2)/\epsilon, & w \in [-\epsilon+d_2, d_2],\\ 1-(w-d_2)/\epsilon, & w \in [d_2, d_2+\epsilon],\\ 0, & \text{oth.,}\end{cases} \tag{53}$$

where d_1, d_2 and $\epsilon > 0$ are real numbers. Figure 1 demonstrates these membership functions for $d_1 = 1$, $d_2 = 2$, and $\epsilon = 1$. To find the boundaries of $C_{x_1(0)}(r)$, we solve the inverse problem

$$1-(w-d_1)^2/\epsilon^2 = r$$

and thus the boundaries are

$$w = d_1 \pm \epsilon\sqrt{1-r}.$$

Therefore, their r-cat boundaries can be described by

$$\begin{aligned} C_{x_1(0)}(r) =& [d_1 - \epsilon\sqrt{1-r}, d_1 + \epsilon\sqrt{1-r}] \\ =& [C_{x_1(0)*}(r), C^*_{x_1(0)(r)}] \\ =& [d_1 - f_1^{[0]}(r), d_1 + g_1^{[0]}(r)]. \end{aligned} \tag{54}$$

Since the unique deterministic value of $x_1(0)(w)$ is $d_1[0] = d_1$, the shape functions in $\mathbb{U}$ are obtained by

$$f_1^{[0]}(r) = g_1^{[0]}(r) = \epsilon\sqrt{1-r}, \tag{55}$$

The shape functions and r-cuts of fuzzy number $x_1(0)$ are depicted in Figure 2.

Similarly, for $x_2(0)$, we have

$$C_{x_2(0)}(r) = [d_2(0) - \epsilon(1-r), d_2(0) + \epsilon(1-r)], \tag{56}$$

and

$$f_2^{[0]}(r) = g_2^{[0]}(r) = \epsilon(1-r). \tag{57}$$

Suppose we trained an NN with Sigmoid activation functions

$$q_i(t) = \frac{1}{1+e^t},$$

and we obtained the weights

$$W = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 8 & -10 \end{bmatrix},$$

biases

$$\vec{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

order,

$$\vec{\alpha} = [0.3, 0.5]^T,$$

and fuzzy parameters

$$\vec{d}(0) = [d_1(0), d_2(0)]^T = [0.4, 0.3]^T, \quad \epsilon = 0.01.$$

It follows from Equation (41) that

$$\begin{aligned} \tilde{d}_1(0) =& 2d_1(0) - 2d_2(0) + 0 = 0.2, \\ \tilde{f}_1^{[0]}(r) =& 2f_1^{[0]} + 2g_2^{[0]} \\ =& 2\epsilon(\sqrt{1-r} + 1 - r), \\ \tilde{g}_1^{[0]}(r) =& 2g_1^{[0]} + 2f_2^{[0]} \\ =& 2\epsilon(\sqrt{1-r} + 1 - r), \end{aligned} \tag{58}$$

and

$$\begin{aligned} \tilde{d}_2(0) =& 8d_1(0) - 10d_2(0) + 1 = 1.2, \\ \tilde{f}_2^{[0]}(r) =& 8f_1^{[0]} + 10g_2^{[0]} \\ =& 2\epsilon(4\sqrt{1-r} + 5(1-r)), \\ \tilde{g}_2^{[0]}(r) =& 8g_1^{[0]} + 10f_2^{[0]} \\ =& 2\epsilon(4\sqrt{1-r} + 5(1-r)). \end{aligned} \tag{59}$$

from Equation (43) we obtain

$$\begin{aligned} \tilde{\tilde{d}}_1(0) =& q_1(0.2) = 0.5498, \\ \tilde{\tilde{f}}_1^{[0]}(r) =& 0.5498 - q_1(0.2 - 2\epsilon(\sqrt{1-r} + 1 - r)), \\ \tilde{\tilde{g}}_1^{[0]}(r) =& q_1(0.2 + 2\epsilon(\sqrt{1-r} + 1 - r)) - 0.5498. \end{aligned} \tag{60}$$

Equation (60) represents a transformation that takes into account the effect of the sigmoid activation function on the shape functions. Clearly, $\tilde{\tilde{f}}_1^{[0]}(1) = \tilde{\tilde{g}}_1^{[0]}(1) = 0$ as expected.

