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RESEARCH ARTICLE

Investigate the solution of an initial Hilfer fractional value problem

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ABSTRACT

This paper aims to investigate sufficient criteria of the existence solution for a new category of nonlinear fractional differential equation under the Hilfer fractional derivative. The primary existence results are achieved by using a modified version of the Krasnoselskii-Dhage fixed-point theorem in the weighted Banach space. Finally, an application is illustrated to test the validity of the findings.

1. Introduction

In numerous mathematical studies, the theory of fractional calculus and its applications have been thoroughly examined. Many real-world problems in science, engineering, and economics are now mathematically modeled with the help of fractional calculus. See the books and research works[1–7] and references therein. Fractional differential equations (FDE) are undoubtedly still a popular topic among writers. The use of the fixed-point theorem approach to obtain the existence and uniqueness results for fractional differential equations is the most favored topic by several scholars. See ref.[8–21] and references therein for the recent development in this area. This motivates researchers to study various fixed-point theorems and their generalizations. See ref.[22–25] and references therein.

In ref.,[26–29] the researchers discussed some existence and optimal control results for various fractional differential equations and inclusions by means of fixed point theorems. Similarly, several scholars have demonstrated a great deal of interest in the theory of fractional differential equations with linear perturbation. See ref.[30–33] and references therein. In ref.,[25] Dhage presented a novel form of the Kransoselskii-type FPT termed the Krasnoselskii-Dhage type FPT by utilizing a nonlinear $\mathcal{D}-$contraction condition. In ref.,[30] Dhage and Lakshmikantham used fixed point theorems (FPT) of the Krasnoselskii type to study the first-order hybrid differential problem with linear perturbations:

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$$\frac{d}{dx}\left[\frac{x(t)}{f(t,x(t))}\right]=g(t,x(t)),\ \ t\in J,$$
$$x(t_0)=x_0\in\mathbb{R},$$

where $J=[t_0,t_0+a]$, for some fixed $t_0\in\mathbb{R}, a>0$ and $f:\mathcal{J}\times\mathbb{R}\to\mathbb{R}\backslash\{0\}$ and $g:\mathcal{J}\times\mathbb{R}\to\mathbb{R}$. Furthermore, Dhage et al.[31] established the existence results for hybrid differential equations with linear perturbations. In,[32] Lu et al. formulated the theory of hybrid type FDE by using linear perturbations of the second kind:

$$D^q\left[x\left(t\right)-f\left(t,x\left(t\right)\right)\right]=g\left(t,x\left(t\right)\right),\qquad t\in J$$
$$x\left(t_0\right)=x_0\in\mathbb{R},$$

where $f,g:\mathcal{J}\times\mathbb{R}\to\mathbb{R}$ and $0<q<1$.

In ref.[33] Akhadkulov et al. utilized this revised version of the Krasnoselskii-Dhage type FPT to obtain the existence results for a hybrid type FDE that incorporates the differential and integral operators of Riemann–Liouville (RL) type:

$$D^\alpha\left[x\left(t\right)-f\left(t,x\left(t\right)\right)\right]=g\left(t,x\left(t\right),I^\beta\left(x\left(t\right)\right)\right),$$
$$t\in J,\ \beta>0$$
$$x\left(t_0\right)=x_0,$$

where $J=[t_0,t_0+a]$, for some fixed $t_0\in\mathbb{R}, a>0$ and $f:\mathcal{J}\times\mathbb{R}\to\mathbb{R}$ and $g:\mathcal{J}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$.

In ref.,[34] Kiataramkul et al. discussed the existing results of the following fractional integro-differential hybrid boundary value problems for differential equations with ψ- Hilfer derivative operator:

$$^{H}D_{a^+}^{\alpha,\rho,\psi}\left[\left(\frac{x\left(t\right)}{g\left(t,x\left(t\right)\right)}-\sum_{i=1}^{n}I_{a^+}^{\beta_i,\psi}h_i\left(t,x\left(t\right)\right)\right)\right]$$
$$=f\left(t,x\left(t\right)\right),$$
$$x\left(a\right)=0,\ x\left(b\right)=m\left(x\right),$$

where $t\in J=[a,b]$, $0<\alpha\le 2,\ \ 0\le\rho\le 1;\ \ \beta_i>0$, for $i=1,2,...,n$, $g:J\times\mathbb{R}\to\mathbb{R}\backslash\{0\}$, $f:J\times\mathbb{R}\to\mathbb{R}, m:J\to\mathbb{R},\ h_i:J\times\mathbb{R}\to\mathbb{R}$ such that $h_i\left(a,0\right)=0,\ i=1,2,...,n$. Further, they obtained the existence result for inclusion ψ- Hilfer fractional integro-differential hybrid boundary value problems by means of fixed point theorem. For some recent development in the field of fractional differential equations readers are encouraged to see[35–43] and references therein.

