Supplemental Materials to "Interband optical conductivities in two-dimensional tilted Dirac bands revisited within the tight-binding model"

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S1. EXPLICIT EXPRESSION OF THE LONGITUDINAL OPTICAL CONDUCTIVITY

Introducing the electromagnetic field, we have

$$\mathcal{H}(\boldsymbol{k}+e\boldsymbol{A}) = \tau_0\left[t\cos(ak_y+aeA_y)+h\sin(ak_y+aeA_y)\right]+\tau_1\sin(ak_x+aeA_x)+\tau_2\cos(ak_y+aeA_y), \tag{S1}$$

from the tight-binding Hamiltonian in Eq.(1) of main text. To the leading order of e,

$$\begin{aligned}
\cos(ak_y+aeA_y) &= \cos(ak_y)\cos(aeA_y)-\sin(ak_y)\sin(aeA_y) = \cos(ak_y)\left[1+\mathcal{O}(e^2)\right]-\sin(ak_y)\left[aeA_y+\mathcal{O}(e^3)\right],\\
\sin(ak_y+aeA_y) &= \sin(ak_y)\cos(aeA_y)+\cos(ak_y)\sin(aeA_y) = \sin(ak_y)\left[1+\mathcal{O}(e^2)\right]+\cos(ak_y)\left[aeA_y+\mathcal{O}(e^3)\right],\\
\sin(ak_x+aeA_x) &= \sin(ak_x)\cos(aeA_x)+\cos(ak_x)\sin(aeA_x) = \sin(ak_x)\left[1+\mathcal{O}(e^2)\right]+\cos(ak_x)\left[aeA_x+\mathcal{O}(e^3)\right].
\end{aligned}$$

As a consequence, we arrive at

$$\mathcal{H}(\boldsymbol{k}+e\boldsymbol{A}) = \mathcal{H}(\boldsymbol{k})+\tau_1 eaA_x\cos(ak_x)-(t\tau_0+\tau_2)eaA_y\sin(ak_y)+h\tau_0 eaA_y\cos(ak_y). \tag{S2}$$

The Green's function in Eq.(8) of main text can be obtained as

$$\begin{aligned}
G(\boldsymbol{k},i\Omega_m) &= \left[\tau_0(i\Omega_m+\mu)-\mathcal{H}(\boldsymbol{k})\right]^{-1} = \frac{\tau_0\left[i\Omega_m+\mu-t\cos(ak_y)-h\sin(ak_y)\right]+\tau_1\sin(ak_x)+\tau_2\cos(ak_y)}{\left[i\Omega_m+\mu-t\cos(ak_y)-h\sin(ak_y)\right]^2-\left[\mathcal{Z}(k_x,k_y)\right]^2}\\
&= \frac{1}{2}\sum_{\lambda=\pm}\frac{1}{i\Omega_m+\mu-\varepsilon_\lambda(k_x,k_y)}\left[\tau_0+\lambda\frac{\tau_1\sin(ak_x)+\tau_2\cos(ak_y)}{\mathcal{Z}(k_x,k_y)}\right]\\
&= \frac{1}{2}\sum_{\lambda=\pm}\frac{\mathcal{P}_\lambda(\boldsymbol{k})}{i\Omega_m+\mu-\varepsilon_\lambda(k_x,k_y)}
\end{aligned} \tag{S3}$$

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with $\mathcal{P}_\lambda(\boldsymbol{k}) = \tau_0 + \lambda\frac{\tau_1\sin(ak_x)+\tau_2\cos(ak_y)}{\mathcal{Z}(k_x,k_y)}$, and

$$G(\boldsymbol{k}, i\Omega_m + \omega + i\eta) = \frac{1}{2}\sum_{\lambda'=\pm}\frac{\mathcal{P}_{\lambda'}(\boldsymbol{k})}{i\Omega_m + \omega + \mu - \varepsilon_\lambda(k_x,k_y) + i\eta} \tag{S4}$$

with $\mathcal{P}_{\lambda'}(\boldsymbol{k}) = \tau_0 + \lambda'\frac{\tau_1\sin(ak_x)+\tau_2\cos(ak_y)}{\mathcal{Z}(k_x,k_y)}$.

After summing over Matsubara frequency Ω_m and a straightforward trace calculation, we express the longitudinal current-current correlation function $\Pi_{jj}(\omega,\mu,t,h)$ as

$$\Pi_{jj}(\omega,\mu,t,h) = e^2\sum_{\lambda=\pm}\sum_{\lambda'=\pm}\int_{-\frac{\pi}{2a}}^{+\frac{\pi}{2a}}\frac{\mathrm{d}k_x}{2\pi}\int_{-\frac{\pi}{a}}^{+\frac{\pi}{a}}\frac{\mathrm{d}k_y}{2\pi}\mathcal{F}_{jj}^{\lambda,\lambda'}(k_x,k_y)\frac{f\left[\varepsilon_\lambda(k_x,k_y)\right] - f\left[\varepsilon_{\lambda'}(k_x,k_y)\right]}{\omega + \varepsilon_\lambda(k_x,k_y) - \varepsilon_{\lambda'}(k_x,k_y) + i\eta}, \tag{S5}$$

where $f(x) = 1/\left\{1+\exp\left[(x-\mu)/(k_BT)\right]\right\}$ is the Fermi distribution function, and the explicit expressions of $\mathcal{F}_{jj}^{\lambda,\lambda'}(k_x,k_y)$ are given as