Finally, we can compute the first iteration by Equation (45)

$$\begin{aligned} d_1(1) =& 0.3d_1(0) + 0.5498 = 0.6698, \\ f_1^{[1]}(r) =& 0.3\epsilon\sqrt{1-r} \\ &+ 0.5498 - q_1(0.2 - 2\epsilon(\sqrt{1-r} + 1 - r)), \\ g_1^{[1]}(r) =& 0.3\epsilon\sqrt{1-r} \\ &+ q_1(0.2 + 2\epsilon(\sqrt{1-r} + 1 - r)) - 0.5498. \end{aligned} \tag{61}$$

Similarly, the solution for $t > 1$ can be obtained from Equations (47), (49) and (51). A realization of the solution for diverse r is depicted in Figs. 3 and 4.

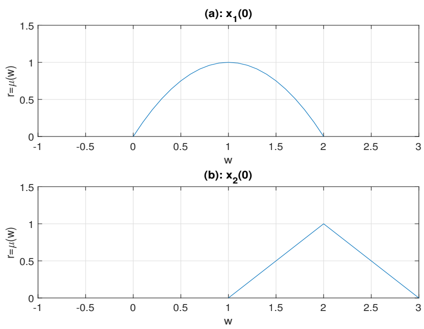

Figure 1. Membership functions of initial values. (a) A parabolic-shaped membership function for $x_1(0)$ (b) a triangular-shaped membership function for $x_2(0)$

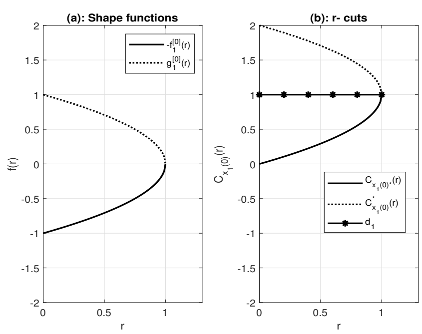

Figure 2. (a) Shape functions and (b) r-cuts of the fuzzy number $x_1(0)$

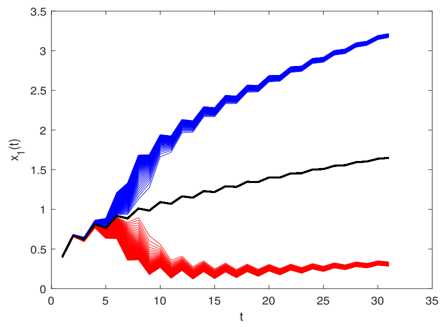

Figure 3. A realization of the solution of a fuzzy system for different values of $r = [0, 0.01, \ldots, 1]$, for first state $[d_1(t), d_1(t) - f_1^{[t]}(r), d_1(0) + g_1^{[t]}(r)]$

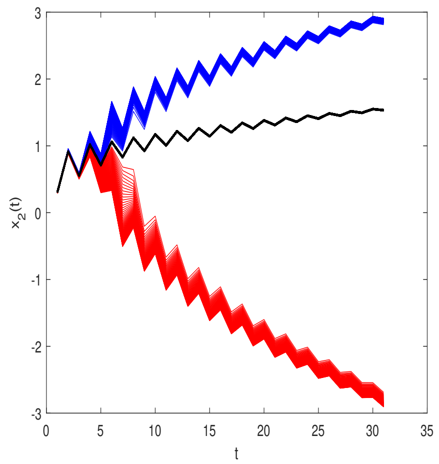

Figure 4. A realization of the solution of a fuzzy system for different values of $r = [0, 0.01, \ldots, 1]$, for second state $[x_2(t), d_2(t) - f_2^{[t]}(r), d_2(t) + g_2^{[t]}(r)]$

