SUPPLEMENTARY MATERIAL FOR “A CONTINUOUS VARIABLE QUANTUM BATTERY WITH WIRELESS AND REMOTE CHARGING”

The density matrix of the continuous variable system has an infinite number of rows and columns, which is detrimental to the study of the properties of the system. The system can also be described by the Wigner functions or characteristic functions, which are complex numbers that correspond one-to-one to the density matrix. Once the characteristic functions are obtained, all the information about the system can be determined. Reference [47] provides a method for calculating the characteristic function of the Gaussian states. For obtaining the characteristic function of the continuous variable quantum batteries in the main text, the expectation values of the operators $\langle\hat{a}_2^\dagger\hat{a}_2\rangle$, $\langle\hat{a}_2^\dagger\hat{a}_2^\dagger\rangle$, $\langle\hat{a}_2\hat{a}_2\rangle$ and $\langle\hat{a}_2^\dagger(t)\pm\hat{a}_2(t)\rangle$ need to be calculated firstly.

The expectation value of the operator does not vary due to the differences in the selected pictures (such as Schr$\ddot{o}$dinger’s picture, Heisenberg’s picture, and Dirac’s picture). The Heisenberg’s picture is chosen for our closed system. The evolution of the operators with time can be calculated analytically by the Laplace transform. The Hamiltonian is

$$H=\hbar\omega_1\hat{a}_1^\dagger\hat{a}_1+\hbar\omega_2\hat{a}_2^\dagger\hat{a}_2+g\hbar(\hat{a}_1^\dagger+\hat{a}_1)(\hat{a}_2^\dagger+\hat{a}_2)+G\hbar(\hat{a}_1^\dagger-\hat{a}_1)(\hat{a}_2^\dagger-\hat{a}_2). \tag{S1}$$

According to the Heisenberg equation of the operator, we can obtain the dynamical equations from Eq. (S1) if the role of the environment has not been taken into account ($\gamma=0$):

$$\begin{aligned}
\frac{d\hat{a}_1^\dagger(t)}{dt}&=i\omega_1\hat{a}_1^\dagger(t)+i(g-G)\hat{a}_2^\dagger(t)+i(g+G)\hat{a}_2(t),\\
\frac{d\hat{a}_1(t)}{dt}&=-i\omega_1\hat{a}_1(t)-i(g+G)\hat{a}_2^\dagger(t)+i(G-g)\hat{a}_2(t),\\
\frac{d\hat{a}_2^\dagger(t)}{dt}&=i\omega_2\hat{a}_2^\dagger(t)+i(g-G)\hat{a}_1^\dagger(t)+i(g+G)\hat{a}_1(t),\\
\frac{d\hat{a}_2(t)}{dt}&=-i\omega_2\hat{a}_2(t)-i(g+G)\hat{a}_1^\dagger(t)+i(G-g)\hat{a}_1(t).
\end{aligned} \tag{S2}$$

The variation of the operators with time in the above equations can be solved by the Laplace transform and its inverse transform. Through performing the Laplace transform for Eqs. (S2), we can obtain

$$\begin{aligned}
p\mathcal{L}[\hat{a}_1^\dagger(t)]-\hat{a}_1^\dagger(0)&=i\omega_1\mathcal{L}[\hat{a}_1^\dagger(t)]+i(g-G)\mathcal{L}[\hat{a}_2^\dagger(t)]+i(g+G)\mathcal{L}[\hat{a}_2(t)],\\
p\mathcal{L}[\hat{a}_1(t)]-\hat{a}_1(0)&=-i\omega_1\mathcal{L}[\hat{a}_1(t)]-i(g+G)\mathcal{L}[\hat{a}_2^\dagger(t)]+i(G-g)\mathcal{L}[\hat{a}_2(t)],\\
p\mathcal{L}[\hat{a}_2^\dagger(t)]-\hat{a}_2^\dagger(0)&=i\omega_2\mathcal{L}[\hat{a}_2^\dagger(t)]+i(g-G)\mathcal{L}[\hat{a}_1^\dagger(t)]+i(g+G)\mathcal{L}[\hat{a}_1(t)],\\
p\mathcal{L}[\hat{a}_2(t)]-\hat{a}_2(0)&=-i\omega_2\mathcal{L}[\hat{a}_2(t)]-i(g+G)\mathcal{L}[\hat{a}_1^\dagger(t)\}+i(G-g)\mathcal{L}[\hat{a}_1(t)].
\end{aligned} \tag{S3}$$