$$\mathcal{F}_{xx}^{\lambda,\lambda'}(k_x,k_y) = \frac{a^2\cos^2(ak_x)}{2}\left\{1 + \lambda\lambda'\frac{\sin^2(ak_x) - \cos^2(ak_y)}{\left[\mathcal{Z}(k_x,k_y)\right]^2}\right\}, \tag{S6}$$

$$\begin{aligned}\mathcal{F}_{yy}^{\lambda,\lambda'}(k_x,k_y) =& \frac{a^2}{2}\left\{\sin^2(ak_y) + \left[t\sin(ak_y) - h\cos(ak_y)\right]^2 + 2(\lambda+\lambda')\frac{\sin(ak_y)\cos(ak_y)\left[t\sin(ak_y) - h\cos(ak_y)\right]}{\mathcal{Z}(k_x,k_y)}\right.\\ &\left.+\lambda\lambda'\left[\left[t\sin(ak_y) - h\cos(ak_y)\right]^2 - \sin^2(ak_y) + \frac{2\sin^2(ak_y)\cos^2(ak_y)}{\left[\mathcal{Z}(k_x,k_y)\right]^2}\right]\right\}.\end{aligned} \tag{S7}$$

After utilizing $\frac{1}{x+i\eta} = \frac{1}{x} - \mathrm{i}\pi\delta(x)$, the real part of the longitudinal optical conductivities (LOCs) are

$$\begin{aligned}&\mathrm{Re}\ \sigma_{jj}(\omega,\mu,t,h) = \frac{\mathrm{i}}{\omega}\Pi_{jj}(\omega,\mu,t,h) = \frac{-\mathrm{Im}\ \Pi_{jj}(\omega,\mu,t,h)}{\omega}\\ =&e^2\sum_{\lambda=\pm}\sum_{\lambda'=\pm}\int_{-\frac{\pi}{2a}}^{+\frac{\pi}{2a}}\frac{\mathrm{d}k_x}{2\pi}\int_{-\frac{\pi}{a}}^{+\frac{\pi}{a}}\frac{\mathrm{d}k_y}{2\pi}\mathcal{F}_{jj}^{\lambda,\lambda'}(\boldsymbol{k})\frac{f\left[\varepsilon_\lambda(k_x,k_y)\right] - f\left[\varepsilon_{\lambda'}(k_x,k_y)\right]}{\omega}\delta\left[\omega + (\lambda-\lambda')\mathcal{Z}(k_x,k_y)\right].\end{aligned} \tag{S8}$$

We focus on the real part of the interband LOCs take the form

$$\mathrm{Re}\ \sigma_{jj}^{\mathrm{IB}}(\omega,\mu,t,h) = e^2\pi\int_{-\frac{\pi}{2a}}^{+\frac{\pi}{2a}}\frac{\mathrm{d}k_x}{2\pi}\int_{-\frac{\pi}{a}}^{+\frac{\pi}{a}}\frac{\mathrm{d}k_y}{2\pi}\mathcal{F}_{jj}^{-,+}(\boldsymbol{k})\frac{f\left[\varepsilon_-(k_x,k_y)\right] - f\left[\varepsilon_+(k_x,k_y)\right]}{\omega}\delta\left[\omega - 2\mathcal{Z}(k_x,k_y)\right]. \tag{S9}$$

S2. ANALYTICAL EXPRESSIONS OF CONVENTIONAL CRITICAL FREQUENCIES AND PARTNER FREQUENCIES BY UTILIZING THE LAGRANGE MULTIPLIER METHOD

The energy dispersion can be recast as

$$\begin{aligned}\tilde{\varepsilon}_\lambda^\kappa\left(\tilde{k}_x,\tilde{k}_y\right) &= \kappa t\sin(a\tilde{k}_x)\varepsilon_0 - \kappa h\cos(a\tilde{k}_y)\varepsilon_0 + \lambda\sqrt{\sin^2(a\tilde{k}_x) + \sin^2(a\tilde{k}_y)}\varepsilon_0\\ &= \kappa tY\varepsilon_0 - \kappa h\sqrt{1-Y^2}\varepsilon_0 + \lambda\sqrt{X^2+Y^2}\varepsilon_0,\end{aligned} \tag{S10}$$

where

$$\tilde{k}_x \equiv \tilde{k}\cos\tilde{\theta}_k \in \left[-\frac{\pi}{2a}, +\frac{\pi}{2a}\right], \tag{S11}$$

$$\tilde{k}_y \equiv \tilde{k}\sin\tilde{\theta}_k \in \left[-\frac{\pi}{2a}, +\frac{\pi}{2a}\right], \tag{S12}$$

which lead to

$$\sin(a\tilde{k}_x) = X \in [-1,+1], \tag{S13}$$

$$\cos(a\tilde{k}_x) = \sqrt{1-X^2} \in [0,+1], \tag{S14}$$

$$\sin(a\tilde{k}_y) = Y \in [-1,+1], \tag{S15}$$

$$\cos(a\tilde{k}_y) = \sqrt{1-Y^2} \in [0,+1]. \tag{S16}$$

$$\tag{S17}$$

For given λ and κ, we find the extreme values by utilizing the Lagrange multiplier method

$$\mathcal{L} = 2\sqrt{X^2+Y^2}\varepsilon_0 + \zeta\left(\kappa t Y\varepsilon_0 - \kappa h\sqrt{1-Y^2}\varepsilon_0 + \lambda\sqrt{X^2+Y^2}\varepsilon_0 - \mu\right), \tag{S18}$$

