An International Journal of Optimization and Control: Theories & Applications
ISSN: 2146-0957 eISSN: 2146-5703
Vol.15, No.3, pp.396-406 (2025)
https://doi.org/10.36922/IJOCTA025090036

RESEARCH ARTICLE

Using infinitesimal symmetries for determining the first Maxwell time of geometric control problem on SH(2)

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ARTICLE INFO

Article History:
Received: February 25, 2025
Revised: April 1, 2025
Accepted: April 14, 2025
Published Online: April 29, 2025

Keywords:
Geometric control theory
Lie algebra
Maxwell time
Special hyperbolic group
Sub-Riemannian geometry

AMS Classification 2010:
93C85; 17B66; 53C17; 22E60; 53C22

ABSTRACT

In this work, we utilize infinitesimal symmetries to compute Maxwell points which play a crucial role in studying sub-Riemannian control problems. By examining the infinitesimal symmetries of the geometric control problem on the SH(2) group, particularly through its Lie algebraic structure, we identify invariant quantities and constraints that streamline the Maxwell point computation.

1. Introduction

Describing locally optimal trajectories is crucial in dynamical systems, control theory, robotics, and other areas where precise motion control is necessary. In Sachkov,[1] Moiseev and Sachkov,[2] Sachkov,[3] several optimal control problems are investigated to determine optimal solutions and analyze the optimality of geodesics, specifically focusing on the first Maxwell time, which indicates how long a geodesic trajectory remains optimal. For more details on sub-riemannian geometry and geometric control theory, we refer the reader to Sachkov[1] and Agrachev et al.[4]

An optimal control problem concerns finding controls that steer the studied system from one state to another while minimizing certain quantities, often related to energy or time. A simple and interesting example is the geometric control problem on the Lie group SH(2) that involves controlling a system whose state space is the special Euclidean group in two dimensions. In Butt,[5] the author makes significant contributions to this problem by presenting optimal solutions derived using Pontryagin's maximum principle (PMP). Additionally, the author analyzes the discrete symmetries of this problem to understand the optimality of geodesics.

Infinitesimal symmetries are transformations that leave the fundamental equations of a system invariant. They are also powerful tools for studying problems in theoretical physics and dynamic systems, providing insight into conserved quantities and invariant structures within a system. By studying the Lie algebra of these symmetries, we can often derive conserved quantities that serve in the computation of some properties of some control systems.

In this work, we extend the analysis presented in Butt[5] by considering a control problem primarily addressed through geometric methods. We build upon both algebraic and geometric techniques Gallier and Quaintance[6] to further investigate this problem. First, we compute the infinitesimal symmetries of the sub-Riemannian problem and then apply these results to the geometric control problem on the Lie group SH(2), whose

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Lie algebra is non-nilpotent and solvable. Rather than relying on nilpotent approximations, which have been used in previous works by Hrdina and Zalabov ́,[7] we directly compute the infinitesimal symmetries. This approach enhances the accuracy of calculating the first Maxwell time and avoids potential errors associated with nilpotent approximations.

Additionally, in Agrachev and Barilari,[8] the authors classify all sub-Riemannian structures on three-dimensional Lie groups using basic differential invariants. Our study, therefore, highlights the significance of the sub-Riemannian problem on the SH(2) group, as it plays a crucial role in the broader context of three-dimensional Lie groups. We focus on the motion of a unicycle, a classical problem in geometric mechanics. These concepts can also be adjusted for use in other nonholonomic mechanical systems; see Jean,[9] Hill and Nurowski,[10] Hermans,[11] Bloch.[12]

Moreover, deep learning techniques (see Mao et al.,[13] Peng et al.[14]) have shown their potential in optimizing complex dynamical systems. Such approaches could offer new perspectives for the analysis of sub-Riemannian structures and optimal control problems.

Sub-Riemannian geodesics are generally not optimal at all times. Each geodesic has a specific point where it ceases to be optimal. At this stage, the role of the infinitesimal symmetries of our problem on the group SH(2) becomes crucial. Using the Lie algebra of symmetries, we identify transformations that map a given geodesic to another geodesic. We show that there is a subgroup of symmetries isomorphic to the group SO(1, 2). Using this action, we determine the set of Maxwell points and, subsequently, the first Maxwell time corresponding to our infinitesimal symmetries. All of this is detailed in Section 4 of our main results.

2. Preliminaries on geometric control theory

In this section, we give some preliminaries on geometric control theory. For more details on the subject, see Agrachev et al.[4]

2.1. Controllability of control affine systems

A control affine system on a manifold M is any differential system of the form

$$\frac{dx}{dt} = X_0(x) + \sum_{i=1}^{m} u_i(t)X_i(x), \qquad (1)$$

where $X_0, X_1, \dots, X_m$ are smooth vector fields on M and $u_i(t)$ are control inputs. Further details on control affine systems can be found in Jurdjevic.[15] Let $\mathcal{F}$ be a family of smooth vector fields on M, then the Lie algebra generated by $\mathcal{F}$ is the smallest Lie algebra that contains $\mathcal{F}$. It is obtained by considering all linear combinations of elements of $\mathcal{F}$, taking all Lie brackets of these, considering all linear combinations of these, and continuing so on. It will be denoted by $Lie(\mathcal{F})$ and its evaluation at any point $q \in M$ will be denoted by $Lie_q(\mathcal{F})$. The following result gives a necessary and sufficient condition for a driftless control affine system to be controllable.

Proposition 1. *(Jurdjevic[15]) The control affine system* $\frac{dx}{dt} = \sum_{i=1}^{m} u_i X_i(x)$ *with* $u = (u_1, \dots, u_m) \in \mathbb{R}^m$ *is controllable if and only if* $Lie_q\mathcal{F} = T_qM$*, for all* $q \in M$.[1]

2.2. Sub-Riemannian problem and optimal solutions

Consider a sub-Riemannian manifold (M, Δ, g), where M is an n-dimensional smooth manifold, Δ is a smooth distribution of rank $k \le n$, and g is a Riemannian metric on Δ; see Sachkov[1] for more details. The sub-Riemannian length of an admissible curve $\gamma(t)$, defined on $[0, t_1]$, is given by:

$$\ell(\gamma) = \int_0^{t_1} \sqrt{g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))}\, dt.$$

