

RESEARCH ARTICLE

Anomalous diffractions of water waves in parity-controlled metagratings

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Supporting Information

Supplementary Note 1. Diffraction mechanism of water-wave metagratings (WMGs)

We replace the uniform refractive index in each slot with a linearly varying refractive index to reduce the parasitic reflections caused by impedance mismatch, as shown in Fig. S1.

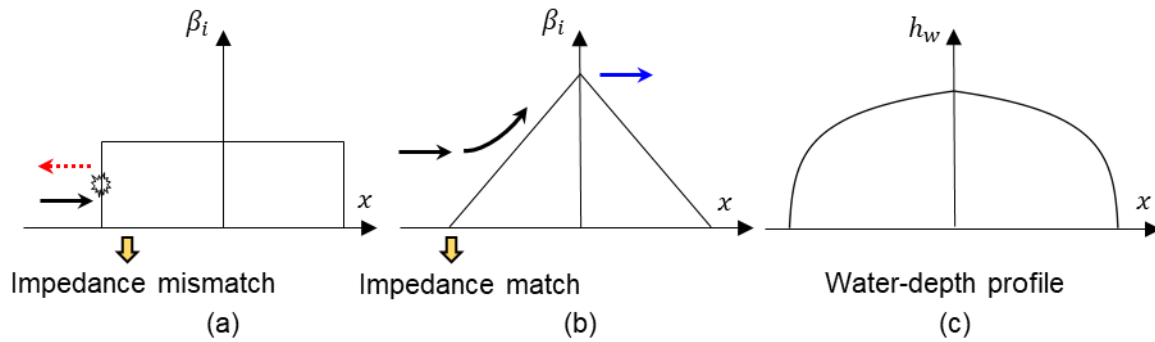


Fig. S1 Schematic of the WMG design method. **(a)** Constant effective index distribution of the WMG unit with impedance mismatch around the interface, β_i is refractive index of each unit cell. **(b)** Linear rescale of the effective index distribution from (a). **(c)** Calculated height profile of the WMG unit.

Supplementary Note 2. Diffraction behaviour of WMGs

Anomalous diffraction in metagratings is summarized as the GSL,

$$k_y^{in} = k_y^{r(t)} - \xi \quad (S1)$$

where k_y^{in} and $k_y^{r(t)}$ are tangential wave vectors of incident and reflected (transmitted) wave, and $\xi = \partial\phi(y)/\partial y$ describes the phase gradient along the meta-grating. Essentially, phase gradient metagratings are periodical supercells array along the interface, and Eq. (S1) cannot complementally determine the direction of anomalous transmitted/reflected waves, especially for incident angle beyond the critical angle. We argue that Eq. (S1) can be modified as follows:

$$k_y^{in} = k_y^{r(t)} - nG \quad (S2)$$

where $G = 2\pi/l$ is the reciprocal lattice vector, θ_{in} is the incident angle, and l is the spatial period. Notably, both ξ and G commonly share the identical magnitude, yet with different physical origins; the former is introduced by the phase gradient, whereas the latter is caused by the periodicity of metagratings. We consider an incident wave from background with $k_y^{in} = k_0 \sin(\theta_{in})$, where θ_{in} is incident angle. The transmitted/reflected waves obey the diffraction law of Eq. (S2), with the maximum diffraction order $N = \text{roundup}\left(-\frac{2k_0}{G}\right) + 1$ while N is a negative

integer. Regardless of the direction of the incident wave, i.e., $k_y^{in} \in [-k_0, k_0]$, the existing diffraction orders of the reflected and transmitted waves belong to $n \in [N, 1]$.