with ζ being the Lagrange multiplier. Equivalently, we have

$$\frac{\partial\mathcal{L}}{\partial X} = \frac{(2+\lambda\zeta)\,X\varepsilon_0}{\sqrt{X^2+Y^2}} = 0, \tag{S19}$$

$$\frac{\partial\mathcal{L}}{\partial Y} = \zeta\kappa t\varepsilon_0 + \zeta\kappa h\frac{Y}{\sqrt{1-Y^2}}\varepsilon_0 + (2+\lambda\zeta)\frac{Y}{\sqrt{X^2+Y^2}}\varepsilon_0 = 0, \tag{S20}$$

$$\frac{\partial\mathcal{L}}{\partial\zeta} = \kappa t Y\varepsilon_0 - \kappa h\sqrt{1-Y^2}\varepsilon_0 + \lambda\sqrt{X^2+Y^2}\varepsilon_0 - \mu = 0. \tag{S21}$$

It is noted that the condition in Eq.(S19) is satisfied only when $X = 0$ and $X \neq 0$, which will be detailedly discussed in the following two subsections.

A. Analytical results for $X = 0$

For $X = 0$,

$$\sin\left(a\tilde{k}_x\right) = 0, \tag{S22}$$

or equivalently

$$\tilde{\theta}_k = \frac{\pi}{2}, \tag{S23}$$

$$\tilde{\theta}_k = \frac{3\pi}{2}. \tag{S24}$$

We list the discussion as follows.

1. *Analytical results for $h = 0$, $t = 0$ and $0 < \mu < (1-t)\varepsilon_0$*

If $h = 0$, $t = 0$ and $\mu > 0$, the conditions in Eq.(S19) and (S20) are simultaneously satisfied only when $(2+\lambda\zeta) = 0$. Then the condition in Eq.(S21) gives rise to

$$\sqrt{X^2+Y^2}\varepsilon_0 = |Y|\varepsilon_0 \equiv \frac{\omega}{2} \geq 0, \tag{S25}$$

$$\lambda\frac{\omega}{2} - \mu = 0, \tag{S26}$$

leading to

$$\omega = \frac{2\mu}{\lambda} \geq 0. \tag{S27}$$

Since $\mu > 0$, we get $\lambda = +1$, then

$$\omega \equiv 2\mu. \tag{S28}$$

2. *Analytical results for $h = 0$, $0 < t < 1$ and $0 < \mu < (1-t)\varepsilon_0$*

From the solution $X = 0$, the condition in Eq.(S19) is automatically satisfied, and the condition in Eq.(S20) can be satisfied if a suitable Lagrange multiplier ζ is chosen. Therefore, we focus on the condition in Eq.(S21), which is independent of ζ. We have $\frac{\omega}{2} = \sqrt{X^2+Y^2}\varepsilon_0 = |Y|\varepsilon_0 = \mathrm{sgn}(Y)Y\varepsilon_0$ for $X = 0$. It is straightforward that the condition in Eq.(S21) reduces to

$$\mathrm{sgn}(Y)\lambda\left[\kappa t Y + \mathrm{sgn}(Y)\lambda Y\right]\varepsilon_0 = \mathrm{sgn}(Y)\lambda\mu \tag{S29}$$

$$Y = \frac{\mathrm{sgn}(Y)\lambda\frac{\mu}{\varepsilon_0}}{1+\mathrm{sgn}(Y)\kappa\lambda t} \tag{S30}$$

$$\frac{\omega}{2} = |Y|\varepsilon_0 = \frac{\mu}{1+\mathrm{sgn}(Y)\kappa\lambda t} \geq 0 \tag{S31}$$

For $h = 0$, $0 < t < 1$ and $\mu > 0$, we have

$$Y_+ = \frac{+\lambda\frac{\mu}{\varepsilon_0}}{1+\kappa\lambda t}, \tag{S32}$$

$$Y_- = \frac{-\lambda\frac{\mu}{\varepsilon_0}}{1-\kappa\lambda t}. \tag{S33}$$

For $\kappa\lambda = +1$, we have

$$Y_+ = \frac{+\lambda\frac{\mu}{\varepsilon_0}}{1+t}; \tag{S34}$$

$$Y_- = \frac{-\lambda\frac{\mu}{\varepsilon_0}}{1-t}. \tag{S35}$$

For $\kappa\lambda = -1$, we have

$$Y_+ = \frac{+\lambda\frac{\mu}{\varepsilon_0}}{1-t}; \tag{S36}$$

$$Y_- = \frac{-\lambda\frac{\mu}{\varepsilon_0}}{1+t}. \tag{S37}$$

When $h = 0$, $0 < t < 1$ and $\mu > 0$, the band index λ must be $+1$. Consequently, we have: for $\kappa = +1$,

$$Y_+ = \frac{+\frac{\mu}{\varepsilon_0}}{1+t}, \tag{S38}$$

$$Y_- = \frac{-\frac{\mu}{\varepsilon_0}}{1-t}, \tag{S39}$$

$$\omega_2^+ = 2|Y_-|\varepsilon_0 = \frac{2\mu}{1-t}, \tag{S40}$$

$$\omega_2^- = 2|Y_+|\varepsilon_0 = \frac{2\mu}{1+t}, \tag{S41}$$

and for $\kappa = -1$

$$Y_+ = \frac{+\frac{\mu}{\varepsilon_0}}{1-t}, \tag{S42}$$

$$Y_- = \frac{-\frac{\mu}{\varepsilon_0}}{1+t}, \tag{S43}$$

$$\omega_1^+ = 2|Y_+|\varepsilon_0 = \frac{2\mu}{1-t}, \tag{S44}$$

$$\omega_1^- = 2|Y_-|\varepsilon_0 = \frac{2\mu}{1+t}. \tag{S45}$$

3. Analytical results for $h > 0$

The condition in Eq.(S19) is automatically satisfied, and the condition in Eq.(S20) can be satisfied if a suitable Lagrange multiplier ζ is chosen. Therefore, we focus on the condition in Eq.(S21), which is independent of ζ. We have $\sqrt{X^2+Y^2} = |Y| = \mathrm{sgn}(Y)Y$ for $X = 0$. It is straightforward that the condition in Eq.(S21) reduces to