Given two points q_0 and q_1 of M, the sub-Riemannian distance between q_0 and q_1 is stated by

$$d(q_0, q_1) = \inf\{\ell(\gamma) \mid \gamma \text{ admissible}, \\ \gamma(0) = q_0, \gamma(t_1) = q_1\}.$$

A sub-Riemannian problem is a control problem on a sub-Riemannian manifold (M, Δ, g) where one seek for admissible curves γ that satisfy the property $\ell(\gamma) = d(q(0), q(t_1))$. Suppose there exists a family of smooth vector fields $X_1, \dots, X_k$ that forms an orthonormal frame on (Δ, g), i.e. $\forall q \in M, \Delta_q = span\{X_1(q), \dots, X_k(q)\}$ and $g_q(X_i(q), X_j(q)) = \delta_{ij}$. Thus, sub-Riemannian minimizers (or optimal solutions) are the solutions of the following optimal control problem on M:

$$\dot x = \sum_{i=1}^{k} u_i X_i(x), \quad u = (u_1, ..., u_k) \in \mathbb{R}^k, \qquad (2)$$

$$x(0) = q_0, \quad x(t_1) = q_1, \qquad (3)$$

$$\ell = \int_0^{t_1} (\sum_{i=1}^{k} u_i^2)^{1/2} dt \longrightarrow \min. \qquad (4)$$

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[1] Here T_qM denotes the tangent space to M at the point q

If we add the condition $\sum_{i=1}^{k} u_i^2 \leq 1$, the system given by equations Equations(2)-(4), becomes:

$$\begin{aligned}
&\dot{x} = \sum_{i=1}^{k} u_i X_i(x), \quad \sum_{i=1}^{k} u_i^2 \leq 1, \\
&x(0) = q_0, \quad x(t_1) = q_1, \\
&t_1 \to \min .
\end{aligned} \tag{5}$$

By Filipov's theorem,[1] the existence of optimal solutions for the optimal control problem Equation (5), is guaranteed.

The Pontryagin Maximum Principle[4] provides necessary conditions for the optimality of trajectory solutions of sub-Riemannian control problems. Following this principle, we first compute trajectories, called extremals, of a dynamic system in the cotangent bundle of the variety M. Then the projections of the extremals on the state space M constitute the optimal solutions and are called geodesics. A point $\gamma(t_1)$ is called a Maxwell point along a geodesic γ if there exists another geodesic $\widetilde{\gamma} \not\equiv \gamma$ such that $\gamma(t_1) = \widetilde{\gamma}(t_1)$. This means that there exists a geodesic $\widehat{\gamma}$ coming to the point $q_1 = \gamma\,(t_1)$ earlier than γ.

3. Computing infinitesimal symmetries of a sub-Riemannian problem

Let (M, Δ, g) be a sub-Riemannian manifold, where M is endowed with a Lie group structure of dimension n that is left-invariant. This means that

$$\begin{aligned}
&\Delta_{ab} = L_{a*}\Delta_b, \\
&g_b(v, w) = g_{ab}(L_{a*}v, L_{a*}w), \quad \forall a, b \in M.
\end{aligned}$$

where L_{a*} denotes the differential of the left translation by a. We consider the sub-Riemannian system Equations(2)-(4) on the Lie group (M, Δ, g) and we assume it is controllable. Then $Lie_q\Delta = T_qM$ for every $q \in M$. Therefore,

$$\begin{aligned}
Lie_q(\Delta) = \operatorname{span}\{&X_1(q), \ldots, X_k(q), \\
&X_{k+1}(q), \ldots, X_n(q)\}
\end{aligned} \tag{6}$$

for all $q \in M$, where Δ is given by the vector fields $X_1, \ldots, X_k$ and $X_{k+1}, \ldots, X_n \in Lie(\Delta)$. Furthermore, suppose the problem admits optimal solutions.

Definition 1. *(Agrachev and Barilari.[16]) Let M be a $2m+1$ dimensional manifold. A sub-Riemannian structure on M is said to be contact if Δ is a contact distribution, i.e. $\Delta = \ker \omega$, where $\omega \in \Lambda^1 M$ satisfies $(\bigwedge^m d\omega) \wedge \omega \neq 0$.*

By reference to Olver,[17] which provides us with an important result regarding the conditions that a symmetry vector must satisfy, we have that:

A vector v is a symmetry of our system if it satisfies the following two conditions:

$$\mathcal{L}_v(\Delta) \subset \Delta \quad \text{and} \quad \mathcal{L}_v(g) = 0. \tag{7}$$

where $\mathcal{L}$ denotes the Lie derivative. In what follows, we outline the necessary steps for computing the symmetry algebra of a system defined by Equations (2)-Eq.(4). We consider $\Delta = \{\Delta_q \subset T_qM \mid q \in M\}$, where $\Delta_q = \operatorname{span}\{X_1(q), \ldots, X_k(q)\}$. Additionally, $\mathrm{g} = \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{a_i dq_i} \otimes \mathrm{dq_i}$, where a_i are some constants, represents a Riemannian metric such that $\mathrm{g(X_i, X_j)} = \delta_{\mathrm{ij}}$. Furtheremore, we suppose that Δ is a contact distribution, meaning that $\ker(\omega) = \Delta$, where $\omega \in \Lambda^1(M)$ is a 1-form in the manifold M. In terms of the local coordinates $q_1, \ldots, q_n$, the contact form can be written as $\omega = \sum_{i=1}^{n} e_i\, dq_i$, where e_i are scalar functions of the coordinates $q_1, \ldots, q_n$. An arbitrary vector field v can be expressed as $v = h_1(q_1, \ldots, q_n)X_1 + \ldots + h_n(q_1, \ldots, q_n)X_n$ and it is a symmetry vector if it satisfies the two preceding conditions Equation (7). We recall that in local coordinates, each X_j is of the form $X_j = \sum_{i=1}^{n} g_i^j \partial q_i$, where g_i^j are some scalar functions defined on M. Our approach begins by calculating $\mathcal{L}_v(X_1), \ldots, \mathcal{L}_v(X_k)$:

$$\begin{aligned}
\mathcal{L}_v(X_j) &= [v, X_j] \\
&= [h_1X_1 + \ldots + h_nX_n, X_j] \\
&= -[X_j, h_1X_1 + \ldots + h_nX_n] \\
&= -h_1[X_j, X_1] - (X_jh_1)X_1 - \ldots \\
&- h_j[X_j, X_j] \\
&- (X_jh_j)X_j \quad - \ldots - h_n[X_j, X_n] \\
&- (X_jh_n)X_n \\
&= -h_1 \sum_{i=1}^{n} s_i \frac{\partial}{\partial q_i} - \left(\sum_{i=1}^{n} g_i^j \frac{\partial h_1}{\partial q_i}\right) \sum_{i=1}^{n} g_i^1 \frac{\partial}{\partial q_i} \\
&\quad - \ldots - \left(\sum_{i=1}^{n} g_i^j \frac{\partial h_j}{\partial q_i}\right) \sum_{i=1}^{n} g_i^j \frac{\partial}{\partial q_i} \\
&\quad - \ldots - h_n \sum_{i=1}^{n} t_i \frac{\partial}{\partial q_i} - \left(\sum_{i=1}^{n} g_i^j \frac{\partial h_n}{\partial q_i}\right) \\
&\sum_{i=1}^{n} g_i^n \frac{\partial}{\partial q_i}
\end{aligned}$$

where s_i and t_i are some scalar functions defined on M. Then, applying the condition $\omega(\mathcal{L}_v(X_j)) = 0$ yields an equation of the form:

$$\sum_{i \neq j}^{n} b_i h_i + \sum_{i=1}^{n} c_i \frac{\partial h_1}{\partial q_i} + \ldots + \sum_{i=1}^{n} d_i \frac{\partial h_n}{\partial q_i} = 0, \tag{8}$$

where b_i, c_i and d_i are some scalar functions. Extending the procedure to the all vector fields

$X_1, \dots, X_k$, this yields a system of order k, consisting of equations of the form Equation (8). Next, we compute the Lie derivative of g. We have:

$$\begin{aligned}\mathcal{L}_v(\mathrm{g}) &= \mathcal{L}_v\left(\sum_{i=1}^{n} a_i\, dq_i \otimes dq_i\right)\\ &= \sum_{i=1}^{n} a_i\, \mathcal{L}_v(dq_i \otimes dq_i)\\ &= \sum_{i=1}^{n} 2a_i\, dq_i\, \mathcal{L}_v(dq_i)\\ &= \sum_{i=1}^{n} 2a_i\, dq_i\, \left(i(v)\, d(dq_i) + d(i(v)\, dq_i)\right)\\ &= \sum_{i=1}^{n} 2a_i\, dq_i\, d(i(v)\, dq_i)\\ &= \sum_{i=1}^{n} 2a_i\, dq_i\, d(dq_i(v))\end{aligned}$$

where $i(v)$, is the interior product. Condition $\mathcal{L}_v(\mathrm{g}) = 0$ provides a system of order $\frac{n(n+1)}{2}$, consisting of equations of the form:

$$\sum_{i=1}^{n} k_i h_i + \sum_{i=1}^{n} l_i \frac{\partial h_1}{\partial q_i} + \ldots + \sum_{i=1}^{n} r_i \frac{\partial h_n}{\partial q_i} = 0, \quad (9)$$

where k_i, l_i, and r_i are some scalar functions. Soving the system associated with Equations (8) and (9), we determine the functions h_i and, consequently, the generators v_i of our Lie algebra of symmetries. As a consequence, we can state the following result:

Proposition 2. *The infinitesimal symmetries of control system Equations (2)-(4) can be identified by calculating the flow associated with the vector field v_i at time s, denoted by $\gamma_i(s,t) = (q_1(s,t), \dots, q_n(s,t))$. This flow is obtained by solving the system of differential equations:*

$$\frac{\partial \gamma_i(s,t)}{\partial s} = v_i(\gamma_i(s,t)), \quad \gamma_i(0,t) = (q_1(t), \dots, q_n(t)).$$

4. Infinitesimal symmetries on SH(2) and the first Maxwell time

4.1. Geometric control problem on SH(2)

The motion of a unicycle on a hyperbolic plane can be described using the following driftless control system

$$\begin{cases} \dot{x} = u_1 \cosh z, \\ \dot{y} = u_1 \sinh z, \\ \dot{z} = u_2, \end{cases} \quad (10)$$

where u_1 is the translational velocity and u_2 is the angular velocity. The configuration and state manifold M of the system is three-dimensional, where, for any point $q = (x, y, z) \in M$, x and y are position variables and z is the angular orientation variable of the unicycle on the hyperbolic plane. A motion $m(x, y, z)$ on the configuration manifold M, parameterized by $x, y, z \in \mathbb{R}$, is a transformation that maps a point $\mathbf{a}(a_1, a_2)$ to a point $\mathbf{b}(b_1, b_2)$, such that:

$$\begin{aligned} b_1 &= a_1 \cosh z + a_2 \sinh z + x, \\ b_2 &= a_1 \sinh z + a_2 \cosh z + y. \end{aligned}$$

Composition of two motions $m_1(x_1, y_1, z_1)$ and $m_2(x_2, y_2, z_2)$ is another motion $m_3(x_3, y_3, z_3)$ given as:

$$m_3(x_3, y_3, z_3) = m_1(x_1, y_1, z_1).m_2(x_2, y_2, z_3)$$

where,

$$\begin{aligned} x_3 &= x_2 \cosh z_1 + y_2 \sinh z_1 + x_1, \\ y_3 &= x_2 \sinh z_1 + y_2 \cosh z_1 + y_1, \\ z_3 &= z_1 + z_2 \end{aligned}$$

The identity motion m_{Id} is given by $x = y = z = 0$, and inverse of a motion $m(x, y, z)$ is given by $m^{-1}(x^1, y^1, z^1)$ where,

$$\begin{aligned} x^1 &= -x \cosh z + y \sinh z, \\ y^1 &= x \sinh z - y \cosh z, \\ z^1 &= -z. \end{aligned}$$