Inspired by the aforementioned work, in this study, we examine the subsequent initial value problem (IVP) employing the Hilfer fractional derivative (HFD) operator:

$$\begin{aligned}&D_{p^+}^{\sigma,\tau}\left[\varphi\left(s\right)-f\left(s,\varphi\left(s\right)\right)\right]=g\left(s,\varphi\left(s\right)\right),\\&\quad s\in\mathcal{J},0<\sigma<1,0\le\tau\le 1,\end{aligned}\tag{1}$$

$$\begin{aligned}&I_{p^+}^{1-\kappa}\left[\varphi\left(s\right)-f\left(s,\varphi\left(s\right)\right)\right]\left(p^+\right)=\varphi_0\in\mathbb{R},\\&\quad\kappa=\sigma+\tau-\sigma\tau,\end{aligned}\tag{2}$$

where $\mathcal{J}=(p,q]$. The non-linear terms $f:\mathcal{J}\times\mathbb{R}\to\mathbb{R}$ and $g:\mathcal{J}\times\mathbb{R}\to\mathbb{R}$ be the given continuous functions. The operator $D_{p^+}^{\sigma,\tau}$ is a HFD operator of order σ and type τ.

The main contributions of the study are as follows:

(i) Studying the initial value problem involving the Hilfer fractional derivatives.

(ii) Investigating the applicability of the Krasnoselskii–Dhage fixed point technique to the proposed problem.

(iii) The problem being discussed in this paper is more general than in the literature, such as for $\tau=1$ it reduces to a problem in the Caputo fractional derivative, and if $\tau=0$ it returns to a problem in the Riemann-–Liouville fractional derivative.

This article is organized as follows: Section 2 presents some essential definitions and lemmas needed throughout the study. Section 3 investigates the main findings. Section 4 provides an application of the results.

2. Preliminary results

Definition 1. ([3]) *For $\sigma\in\mathbb{R}^+$, the RL-fractional integral of a function $h(u)$ is given as;*

$$\left(I_{p^+}^{\sigma}h\right)(u)=\frac{1}{\Gamma\left(\sigma\right)}\int_p^u\frac{h\left(s\right)}{\left(u-s\right)^{1-\sigma}}ds,\qquad (u>p).$$

Definition 2. ([3]) *For $\sigma\in[m-1,m),\ m\in\mathbb{Z}^+$, the RL-fractional derivative of a function $h(u)$ is given as;*

$$\left(D_{p^+}^{\sigma}h\right)(u)=\frac{1}{\Gamma\left(m-\sigma\right)}\left(\frac{d}{du}\right)^m\cdot\int_p^u\frac{h\left(s\right)}{\left(u-s\right)^{\sigma-m+1}}ds,\qquad (u>p).$$

Definition 3. ([1]) *The HFD of order $\sigma\in(0,1)$ and type $\tau\in[0,1]$ of a function $h(u)$ is given as;*

$$\left(D_{p^+}^{\sigma,\tau}h\right)(u)=\left(I_{p^+}^{\tau(1-\sigma)}D\left(I_{p^+}^{(1-\tau)(1-\sigma)}h\right)\right)(u),$$

where $D=\frac{d}{du}$.

Let $0<p<q<\infty$, and let $\mathcal{C}[p,q]$ denotes a Banach space of all continuous mappings from $[p,q]$ into $\mathbb{R}$ with the maximum norm

$$\|u\|_{\mathcal{C}}=\max\left\{|u\left(s\right)|:s\in[p,q]\right\}.$$

For $0\le\kappa=\sigma+\tau-\sigma\tau<1$, we define $\mathcal{C}_{1-\kappa}\left[p,q\right]$, the weighted space of the continuous functions ϕ as

$$\mathcal{C}_{1-\kappa}[p,q]=\left\{\phi(u):[p,q]\to\mathbb{R}\mid (u-p)^{1-\kappa}\phi(u)\in\mathcal{C}[p,q]\right\}, \quad (3)$$

where $[p,q]$ is the finite interval.

Obviously, $\mathcal{C}_{1-\kappa}[p,q]$ is the weighted Banach space with the norm

$$\|\phi\|_{\mathcal{C}_{1-\kappa}}=\left\|(u-p)^{1-\kappa}\phi(u)\right\|_{\mathcal{C}}.$$

At the same time, we define a Banach space

$$\mathcal{C}_{\kappa}^{m}[p,q]:=\left\{\Theta\in\mathcal{C}^{m-1}[p,q]:\Theta^{(m)}\in\mathcal{C}_{\kappa}[p,q]\right\},$$

with the norm

$$\|\Theta\|_{\mathcal{C}_{\kappa}^{m}}=\sum_{k=0}^{m-1}\left\|\Theta^{(k)}\right\|_{\mathcal{C}}+\left\|\Theta^{(m)}\right\|_{\mathcal{C}_{\kappa}}, \qquad m\in\mathbb{N}.$$

Also, $\mathcal{C}_{\kappa}^{0}[p,q]:=\mathcal{C}_{\kappa}[p,q]$.

Lemma 1. ([3,21]) *If $\sigma>0$ and $\tau>0$, there exists*

$$\left[I_{p^{+}}^{\sigma}(t-p)^{\tau-1}\right](u)=\frac{\Gamma(\tau)}{\Gamma(\tau+\sigma)}(u-p)^{\tau+\sigma-1}$$

and

$$\left[D_{p^{+}}^{\sigma}(t-p)^{\sigma-1}\right](u)=0, \qquad 0<\sigma<1.$$

Lemma 2. ([3, 21]) *If $\sigma>0,\tau>0$, and $\phi\in\mathcal{L}^{1}(p,q)$, for $u\in[p,q]$, there exist the following properties,*

$$\left(I_{p^{+}}^{\sigma}I_{p^{+}}^{\tau}\phi\right)(u)=\left(I_{p^{+}}^{\sigma+\tau}\phi\right)(u)$$

and

$$\left(D_{p^{+}}^{\sigma}I_{p^{+}}^{\sigma}\phi\right)(u)=\phi(u).$$

In particular, if $\phi\in\mathcal{C}_{\kappa}[p,q]$ or $\phi\in\mathcal{C}[p,q]$, then above equalities hold at each $u\in(p,q]$ or $u\in[p,q]$, respectively.

