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## Several basic problems in plastic theory of geomaterials

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**Abstract** Based on the basic mechanical properties of geomaterials, it was proven that the Drucker Postulate and the classical theory of plasticity can not be applied to geomaterials. Moreover, several basic problems of plastic theory of geomaterials were discussed. Based on the strict theoretical analysis, the following have been proven: the single yield surface model based on the classical theory of plasticity is unsuitable for geomaterials whether the rule of associated flow is applied or not; the yield surface of geomaterials is not unique, and its number is the same as the freedoms of plastic strain increment; the yield surface is not convex; and the rule of associated flow is unsuitable for geomaterials.

**Keywords** constitutive relation, geomaterials, classical plastic theory, Drucker Postulate, yield surface, flow rule

### 1 Introduction

The geomaterial constitutive models are always built on the classical theory of plasticity, such as the famous Cam-clay model [1]. The following basic mechanical properties obtained from a large number of geotechnical tests can not be described by the models:

1) Based on the classical theory of plasticity, the direction of plastic strain increment is determined only by stress states. However, a number of experimental results [2,3] show that the direction of plastic strain increment is significantly influenced by stress increment; that is, geomaterials do not satisfy the assumption of uniqueness of the direction of plastic strain increment and stress.

2) The rotation of principal stress axes will cause plastic strain even though the values of principal stresses are constant [4].

3) The elastoplastic model with single yield surface based on the classical theory of plasticity can not describe the dilatancy of geomaterials reasonably [5].

To overcome the above disadvantages, a lot of work has been done by scholars over the world. Some constitutive models which disobey the classical theory of plasticity, such as double yield surfaces model [5] were presented. The rule of non-associated flow was used to remedy excessive dilatancy [6]. The classical theory of plasticity is based on the postulation of plasticity. Huang [7] found that the Drucker Postulate and Illiushin Postulate are independent of the laws of thermodynamics. So these two postulates would not be satisfied by some materials.

Based on the basic mechanical properties of geomaterials, it was proven that the Drucker Postulate and the classical theory of plasticity principle are not suitable for geomaterials. In this paper, several basic problems of plastic theory of geomaterials were discussed.

### 2 Drucker Postulate unsuitable for geomaterials

The Drucker Postulate, the work of arbitrary additional stress circulation is not negative, can be expressed as

$$W_D = \oint (\sigma - \sigma_0) d\varepsilon \geq 0.$$

An additional stress circulation is illustrated in Fig. 1. During additional stress circulation, the work caused by elastic deformation is zero, the plastic deformation will not be generalized at the stage of elastic loading,  $A_0 \rightarrow A$  and the stage of unloading,  $A_1 \rightarrow A_0$ . Then

$$W_D = \int_A^{A_1} (\sigma - \sigma_0) d\varepsilon^p = (\sigma - \sigma_0) d\varepsilon^p + \frac{1}{2} d\sigma d\varepsilon^p. \quad (1)$$

When the point  $A_0(\sigma_0)$  and point  $A(\sigma)$  are not superpositioned, the second term in the right side of Eq.(1) can be ignored, then

$$W_D = (\sigma - \sigma_0) d\varepsilon^p. \quad (2)$$

The elastoplastic models of geomaterials could be

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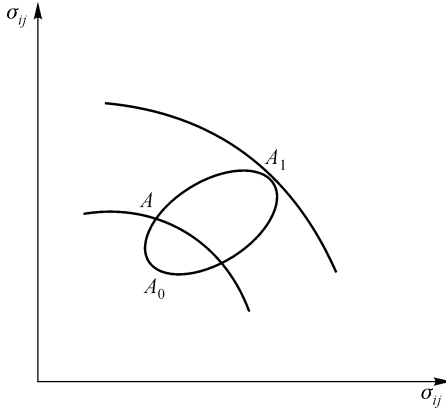


Fig. 1 Additional stress circulation

formulated in the plane of  $p$  (mean stress)– $q$  (deviatoric stress):

$$\begin{pmatrix} d\varepsilon_v^p \\ d\varepsilon_s^p \end{pmatrix} = \begin{pmatrix} E & B \\ C & D \end{pmatrix} \begin{pmatrix} dp \\ dq \end{pmatrix}. \quad (3)$$

$d\varepsilon_v^p$ ,  $d\varepsilon_s^p$  are the plastic volumetric strain and plastic deviatoric strain increments respectively.