The motions of the pseudo-Euclidean plane exhibit a group structure, with composition serving as the group operation. This group, known as the special hyperbolic group SH(2), also possesses a smooth manifold structure, which qualifies it as a Lie group. One can equivalently formulate problem Equation (10) as a sub-Riemannian problem on SH(2):

$$\begin{aligned} &\dot{q} = u_1 X_1(q) + u_2 X_2(q), \\ &q \in M = \mathrm{SH}(2), \quad u = (u_1, u_2) \in \mathbb{R}^2, \\ &q(0) = q_0 = m_{Id}, \quad q(t_1) = q_1, \\ &J = \frac{1}{2}\int_0^{t_1} \left(u_1^2 + u_2^2\right) dt \to \min \end{aligned} \quad (11)$$

where,

$$X_1 = \cosh z\, \partial_x + \sinh z\, \partial_y, \quad X_2 = \partial_z.$$

The problem under consideration is formulated on a sub-Riemannian manifold (M, Δ, g). Here, M represents the underlying smooth manifold, $\Delta = \mathrm{span}\{X_1, X_2\}$ denotes a distribution of rank 2, and g is a Riemannian metric defined on Δ such that $\mathrm{g(X_i, X_j)} = \delta_{\mathrm{ij}}$, explicitly $g = dx^2 - dy^2 + dz^2$. The vector fields X_1 and X_2, which span the distribution Δ, are left-invariant with respect to the group structure associated with M. One can see that the family

$\mathcal{F}=\{u_1X_1+u_2X_2 | u=(u_1,u_2)\in\mathbb{R}^2\}$ is symmetric. Computing the vector field $X_3=[X_1,X_2]$, we get $X_3=[X_1,X_2]=-\sinh z\partial_x-\cosh z\partial_y$. Since X_1, X_2, and X_3 are linearly independent, we have

$$Lie_q(\mathcal{F})=span(X_1(q),X_2(q),X_3(q))=T_q(SH(2)),$$

for every point q. This implies that system Equation (11), is controllable. After transforming this system into an equivalent time-optimal problem with controls
$u_1^2+u_2^2\leqslant 1$, one can observe that

$$|u_1X_1+u_2X_2|\leqslant c|q|,\quad q\in M.$$

This gives the inequality $|q(t)|\leqslant q_0\exp(tc)$. Therefore, the attainable sets satisfy the following a priori bound:

$$\mathcal{A}_{q_0}(\leq t_1)\subset\{q\in M \mid |q|\leqslant q_0\exp(t_1c)\}.$$

Thus, according to Filipov's theorem (^{1}), optimal controls exist.

4.2. Computing extremal trajectories

We begin by applying the Pontryagin Maximum Principle,[4] which allows to determine the extremal trajectories of system Equation(11). For this purpose, we define the functions $h_i(\lambda)=\langle\lambda,X_i\rangle\quad i=1,2,3$, where $\lambda\in T^*M$. These functions (h_1,h_2,h_3), form a coordinate system on the fibers of T^*M. Consequently, we adopt the global coordinates (q,h_1,h_2,h_3) to describe the structure of T^*M. The Hamiltonian from the Pontryagin Maximum Principle (PMP) is given by:

$$h_u^v(\lambda)=\frac{\nu}{2}\left(u_1^2+u_2^2\right)+u_1h_1(\lambda)+u_2h_2(\lambda),$$

where, $u=(u_1,u_2)\in\mathbb{R}^2$, and $v\in\{-1,0\}$. Then the Pontryagin maximum principle[4] for the problem under consideration reads as follows.

Theorem 1. *(^{4}) Let $u(t)$ and $q(t),t\in[0,t_1]$, be an optimal control and the corresponding optimal trajectory in problem Equation (11). Then there exist a Lipschitzian curve $\lambda_t\in T^*M,\pi(\lambda_t)=q(t),t\in[0,t_1]$, and a number $v\in\{-1,0\}$ for which the following conditions hold for almost all $t\in[0,t_1]$:*

$$\begin{gathered}\dot{\lambda}_t=\vec{h}_{u(t)}^v(\lambda_t),\\ h_{u(t)}^{\nu}(\lambda_t)=\max_{u\in\mathbb{R}^2}h_u^v(\lambda_t),\\ (v,\lambda_t)\neq 0\end{gathered}\tag{12}$$

Our analysis considers two distinct cases:

- Abnormal case

In this case $v=0$. Thus, the Hamiltonian of PMP for the system takes the form:
$h_u(\lambda)=u_1h_1(\lambda)+u_2h_2(\lambda)$

$$\begin{aligned}\dot{h_1}&=\{h_u(\lambda),h_1\}\\&=u_1\{h_1,h_1\}+u_2\{h_2,h_1\}\\&=u_2h_3\\ \dot{h_2}&=-u_1h_3\end{aligned}$$

Then, Abnormal extremals satisfy the Hamiltonian system:

$$\begin{aligned}\dot{h_1}&=u_2h_3\\ \dot{h_2}&=-u_1h_3\\ \dot{q}&=u_1X_1+u_2X_2\end{aligned}$$

and the hypothesis $h_u(\lambda)\longrightarrow max$ implies

$$h_1(\lambda_t)=h_2(\lambda_t)=0$$

Therefore, condition Equation (12) gives $h_3(\lambda_t)\neq 0$ and the first two equations of the Hamiltonian system yield $u_1(t)=u_2(t)=0$. So, abnormal trajectories are constant.

- Normal case

The maximality condition

$$u_1h_1+u_2h_2-\tfrac{1}{2}(u_1^2+u_2^2)\longrightarrow max$$

yields $u_1=h_1$ and $u_2=h_2$, and then the Hamiltonnianis $H=\frac{1}{2}(h_1^2+h_2^2)$. Thus, we get the system:

$$\begin{aligned}\dot{h_1}&=h_2h_3\\ \dot{h_2}&=-h_1h_3\\ \dot{h_3}&=h_1h_2\\ \dot{q}&=h_1X_1+h_2X_2\end{aligned}$$

and Hamiltonian system in the normal case is given by the equations:

$$\dot{h}_1=h_2h_3,\quad \dot{h}_2=-h_1h_3,\quad \dot{h}_3=h_1h_2\tag{13}$$

$$\dot{x}=h_1\cosh z,\quad \dot{y}=h_1\sinh z,\quad \dot{z}=h_2.\tag{14}$$

In this case, the initial covector λ lies on the initial cylinder defined by

$$C=T_{q_0}^*M\cap\{H(\lambda)=\frac{1}{2}\}.$$

This set can be explicitly described as

$$C=\{(h_1,h_2,h_3)\in\mathbb{R}^3 \mid h_1^2+h_2^2=1\},$$

representing a cylinder in the cotangent space where the Hamiltonian takes the constant value $\frac{1}{2}$. We introduce the following change of variables:

$$h_1=\cos(\alpha),\quad h_2=\sin(\alpha).$$

With these coordinates, the vertical system Equation (13) satisfies the equations:

$$\dot{h}_3 = \frac{1}{2}\sin(2\alpha), \quad \dot{\alpha} = h_3.$$

Next, we introduce another change of variables:

$$\gamma = 2\alpha \in \mathbb{R}/4\pi\mathbb{Z}, \quad c = 2h_3 \in \mathbb{R}.$$