5.2. Data-driven dynamical systems for time series

In this section, we apply System Equation (2) for time series prediction. It is illustrated with a simple example of a 2-dimensional system, noting that high-dimensional systems require further study. The System Equation (2) can be written as

$$x_i(t) = \mathfrak{D}^{-(1-\vec{\alpha})} x_i(t-1) + q_i(w_{i1}x_1(t-1) + w_{i2}x_2(t-1) + p_i), \tag{62}$$

where

$$\mathfrak{D}^{-(1-\vec{\alpha})}(t-1)x_i(t-1) := \frac{\alpha_i}{\Gamma(1-\alpha_i)} \sum_{s=0}^{t-1} \left(\frac{\Gamma(t-s-\alpha_i)}{\Gamma(t-s+1)} \right) x_i(s), \tag{63}$$

for $i = 1, 2$. Considering that

$$\frac{-\alpha_i \Gamma(t-s+\alpha_i)}{\Gamma(1-\alpha_i)} = \prod_{j=0}^{t-s-1} (j - \alpha_i),$$

we can compute the derivative with respect to α_i, yielding

$$\frac{\partial}{\partial \alpha_k} \left(\mathfrak{D}^{-(1-\vec{\alpha})} x_i(t-1) \right) = \delta_{ki} \sum_{s=0}^{t-1} \frac{1}{(t-s)!} \sum_{r=0}^{t-s-1} \prod_{\substack{j=0 \\ j \neq r}}^{t-s-1} (j - \alpha_i) x_i(s). \tag{64}$$

Therefore, the rate of change of x_i with respect to α_i can be calculated recursively by

$$\frac{\partial x_i(t)}{\partial \alpha_k} = \frac{\partial \mathfrak{D}^{-(1-\alpha_i)} x_i(t-1)}{\partial \alpha_k} + \sum_{s=1}^{t-1} \frac{\partial \mathfrak{D}^{-(1-\alpha_i)} x_i(t-1)}{\partial x_i(s)} \frac{\partial x_i(s)}{\partial \alpha_k} + \sum_{l=1}^{2} \frac{\partial q_i(z_i(t-1))}{\partial x_l(t-1)} \frac{\partial x_l(t-1)}{\partial \alpha_k} \tag{65}$$

for $t = 1, \ldots, T$, and $i, j, k = 1, 2$.

Let us denote by θ the other training parameters of the NN, where

$$\theta = \{w_{ij}, p_i : i, j = 1, 2\}.$$

The derivative of x_i with respect to θ can be calculated as

$$\frac{\partial x_i(t)}{\partial \theta} = \sum_{s=1}^{t-1} \frac{\partial \mathfrak{D}^{-(1-\alpha_i)} x_i(t-1)}{\partial x_i(s)} \frac{\partial x_i(s)}{\partial \theta} + \sum_{l=1}^{2} \frac{\partial q_i(z_i(t-1))}{\partial x_l(t-1)} \frac{\partial x_l(t-1)}{\partial \theta} + q_i'(z_i(t-1)) \frac{\partial z_i(t-1)}{\partial \theta}, \tag{66}$$

where

$$z_i(t-1) = w_{i1}x_1(t-1) + w_{i1}x_2(t-1) + p_i.$$

Each term in Equation (66) can be computed using the following formulas:

$$\frac{\partial \mathfrak{D}^{-(1-\vec{\alpha})} x_i(t-1)}{\partial x_i(s)} = \frac{\alpha_i}{\Gamma(1-\alpha_i)} \frac{\Gamma(t-s-\alpha_i)}{\Gamma(t-s+1)}, \tag{67}$$

$$\frac{\partial q_i(z_i(t-1))}{\partial x_j(t-1)} = q_i'(z_i(t-1)) w_{ij}, \tag{68}$$

$$\frac{\partial z_i(t-1)}{\partial w_{kj}} = \delta_{ik} x_j(t-1), \tag{69}$$

and

$$\frac{\partial z_i(t-1)}{\partial p_k} = \delta_{ik}, \tag{70}$$

for $i, k, j = 1, 2$, where δ_{ik} denotes the Kronecker delta function.