Owing to the perfect rigid-body structure of the WMGs, these slots could be regarded as water-wave waveguides. The perfect rigid body can avoid wave coupling between adjacent units. The forward and backward waves propagating in the slots interfere to form standing waves stemming from the multiple reflection effect of incident rays in the WMGs. We define the number of times waves pass through the medium as L . When incident waves directly pass through the WMG, i.e., $L = 1$, the phase delay is $\phi_j = \beta_j k_0 d$ in j -th slot, while the β_j and $\Delta\phi = \phi_{j+1} - \phi_j = 2\pi/m$ represent the effective refractive index and phase difference of adjacent slots per period, respectively. According to Eq. (S2), the phase gradient of the n -th diffraction order could be equivalent to $\xi = nG$, then the phase difference of two adjacent slots is $\Delta\phi_n = 2\pi n/m$. For the case of $L = n = 1$, Eq. (S2) can be expressed as

$$k_y^{in} = k_y^t - G = k_y^t - \xi \quad (S3)$$

which is exactly the governing equation of GSL. In this case, the incident wave ($k_y^{in} \in [-k_0, k_0 - \xi]$) follows GSL. When the incident angle is larger than the critical angle $k_y^{in} = k_0 - \xi$, the channel of $n = 1$ diffraction order will be closed, then the incident wave cannot directly transmit the WMG. Therefore, waves will internally reflect at transmitted interface and start another propagation process [$L = 2$, yellow dashed arrows in Fig. 1(c) in main text], resulting in

phase delay of $2\phi_j$ and a phase difference of $\Delta\phi = 2 \times 2\pi/m$. $\Delta\phi_n = \frac{2\pi n}{m} \leq 0$ due to the remaining diffraction orders are $n \in [N, 0]$. Consequently, waves cannot couple to these diffraction orders by means of $\Delta\phi = \Delta\phi_n$. However, when a phase wrap of 2π is applied to $\Delta\phi$, i.e., $\Delta\phi - 2\pi$ (2π phase wrap is enough for wave to couple to higher-order (n) in a WMG with slots supporting fundamental waveguide modes), the phase difference is still equivalently effective. Therefore, when $\Delta\phi - 2\pi = \Delta\phi_n$, the reflected wave with the n -th diffraction order will occur (see the red arrows in Fig. 1(c) in main text). Otherwise, there will be a third-time propagation process ($L = 3$) in slots via internal reflection at the interface. Similarly, if $\Delta\phi - 2\pi = \Delta\phi_n$, there will be a transmitted wave of the n -th diffraction order, otherwise the fourth time propagation ($L = 4$) will happen and so on. Generally, the relationship between L and n is $2\pi L/m - 2\pi = 2\pi n/m$, i.e.

$$L = m + n \quad (S4)$$

As $L > 0$ and the maximum diffraction order is N , the number of slots is required to meet $m \geq 1 - N$. When L is odd, the incident wave will couple to the corresponding transmitted wave of the higher orders; whereas when L is even, it will couple to the corresponding reflected wave of the higher orders. Consequently, by combining Eq. (S2) and Eq. (S4), the diffraction law in a WMG is summarized as

$$\begin{cases} k_y = k_y^t - nG, & L \text{ is odd} \\ k_y = k_y^r - nG, & L \text{ is even} \end{cases} \quad (S5)$$

The incident wave below the critical angle ($L = 1$) will couple to the transmitted wave of the lowest order $n = 1$. For the incident wave beyond the critical angle, multiple reflection happens inside the WMG in turn, i.e., $L = 1 \rightarrow 2 \rightarrow 3 \dots$. If waves pass through WMG with odd/even propagation number, the transmission/reflection can be generated, with the diffraction order determined by Eq. (S5), as shown in Fig. 1(c) in main text. Furthermore, we can reverse the transmission or reflection of diffraction order by adjusting the parity of the number of slots m in each supercell. In other words, the diffraction order is correlated with the anomalous reflection (refraction) when m is odd (even).

Supplementary Note 3. Detailed parameters of two types of WMGs

Detailed parameters of the two types of WMGs are shown in Table S1.

Table S1 Detailed parameters of two types of WMGs.