Lemma 3. ([3,21]) *Let $\sigma>0$ and $0\le\kappa<1$. Then $I_{p^{+}}^{\sigma}$ is bounded from $\mathcal{C}_{\kappa}[p,q]$ into $\mathcal{C}_{\kappa}[p,q]$.*

Lemma 4. ([3,21]) *Let $\sigma>0$ and $0\le\kappa<1$. If $\phi\in\mathcal{C}_{\kappa}[p,q]$ and $I_{p^{+}}^{1-\sigma}\phi\in\mathcal{C}_{\kappa}^{1}[p,q]$, then*

$$I_{p^{+}}^{\sigma}D_{p^{+}}^{\sigma}\phi(u)=\phi(u)-\frac{I_{p^{+}}^{1-\sigma}\phi(p)}{\Gamma(\sigma)}(u-p)^{\sigma-1},$$

for all $u\in(p,q]$.

Lemma 5. ([3,21]) *If $0\le\kappa<1$ and $\phi\in\mathcal{C}_{\kappa}[p,q]$, then*

$$I_{p^{+}}^{\sigma}\phi(p):=\lim_{u\to p^{+}}I_{p^{+}}^{\sigma}\phi(u)=0, \qquad 0\le\kappa<\sigma.$$

Now, we present the following weighted spaces which are required in our main result.

$$\mathcal{C}_{1-\kappa}^{\sigma,\tau}=\left\{\phi\in\mathcal{C}_{1-\kappa}[p,q],\ D_{p^{+}}^{\sigma,\tau}\phi\in\mathcal{C}_{1-\kappa}[p,q]\right\}$$

and

$$\mathcal{C}_{1-\kappa}^{\kappa}=\left\{\phi\in\mathcal{C}_{1-\kappa}[p,q],\ D_{p^{+}}^{\kappa}\phi\in\mathcal{C}_{1-\kappa}[p,q]\right\}.$$

It is obvious that

$$\mathcal{C}_{1-\kappa}^{\kappa}[p,q]\subset\mathcal{C}_{1-\kappa}^{\sigma,\tau}[p,q].$$

Lemma 6. ([3, 21]) *Let $\sigma,\tau>0$ and $\kappa=\sigma+\tau-\sigma\tau$. If $f\in\mathcal{C}_{1-\kappa}^{\kappa}$, then*

$$I_{p^{+}}^{\kappa}D_{p^{+}}^{\kappa}f=I_{p^{+}}^{\sigma}D_{p^{+}}^{\sigma,\tau}f,$$

$$D_{p^{+}}^{\sigma}I_{p^{+}}^{\sigma}f=D_{p^{+}}^{\tau(1-\sigma)}f.$$

Lemma 7. ([3, 21]) *Let $h\in\mathcal{L}^{1}[p,q]$ and $D_{p^{+}}^{\tau(1-\sigma)}h\in\mathcal{L}^{1}[p,q]$ exists, then*

$$D_{p^{+}}^{\sigma,\tau}I_{p^{+}}^{\sigma}h=I_{p^{+}}^{\tau(1-\sigma)}D_{p^{+}}^{\tau(1-\sigma)}h.$$

Lemma 8. ([14, 21]). *Let a real-valued function ϕ defined on $(p,q]\times\mathbb{R}$ be such that, for any $\varphi\in\mathcal{C}_{1-\kappa}[p,q]$, $\phi(\cdot,\varphi(\cdot))\in\mathcal{C}_{1-\kappa}[p,q]$. Then $\varphi\in\mathcal{C}_{1-\kappa}^{\kappa}[p,q]$ is a solution of IVP:*

$$D_{p^{+}}^{\sigma,\tau}\varphi(s)=\phi(s,\varphi(s)), \qquad 0<\sigma<1, 0\le\tau\le1,$$

$$I_{p^{+}}^{1-\kappa}\varphi(p^{+})=\varphi_{0}, \qquad \kappa=\sigma+\tau-\sigma\tau,$$

iff φ verifies the next integral equation:

$$\varphi(s)=\frac{\varphi_{0}(s-p)^{\kappa-1}}{\Gamma(\kappa)}+\frac{1}{\Gamma(\sigma)}\cdot\int_{p}^{s}(s-w)^{\sigma-1}\phi(w,u(w))\,dw.$$

Now, the following are some of the concepts, which we will be employed in the main result.

Let $\mathcal{C}(\mathcal{J}\times\mathbb{R},\mathbb{R})$ represents the category of continuous functions $\theta:\mathcal{J}\times\mathbb{R}\to\mathbb{R}$ and the kind of mappings $\phi:\mathcal{J}\times\mathbb{R}\to\mathbb{R}$, where the map

(1) $s\mapsto\phi(s,w)$ is measurable $\forall\ w\in\mathbb{R}$,
(2) $s\mapsto\phi(s,w)$ is continuous $\forall\ w\in\mathbb{R}$.