The derivative propagation is initialized by:

$$\frac{\partial x_i(1)}{\partial w_{kj}} = \delta_{ik} q_k'(z_k(0)) x_j(0), \quad i, j, k = 1, 2, \tag{71}$$

and

$$\frac{\partial x_i(1)}{\partial p_j} = \delta_{ij} q_i'(z_i(0)), \quad i,j = 1,2. \tag{72}$$

Here, T represents the memory length of the NN.

The incommensurate NN Equation (62) is trained to obtain $\vec{x}(T)$. Inputs are $\vec{x}^{\{s\}}(0) = \vec{i}(s)$ for $s = 0, \ldots, n$, (where n is the number of data points). In time series analysis, this can be interpreted as predicting the time series at $s + T$.

The goal is $x_i^{\{s\}}(T)$ to match real data $y_i^{\{s\}}$, Corresponds to predicting $\vec{i}(s+c)$, (with $c = 1$ or $c = T$). As is conventional, the training energy function is minimized:

$$E^{\text{Train}} = \frac{1}{m}\sum_{s=1}^{m}\sum_{i=1}^{2}(x_i^{\{s\}}(T) - y_i^{\{s\}})^2,$$

where m denotes the number of training data points. The derivative of the training energy is:

$$\frac{dE^{\text{Train}}}{d\theta} = \frac{2}{m}\sum_{s=1}^{m}\sum_{i=1}^{2}(x_i^{\{s\}}(T) - y_i^{\{s\}})\frac{dx_i^{\{s\}}(T)}{d\theta}$$

To train the NN, we use a mini-batch SGD method[29] combined with the ADAM optimization algorithm.

Remark 5. *As shown in the prior example, even minor input value deviations can cause significant output variations when T is large. Moreover, increasing T may raise computational costs. Notably, these issues are mitigated for small T, allowing results to be calculated efficiently.*

The time series data described by

$$\vec{y}(t) = \begin{pmatrix} \cos(0.02t) \\ (e^{0.01t} - 2)/(1 - 2e^{0.01t}) \end{pmatrix}$$

for $t = 1, \ldots, 367$, is used, as shown in Figure 5. The aim is to provide a local prediction based on previous local data. 240 data points are used for training and the remaining data points are used for testing and validation. The activation function $q_i(t) = \tanh(t)$ $i = 1, 2$, is used.

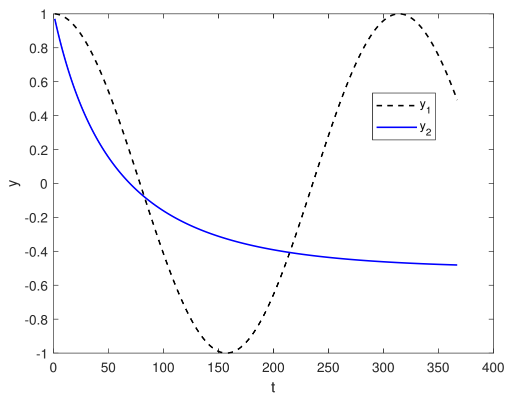

Figure 5. Time series data

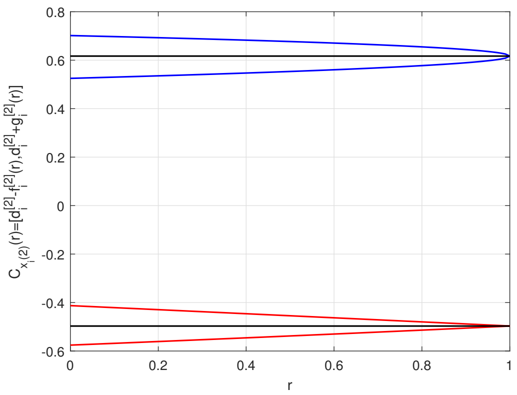

Figure 6. The fuzzy output for fuzzy input data at $s = 365$. The expected value is $\vec{y}(365) = [0.5090, -0.4804]^T$