Type	a_0/mm	w/mm	l/mm	d/mm	λ_0/mm
$m = 3$	8	2	30	30	30
$m = 4$	8	2	40	40	40

Supplementary Note 4. Coupled-mode theory for the water waves phase gradient meta-grating (WMG)

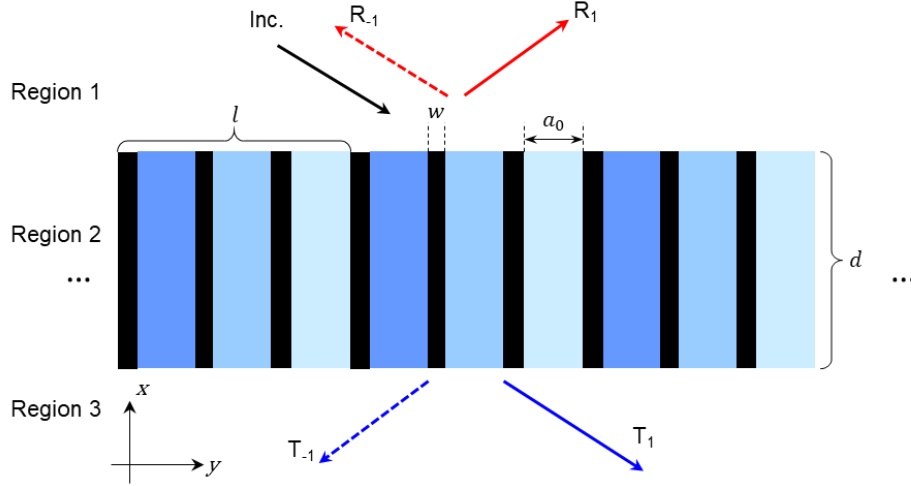


Fig. S2 Structure of the type of $m = 3$ WMG, gradient blue represents the gradient water depth, black is perfect rigid body.

As shown in Fig. S2, the proposed WMG is composed of m unit cells ($m = 3$ in this case) with widths of a_0 . The periodic length is $l = m(a_0 + w)$, and the thickness of the WMG is d . Each unit cell is perforated by a slot with a width of a_0 , while w is the width of walls rising above the water. Each period with m individual unit cells should cover 2π abrupt phase change, and the phase difference of two adjacent unit cells is $\Delta\phi = 2\pi/m$. In order to reduce impedance mismatch, the water depth in the WMGs should be gradient. The effective refractive index of each slot varies linearly, while the maximum refractive index of each slot is $\beta_{m,j} = 1 + 2(j-1)\lambda/md$. Then the equivalent refractive index of each slot is $\beta_j = (1 + \beta_{m,j})/2$. For the convenience of calculation, we use the equivalent refractive index n_j instead of a linearly varying refractive index for the following derivation. When water waves with an incident angle of θ_{in} propagate into WMG, the entire system is divided into three regions: the incident region (region 1), the WMG region (region 2) and the transmitted region (region 3). In the incident side, the total vertical displacement η of water waves is the superposition of the incident wave and the reflected waves of all the diffraction orders,

$$\eta_1 = \sum_n [\delta_{n,0} \exp(ik_x^n x) + r_n \exp(-ik_x^n x)] \exp(i\alpha_n y) \quad (S6)$$

where $\delta_{n,0}$ is the Kronecker delta, $k_x^n = \sqrt{k_0^2 - \alpha_n^2}$, $\alpha_n = k_0 \sin \theta_{in} + nG$ and $G = 2\pi/l$, with $k_0 = 2\pi/\lambda_0$ is the wavevector in free space. r_n is the reflection coefficient of the n -th diffraction order. In outgoing side, the transmitted wave includes all the transmission components of diffraction orders,

$$\eta_3 = \sum_n t_n \exp[ik_x^n (x - d)] \exp(i\alpha_n y) \quad (S7)$$

where t_n is the transmission coefficient of the n -th diffraction order. In the WMG region, as the width of the unit cell is much smaller than wavelength, only the fundamental mode is considered. Then the vertical displacement of water waves in unit cell is the standing wave,

$$\eta_{2,j} = a_j \exp(ik_j x) + b_j \exp(-ik_j (x - d)), (j = 1, 2 \dots m) \quad (S8)$$