Moreover, the space $\mathcal{C}(\mathcal{J}\times\mathbb{R},\mathbb{R})$ is referred to be the mappings of Carathéodory type on $\mathcal{J}\times\mathbb{R}$, which are Lebesgue integrable functions when bounded by a Lebesgue integrable function on $\mathcal{J}$.

Definition 4. ([22–24]) *A function $\Upsilon:\mathbb{R}_{+}\to\mathbb{R}_{+}$ which is upper semi-continuous and non-decreasing in nature is said to be a $\mathcal{D}$-function if $\Upsilon(0)=0$. Moreover, $\mathfrak{D}$ represents the category of each $\mathcal{D}$-functions on $\mathbb{R}_{+}$.*

Definition 5. ([22–24]) *For an operator $\Delta:\mathcal{C}_{1-\kappa}(\mathcal{J},\mathbb{R})\to\mathcal{C}_{1-\kappa}(\mathcal{J},\mathbb{R})$, if there is a $\mathcal{D}$-function $\Upsilon_{\Delta}\in\mathfrak{D}$, where*

$$\|\Delta v-\Delta w\|\le\Upsilon_{\Delta}(\|v-w\|),$$

for all $v,w\in\mathcal{C}_{1-\kappa}(\mathcal{J},\mathbb{R})$, where $0<\Upsilon_{\Delta}(\alpha)<\alpha$ for all $\alpha>0$, then the operator Δ is referred as a nonlinear $\mathcal{D}$-contraction on $\mathcal{C}_{1-\kappa}(\mathcal{J},\mathbb{R})$.

Theorem 1. ([25, 33]) *Let $\mathcal{K} \subset \mathcal{C}_{1-\kappa}(\mathcal{J},\mathbb{R})$ be closed, convex, and bounded in nature. Let the operators $\Delta : \mathcal{C}_{1-\kappa}(\mathcal{J},\mathbb{R}) \to \mathcal{C}_{1-\kappa}(\mathcal{J},\mathbb{R})$ and $\Omega : \mathcal{K} \to \mathcal{C}_{1-\kappa}(\mathcal{J},\mathbb{R})$ satisfying:*

(1) *Δ is $\mathcal{D}$-contraction of nonlinear type,*
(2) *Ω is completely continuous and*
(3) *$u = \Delta u + \Omega w \Rightarrow u \in \mathcal{K}$ for all $w \in \mathcal{K}$.*

Then, the mapping $\Delta + \Omega$ owns a fixed point in $\mathcal{K}$.

Lemma 9. *Let a function $f : \mathcal{J} \times \mathbb{R} \to \mathbb{R}$ such that, for any $\varphi \in \mathcal{C}_{1-\kappa}[p,q]$, $f(\cdot,\varphi(\cdot)) \in \mathcal{C}_{1-\kappa}[p,q]$. Then, $\varphi \in \mathcal{C}_{1-\kappa}[p,q]$ is a solution of fractional IVP*

$$D^{\sigma,\tau}_{p^+}[\varphi(s) - f(s,\varphi(s))] = z(s), \qquad s \in \mathcal{J}, \tag{4}$$

$$I^{1-\kappa}_{p^+}[\varphi(s) - f(s,\varphi(s))](p^+) = \varphi_0, \quad \kappa = \sigma+\tau-\sigma\tau, \tag{5}$$

if and only if $\varphi(s)$ satisfies the following VIE

$$\varphi(s) = \frac{\varphi_0(s-p)^{\kappa-1}}{\Gamma(\kappa)} + f(s,\varphi(s)) + \frac{1}{\Gamma(\sigma)}\int_p^s (s-w)^{\sigma-1} z(w)\,dw. \tag{6}$$

Proof. Let φ be a solution of (4)–(5). Consequently, by Lemma 4, we obtain

$$\begin{aligned} I^{\sigma}_{p^+}D^{\sigma,\tau}_{p^+}[\varphi(s) - f(s,\varphi(s))] &= \varphi(s) - f(s,\varphi(s)) \\ &\quad - \frac{I^{1-\kappa}_{p^+}[\varphi(s) - f(s,\varphi(s))](p^+)}{\Gamma(\kappa)}(s-p)^{\kappa-1} \\ &= I^{\sigma}_{p^+}z(s). \end{aligned}$$

Then,

$$\begin{aligned} \varphi(s) - f(s,\varphi(s)) &= I^{\sigma}_{p^+}z(s) + \frac{I^{1-\kappa}_{p^+}[\varphi(s) - f(s,\varphi(s))](p^+)}{\Gamma(\kappa)}(s-p)^{\kappa-1} \\ &= I^{\sigma}_{p^+}z(s) + \frac{\varphi_0(s-p)^{\kappa-1}}{\Gamma(\kappa)}. \end{aligned}$$

So,

$$\varphi(s) = \frac{\varphi_0(s-p)^{\kappa-1}}{\Gamma(\kappa)} + f(s,\varphi(s)) + I^{\sigma}_{p^+}z(s) \tag{7}$$

$$= \frac{\varphi_0(s-p)^{\kappa-1}}{\Gamma(\kappa)} + f(s,\varphi(s)) + \frac{1}{\Gamma(\sigma)}\int_p^s (s-w)^{\sigma-1} z(w)\,dw, \quad s \in \mathcal{J}.$$

Thus, Equation (6) holds. Now, to demonstrate the sufficiency. Let $\varphi \in \mathcal{C}^{\kappa}_{1-\kappa}[p,q]$ satisfy Equation (6), which can be expressed as (7).