$$\kappa t Y - \kappa h\sqrt{1-Y^2} + \mathrm{sgn}(Y)\lambda Y = \frac{\mu}{\varepsilon_0}, \tag{S46}$$

which leads to

$$[\mathrm{sgn}(Y)\lambda+\kappa t]\,Y-\frac{\mu}{\varepsilon_0}=\kappa h\sqrt{1-Y^2}, \tag{S47}$$

namely,

$$\kappa\,[\mathrm{sgn}(Y)\lambda+\kappa t]\,Y-\kappa\frac{\mu}{\varepsilon_0}=[\mathrm{sgn}(Y)\kappa\lambda+t]\,Y-\kappa\frac{\mu}{\varepsilon_0}=h\sqrt{1-Y^2}\geq 0, \tag{S48}$$

Consequently, we have

$$[\mathrm{sgn}(Y)\kappa\lambda+t]^2\,Y^2-2\kappa\frac{\mu}{\varepsilon_0}\,[\mathrm{sgn}(Y)\kappa\lambda+t]\,Y+\frac{\mu^2}{\varepsilon_0^2}=h^2-h^2Y^2, \tag{S49}$$

$$[\mathrm{sgn}(Y)\kappa\lambda+t]\,Y-\kappa\frac{\mu}{\varepsilon_0}\geq 0. \tag{S50}$$

which can further reduce to

$$\left\{[1+\mathrm{sgn}(Y)\kappa\lambda t]^2+h^2\right\}Y^2-2\mathrm{sgn}(Y)\lambda\frac{\mu}{\varepsilon_0}\,[1+\mathrm{sgn}(Y)\kappa\lambda t]\,Y-\left(h^2-\frac{\mu^2}{\varepsilon_0^2}\right)=0, \tag{S51}$$

$$\mathrm{sgn}(Y)\kappa\lambda\,[1+\mathrm{sgn}(Y)\kappa\lambda t]\,Y\geq\kappa\,\frac{\mu}{\varepsilon_0}. \tag{S52}$$

The solutions take

$$\begin{aligned}
Y_\pm&=\frac{2\mathrm{sgn}(Y)\lambda\frac{\mu}{\varepsilon_0}\,[1+\mathrm{sgn}(Y)\kappa\lambda t]\pm\sqrt{\left\{2\mathrm{sgn}(Y)\lambda\frac{\mu}{\varepsilon_0}\,[1+\mathrm{sgn}(Y)\kappa\lambda t]\right\}^2+4\left\{[1+\mathrm{sgn}(Y)\kappa\lambda t]^2+h^2\right\}\left(h^2-\frac{\mu^2}{\varepsilon_0^2}\right)}}{2\left\{[1+\mathrm{sgn}(Y)\kappa\lambda t]^2+h^2\right\}}\\
&=\pm\frac{\sqrt{4h^2\left\{[1+\mathrm{sgn}(Y)\kappa\lambda t]^2+\left(h^2-\frac{\mu^2}{\varepsilon_0^2}\right)\right\}}\pm 2\mathrm{sgn}(Y)\lambda\frac{\mu}{\varepsilon_0}\,[1+\mathrm{sgn}(Y)\kappa\lambda t]}{2\left\{[1+\mathrm{sgn}(Y)\kappa\lambda t]^2+h^2\right\}}\\
&=\pm\frac{4h^2\left\{[1+\mathrm{sgn}(Y)\kappa\lambda t]^2+h^2-\frac{\mu^2}{\varepsilon_0^2}\right\}-4\frac{\mu^2}{\varepsilon_0^2}\,[1+\mathrm{sgn}(Y)\kappa\lambda t]^2}{2\left\{[1+\mathrm{sgn}(Y)\kappa\lambda t]^2+h^2\right\}\left\{\sqrt{4h^2\left\{[1+\mathrm{sgn}(Y)\kappa\lambda t]^2+\left(h^2-\frac{\mu^2}{\varepsilon_0^2}\right)\right\}}\mp 2\mathrm{sgn}(Y)\lambda\frac{\mu}{\varepsilon_0}\,[1+\mathrm{sgn}(Y)\kappa\lambda t]\right\}}\\
&=\pm\frac{4\left\{[1+\mathrm{sgn}(Y)\kappa\lambda t]^2+h^2\right\}\left(h^2-\frac{\mu^2}{\varepsilon_0^2}\right)}{2\left\{[1+\mathrm{sgn}(Y)\kappa\lambda t]^2+h^2\right\}\left\{\sqrt{4h^2\left\{[1+\mathrm{sgn}(Y)\kappa\lambda t]^2+\left(h^2-\frac{\mu^2}{\varepsilon_0^2}\right)\right\}}\mp 2\mathrm{sgn}(Y)\lambda\frac{\mu}{\varepsilon_0}\,[1+\mathrm{sgn}(Y)\kappa\lambda t]\right\}}\\
&=\pm\frac{\left(h^2-\frac{\mu^2}{\varepsilon_0^2}\right)}{\sqrt{h^2\left\{[1+\mathrm{sgn}(Y)\kappa\lambda t]^2+\left(h^2-\frac{\mu^2}{\varepsilon_0^2}\right)\right\}}\mp\mathrm{sgn}(Y)\lambda\frac{\mu}{\varepsilon_0}\,[1+\mathrm{sgn}(Y)\kappa\lambda t]},
\end{aligned} \tag{S53}$$

and

$$\mathrm{sgn}(Y_\pm)\kappa\lambda\,[1+\mathrm{sgn}(Y_\pm)\kappa\lambda t]\,Y_\pm-\kappa\frac{\mu}{\varepsilon_0}\geq 0. \tag{S54}$$