The Laplace transform is defined as $\mathcal{L}[f(t)]=\int_0^{+\infty}f(t)e^{-pt}dt$ for function $f(t)$. The Laplace transform of $\hat{a}_1^+(t)$, $\hat{a}_1(t)$, $\hat{a}_2^+(t)$ and $\hat{a}_2(t)$ can be obtained from Eqs. (S3). From the inverse Laplace transform, we can obtain

$$\begin{aligned}
\hat{a}_1(t)&=q_1(t)\hat{a}_1^\dagger(0)+q_2(t)\hat{a}_1(0)+q_3(t)\hat{a}_2^\dagger(0)+q_4(t)\hat{a}_2(0),\\
\hat{a}_2(t)&=f_1(t)\hat{a}_1^\dagger(0)+f_2(t)\hat{a}_1(0)+f_3(t)\hat{a}_2^\dagger(0)+f_4(t)\hat{a}_2(0).
\end{aligned} \tag{S4}$$

The conditions that the functions $f_j(t)$ and $q_j(t)$ $(j = 1, 2, 3, 4)$ need to satisfy are

$$
\begin{aligned}
\mathcal{L}[q_1(t)] &= \frac{1}{Z}[2i(g^2 - G^2)\omega_2], \\
\mathcal{L}[q_2(t)] &= \frac{1}{Z}\{p^3 - i\omega_1 p^2 + (\omega_2^2 - 4Gg)p + i[2\omega_2(G^2 + g^2) - \omega_1\omega_2^2]\}, \\
\mathcal{L}[q_3(t)] &= \frac{1}{Z}[-i(G+g)p^2 + (G+g)(\omega_2 - \omega_1)p + i(G+g)(4Gg - \omega_1\omega_2)], \\
\mathcal{L}[q_4(t)] &= \frac{1}{Z}[-i(g-G)p^2 - (g-G)(\omega_2 + \omega_1)p + i(g-G)(4Gg + \omega_1\omega_2)], \\
\mathcal{L}[f_1(t)] &= \frac{1}{Z}[-i(G+g)p^2 + (G+g)(\omega_1 - \omega_2)p + i(G+g)(4Gg - \omega_1\omega_2)], \\
\mathcal{L}[f_2(t)] &= \frac{1}{Z}[-i(g-G)p^2 - (g-G)(\omega_2 + \omega_1)p + i(g-G)(4Gg + \omega_1\omega_2)], \\
\mathcal{L}[f_3(t)] &= \frac{1}{Z}[2i(g^2 - G^2)\omega_1], \\
\mathcal{L}[f_4(t)] &= \frac{1}{Z}\{p^3 - i\omega_2 p^2 + (\omega_1^2 - 4Gg)p + i[2\omega_1(G^2 + g^2) - \omega_2\omega_1^2]\},
\end{aligned}
\tag{S5}
$$

with $Z = p^4 + (\omega_1^2 + \omega_2^2 - 8Gg)p^2 + [16G^2g^2 - 4\omega_1\omega_2(G^2 + g^2) + \omega_1^2\omega_2^2]$. Based on the residue theorem, one can solve $Z = 0$ with p and find out the corresponding pole point. The covariance matrix of the Gaussian states and the expectation values of $\hat{a}_2^\dagger(t) \pm \hat{a}_2(t)$ can be solved by Eqs. (S4). Therefore, the dynamics of the closed system can be determined.