By applying $D^{\kappa}_{p^+}$ to either sides of (7), it can be deduced from Lemma 1, Lemma 6, and Definition 2 that

$$D^{\kappa}_{p^+}[\varphi(s) - f(s,\varphi(s))] = D^{\tau(1-\sigma)}_{p^+}z(s). \tag{8}$$

From (8), and the hypothesis $D^{\kappa}_{p^+}[\varphi - f(\cdot,\varphi(\cdot))] \in \mathcal{C}_{1-\kappa}[a,b]$, we have

$$DI^{1-\tau(1-\sigma)}_{p^+}z = D^{\tau(1-\sigma)}_{p^+}z \in \mathcal{C}_{1-\kappa}[a,b]. \tag{9}$$

Also, since $z \in \mathcal{C}_{1-\kappa}[a,b]$, by Lemma 3,

$$I^{1-\tau(1-\sigma)}_{p^+}z \in \mathcal{C}_{1-\kappa}[a,b]. \tag{10}$$

It follows from equations (9) and (10) that

$$I^{1-\tau(1-\sigma)}_{p^+}z \in \mathcal{C}^{1}_{1-\kappa}[a,b].$$

Thus, z and $I^{1-\tau(1-\sigma)}_{p^+}z$ verify the conditions of Lemma 4.

Next, applying $I^{1-\tau(1-\sigma)}_{p^+}$ to either sides of Equation (8), and using Def. 3, and Lemma 4, it implies that

$$\begin{aligned} D^{\sigma,\tau}_{p^+}[\varphi(s) - f(s,\varphi(s))] &= z(s) \\ &\quad - \frac{\left[I^{1-\tau(1-\sigma)}_{p^+}z(s)\right](p^+)}{\Gamma(\tau(1-\sigma))}(s-p)^{\tau(1-\sigma)-1}. \end{aligned} \tag{11}$$

Since, $1-\kappa < 1-\tau(1-\sigma)$, using Lemma 5, we get

$$\left[I^{1-\tau(1-\sigma)}_{p^+}z(s)\right](p^+) = 0.$$

Hence, (11) reduces to

$$D^{\sigma,\tau}_{p^+}[\varphi(s) - f(s,\varphi(s))] = z(s), \qquad s \in \mathcal{J}.$$

Now, in order to show that equation (5) also holds, apply $I^{1-\kappa}_{p^+}$ to either sides of (7), then by means of Lemma 1 and Lemma 2, we get

$$I^{1-\kappa}_{p^+}\varphi(s) = \varphi_0 + I^{1-\kappa}_{p^+}f(s,\varphi(s)) + I^{1-\kappa}_{p^+}I^{\sigma}_{p^+}z(s).$$

Since, $1-\kappa < 1-\tau(1-\sigma)$ it follows from Lemma 5, when taking the limit as $s \to p^+$

$$I^{1-\kappa}_{p^+}[\varphi(s) - f(s,\varphi(s))](p^+) = \varphi_0.$$

This completes the proof. □

3. Main result

Taking into account the following theories, we can demonstrate the main result:

Q1: Following weak contraction condition holds for the function $f(s,\cdot)$

$$|f(s,v) - f(s,w)| \le \psi_{\mathcal{D}}(|v-w|),$$

for all $s \in \mathcal{J}$ and $v,w \in \mathbb{R}$, where $\psi_{\mathcal{D}}(s)$ is a $\mathcal{D}-$function.

Q2: For every $s \in \mathcal{J}$ and $v \in \mathbb{R}$, there is a continuous function $\xi \in \mathcal{C}_{1-\kappa}(\mathcal{J},\mathbb{R})$, where

$$|g(s,v)| \le \xi(s).$$

Theorem 2. *Suppose that the assumptions (Q1) and (Q2) are fulfilled. If $(q-p)^{1-\kappa} < 1$, then the IVP (1)–(2) possesses a solution in $\mathcal{C}_{1-\kappa}(\mathcal{J},\mathbb{R})$.*

Proof. Let $\mathcal{K} = \left\{\varphi \in \mathcal{C}_{1-\kappa}(\mathcal{J},\mathbb{R}) : \|\varphi\|_{\mathcal{C}_{1-\kappa}} \le \eta\right\}$, where

$$\eta \ge \frac{\frac{|\varphi_0|}{\Gamma(\kappa)} + \Lambda + \frac{(q-p)^{\sigma}\beta(\kappa,\sigma)}{\Gamma(\sigma)}\|\xi\|_{\mathcal{C}_{1-\kappa}}}{1-(q-p)^{1-\kappa}},$$

$(q-p)^{1-\kappa} < 1$ and $\Lambda = \max_{t\in\mathcal{J}}(s-p)^{1-\kappa} f(s,0)$. Certainly, $\mathcal{K} \subset \mathcal{C}_{1-\kappa}(\mathcal{J},\mathbb{R})$ which is closed, convex, and bounded in nature.