4. Analytical results for $h>0$, $0\leq t<1$ and $\mu=0$

For $h>0$, $0\leq t<1$ and $\mu=0$, we have

$$Y_+=+\frac{h}{\sqrt{(1+\kappa\lambda t)^2+h^2}},\qquad\frac{+\kappa\lambda\,(1+\kappa\lambda t)\,h}{\sqrt{(1+\kappa\lambda t)^2+h^2}}\geq 0; \tag{S55}$$

$$Y_-=-\frac{h}{\sqrt{(1-\kappa\lambda t)^2+h^2}},\qquad\frac{-\kappa\lambda\,(1-\kappa\lambda t)\,h}{\sqrt{(1-\kappa\lambda t)^2+h^2}}\geq 0. \tag{S56}$$

For $\kappa\lambda = +1$, we have

$$Y_+ = +\frac{h}{\sqrt{(1+t)^2+h^2}}, \qquad \frac{(1+t)\,h}{\sqrt{(1+t)^2+h^2}} \geq 0; \tag{S57}$$

$$Y_- = -\frac{h}{\sqrt{(1-t)^2+h^2}}, \qquad \frac{-(1-t)\,h}{\sqrt{(1-t)^2+h^2}} < 0 \ (\not\geq 0). \tag{S58}$$

If $(\kappa,\lambda) = (+1,+1)$ or $(\kappa,\lambda) = (-1,-1)$, we further have

$$Y_+ = +\frac{h}{\sqrt{(1+t)^2+h^2}}, \tag{S59}$$

$$\omega_1^{\pm} = 2|Y_+|\varepsilon_0 = +\frac{2h\varepsilon_0}{\sqrt{(1+t)^2+h^2}}, \tag{S60}$$

$$\omega_2^{\pm} = 2|Y_+|\varepsilon_0 = +\frac{2h\varepsilon_0}{\sqrt{(1+t)^2+h^2}}. \tag{S61}$$

5. *Analytical results for $h>0$, $0\leq t<1$ and $0<\mu\leq \frac{h}{\sqrt{1+h^2}}\varepsilon_0 < \frac{h}{\sqrt{t^2+h^2}}\varepsilon_0$*

From the expressions

$$Y_\pm = \pm\frac{\left(h^2-\frac{\mu^2}{\varepsilon_0^2}\right)}{\sqrt{h^2\left\{\left[1+\mathrm{sgn}(Y_\pm)\kappa\lambda t\right]^2+\left(h^2-\frac{\mu^2}{\varepsilon_0^2}\right)\right\}}\mp \mathrm{sgn}(Y_\pm)\lambda\frac{\mu}{\varepsilon_0}\left[1+\mathrm{sgn}(Y_\pm)\kappa\lambda t\right]}, \tag{S62}$$

and

$$\mathrm{sgn}(Y_\pm)\kappa\lambda\left[1+\mathrm{sgn}(Y_\pm)\kappa\lambda t\right]Y_\pm - \kappa\frac{\mu}{\varepsilon_0} \geq 0, \tag{S63}$$

we obtain that, for $h>0$, $0\leq t<1$ and $0\leq\mu\leq\frac{h}{\sqrt{1+h^2}}\varepsilon_0 < \frac{h}{\sqrt{t^2+h^2}}\varepsilon_0$,

$$\left(h^2-\frac{\mu^2}{\varepsilon_0^2}\right) > 0, \tag{S64}$$

$$\sqrt{4h^2\left\{\left[1+\mathrm{sgn}(Y_\pm)\kappa\lambda t\right]^2+\left(h^2-\frac{\mu^2}{\varepsilon_0^2}\right)\right\}} > \left|2\mathrm{sgn}(Y_\pm)\lambda\frac{\mu}{\varepsilon_0}\left[1+\mathrm{sgn}(Y_\pm)\kappa\lambda t\right]\right| = 2\frac{\mu}{\varepsilon_0}\left|1+\mathrm{sgn}(Y_\pm)\kappa\lambda t\right|. \tag{S65}$$

Consequently, we have

$$\mathrm{sgn}\left(Y_\pm\right) = \pm, \tag{S66}$$

and

$$\pm\,\kappa\lambda\left(1\pm\kappa\lambda t\right)Y_\pm - \kappa\frac{\mu}{\varepsilon_0} \geq 0. \tag{S67}$$