Supposing an additional stress circulation in the  $p$ – $q$  plane, and the coordinates for  $A_0$ ,  $A$ ,  $A_1$  are  $(p_0, q_0)$ ,  $(p, q)$ ,  $(p + dp, q + dq)$ , Eq. (2) can be rewritten as

$$W_D = (\sigma - \sigma_0)d\varepsilon^p = (p - p_0)d\varepsilon_v^p + (q - q_0)d\varepsilon_s^p. \quad (4)$$

The dilatancy of geomaterials is recognized by the geomechanics circle. The general dilatancy phenomenon includes dilatancy (volumetric swell caused by deviatoric stress, and  $B < 0$  will be satisfied in Eq. (3)) and negative dilatancy (volumetric contraction caused by deviatoric stress, and  $B > 0$  will be satisfied in Eq. (3)).

### 2.1 Drucker Postulate unsuitable for geomaterials under condition of dilatancy

The key point coordinates of a special stress circulation in the  $p$ – $q$  plane are assumed as  $A_0(p_0, q_0)$ ,  $A(p, q_0)$ ,  $A_1(p, q_0 + dq)$ , illustrated in Fig. 2. Thus

$$\begin{aligned} W_D &= (\sigma - \sigma_0)d\varepsilon^p \\ &= (p - p_0)d\varepsilon_v^p + (q - q_0)d\varepsilon_s^p \\ &= (p - p_0)d\varepsilon_v^p + 0 \cdot d\varepsilon_s^p \\ &= (p - p_0)d\varepsilon_v^p \\ &= (p - p_0)(E \cdot 0 + B \cdot dq) \\ &= B(p - p_0)dq. \end{aligned} \quad (5)$$

At the loading stage  $A \rightarrow A_1$ ,  $dq > 0$ ,  $p - p_0 > 0$ ,  $B < 0$  (under the condition of dilatancy), then

$$W_D = B(p - p_0)dq < 0. \quad (6)$$

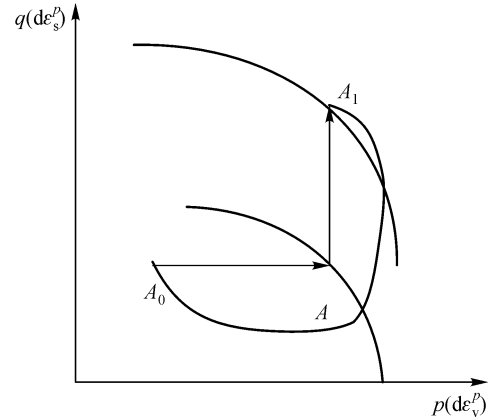


Fig. 2 Illustration of special loading path in  $p$ - $q$  plane

Equation (6) means that the Drucker Postulate will not be satisfied under the condition of dilatancy, that is, the following equation will not be satisfied under the condition of dilatancy:

$$W_D = (\sigma - \sigma_0)d\varepsilon^p > 0. \quad (7)$$

### 2.2 Drucker Postulate unsuitable for geomaterials under condition of negative dilatancy

Under the condition of negative dilatancy,  $B > 0$ , and other geomaterial parameters will satisfy  $E > 0$ ,  $D > 0$ ,  $C < 0$ , So

$$0 > \frac{E}{C} \neq \frac{B}{D} > 0. \quad (8)$$

The last equation shows that the elements in each line of the plastic coefficients matrix in Eq. (3) are not in proportion to each other. The plastic strain increment could be expressed by two linearly independent base vectors:

$$\begin{pmatrix} d\varepsilon_v^p \\ d\varepsilon_s^p \end{pmatrix} = k_1 \xi_1 + k_2 \xi_2 \quad (9)$$

The last equation shows that the potential directions of plastic increment are unlimited, and the uniqueness relation between the direction of plastic strain increment of geomaterials and stress state is nonexistent. The direction of plastic strain increment is illustrated in Fig. 3. Let the exterior normal direction of yield surface  $\mathbf{n}$  coincide with the base vector  $\xi_1$ . As another base vector  $\xi_2$  is linearly independent of  $\xi_1$ ,  $\xi_2$  should not coincide with  $\mathbf{n}(\xi_1)$ . Certainly many initial stress states  $A_0$  on the same side of the exterior normal of yield surface  $\mathbf{n}$  with  $\xi_2$  could be found, for them the following equation could be satisfied:

$$W_D = (\sigma - \sigma_0)d\varepsilon^p = k_3 \cdot \bar{A}_0 \mathbf{A} \cdot \xi_2 < 0.$$

So the Drucker Postulate will not be satisfied also under the condition of negative dilatancy.