Now, in order to establish the existence result for the IVP (1)–(2), let us consider the generalized IVP involving HFD operator of order σ and type τ, for $z \in \mathcal{K}$, as follows:

$$D^{\sigma,\tau}_{p^+}[\varphi(s) - f(s,\varphi(s))] = g(s,z(s)), \qquad s \in \mathcal{J}, \tag{12}$$

$$I^{1-\kappa}_{p^+}[\varphi(s) - f(s,\varphi(s))](p^+) = \varphi_0, \tag{13}$$

where $0<\sigma<1, 0\le\tau\le 1$ and $\kappa = \sigma+\tau-\sigma\tau$, $\mathcal{J} = [p,q]$, for some fixed $p,q \in \mathbb{R}^+$ and $f,g \in \mathcal{C}(\mathcal{J}\times\mathbb{R},\mathbb{R})$.

Assume that the hypotheses (Q1) and (Q2) hold for the functions f and g. Using Lemma 9, we get the equivalent non-linear VIE to the IVP (12)–(13) as

$$\varphi(s) = \frac{\varphi_0}{\Gamma(\kappa)}(s-p)^{\kappa-1} + f(s,\varphi(s)) + \frac{1}{\Gamma(\sigma)}\int_p^s (s-w)^{\sigma-1} g(s,z(w))\,dw. \tag{14}$$

Consider the operators $\Delta : \mathcal{C}_{1-\kappa}(\mathcal{J},\mathbb{R}) \to \mathcal{C}_{1-\kappa}(\mathcal{J},\mathbb{R})$ and $\Omega : \mathcal{K} \to \mathcal{C}_{1-\kappa}(\mathcal{J},\mathbb{R})$ defined as follows:

$$(\Delta\varphi)(s) = \frac{\varphi_0}{\Gamma(\kappa)}(s-p)^{\kappa-1} + f(s,\varphi(s)), \quad s \in \mathcal{J}, \tag{15}$$

and

$$(\Omega z)(s) = \frac{1}{\Gamma(\sigma)}\int_p^s (s-w)^{\sigma-1} g(w,z(w))\,dw, \tag{16}$$

$s \in \mathcal{J}$.

Hence, the equation (14) can be transformed as

$$\varphi(s) = \Delta\varphi(s) + \Omega z(s), \qquad s \in \mathcal{J}. \tag{17}$$

Next, we will show that the operators Δ and Ω satisfy all the conditions of Theorem 1. The proof is divided into the following steps.

Step I: Operator Δ is non-linear $\mathcal{D}-$contraction.

Let $v,w \in \mathcal{C}_{1-\kappa}(\mathcal{J},\mathbb{R})$, by hypothesis (Q1), we have

$$\begin{aligned}
\left|(s-p)^{1-\kappa}\right.&\left.(\Delta v(s) - \Delta w(s))\right| \\
&= (s-p)^{1-\kappa}|f(s,v(s)) - f(s,w(s))| \\
&\le (s-p)^{1-\kappa}\psi_D(|v(s)-w(s)|) \\
&\le (s-p)^{1-\kappa}\psi_D \\
&\quad\cdot\left((s-p)^{\kappa-1}\|v-w\|_{\mathcal{C}_{1-\kappa}}\right).
\end{aligned}$$

Applying a maximum over s gives

$$\|\Delta v - \Delta w\|_{\mathcal{C}_{1-\kappa}} \le \psi_D\left(\|v-w\|_{\mathcal{C}_{1-\kappa}}\right).$$

Hence, the operator Δ is a non-linear contraction.

Step II: Operator Ω is a continuous and compact on $\mathcal{K}$ into $\mathcal{C}_{1-\kappa}(\mathcal{J},\mathbb{R})$.

First, to show Ω is continuous on $\mathcal{K}$, consider a sequence $\{z_n\}$ in $\mathcal{K}$ converging to $z \in \mathcal{K}$. Then, it implies from the Lebesgue-dominated convergence theorem that

$$\begin{aligned}
&\lim_{n\to\infty}\Omega z_n(s) \\
&= \lim_{n\to\infty}\frac{1}{\Gamma(\sigma)}\int_p^s (s-w)^{\sigma-1} g(w,z_n(w))\,dw \\
&= \frac{1}{\Gamma(\sigma)}\int_p^s (s-w)^{\sigma-1}\lim_{n\to\infty} g(w,z_n(w))\,dw \\
&= \frac{1}{\Gamma(\sigma)}\int_p^s (s-w)^{\sigma-1} g(w,z(w))\,dw \\
&= \Omega z(s), \qquad \forall s \in \mathcal{J}.
\end{aligned}$$

This proves the continuity for Ω. Next, we prove that $\Omega\mathcal{K}$ is a uniformly bounded and equicontinuous set in $\mathcal{K}$ which implies the compactness of Ω on $\mathcal{K}$.