where a_j and b_j are amplitude coefficients of the forward and backward waves and $k_j = \beta_j k_0$ is the wave vector in the j -th unit cell. We assume the flow of water wave is $\vec{\zeta}$ and p is the hydrostatic pressure ($p = \rho g \eta$), then $\vec{\zeta} = \frac{i u}{\omega \rho} (\nabla p) = \frac{i u}{\omega \rho} (\rho g \nabla \eta) \Rightarrow \zeta_x = \frac{i g u}{\omega} \frac{\partial \eta}{\partial x}$, where ρ is the fluid density and g is the gravitational acceleration, $u = \tanh(k_0 \bar{h})/k_0$ is the reduced water depth. Then the corresponding x -components of the flow of water waves are written as,

$$\zeta_{1x} = \frac{-gu_0}{\omega} \sum_n k_x^n [\delta_{n,0} \exp(ik_x^n x) - r_n \exp(-ik_x^n x)] \exp(i\alpha_n y) \quad (S9)$$

$$\zeta_{2x,j} = \frac{-gu_j k_j}{\omega} [a_j \exp(ik_j x) - b_j \exp(-ik_j(x-d))], (|y - y_j| \leq \frac{a_0}{2}) \quad (S10)$$

$$\zeta_{2x,j} = 0, \left(\frac{a_0}{2} \leq |y - y_j| \leq \frac{a_0 + w}{2}\right) \quad (S11)$$

$$\zeta_{3x} = \frac{-gu_0}{\omega} \sum_n k_x^n t_n \exp[i k_x^n (x-d)] \exp(i\alpha_n y) \quad (S12)$$

where $y_j = (j-1)(a_0 + w)$. By applying the continuous boundary conditions of vertical displacement η and flow of water waves ζ at $x = 0$ and $x = d$,

$$\eta_1(0) = \eta_2(0), (|y - y_j| \leq \frac{a_0}{2}) \quad (S13)$$

$$\zeta_{1x}(0) = \zeta_{2x,j}(0), (|y - y_j| \leq \frac{a_0}{2}) \quad (S14)$$

$$\zeta_{1x}(0) = 0, \left(\frac{a_0}{2} \leq |y - y_j| \leq \frac{a_0 + w}{2}\right) \quad (S15)$$

$$\eta_3(d) = \eta_2(d), (|y - y_j| \leq \frac{a_0}{2}) \quad (S16)$$

$$\zeta_{3x}(d) = \zeta_{2x,j}(d), (|y - y_j| \leq \frac{a_0}{2}) \quad (S17)$$

$$\zeta_{3x}(d) = 0, \left(\frac{a_0}{2} \leq |y - y_j| \leq \frac{a_0 + w}{2}\right) \quad (S18)$$

we have

$$\sum_n (\delta_{n,0} + r_n) \exp(i\alpha_n y) = a_j + b_j q_j, (|y - y_j| \leq \frac{a_0}{2}) \quad (S19)$$

$$\frac{-gu_0}{\omega} \sum_n k_x^n [\delta_{n,0} - r_n] \exp(i\alpha_n y) = \frac{-gu_j}{\omega} k_j (a_j - b_j q_j), (|y - y_j| \leq \frac{a_0}{2}) \quad (S20)$$

$$\frac{-gu_0}{\omega} \sum_n k_x^n [\delta_{n,0} - r_n] \exp(i\alpha_n y) = 0, \left(\frac{a_0}{2} \leq |y - y_j| \leq \frac{a_0 + w}{2}\right) \quad (S21)$$

$$\sum_n t_n \exp(i\alpha_n y) = a_j q_j + b_j, (|y - y_j| \leq \frac{a_0}{2}) \quad (S22)$$

$$\frac{-gu_0}{\omega} \sum_n k_x^n t_n \exp(i\alpha_n y) = \frac{-gu_j}{\omega} (a_j q_j - b_j), (|y - y_j| \leq \frac{a_0}{2}) \quad (S23)$$

$$\frac{-gu_0}{\omega} \sum_n k_x^n t_n \exp(i\alpha_n y) = 0, \left(\frac{a_0}{2} \leq |y - y_j| \leq \frac{a_0 + w}{2}\right) \quad (S24)$$

where $q_j = \exp(ik_j d)$. By integrating the Eq. (S19) and Eq. (S22) with dy at the region $|y - y_j| \leq \frac{a_0}{2}$, we can get,

$$\sum_n s_n (\delta_{n,0} + r_n) \exp(i\alpha_n y_j) = a_j + b_j q_j \quad (S25)$$