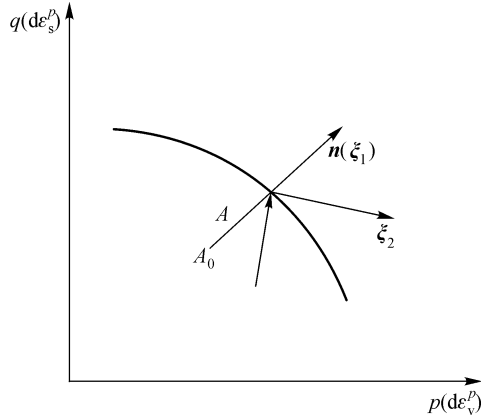


Fig. 3 Strain increment direction in  $p$ - $q$  plane

Dilatancy is the basic mechanical characteristic of geomaterials. The Drucker Postulate will not be satisfied for dilatancy or negative dilatancy, so the Drucker Postulate can not be applied to geomaterials.

### 3 Classic theory of plasticity unsuitable for geomaterials

The core idea of the classical theory of plasticity is the uniqueness relation between the direction of plastic strain increment and stress state, namely the unique plastic potential function  $Q$  exists in stress space, the plastic strain increment can be expressed as

$$d\boldsymbol{\varepsilon}^p = d\lambda \frac{\partial Q}{\partial \boldsymbol{\sigma}}, \quad (10)$$

where  $d\lambda$  is the plastic coefficient, means the magnitude of plastic strain increment.

According to Section 2.2, Eq. (8) shows that the elements in each line of the plastic coefficients matrix in Eq. (3) are not in proportion to each other under the condition of negative dilatancy. The plastic strain increment should be expressed by two linearly independent base vectors, which means that the potential directions of plastic increment are unlimited and the uniqueness relation between the direction of plastic strain increment of geomaterials and stress state can not be satisfied. Namely, the plastic strain increment can not be expressed in the form of Eq. (10) by unique plastic potential surface. So the classical theory of plasticity can not be applied to geomaterials under the condition of negative dilatancy.

Under the condition of dilatancy, the plastic coefficients in Eq. (3) should satisfy  $B < 0$ ,  $E > 0$ ,  $D > 0$ ,  $C < 0$ . Assuming  $E/C = B/D = I < 0$ , two stress increments in arbitrary stress state are chosen:

1)  $dp = 0$ ,  $dq = k$ , the corresponding plastic strain increment will be

$$d\boldsymbol{\varepsilon}_1^p = \begin{pmatrix} d\varepsilon_{v1}^p \\ d\varepsilon_{s1}^p \end{pmatrix} = \begin{pmatrix} E & B \\ C & D \end{pmatrix} \begin{pmatrix} dp \\ dq \end{pmatrix} = Dk \begin{pmatrix} I \\ 1 \end{pmatrix}, \quad (11)$$

2)  $dp = k$ ,  $dq = 0$ , the corresponding plastic strain increment will be

$$\begin{aligned} d\boldsymbol{\varepsilon}_2^p &= \begin{pmatrix} d\varepsilon_{v2}^p \\ d\varepsilon_{s2}^p \end{pmatrix} = \begin{pmatrix} E & B \\ C & D \end{pmatrix} \begin{pmatrix} dp \\ dq \end{pmatrix} \\ &= Ck \begin{pmatrix} I \\ 1 \end{pmatrix} = \frac{C}{D} d\boldsymbol{\varepsilon}_1^p, \quad \frac{C}{D} < 0. \end{aligned} \quad (12)$$

The last equation means that the two plastic strain increment directions in the same stress state are inverse, namely the uniqueness relation between the direction of plastic strain increment of geomaterials and stress state is nonexistent.

On the contrary, if  $E/C = B/D = I < 0$  is not satisfied, namely the elements in each line of the plastic coefficients matrix in Eq. (3) are not in proportion to each other. According to the first part of this section, it could be concluded that the potential directions of plastic increment are unlimited and the uniqueness relation between the direction of plastic strain increment of geomaterials and stress state is nonexistent. Namely the plastic strain increment can not be expressed in the form of Eq. (10) by unique plastic potential surface. So the classical theory of plasticity can not be applied to geomaterials under the condition of dilatancy.