It follows from the hypothesis (Q2) that

$$\begin{aligned}
&\left|(s-p)^{1-\kappa}\Omega z(s)\right| \\
&= \left|\frac{(s-p)^{1-\kappa}}{\Gamma(\sigma)}\int_p^s (s-w)^{\sigma-1} g(w,z(w))\,ds\right| \\
&\le \frac{(s-p)^{1-\kappa}}{\Gamma(\sigma)}\int_p^s (s-w)^{\sigma-1}|g(w,z(w))|\,ds \\
&\le \frac{(s-p)^{1-\kappa}}{\Gamma(\sigma)}\int_p^s (s-w)^{\sigma-1}\xi(w)\,ds
\end{aligned}$$

$$\begin{aligned}&\left|(s-p)^{1-\kappa}\Omega z(s)\right|\\&\leq \frac{(s-p)^{1-\kappa}}{\Gamma(\sigma)}\int_p^s (s-w)^{\sigma-1}(s-p)^{\kappa-1}\|\xi\|_{C_{1-\kappa}}\,ds\\&\leq \frac{(q-p)^{\sigma}\beta(\kappa,\sigma)}{\Gamma(\sigma)}\|\xi\|_{C_{1-\kappa}},\quad \forall\ s\in\mathcal{J}.\end{aligned}$$

Taking maximum over s, we get

$$\|\Omega z\|_{C_{1-\kappa}}\leq \frac{(q-p)^{\sigma}\beta(\kappa,\sigma)}{\Gamma(\sigma)}\|\xi\|_{C_{1-\kappa}}.$$

This proves the uniform boundedness of the operator Ω on $\mathcal{K}$. Let $s_1,s_2\in\mathcal{J}$ with $s_1<s_2$. For any $z\in\mathcal{K}$, one has

$$\begin{aligned}&\left|(s_1-p)^{1-\kappa}\Omega z(s_1)-(s_2-p)^{1-\kappa}\Omega z(s_2)\right|\\&=\left|\frac{(s_1-p)^{1-\kappa}}{\Gamma(\sigma)}\int_p^{s_1}(s_1-w)^{\sigma-1}g(w,z(w))\,dw\right.\\&\quad\left.-\frac{(s_2-p)^{1-\kappa}}{\Gamma(\sigma)}\int_p^{s_2}(s_2-w)^{\sigma-1}g(w,z(w))\,dw\right|\\&\leq \frac{\|\xi\|_{C_{1-\kappa}}\beta(\kappa,\sigma)}{\Gamma(\sigma)}\left|(s_1-p)^{1-\kappa}(s_1-p)^{\sigma+\kappa-1}\right.\\&\quad\left.-(s_2-p)^{1-\kappa}(s_2-p)^{\sigma+\kappa-1}\right|\\&\leq \frac{\|\xi\|_{C_{1-\kappa}}\beta(\kappa,\sigma)}{\Gamma(\sigma)}\left|(s_1-p)^{\sigma}-(s_2-p)^{\sigma}\right|.\end{aligned}$$

Thus, there exists a $\varepsilon>0$, for some η such that

$$\begin{aligned}|s_2-s_1|\leq\varepsilon\ \Rightarrow\ &\left|(s_1-p)^{1-\kappa}\Omega z(s_1)\right.\\&\left.-(s_2-p)^{1-\kappa}\Omega z(s_2)\right|\leq\eta,\end{aligned}$$

for all $s_2,s_1\in\mathcal{J}$ and $z\in\mathcal{K}$. This proves the equicontinuity of $\Omega\mathcal{K}$ in $\mathcal{C}_{1-\kappa}(\mathcal{J},\mathbb{R})$ and hence, by Arzela–Ascoli theorem, it is compact.

Step III: There exists $\varphi\in\mathcal{K}$ such that $\varphi=\Delta\varphi+\Omega z$ for all $z\in\mathcal{K}$.

Let $\varphi\in\mathcal{C}_{1-\kappa}(\mathcal{J},\mathbb{R})$ and $z\in\mathcal{K}$, such that $\varphi=\Delta\varphi+\Omega z$. Using Hypothesis (Q1), we can write

$$\begin{aligned}&\left|(s-p)^{1-\kappa}\varphi(s)\right|\\&=\left|(s-p)^{1-\kappa}(\Delta\varphi(s)+\Omega z(s))\right|\\&\leq\left|(s-p)^{1-\kappa}\Delta\varphi(s)\right|+\left|(s-p)^{1-\kappa}\Omega z(s)\right|\\&\leq\frac{|\varphi_0|}{\Gamma(\kappa)}+\left|(s-p)^{1-\kappa}f(s,\varphi(s))\right|\\&\quad+\left|\frac{(s-p)^{1-\kappa}}{\Gamma(\sigma)}\int_p^s(s-w)^{\sigma-1}g(w,z(w))\,dw\right|\end{aligned}$$

$$\begin{aligned}&\left|(s-p)^{1-\kappa}\varphi(s)\right|\\&\leq\frac{|\varphi_0|}{\Gamma(\kappa)}+(s-p)^{1-\kappa}\left|f(s,\varphi(s))-f(s,0)\right|\\&\quad+(s-p)^{1-\kappa}\left|f(s,0)\right|\\&\quad+\left|\frac{(s-p)^{1-\kappa}}{\Gamma(\sigma)}\int_p^s(s-w)^{\sigma-1}g(w,z(w))\,dw\right|\\&\leq\frac{|\varphi_0|}{\Gamma(\kappa)}+(q-p)^{1-\kappa}\psi_D\left(\|\varphi\|_{C_{1-\kappa}}\right)+\Lambda\\&\quad+\frac{(q-p)^{\sigma}\beta(\kappa,\sigma)}{\Gamma(\sigma)}\|\xi\|_{C_{1-\kappa}}\\&\leq\frac{|\varphi_0|}{\Gamma(\kappa)}+(q-p)^{1-\kappa}\psi_D(\eta)+\Lambda\\&\quad+\frac{(q-p)^{\sigma}\beta(\kappa,\sigma)}{\Gamma(\sigma)}\|\xi\|_{C_{1-\kappa}}\\&\leq\frac{|\varphi_0|}{\Gamma(\kappa)}+(q-p)^{1-\kappa}\eta+\Lambda\\&\quad+\frac{(q-p)^{\sigma}\beta(\kappa,\sigma)}{\Gamma(\sigma)}\|\xi\|_{C_{1-\kappa}}.\end{aligned}$$