Finally, we obtain that the equation describing our vertical system corresponds to a mathematical pendulum given by:

$$\dot{\gamma} = c, \quad \dot{c} = -\sin(\gamma).$$

The total energy integral of the pendulum obtained is given as: $E = \frac{c^2}{2} - \cos(\gamma) = 2h_3^2 - h_1^2 + h_2^2$, according to this energy, the cylinder C can be decomposed in the following way. $C = \bigcup_{i=1}^{5} C_i$, where

$$\begin{aligned}
C_1 &= \{\lambda \in C \mid E \in (-1,1)\}\\
C_2 &= \{\lambda \in C \mid E \in (1,\infty)\}\\
C_3 &= \{\lambda \in C \mid E = 1, c \neq 0\}\\
C_4 &= \{\lambda \in C \mid E = -1\}\\
C_5 &= \{\lambda \in C \mid E = 1\}
\end{aligned}$$

For a detailed derivation of the solutions to our system, refer to Butt(5).

4.3. Infinitesimal symmetries

In this subsection, we compute the symmetry algebra of the control system Equation (11).

Proposition 3. *Infinitesimal symmetries of the control system Equation (11) form a Lie algebra generated (over $\mathbb{R}$) by the vector fields:*

$$\begin{aligned}
v_1 &= -x\partial_y - y\partial_x - \partial_z,\\
v_2 &= \partial_x,\\
v_3 &= \partial_y.
\end{aligned}$$

Proof. It is clear that Δ constitutes a contact distribution. Indeed, $\Delta = \ker(\omega)$, where $\omega = \cosh(z)dy - \sinh(z) \in \Lambda^1(M)$ and we have $d\omega \wedge \omega \neq 0$. Let $v = f_1(x,y,z)X_1 + f_2(x,y,z)X_2 + f_3(x,y,z)X_3$ be a vector field. The conditions $\omega(\mathcal{L}_v(X_1)) = 0$ and $\omega(\mathcal{L}_v(X_2)) = 0$, lead to the following system:

$$f_2 + \cosh z\frac{\partial f_3}{\partial x} + \sinh z\frac{\partial f_3}{\partial y} = 0, \tag{15}$$

$$f_1 - \frac{\partial f_3}{\partial z} = 0. \tag{16}$$

Moreover, the Lie derivative of the metric g has the form:

$$\begin{aligned}
\mathcal{L}_v(\mathrm{g}) =& 2(\cosh z\frac{\partial f_1}{\partial x} - \sinh z\frac{\partial f_3}{\partial x})dx^2\\
&+ 2(\cosh z\frac{\partial f_3}{\partial y} - \sinh z\frac{\partial f_1}{\partial y})dy^2\\
&+ 2(\frac{\partial f_2}{\partial z})dz^2\\
&+ (2\cosh z\frac{\partial f_1}{\partial y} - 2\sinh z\frac{\partial f_3}{\partial y}\\
&- 2\sinh z\frac{\partial f_1}{\partial x} + 2\cosh z\frac{\partial f_3}{\partial x})dxdy\\
&+ (2f_1\sinh z + 2\cosh z\frac{\partial f_1}{\partial z} - 2f_3\cosh z\\
&- 2\sinh z\frac{\partial f_3}{\partial z} + 2\frac{\partial f_2}{\partial x})dxdz\\
&+ (-2\sinh z\frac{\partial f_1}{\partial z} - 2f_1\cosh z\\
&+ 2\cosh z\frac{\partial f_3}{\partial z} + 2f_3\sinh z + 2\frac{\partial f_2}{\partial y})dzdy
\end{aligned}$$

Now the relation $\mathcal{L}_v(\mathrm{g}) = 0$ implies:

$$\cosh z\frac{\partial f_1}{\partial x} - \sinh z\frac{\partial f_3}{\partial x} = 0, \tag{17}$$

$$\cosh z\frac{\partial f_3}{\partial y} - \sinh z\frac{\partial f_1}{\partial y} = 0, \tag{18}$$

$$\frac{\partial f_2}{\partial z} = 0, \tag{19}$$

and

$$\begin{aligned}
&2\cosh z\frac{\partial f_1}{\partial y} - 2\sinh z\frac{\partial f_3}{\partial y}\\
&-2\sinh z\frac{\partial f_1}{\partial x} + 2\cosh z\frac{\partial f_3}{\partial x} = 0,
\end{aligned}$$

and

$$\begin{aligned}
&2f_1\sinh z + 2\cosh z\frac{\partial f_1}{\partial z} - 2f_3\cosh z\\
&\qquad -2\sinh z\frac{\partial f_3}{\partial z} + 2\frac{\partial f_2}{\partial x} = 0,
\end{aligned}$$

and

$$\begin{aligned}
&-2\sinh z\frac{\partial f_1}{\partial z} - 2f_1\cosh z + 2\cosh z\frac{\partial f_3}{\partial z}\\
&\qquad +2f_3\sinh z + 2\frac{\partial f_2}{\partial y} = 0.
\end{aligned}$$

Using equation Equation (16), we have $f_1 = \frac{\partial f_3}{\partial z}$. By substituting f_1 into Equations (17) and (18), we obtain the following:

$$\begin{aligned}
\cosh z\frac{\partial^2 f_3}{\partial z\partial x} - \sinh z\frac{\partial f_3}{\partial x} &= 0,\\
\cosh z\frac{\partial f_3}{\partial y} - \sinh z\frac{\partial^2 f_3}{\partial z\partial y} &= 0.
\end{aligned} \tag{20}$$

Thus
$$\frac{\partial f_3}{\partial x} = c_1 \cosh z + f(x,y),$$
$$\frac{\partial f_3}{\partial y} = c_2 \sinh z + g(x,y).$$
By integrating Equations (17) and (18) with respect to x and y, respectively, we obtain:

$$\cosh z f_1 - \sinh z f_3 = 0,$$
$$\cosh z f_3 - \sinh z f_1 = 0.$$

Then we get:

$$\cosh z \frac{\partial f_3}{\partial z} - \sinh z f_3 = 0,$$
$$\cosh z f_3 - \sinh z \frac{\partial f_3}{\partial z} = 0.$$