$$\sum_n s_n t_n \exp(i\alpha_n y_j) = a_j q_j + b_j \quad (S26)$$

where $s_n = \text{sinc}(\alpha_n a_0/2)$. Furthermore, by multiplying both sides of Eqs. (S20-S21) and Eqs. (S23-S24) with $\exp(-i\alpha_n y)$, and then integrating both sides with dy at the region $0 \leq y \leq l$, we can obtain

$$\frac{-gu_0k_x^n}{\omega}(\delta_{n,0} - r_n) = \frac{-g}{\omega} \sum_{j=1}^m k_j u_j s_n \gamma (a_j - b_j q_j) \exp(-i\alpha_n y_j) \quad (\text{S27})$$

$$\frac{-gu_0t_nk_x^n}{\omega} = \frac{-g}{\omega} \sum_{j=1}^m k_j u_j s_n \gamma (a_j q_j - b_j) \exp(-i\alpha_n y_j) \quad (\text{S28})$$

where $\gamma = a_0/l$ with $l = m(a_0 + w)$. By solving Eq. (S25) through Eq. (S28), we can analytically obtain all unknown coefficients of r_n and t_n , as shown with black and red lines in Fig. 2(c-e).

Supplementary Note 5. The integer parity effect of WMG

We simulated the reflection and transmission of water waves passing through $m = 5$ and $m = 6$ types of WMGs with different incident angles ($-45^\circ, 45^\circ$). We still take WMGs with $\xi = k_0$, in which the maximum diffraction order is $N = -1$ and the critical angle is $\theta_1 = 0^\circ$. The wavelength λ_0 , thickness d and period l for $m = 5$ WMG are all 3 cm, while for $m = 6$ WMG are all 4 cm.

When $\theta_{in} > \theta_1$, incident waves are manipulated into two diffraction orders, i.e., $n = 0$ order and $n = -1$ order. The higher diffraction order is preferred owing to the minimum propagation number. Hence, for $\theta_{in} > \theta_1$, the effective diffraction order is the $n = -1$ order and the corresponding propagation number of WMG with m slots are $L = m - 1$. Based on Eqs. (S4) and (S5), when m is odd, e.g., $m = 5$, the propagation number L is even, which leads to the reflection of the $n = -1$ order. On the other hand, when m is even, e.g., $m = 6$, the propagation number L is odd, which results in the transmission of the $n = -1$ order.

The field patterns when incident angle $\theta_{in} = \pm 30^\circ$ are shown in Fig. S3(a, d). As shown in Fig. S3(a) panel i, for the type of $m = 5$, when the incident angle is $\theta_{in} = -30^\circ$, WMG is modulated to reflection mode, incident waves are manipulated to produce diffraction order of $n = 1$ (R_1) called anomalous reflection. In Fig. S3(a) panel ii, when the incident angle of water waves is $\theta_{in} = 30^\circ$, the transmission mode is turned on, the anomalous reflection (R_{-1}) vanishes, and is replaced by anomalous refraction (T_{-1}). Figure S3 (b) panels (i-ii) illustrate that for the type of $m = 6$, the water waves incident from both directions undergoes anomalous refraction (T_1 for $\theta_{in} = -30^\circ$ and T_{-1} for $\theta_{in} = 30^\circ$). In other words, the transmission mode is activated for both $\theta_{in} = \pm 30^\circ$ when $m = 6$. Therefore, it is feasible to modulate the reflection/transmission mode of WMGs through varying the number of unit cells in each supercell.