Applying maximum over s, we get

$$\begin{aligned}\|\varphi\|_{C_{1-\kappa}}&\leq\frac{|\varphi_0|}{\Gamma(\kappa)}+(q-p)^{1-\kappa}\eta+\Lambda\\&\quad+\frac{(q-p)^{\sigma}\beta(\kappa,\sigma)}{\Gamma(\sigma)}\|\xi\|_{C_{1-\kappa}}\\&\leq\eta.\end{aligned}$$

Hence, $\varphi\in\mathcal{K}$. Thus, all the three assumptions of the Theorem 1 are hold, which implies that there exists $\varphi\in\mathcal{K}$ such that $\Delta\varphi+\Omega\varphi=\varphi$. Thus, the operator $\Delta+\Omega$ has a fixed point in $\mathcal{K}$. Hence, the IVP (1)–(2) owns a solution in $\mathcal{C}_{1-\kappa}(\mathcal{J},\mathbb{R})$. ☐

Remark 1. *For $\tau=1$ the IVP (1)–(2) reduces to a problem with the fractional derivative operator of Caputo type. Hence, the results obtained in the Theorem 2 coincide with the results published in.*[32, 33]

4. Application

Consider the following IVP:

$$D_{0^+}^{\frac{1}{2},\frac{1}{2}}\left[\varphi(s)-\ln(1+|\varphi(s)|)\right]=\frac{1}{s^2+1}\sin(|\varphi(s)|), \tag{18}$$

$$s\in\mathcal{J}=(0,\frac{1}{2}],$$

$$I_{0^+}^{1-\frac{3}{4}}\left[\varphi(s)-\ln(1+|\varphi(s)|)\right](0^+)=2, \tag{19}$$

Here, $\sigma=\frac{1}{2},\tau=\frac{1}{2},\kappa=\frac{3}{4},p=0,q=\frac{1}{2},\varphi_0=2,$

$$f(s,\varphi(s))=\ln(1+|\varphi(s)|),$$

and

$$g\left(s,\varphi\left(s\right)\right)=\frac{1}{s^2+1}\sin(|\varphi(s)|).$$

Hence, we find that

$$\left|f\left(s,\varphi\right)-f\left(s,\bar{\varphi}\right)\right|\leq\left|\ln(1+|\varphi|)-\ln(1+|\bar{\varphi}|)\right|$$

$$\begin{aligned}\left|f\left(s,\varphi\right)-f\left(s,\bar{\varphi}\right)\right| &= \ln\frac{1+|\varphi|}{1+|\bar{\varphi}|}\\ &= \ln\left(1+\frac{|\varphi|-|\bar{\varphi}|}{1+|\bar{\varphi}|}\right)\\ &\leq \ln\left(1+\frac{|\varphi-\bar{\varphi}|}{1+|\bar{\varphi}|}\right)\\ &\leq \ln\left(1+|\varphi-\bar{\varphi}|\right)\\ &\leq \psi_{\mathcal{D}}\left(|\varphi-\bar{\varphi}|\right),\end{aligned}$$

for each $s\in(0,\frac{1}{2}]$, where $\psi_{\mathcal{D}}(s)=\ln(1+s)$ is a $\mathcal{D}-$function, such that $0<\psi_{\mathcal{D}}(0)=0$ and $\psi_{\mathcal{D}}(s)<s$, that is $h(s)=s-\ln(1+s)$, we have $h'(s)=1-\frac{1}{1+s}=s>0$, for $s\in(0,\frac{1}{2}]$. Also,

$$\left|g\left(s,\varphi\left(s\right)\right)\right|\leq\frac{1}{s^2+1}=\xi(s).$$

Additionally, $(q-p)^{1-\kappa}\approx 0.84<1$. Therefore, all assumptions of Theorem 2 are satisfied, it implies that the IVP (18)–(19) possesses a solution in $\mathcal{C}_{1-\kappa}\left(\mathcal{J},\mathbb{R}\right)$.

5. Conclusions

In this article, a Hilfer FDE with linear perturbations of the second type under a certain initial condition was discussed for the existence of the solution. An equivalent VIE is obtained for the IVP (1)–(2). The main result on the existence of the solution to IVP was established employing the Krasnoselskii–Dhage type fixed-point theorem in the weighted Banach space. At the end, an application was given to check the validity of the outcomes. The initial value problem discussed in this paper along with the fractional derivative operator considered in the IVP (1)–(2) generalizes the existing results available in the literature. In IVP (1)–(2) if we put $\tau=1$, then the results obtained in the paper coincide with the results published in.[32,33] However, it could be interesting to see the impact of the inclusion of the control term in IVP (1)–(2). This idea may be an open problem for future investigations.

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Conflict of interest

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