For the open system ($\gamma \neq 0$), it is convenient to use the master equations. The variation of the expectation value of the operator $\langle\widehat{O}\rangle$ with time can be calculated by $\frac{d\langle\widehat{O}\rangle}{dt} = \mathrm{Tr}\left[\widehat{O}\frac{d\hat{\varrho}(t)}{dt}\right]$, where $\widehat{O}$ is an operator that does not obviously contain time. If we take the environment into account, the expectation values of the operators $\langle\hat{a}_m^\dagger\hat{a}_n^\dagger\rangle, \langle\hat{a}_m^\dagger\hat{a}_n\rangle$ and $\langle\hat{a}_m\hat{a}_n\rangle$ $(m, n = 1$ or $2)$ are governed by the following ordinary differential equations:

$$
\begin{aligned}
\frac{d\langle\hat{a}_1^\dagger\hat{a}_1\rangle}{dt} &= -i(G+g)(\langle\hat{a}_1^\dagger\hat{a}_2^\dagger\rangle - \langle\hat{a}_1\hat{a}_2\rangle) - i(g-G)(\langle\hat{a}_1^\dagger\hat{a}_2\rangle - \langle\hat{a}_1\hat{a}_2^\dagger\rangle) - \gamma\langle\hat{a}_1^\dagger\hat{a}_1\rangle + \gamma n_{th}, \\
\frac{d\langle\hat{a}_2^\dagger\hat{a}_2\rangle}{dt} &= -i(G+g)(\langle\hat{a}_1^\dagger\hat{a}_2^\dagger\rangle - \langle\hat{a}_1\hat{a}_2\rangle) - i(G-g)(\langle\hat{a}_1^\dagger\hat{a}_2\rangle - \langle\hat{a}_1\hat{a}_2^\dagger\rangle), \\
\frac{d\langle\hat{a}_1^\dagger\hat{a}_2^\dagger\rangle}{dt} &= i(\omega_1 + \omega_2)\langle\hat{a}_1^\dagger\hat{a}_2^\dagger\rangle + i(G+g)(\langle\hat{a}_1^\dagger\hat{a}_1\rangle + \langle\hat{a}_2^\dagger\hat{a}_2\rangle + 1) + i(g-G)(\langle\hat{a}_1^{\dagger 2}\rangle + \langle\hat{a}_2^{\dagger 2}\rangle) - \gamma/2\langle\hat{a}_1^\dagger\hat{a}_2^\dagger\rangle, \\
\frac{d\langle\hat{a}_1\hat{a}_2\rangle}{dt} &= -i(\omega_1 + \omega_2)\langle\hat{a}_1\hat{a}_2\rangle - i(G+g)(\langle\hat{a}_1^\dagger\hat{a}_1\rangle + \langle\hat{a}_2^\dagger\hat{a}_2\rangle + 1) - i(g-G)(\langle\hat{a}_1^2\rangle + \langle\hat{a}_2^2\rangle) - \gamma/2\langle\hat{a}_1\hat{a}_2\rangle, \\
\frac{d\langle\hat{a}_1^\dagger\hat{a}_2\rangle}{dt} &= i(\omega_1 - \omega_2)\langle\hat{a}_1^\dagger\hat{a}_2\rangle - i(G+g)(\langle\hat{a}_1^{\dagger 2}\rangle - \langle\hat{a}_2^2\rangle) - i(g-G)(\langle\hat{a}_1^\dagger\hat{a}_1\rangle - \langle\hat{a}_2^\dagger\hat{a}_2\rangle) - \gamma/2\langle\hat{a}_1^\dagger\hat{a}_2\rangle, \\
\frac{d\langle\hat{a}_1\hat{a}_2^\dagger\rangle}{dt} &= i(\omega_2 - \omega_1)\langle\hat{a}_1\hat{a}_2^\dagger\rangle - i(G+g)(\langle\hat{a}_2^{\dagger 2}\rangle - \langle\hat{a}_1^2\rangle) + i(g-G)(\langle\hat{a}_1^\dagger\hat{a}_1\rangle - \langle\hat{a}_2^\dagger\hat{a}_2\rangle) - \gamma/2\langle\hat{a}_1\hat{a}_2^\dagger\rangle, \\
\frac{d\langle\hat{a}_1^2\rangle}{dt} &= -2i\omega_1\langle\hat{a}_1^2\rangle - 2i(G+g)\langle\hat{a}_1\hat{a}_2^\dagger\rangle - 2i(g-G)\langle\hat{a}_1\hat{a}_2\rangle - \gamma\langle\hat{a}_1^2\rangle, \\
\frac{d\langle\hat{a}_1^{\dagger 2}\rangle}{dt} &= 2i\omega_1\langle\hat{a}_1^{\dagger 2}\rangle + 2i(G+g)\langle\hat{a}_1^\dagger\hat{a}_2\rangle + 2i(g-G)\langle\hat{a}_1^\dagger\hat{a}_2^\dagger\rangle - \gamma\langle\hat{a}_1^{\dagger 2}\rangle, \\
\frac{d\langle\hat{a}_2^2\rangle}{dt} &= -2i\omega_2\langle\hat{a}_2^2\rangle - 2i(G+g)\langle\hat{a}_1^\dagger\hat{a}_2\rangle - 2i(g-G)\langle\hat{a}_1\hat{a}_2\rangle, \\
\frac{d\langle\hat{a}_2^{\dagger 2}\rangle}{dt} &= 2i\omega_2\langle\hat{a}_2^{\dagger 2}\rangle + 2i(G+g)\langle\hat{a}_1\hat{a}_2^\dagger\rangle + 2i(g-G)\langle\hat{a}_1^\dagger\hat{a}_2^\dagger\rangle.
\end{aligned}
\tag{S6}
$$