### 4 Several key problems in plastic theory of geomaterials

In Section 3, it was proven that the classical theory of plasticity can not be applied to geomaterials. Obviously, the single yield surface model with the rule of associated flow, which is based on the classical theory of plasticity, can not reflect the basic mechanical characteristics of geomaterials and be unreasonable for geomaterials. When the single yield surface model with the rule of nonassociated flow is adopted, the excessively large dilatancy is improved. However, the single yield surface model with the rule of nonassociated flow is based on the uniqueness relation between the direction of plastic strain increment and stress state, although the direction of plastic strain increment is not normal to the yield surface. This uniqueness relation was justified to be unsuitable for geomaterials, so the single yield surface model with the rule of nonassociated flow can not be applied to geomaterials also.

The merit of multi yield surface model is that the uniqueness relation between the direction of plastic strain increment and stress state will not be satisfied while the materials completely yield. But the uniqueness relation

between the direction of plastic strain increment and stress state will exist in the partial yield zone similar to the single yield model. Obviously this is unreasonable, no experiment shows a zone with the uniqueness relation between the direction of plastic strain increment and stress state exists in geomaterials.

While the principal stress axes rotation can not be ignored, the principal axes of plastic strain increment will not coincide with that of stress; obviously, it is unsuitable to describe plastic deformation in the principal stress space under this condition. In general, the plastic strain increment can be decomposed into six components in six linearly independent directions respectively. Thus, in general stress space, the plastic strain increment of geomaterials can be expressed as

$$d\boldsymbol{\varepsilon}^p = \sum_{k=1}^6 d\lambda_k \frac{\partial Q_k}{\partial \boldsymbol{\sigma}}, \quad (13)$$

where  $Q_k$  ( $k=1,2,\dots,6$ ) are six linearly independent plastic potential functions,  $d\lambda_k$  ( $k=1,2,\dots,6$ ) are six corresponding plastic coefficients respectively.

Based on the above analysis, the number of plastic potential surface is the same as that of the freedoms of plastic strain increment. The maximum number of plastic potential surface is six. Then the number of plastic potential surface is three in principal stress space. It will be two in the  $p$ - $q$  plane. The number of yield surfaces should be equal to the number of plastic potential surfaces and be the same as that of the freedoms of plastic strain increment also.

One disputed problem in multi yield surface model of geomaterials is whether the associated flow rule can be applied or not. Let the two yield surface for two yield surface model be  $f_1(p,q)=0$ ,  $f_2(p,q)=0$ , if the associated flow rule is adopted, then the plastic coefficients  $B$  and  $C$  in Eq.(3) will be

$$B = C = \frac{1}{A_1} \frac{\partial f_1}{\partial p} \frac{\partial f_1}{\partial q} + \frac{1}{A_2} \frac{\partial f_2}{\partial p} \frac{\partial f_2}{\partial q}, \quad (14)$$

where  $A_1$ ,  $A_2$  are plastic modules.

The plastic coefficient  $C$  should satisfy  $C < 0$ , but  $B$  will satisfy  $B < 0$  in the condition of dilatancy and  $B > 0$  under the condition of negative dilatancy. So geomaterials should not satisfy  $B = C$  in general case.  $B = C$  must be satisfied if the associated flow rule is adopted in multi yield surface model, so the associated flow rule can not be applied to geomaterials.

In undrained experiments, the effective stress path in the  $p$ - $q$  plane can be regarded as the volumetric yield surface

of geomaterials. The undrained effective stress path will be convex if negative dilatancy appears in the phase transformation point; but the undrained effective stress path will be not convex [8] if the dilatancy appears in the phase transformation point. Thus, the yield surface may not be convex.

## 5 Conclusion

Based on the basic mechanical properties of geomaterials, several basic problems in the plastic theory of geomaterials have been deeply studied. The following conclusions are obtained:

The Drucker Postulate can not be applied to geomaterials; the classical theory of plasticity is not suitable for geomaterials; the single yield surface model which is based on the classical theory of plasticity is unsuitable for geomaterials whether the rule of associated flow is applied or not; the yield surface of geomaterials is not unique, and its number is the same as the freedoms of plastic strain increment; the yield surface might not be convex; the rule of associated flow can not be applied to geomaterials.

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