The relationship between incident angle and the reflection/transmission of diffraction orders is shown in Fig. S3 (c-e), where curves represent the numerical results. It is concluded that the reflection and transmission of the $n = -1$ order in WMGs not only happen at $\theta_{in} = \pm 30^\circ$, but occurs in a wide incident range. For the case of $m = 5$ (see Fig. S3(c-d)), reflection (R) of $n = 1$ order is more than 90% for $\theta_{in} \in [0^\circ, 45^\circ]$, while no reflection of $n = -1$ order for $\theta_{in} \in [-45^\circ, 0^\circ]$. Meanwhile, transmission (T) of $n = 1$ order is more than 90% for $\theta_{in} \in [-45^\circ, 0^\circ]$, while no transmission of $n = -1$ order for $\theta_{in} \in [0^\circ, 45^\circ]$. In other words, for the case of $m = 5$, water waves are manipulated to produce both anomalous reflection and anomalous refraction for $\theta_{in} \in [-45^\circ, 45^\circ]$. For the case of $m = 6$ [see Fig. S3 (e)], transmission of $n = 1$ order, the same as $m = 5$, is more than 90% for $\theta_{in} \in [-45^\circ, 0^\circ]$. For $\theta_{in} \in [0^\circ, 45^\circ]$, different from $m = 5$, transmission of $n = -1$ order is more than 90% instead of reflection. Therefore, for the case of $m = 6$, water waves are manipulated to produce anomalous refraction for all incident angles between $\theta_{in} \in [-45^\circ, 45^\circ]$.

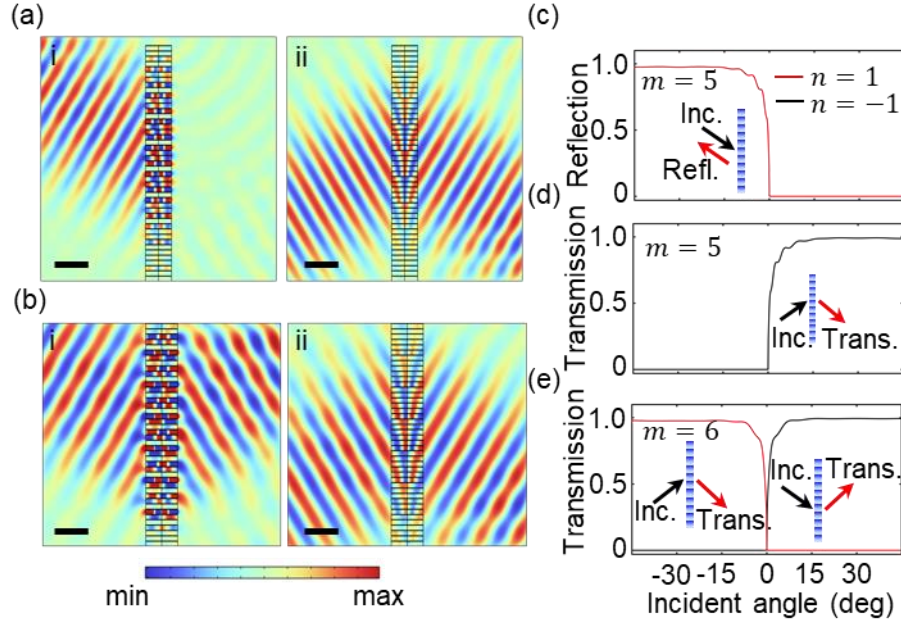


Fig. S3 The manipulative effect of WMGs on water waves and the simulation results of two types of WMGs, water waves are incident at two angles $\theta_{in} = \pm 30^\circ$. **(a)** WMGs of type $m = 5$, water waves enter in WMG region at an angle of incidence of **(i)** $\theta_{in} = -30^\circ$ and **(ii)** $\theta_{in} = 30^\circ$, respectively. **(b)** WMGs of type $m = 6$, water waves enter in WMG region at an angle of incidence of **(i)** $\theta_{in} = -30^\circ$ and **(ii)** $\theta_{in} = 30^\circ$, respectively. The scale bar is 6 cm. **(c-e)** The relationship between transmission/reflection and incident angle of 1 and -1 diffraction orders in WMGs. **(c)(d)** The WMGs of $m = 5$. **(e)** The WMGs of $m = 6$. The curves and points represent analytical and numerical results, respectively.

Supplementary Note 6. Propagation loss measurement

We produced array gratings consisting of each individual slot, and measured peak-to-peak values of the incident wave (V_i) and transmitted wave (V_t) with the help of digital wave height gauge (YWH200-D). Table S2 and S3 show the data measured by digital wave height gauge.

Table S2 Peak-to-peak values with the case of $m = 3$.