Therefore, we deduce that :

$$f_3 = c_1 x \cosh z + c_2 y \sinh z + c_3 \cosh z + c_4 \sinh z,$$
$$f_1 = c_1 x \sinh z + c_2 y \cosh z + c_3 \sinh z + c_4 \cosh z.$$

Using Equation (15) we get $f_2 = -c_1$ and $c_2 = -c_1$. Hence, we obtain the following:

$$f_1 = c_1 x \sinh z - c_1 y \cosh z + c_3 \sinh z + c_4 \cosh z$$
$$f_2 = -c_1$$
$$f_3 = c_1 x \cosh z - c_1 y \sinh z + c_3 \cosh z + c_4 \sinh z$$

For $c_1 = 1$, $c_3 = c_4 = 0$, we find
$v_1 = -x\partial_y - y\partial_x - \partial_z$. □

Proposition 4. *The action of the flow of the infinitesimal transformation $v_1 = -x\partial_y - y\partial_x - \partial_z$, at the time s maps a geodesic with the initial condition $x(0) = y(0) = z(0) = 0$ to another geodesic.*

$$\begin{aligned} x &\mapsto x\cosh(s) - y\sinh(s),\\ y &\mapsto y\cosh(s) - x\sinh(s), \qquad (21)\\ z &\mapsto z - s. \end{aligned}$$

4.4. First Maxwell time corresponding to infinitesimal symmetries

The transformation Equation (??) can be viewed as the result of composing two distinct motions: the first motion $m(x, y, z)$, which represents the geodesic under consideration, and a second motion $m(0, 0, -s)$. Their combination results in a new motion $m(x', y', z')$, corresponding to a different geodesic, given by:

$$m(x', y', z') = m(0, 0, -s) \cdot m(x, y, z)$$

The motion $m(0, 0, -s)$ is a transformation in the hyperbolic plane, mapping each point (a_1, a_2) to another point (b_1, b_2), such that:

$$b_1 = a_1 \cosh s - a_2 \sinh s,$$
$$b_2 = a_2 \cosh s - a_1 \sinh s.$$

This motion can be represented in matrix form as follows:

$$N = \begin{pmatrix} \cosh(s) & -\sinh(s) & 0 \\ -\sinh(s) & \cosh(s) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

It is clear that the group generated by the matrix N is isomorphic to SO(1, 2), the Lorentz group. Therefore, Proposition (4) yields the following result:

Corollary 1. *For each $R \in \mathrm{SO}(1, 2)$, the map*

$$(x, y, z) \longmapsto R \cdot (x, y, z),$$

which represents a motion in the group M, maps geodesics starting at the origin to other geodesics. This map is defined by the composition of these two motions.

It can be noted that the set of fixed points of this action is:

$$\mathrm{S} = \{(0, 0, \mathrm{z}) \quad |\mathrm{z} \in \mathbb{R}\}$$

Proposition 5. *The intersection of our geodesics with the set* S *provides the set of Maxwell points. Consequently, the following result is obtained:*

1. *If $\lambda \in C_1 \cup C_3 \cup C_4$, the geodesics do not intersect* S *for any $t > 0$.*
2. *If $\lambda \in C_2$, the set of Maxwell points is given by: $MAX = \{(0, 0, z(4nk_0K) \quad k_0 \in (0, 1)\}$*
3. *If $\lambda \in C_5$, the set of Maxwell points is given by: $MAX = \{(0, 0, z(t)) \quad t > 0\}$*

Proof. (1) We begin with the case where $\lambda = (\varphi, k) \in C_1$, the geodesics intersect the set S if and only if: $x_t = y_t = 0$. Let $L_1 = x_t + y_t$, and $L_2 = x_t - y_t$, we then obtain the following expressions for L_1 and L_2:

$$L_1 = \omega\big((E(\varphi_t) - E(\varphi)) - k(sn(\varphi_t) - sn(\varphi))\big),$$
$$L_2 = \frac{1}{\omega(1 - k^2)}\big((E(\varphi_t) - E(\varphi)) + k(sn(\varphi_t) - sn(\varphi))\big).$$

The equations $L_1 = 0$ and $L_2 = 0$ hold if and anly if $(E(\varphi_t) - E(\varphi)) = 0$. This condition cannot be satisfied for all $t > 0$. Therefore, the geodesics do not intersect the set S. In the case where $\lambda \in C_3$, we have $L_1 = \frac{1}{\omega}(\varphi_t - \varphi)$ and $L_2 = 2\omega(tanh(\varphi_t) - tanh(\varphi))$, since both equations cannot be obtained for each $t > 0$, this leads to our result. In the last case, it is clear that there is no intersection between our geodesic and S.

(2) If $\lambda \in C_2$. It is assumed that $L_1 = x_t + y_t$ and $L_2 = x_t - y_t$, which gives us the following expressions:

$$L_1 = \omega\Big(-\left(E(\psi_t) - E(\psi)\right) + k'^2(\psi_t - \psi) \\ +k\big(sn(\psi_t) - sn(\psi)\big)\Big),$$

$$L_2 = \tfrac{1}{\omega(1-k^2)}\Big(\left(E(\psi_t) - E(\psi)\right) - k'^2(\psi_t - \psi) \\ +k\big(sn(\psi_t) - sn(\psi)\big)\Big).$$

We have $L_1 = 0$ and $L_2 = 0$. if and only if both equations, $h_1 = 2k(\big(sn(\psi_t) - sn(\psi)\big) = 0$ and $h_2 = (E(\psi_t) - E(\psi)) - k'^2(\psi_t - \psi) = 0$, hold. The equation $h_1 = 0$ holds exclusively when $t = 4nkK$ with $k \in (0, 1)$. Here $K(k) = \int_0^{\pi/2} \frac{dt}{\sqrt{1-k^2\sin^2 t}}$, $k'^2 = 1 - k^2$, for these given values, it follows that $h_2(4nkK) = 4nE(k) - 4nk'^2K$. We subsequently take $g(k) = E(k) - k'^2K$ $k \in [0, 1)$ we observe that its derivative $g'(k) = kK(k)$, indicating g increases on the interval $[0, 1)$, with $g([0, 1)) = [0, 1)$. Therefore, there exists $k_0 \in (0, 1)$ such that $g(k_0) = 0$. Since $z(4nk_0K) = 0$, the set of Maxwell points is reduced to $\{(0, 0, 0)\}$.