Therefore,

$$Y_\pm = \pm\frac{\left(h^2-\frac{\mu^2}{\varepsilon_0^2}\right)}{\sqrt{h^2\left\{\left(1\pm\kappa\lambda t\right)^2+\left(h^2-\frac{\mu^2}{\varepsilon_0^2}\right)\right\}}-\lambda\frac{\mu}{\varepsilon_0}\left(1\pm\kappa\lambda t\right)}, \tag{S68}$$

and

$$\pm\,\kappa\lambda\left(1\pm\kappa\lambda t\right)Y_\pm - \kappa\frac{\mu}{\varepsilon_0} \geq 0. \tag{S69}$$

$$Y_{\pm} = \pm \frac{\left(h^2 - \frac{\mu^2}{\varepsilon_0^2}\right)}{\sqrt{h^2\left\{(1 \pm \kappa\lambda t)^2 + \left(h^2 - \frac{\mu^2}{\varepsilon_0^2}\right)\right\}} - \lambda \frac{\mu}{\varepsilon_0}(1 \pm \kappa\lambda t)}$$
$$= \pm \frac{\sqrt{\frac{\left(h^2 - \frac{\mu^2}{\varepsilon_0^2}\right)}{h^2}}}{\sqrt{1 + \frac{(1 \pm \kappa\lambda t)^2}{\left(h^2 - \frac{\mu^2}{\varepsilon_0^2}\right)}} - \lambda \frac{\mu}{\varepsilon_0 h}\sqrt{\frac{(1 \pm \kappa\lambda t)^2}{\left(h^2 - \frac{\mu^2}{\varepsilon_0^2}\right)}}}, \tag{S70}$$

and

$$\pm \kappa\lambda (1 \pm \kappa\lambda t) Y_{\pm} - \kappa \frac{\mu}{\varepsilon_0} \geq 0. \tag{S71}$$

For $\kappa\lambda = +1$, we have

$$Y_{\pm} = \pm \frac{\left(h^2 - \frac{\mu^2}{\varepsilon_0^2}\right)}{\sqrt{h^2\left\{(1 \pm t)^2 + \left(h^2 - \frac{\mu^2}{\varepsilon_0^2}\right)\right\}} - \lambda \frac{\mu}{\varepsilon_0}(1 \pm t)}, \tag{S72}$$

and

$$(1 \pm t) \frac{\left(h^2 - \frac{\mu^2}{\varepsilon_0^2}\right)}{\sqrt{h^2\left\{(1 \pm t)^2 + \left(h^2 - \frac{\mu^2}{\varepsilon_0^2}\right)\right\}} - \lambda \frac{\mu}{\varepsilon_0}(1 \pm t)} - \kappa \frac{\mu}{\varepsilon_0} \geq 0. \tag{S73}$$

If $(\kappa, \lambda) = (+1, +1)$, we have

$$Y_{\pm} = \pm \frac{\left(h^2 - \frac{\mu^2}{\varepsilon_0^2}\right)}{\sqrt{h^2\left\{(1 \pm t)^2 + \left(h^2 - \frac{\mu^2}{\varepsilon_0^2}\right)\right\}} - \frac{\mu}{\varepsilon_0}(1 \pm t)}, \tag{S74}$$

and

$$(1 \pm t) \frac{\left(h^2 - \frac{\mu^2}{\varepsilon_0^2}\right)}{\sqrt{h^2\left\{(1 \pm t)^2 + \left(h^2 - \frac{\mu^2}{\varepsilon_0^2}\right)\right\}} - \frac{\mu}{\varepsilon_0}(1 \pm t)} - \frac{\mu}{\varepsilon_0} \geq 0. \tag{S75}$$

$$\omega_2^+ = 2|Y_-|\varepsilon_0 = \frac{2\left(h^2 - \frac{\mu^2}{\varepsilon_0^2}\right)}{\sqrt{h^2\left\{(1 - t)^2 + \left(h^2 - \frac{\mu^2}{\varepsilon_0^2}\right)\right\}} - \frac{\mu}{\varepsilon_0}(1 - t)}\varepsilon_0, \tag{S76}$$

$$\omega_2^- = 2|Y_+|\varepsilon_0 = \frac{2\left(h^2 - \frac{\mu^2}{\varepsilon_0^2}\right)}{\sqrt{h^2\left\{(1 + t)^2 + \left(h^2 - \frac{\mu^2}{\varepsilon_0^2}\right)\right\}} - \frac{\mu}{\varepsilon_0}(1 + t)}\varepsilon_0. \tag{S77}$$

If $(\kappa, \lambda) = (-1, -1)$, we have

$$Y_{\pm} = \pm \frac{\left(h^2 - \frac{\mu^2}{\varepsilon_0^2}\right)}{\sqrt{h^2\left\{(1 \pm t)^2 + \left(h^2 - \frac{\mu^2}{\varepsilon_0^2}\right)\right\}} + \frac{\mu}{\varepsilon_0}(1 \pm t)}, \tag{S78}$$

and

$$(1\pm t)\frac{\left(h^2-\frac{\mu^2}{\varepsilon_0^2}\right)}{\sqrt{h^2\left\{(1\pm t)^2+\left(h^2-\frac{\mu^2}{\varepsilon_0^2}\right)\right\}}+\frac{\mu}{\varepsilon_0}(1\pm t)}+\frac{\mu}{\varepsilon_0}\geq 0. \tag{S79}$$

$$\omega_1^+=2|Y_-|\varepsilon_0=\frac{2\left(h^2-\frac{\mu^2}{\varepsilon_0^2}\right)}{\sqrt{h^2\left\{(1-t)^2+\left(h^2-\frac{\mu^2}{\varepsilon_0^2}\right)\right\}}+\frac{\mu}{\varepsilon_0}(1-t)}\varepsilon_0, \tag{S80}$$

$$\omega_1^-=2|Y_+|\varepsilon_0=\frac{2\left(h^2-\frac{\mu^2}{\varepsilon_0^2}\right)}{\sqrt{h^2\left\{(1+t)^2+\left(h^2-\frac{\mu^2}{\varepsilon_0^2}\right)\right\}}+\frac{\mu}{\varepsilon_0}(1+t)}\varepsilon_0. \tag{S81}$$