	1	2	3
V_i/mm	0.12	0.1	0.12
V_t/mm	0.08	0.05	0.05

Table S3 Peak-to-peak values with the case of $m = 4$.

	1	2	3	4
V_i/mm	0.12	0.1	0.11	0.1
V_t/mm	0.07	0.05	0.05	0.05

Obviously, the intensity of the water waves is weakened by half by gratings, i.e. $\frac{I}{I_0} = \left(\frac{V_t}{V_i}\right)^2 = 0.25$.

Let α is extinction coefficient and d is the width of WMGs,

$$I = I_0 e^{-\alpha d} \quad (\text{S29})$$

And the relationship between α and loss Γ is

$$\alpha = \frac{4\pi\Gamma}{\lambda} \quad (\text{S30})$$

we can obtain $\Gamma = 0.11$.

Supplementary Note 7. Other diffraction orders and iso-frequency contours

We analyze the cases of $m = 4$ and $\xi = 0.8k_0$ in detail, as shown in Fig. S4. The iso-frequency contours are employed to intuitively uncover the underlying physics. In Fig. S4 (a), if there is no phase shift along the interface, the iso-frequency contours in free space are merely indicated by two identical black circles, where the left one is for the incident wave, while the right one is for the refracted wave. Once the phase shifts are introduced and water waves propagate to WMG at an incident angle of 30° , then at the refracted side, the black circle will be shifted up with a displacement of ξ , as marked by the red circle. Then the direction of the transmitted wave is marked by red arrow. As for Fig. S4(c), water waves propagate to WMG at an incident angle of -30° , the black circle will be shifted down with a displacement of ξ , as marked by the green circle. Then the direction of the transmitted wave is marked by a green arrow. In Fig. S4(e), water waves propagate to WMG at an incident angle of 50° , the black circle will be shifted up with displacements of ξ and 2ξ , as marked by the red and blue circles. Then the direction of the transmitted/reflected wave is marked by red/blue arrow. The field patterns are shown in Fig. S4(b, d, f), which prove the analysis above.

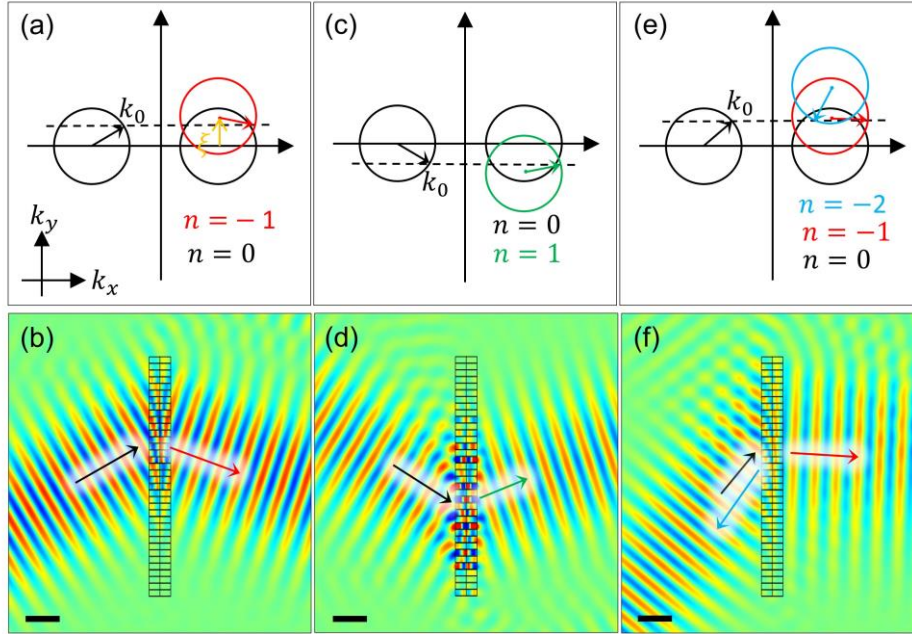


Fig. S4 Iso-frequency contours and field pattern with incident angle of **(a-b)** $\theta_{in} = 30^\circ$, **(c-d)** $\theta_{in} = -30^\circ$ and **(e-f)** $\theta_{in} = 50^\circ$. The scale bar is 6 cm.