(3) If $\lambda \in C_5$ For each t, we have $x_t = 0$ and $y_t = 0$, with $z_t \neq 0$. Subsequently, for each strictly positive t, our geodesic intersects S. For more details on elliptic functions (see Olver[18])

□

Corollary 2. *The first Maxwell time T_1^{Max} where our geodesic loses optimality corresponding to this action is given as:*

$$\lambda \in C_1 \cup C_3 \cup C_4 \Longrightarrow T_1^{Max} = +\infty$$
$$\lambda \in C_2 \Longrightarrow T_1^{Max} = 4k_0K(k_0), \quad k_0 \in (0,1)$$
$$\lambda \in C_5 \Longrightarrow T_1^{Max} = t_0, \quad t_0 \neq 0$$

Proof. Whenever our geodesics do not intersect the set S they are optimal, and as a result $T_1^{Max} = +\infty$. In case 2, we assign the 1 to n to find the first Maxwell time. In case 5, there is always an intersection; consequently, our geodesic is no longer optimal. The first Maxwell time is a certain non-zero $t \neq 0$. □

Here, we include several figures and some numerical verification to better understand our method.

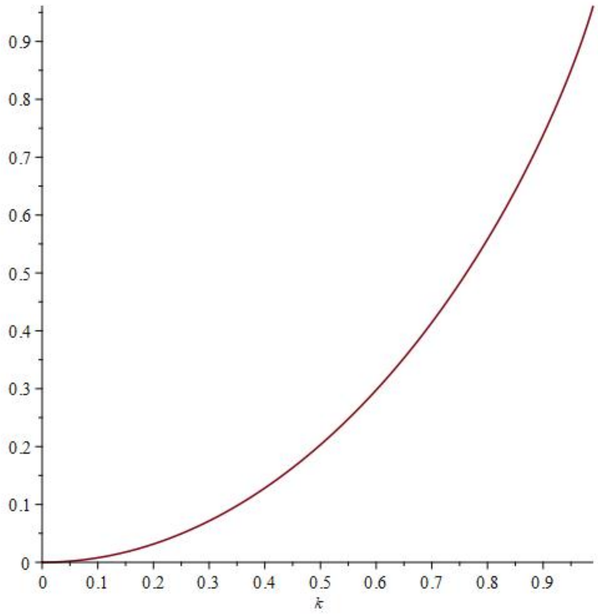

Figure 1. The graph of the function g

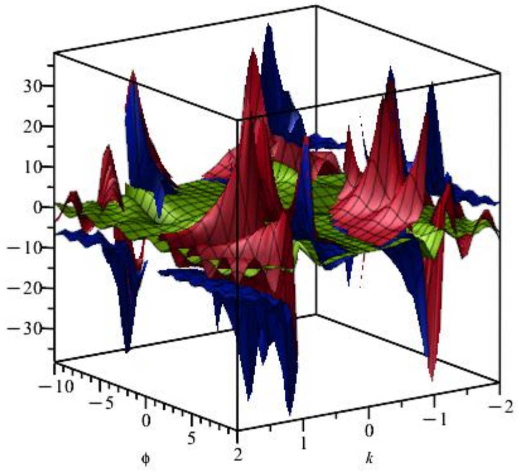

Figure 2. Local minimizer, in the case $\lambda = (\varphi, k) \in C_1$

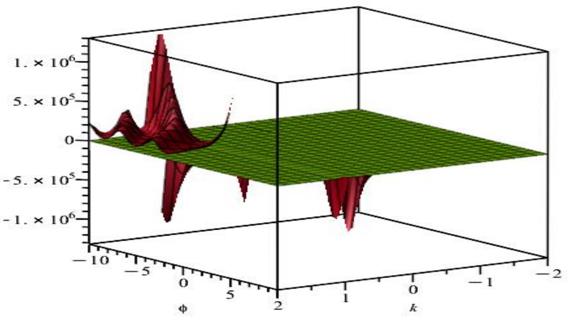

Figure 3. The trajectory's symmetric in the case $\lambda \in C_1$

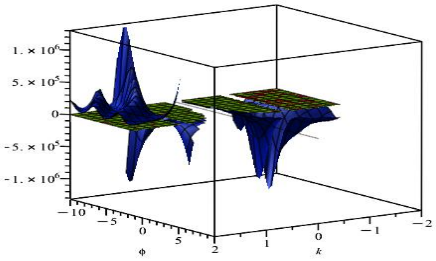

Figure 4. The trajectory $(x(t), y(t))$ and its symmetry, in the case $\lambda \in C_1$

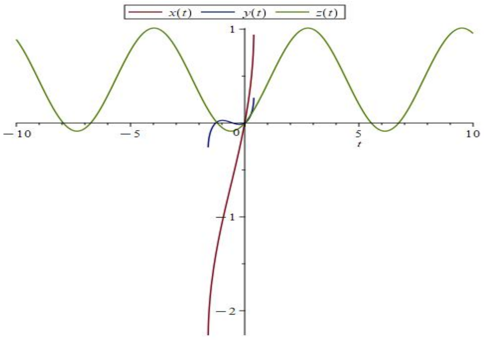

Figure 5. The local minimizer for numerical values, where $\lambda \in C_1$

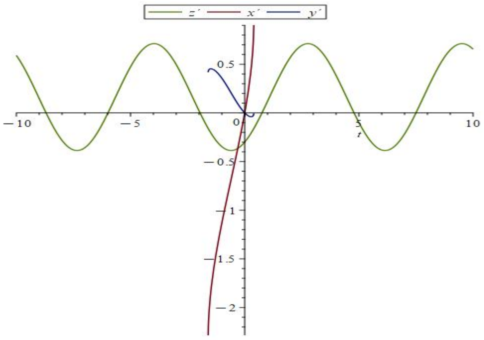

Figure 6. The trajectory's symmetric for numerical values, where $\lambda \in C_1$

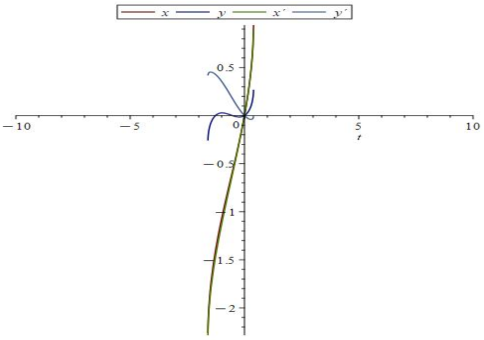

Figure 7. The trajectory $(x(t), y(t))$ and its symmetry for numerical values, where $\lambda \in C_1$