The evolution of $\langle\hat{a}_m^\dagger\hat{a}_n^\dagger\rangle$, $\langle\hat{a}_m^\dagger\hat{a}_n\rangle$ and $\langle\hat{a}_m\hat{a}_n\rangle$ for the open system can be obtained by solving the system with 10 ordinary differential equations (ODEs). The change of the expectation values of the operators $\langle\hat{a}_2^\dagger\hat{a}_2\rangle$, $\langle\hat{a}_2^\dagger\hat{a}_2^\dagger\rangle$ and $\langle\hat{a}_2\hat{a}_2\rangle$ with time can be calculated from Eqs. (S6). The expectation values of $\langle\hat{a}_2^\dagger(t) \pm \hat{a}_2(t)\rangle$ can be obtained by solving the following equations

$$\begin{aligned}
\frac{d\langle\hat{a}_1\rangle}{dt} &= -i\omega_1\langle\hat{a}_1\rangle - ig(\langle\hat{a}_2^\dagger\rangle + \langle\hat{a}_2\rangle) - iG(\langle\hat{a}_2^\dagger\rangle - \langle\hat{a}_2\rangle) - \gamma/2\langle\hat{a}_1\rangle,\\
\frac{d\langle\hat{a}_1^\dagger\rangle}{dt} &= i\omega_1\langle\hat{a}_1^\dagger\rangle + ig(\langle\hat{a}_2^\dagger\rangle + \langle\hat{a}_2\rangle) - iG(\langle\hat{a}_2^\dagger\rangle - \langle\hat{a}_2\rangle) - \gamma/2\langle\hat{a}_1^\dagger\rangle,\\
\frac{d\langle\hat{a}_2\rangle}{dt} &= -i\omega_2\langle\hat{a}_2\rangle - ig(\langle\hat{a}_1^\dagger\rangle + \langle\hat{a}_1\rangle) - iG(\langle\hat{a}_1^\dagger\rangle - \langle\hat{a}_1\rangle),\\
\frac{d\langle\hat{a}_2^\dagger\rangle}{dt} &= i\omega_2\langle\hat{a}_2^\dagger\rangle + ig(\langle\hat{a}_1^\dagger\rangle + \langle\hat{a}_1\rangle) - iG(\langle\hat{a}_1^\dagger\rangle - \langle\hat{a}_1\rangle). \qquad \text{(S7)}
\end{aligned}$$

Based on such calculations, the characteristic functions of the Gaussian states for the open system can be determined. It is noted that the evolution of the above expectation values for the closed system can also be investigated by the master equations under the condition $\gamma = 0$. The results are consistent with that given by the method of the Laplace transform.