After training, the following weights is obtained for $T = 2$:

$$W = \begin{pmatrix} 0.9662264009849 & -0.0067547002337 \\ -0.0067547423047 & 0.9662262855277 \end{pmatrix},$$

$$\vec{p} = \begin{pmatrix} 0.007754739375402 \\ 0.007754742992966 \end{pmatrix},$$

and

$$\vec{\alpha} = \begin{pmatrix} 0.1506754746276 \\ 0.0993245241422 \end{pmatrix}.$$

Using these values, the energy function was reduced for the training data to

$$E^{\text{Train}} = 0.0050824858689,$$

and for the test data to

$$E^{\text{Test}} = 0.005257048048326.$$

The precision of the random test data points is demonstrated in Table 4, which shows an acceptable local prediction.

Remark 6. *A hyperbolic (non-periodic) function was used to predict a periodic signal. Periodic activation functions like* sin *may better minimize the energy function for periodic data. However, real-world scenarios often require a single model to handle diverse datasets, making it valuable to achieve good results with a fixed activation function.*

Finally, Algorithm 1 is used to obtain the fuzzy output of this RNN for fuzzy input data at $t = 365$, with $\epsilon = 0.1$ and shape functions defined by Equations (55) and (57). Figure 6 shows $C_{\vec{x}(T)}(r)$.

The real outputs are $\vec{y}(365) = [0.5090, -0.4804]^T$. We find that the fuzzy results still distinguish these two values, even though input uncertainty introduces ambiguity.

Table 4. Exact output end local prediction

t	$y_i^{\{t\}}$	$x_i^{\{t\}}(T)$
$t = 341$	0.8416	0.8489
	−0.4955	−0.4751
$t = 271$	0.7133	0.6651
	−0.4711	−0.4489
$t = 82$	−0.0731	−0.0891
	−0.0730	−0.0818

6. Conclusion

We proposed incommensurate fractional nabla discrete systems for a type of RNNs. These systems become classical neural networks when all orders are zero, and classical RNNs when all orders are one.

Using fuzzy theory, we analyzed how input data uncertainty affects output data. First, we revisited the definition of fuzzy numbers and added a condition for the uniqueness of their deterministic parts. We then used shape functions (instead of membership functions) to handle operations on fuzzy numbers. We found that most discrete fractional nabla difference equations have a unique H-difference solution.

The GH-difference was introduced because the H-difference isn't defined for many uncertain numbers. However, our results show that in the context of difference equations, we might not need to extend the difference definition for other uncertain numbers. So, we introduced the concept of "H-differenceable." Whether this holds in continuous cases is unknown and needs future study.

Based on our analysis, we developed a programmable algorithm to find fuzzy solutions for the incommensurate RNNs. Examples showed that nonlinearity deforms the shape functions—an interesting finding.

We trained a 2D incommensurate NN for local time series prediction. However, training the algorithms for higher-dimensional incommensurate NNs require further dedicated study. The structure and derivatives of these NNs rely on recursive calculations with previous datasets, making higher recursion levels computationally complex. This suggests that higher recursion may not be efficient for training. Additionally, our first example showed that small input data inaccuracies can cause results to deviate from expected values, implying higher recursion might not be effective either.

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Conflict of interest

The authors declare that they have no conflict of interest to disclose.

Author contributions

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Investigation: All authors
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Writing–original draft: Babak Shiri, Ehsan Dadkhah Khiabani
Writing–review & editing: Dumitru Baleanu

Availability of data

Data used in this research are available on demand from the authors.

AI tools statement

All authors confirm that no AI tools were used in the preparation of this manuscript.

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