B. Analytical results for $X\neq 0$

For $X\neq 0$, the condition in Eq.(S19) requires that $(2+\lambda\zeta)=0$. Consequently, the condition in Eq.(S20) reduces to

$$t+h\frac{Y}{\sqrt{1-Y^2}}=0. \tag{S82}$$

1. *Analytical results for $h=0$, $t=0$ and $\mu>0$*

For $h=0$, $t=0$ and $\mu>0$, the condition in Eq.(S82) is automatically satisfied, and Y is not constrained.

$$\omega'=2\sqrt{X^2+Y^2}\varepsilon_0=2\lambda\mu\geq 0. \tag{S83}$$

For $\mu>0$, we have $\lambda=+1$. As a consequence,

$$\omega'_j=2\mu. \tag{S84}$$

2. *Analytical results for $h>0$, $t=0$ and $\mu\geq 0$*

For $h>0$, $t=0$ and $\mu\geq 0$, the condition in Eq.(S82) requires $Y=0$, and hence we have the partner frequencies

$$\omega'=2\sqrt{X^2+Y^2}\varepsilon_0=2|Y|\varepsilon_0=2\kappa\lambda\left[h\varepsilon_0+\kappa\mu\right]\geq 0. \tag{S85}$$

For $0<\mu<h\varepsilon_0$, if $\kappa\lambda=+1$,

$$\omega'=2\left[h\varepsilon_0+\kappa\mu\right]\geq 0, \tag{S86}$$

if $\kappa\lambda=-1$,

$$\omega'=-2\left[h\varepsilon_0+\kappa\mu\right]<0\ (\not\geq 0). \tag{S87}$$

In brief, for $t=0$, $h>0$ and $0<\mu<\sqrt{t^2+h^2}\varepsilon_0$, if $\kappa\lambda=+1$,

$$\omega'=2\sqrt{X^2+Y^2}\varepsilon_0=2\kappa\lambda\left[\sqrt{h^2+t^2}\varepsilon_0+\kappa\mu\right]\geq 0. \tag{S88}$$

For $(\kappa,\lambda)=(-1,-1)$

$$\omega'_1=2\left|h\varepsilon_0-\mu\right|,\qquad h\varepsilon_0-\mu\geq 0, \tag{S89}$$

and for $(\kappa,\lambda)=(+1,+1)$,

$$\omega'_2=2\left|h\varepsilon_0+\mu\right|,\qquad h\varepsilon_0+\mu\geq 0. \tag{S90}$$

In brief,

$$\omega'_j=2\left|h\varepsilon_0+(-1)^j\mu\right|,\qquad h\varepsilon_0+(-1)^j\mu\geq 0. \tag{S91}$$

3. Analytical results for $h = 0$, $0 < t < 1$ and $\mu > 0$

For $h = 0$, $0 < t < 1$ and $\mu > 0$, the conditions in Eq.(S19) and Eq.(S20)—or equivalently the condition in Eq.(S82)—can not be satisfied, indicating that there is no extreme value of ω'_j.

4. Analytical results for $h > 0$, $0 < t < 1$ and $\mu \geq 0$

For $h > 0$ and $0 < t < 1$, if $-1 < Y < 1$, we have

$$\sqrt{1 - Y^2} = -\frac{h}{t} Y \geq 0. \tag{S92}$$

Further, we have

$$-1 < Y \leq 0, \tag{S93}$$

$$1 - Y^2 = \frac{h^2}{t^2} Y^2. \tag{S94}$$

The solution takes

$$Y = -\frac{t}{\sqrt{h^2 + t^2}}. \tag{S95}$$

From the condition in Eq.(S21),

$$\sqrt{X^2 + Y^2}\varepsilon_0 = \lambda \left[\mu - \kappa t Y \varepsilon_0 + \kappa h \sqrt{1 - Y^2}\varepsilon_0\right] = \lambda \left[\mu + \kappa \sqrt{h^2 + t^2}\varepsilon_0\right] = \kappa\lambda \left[\sqrt{h^2 + t^2}\varepsilon_0 + \kappa\mu\right] \geq 0. \tag{S96}$$

Finally, we have the partner frequencies

$$\omega' = 2\sqrt{X^2 + Y^2}\varepsilon_0 = 2\kappa\lambda \left[\sqrt{h^2 + t^2}\varepsilon_0 + \kappa\mu\right] \geq 0. \tag{S97}$$

For $0 < \mu < \sqrt{t^2 + h^2}\varepsilon_0$, if $\kappa\lambda = +1$,

$$\omega' = 2\sqrt{X^2 + Y^2}\varepsilon_0 = 2 \left[\sqrt{h^2 + t^2}\varepsilon_0 + \kappa\mu\right] \geq 0, \tag{S98}$$

if $\kappa\lambda = -1$,

$$\omega' = 2\sqrt{X^2 + Y^2}\varepsilon_0 = -2 \left[\sqrt{h^2 + t^2}\varepsilon_0 + \kappa\mu\right] < 0 \ (\not\geq 0). \tag{S99}$$