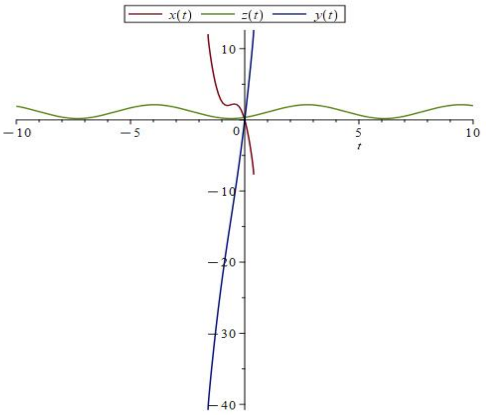

Figure 8. The local minimizer for numerical values, where $\lambda = (\varphi, k) \in C_2$

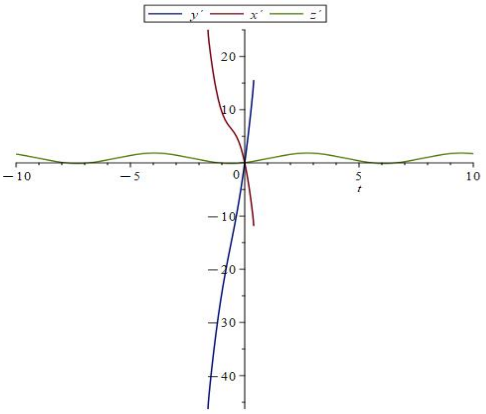

Figure 9. The trajectory's symmetric for numerical values, where $\lambda \in C_2$

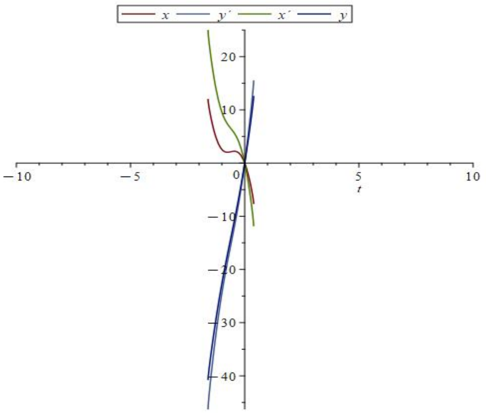

Figure 10. The trajectory and its symmetry for numerical value, where $\lambda \in C_2$

Table 1. Numerical verification of the Maxwell points

k_0	10^{-3}	10^{-12}	10^{-15}
$t = 4k_0K(k_0)$	6.283×10^{-3}	6.283×10^{-12}	6.283×10^{-15}
x_t	$-1.25\times10^{-5}-8.37\times10^{-9}i$	1.89×10^{-12}	2.03×10^{-15}
y_t	$-6.28\times10^{-3}-8.37\times10^{-6}i$	-6.43×10^{-12}	-6.44×10^{-15}

To simulate the dynamics of the system, we used the RK45 numerical integration method, which is an adaptive Runge-Kutta method of 4th and 5th order. This method solves the differential equations governing the system while adjusting the time step based on the required precision. This approach was chosen for its robustness and efficiency in handling complex nonlinear systems. Using this method, we were able to obtain the trajectories and detect the moments when the trajectories come sufficiently close, thus identifying the Maxwell times.

Table 2. Maxwell times via numerical integration

Index	Maxwell Time (Numerical)
1	0.02013
2	0.0401
3	0.0201
4	0.0201
5	0.0401

The Figures 2, 3, and 4 illustrate the trajectory and its symmetry without numerical values, highlighting their intersections. Figures 5 and 6 introduce numerical parameters to visualize the trajectory and its symmetric counterpart, while Figure 7 confirms the absence of intersections for strictly positive times. In the second case, Figure 10 reveals an intersection closer to 0 with a minimized value of k_0. Numerically, Table 1 provides the Maxwell times for each k_0, whenever x_t and y_t approach 0, indicating an intersection between the trajectory and its symmetric counterpart which subsequently leads to a loss of optimality. Table 2 presents the Maxwell times obtained through numerical integration. Our method, which leverages both geometric and algebraic properties, allows us to determine the symmetry algebra of the system. This structure enables us to efficiently identify the points where the trajectory loses its optimality, while reducing computational costs. Moreover, by comparing the two tables, we observe that the Maxwell times in Table 1 are more precise and inferred compared to those obtained numerically.

5. Conclusion

In this work, we computed the infinitesimal symmetries of a sub-Riemannian problem under certain conditions. We then applied this approach to our sub-Riemannian problem on the special hyperbolic Lie group SH(2), where we determined the infinitesimal symmetries of the geometric control problem by leveraging the structure of the associated Lie algebra. This allowed us to identify the Maxwell points and, consequently, compute the first Maxwell time corresponding to these symmetries, which, in turn, enabled us to study the loss of optimality along a geodesic.

The methods used in this study could be particularly useful for analyzing other sub-Riemannian problems associated with a solvable Lie group. If the group is not solvable, one can proceed using the nilpotent approximation, which has been widely used in several works. An interesting direction for future research would be to explore how the methodology developed for SH(2) can be extended to higher-dimensional Lie groups or more complex distributions.

Moreover, the theoretical results obtained in this work have practical implications in robotics, control engineering, and physics. Many real-world systems, such as unicycle-type models, evolve in non-Euclidean spaces, making geometric approaches essential. In particular, determining Maxwell time enables real-time trajectory adjustments, optimizing efficiency and energy consumption in applications like mobile robotics. While this study focuses on the theoretical aspects, future work could include numerical experiments to further illustrate these benefits in concrete scenarios.

Acknowledgments

None.

Funding

None.

Conflict of interest

The authors declare that they have no conflict of interest regarding the publication of this article.

Author contributions

Conceptualization: All authors
Formal analysis: All authors
Methodology: All authors
Writing–original draft: All authors
Writing–review & editing: All authors

Availability of data

Not applicable.

Further disclosure

A preliminary summary of this work was presented at the International Conference on Mathematics and Decision, held at the UM6P Vanguard Center (UM6P, Rabat, December 17-20, 2024).

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