In brief, for $0 \leq t < 1$, $h > 0$ and $0 < \mu < \sqrt{t^2 + h^2}\varepsilon_0$, if $\kappa\lambda = +1$,

$$\omega' = 2\sqrt{X^2 + Y^2}\varepsilon_0 = 2\kappa\lambda \left[\sqrt{h^2 + t^2}\varepsilon_0 + \kappa\mu\right] \geq 0. \tag{S100}$$

For $(\kappa, \lambda) = (+1, +1)$,

$$\omega'_2 = 2\left|\sqrt{h^2 + t^2}\varepsilon_0 + \mu\right|, \qquad \sqrt{h^2 + t^2}\varepsilon_0 + \mu \geq 0, \tag{S101}$$

and for $(\kappa, \lambda) = (-1, -1)$

$$\omega'_1 = 2\left|\sqrt{h^2 + t^2}\varepsilon_0 - \mu\right|, \qquad \sqrt{h^2 + t^2}\varepsilon_0 - \mu \geq 0. \tag{S102}$$

C. Explicit values of conventional critical frequencies and partner frequencies

In the following, we provide explicit values of both the conventional critical frequencies and their partner frequencies, as specified by the parameters in the main text. This is done to demonstrate quantitative consistency between the analytical expressions derived in the preceding three subsections and the full numerical calculations in the main text.

1. untilted cases

When $t = 0.0, h = 0.0, \mu = 0.45\ \varepsilon_0$,

$$\omega_1^{\pm} = \frac{2\mu}{1 \mp t} \equiv \omega_1 = 0.9\ \varepsilon_0, \tag{S103}$$
$$\omega_2^{\pm} = \frac{2\mu}{1 \mp t} \equiv \omega_2 = 0.9\ \varepsilon_0, \tag{S104}$$
$$\omega_1' = 2\mu = 0.9\ \varepsilon_0, \tag{S105}$$
$$\omega_2' = 2\mu = 0.9\ \varepsilon_0. \tag{S106}$$

When $t = 0.0, h = 0.6, \mu = 0.0\ \varepsilon_0$,

$$\omega_1^{\pm} = \frac{2h\varepsilon_0}{\sqrt{(1+t)+h^2}} \equiv \omega_1 = 1.02899\ \varepsilon_0, \tag{S107}$$
$$\omega_2^{\pm} = \frac{2h\varepsilon_0}{\sqrt{(1+t)+h^2}} \equiv \omega_2 = 1.02899\ \varepsilon_0, \tag{S108}$$
$$\omega_1' = 2h\varepsilon_0 = 1.2\ \varepsilon_0, \tag{S109}$$
$$\omega_2' = 2h\varepsilon_0 = 1.2\ \varepsilon_0. \tag{S110}$$

When $t = 0.0, h = 0.6, \mu = 0.15\ \varepsilon_0$, we have

$$\omega_1^{+} = 1.24103\ \varepsilon_0, \tag{S111}$$
$$\omega_1^{-} = 1.24103\ \varepsilon_0, \tag{S112}$$
$$\omega_2^{+} = 0.799856\ \varepsilon_0, \tag{S113}$$
$$\omega_2^{-} = 0.799856\ \varepsilon_0, \tag{S114}$$
$$\omega_1' = 2\left[h\varepsilon_0 + \mu\right] = 1.5\ \varepsilon_0, \tag{S115}$$
$$\omega_2' = 2\left[h\varepsilon_0 - \mu\right] = 0.9\ \varepsilon_0. \tag{S116}$$

2. tilted cases

When $t = 0.3, h = 0.0, \mu = 0.45\ \varepsilon_0$, we have

$$\omega_1^{+} = 1.28571\ \varepsilon_0, \tag{S117}$$
$$\omega_1^{-} = 0.69231\ \varepsilon_0, \tag{S118}$$
$$\omega_2^{+} = 1.28571\ \varepsilon_0, \tag{S119}$$
$$\omega_2^{-} = 0.69231\ \varepsilon_0. \tag{S120}$$

It is emphasized that there is no extreme value of ω_j' in this case.
When $t = 0.3, h = 0.6, \mu = 0.0\ \varepsilon_0$, we have

$$\omega_1^{+} = 1.30158\ \varepsilon_0, \tag{S121}$$
$$\omega_1^{-} = 0.838116\ \varepsilon_0, \tag{S122}$$
$$\omega_2^{+} = 1.30158\ \varepsilon_0, \tag{S123}$$
$$\omega_2^{-} = 0.838116\ \varepsilon_0, \tag{S124}$$
$$\omega_1' = 2\sqrt{h^2+t^2}\varepsilon_0 = 1.34164\ \varepsilon_0, \tag{S125}$$
$$\omega_2' = 2\sqrt{h^2+t^2}\varepsilon_0 = 1.34164\ \varepsilon_0. \tag{S126}$$

When $t = 0.3, h = 0.6, \mu = 0.15\ \varepsilon_0$, we have

$$\omega_1^+ = 1.53130\ \varepsilon_0, \tag{S127}$$

$$\omega_1^- = 1.02375\ \varepsilon_0, \tag{S128}$$

$$\omega_2^+ = 1.03718\ \varepsilon_0, \tag{S129}$$

$$\omega_2^- = 0.64326\ \varepsilon_0, \tag{S130}$$

$$\omega_1' = 2\left[\sqrt{h^2 + t^2}\varepsilon_0 + \mu\right] = 1.64164\ \varepsilon_0, \tag{S131}$$

$$\omega_2' = 2\left[\sqrt{h^2 + t^2}\varepsilon_0 - \mu\right] = 1.04164\ \varepsilon_0. \tag{S132}$$