Supplementary Note 8. Phase changes in the slots

In each unit cell, the phase delay is

$$\phi(\lambda, n_{m,j}) = \phi_0 + \frac{2\pi}{\lambda} \beta_j d = \phi_0 + \frac{2\pi}{\lambda} \frac{\beta_{m,j} + 1}{2} d \quad (\text{S31})$$

Since the frequency of water waves $f = \frac{v}{\lambda}$, the phase changes in the slots can be obtained (see Fig. S5), which is perfectly matched with simulation results [see Fig. 5(b) in main text]. The phase delay conforms to a distribution resembling a quadratic curve in the case of several frequencies, characterized by a smaller phase delay at both ends and a larger one in the middle region (as shown in Fig. S5(b)).

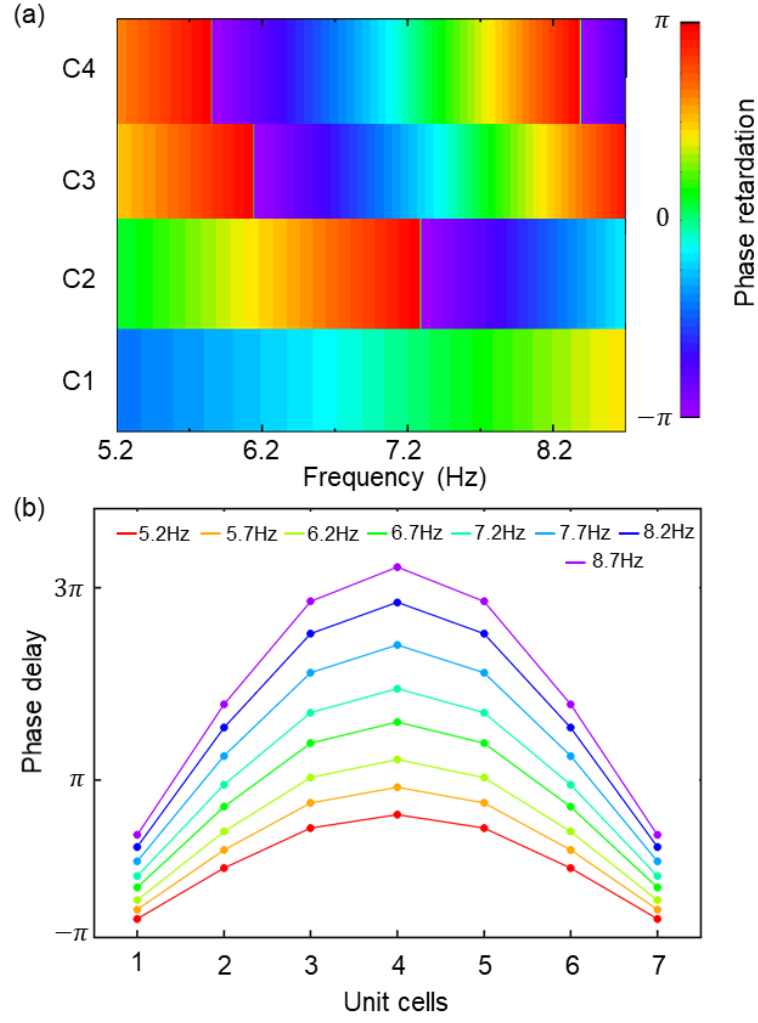


Fig. S5 (a) Phase changes in each unit cell. **(b)** Relationship between phase delay and unit cells for different frequencies.

Supplementary Note 9. Focusing effect for different frequencies in the experiment

Fig. S6(a-h) shows the effect of WMGs for water waves focusing at the effective frequency of 5.2-8.7 Hz while the step size is 0.5 Hz. The shadows on the left and right sides are perfectly matched layers (PML) that absorb water waves and avoid reflections at the border. Fig. S6(i) shows the normalized amplitude along the y-direction where the focal point is located. It is concluded that the focusing effect is exhibited in the range of 5.7-8.7 Hz in the simulation.

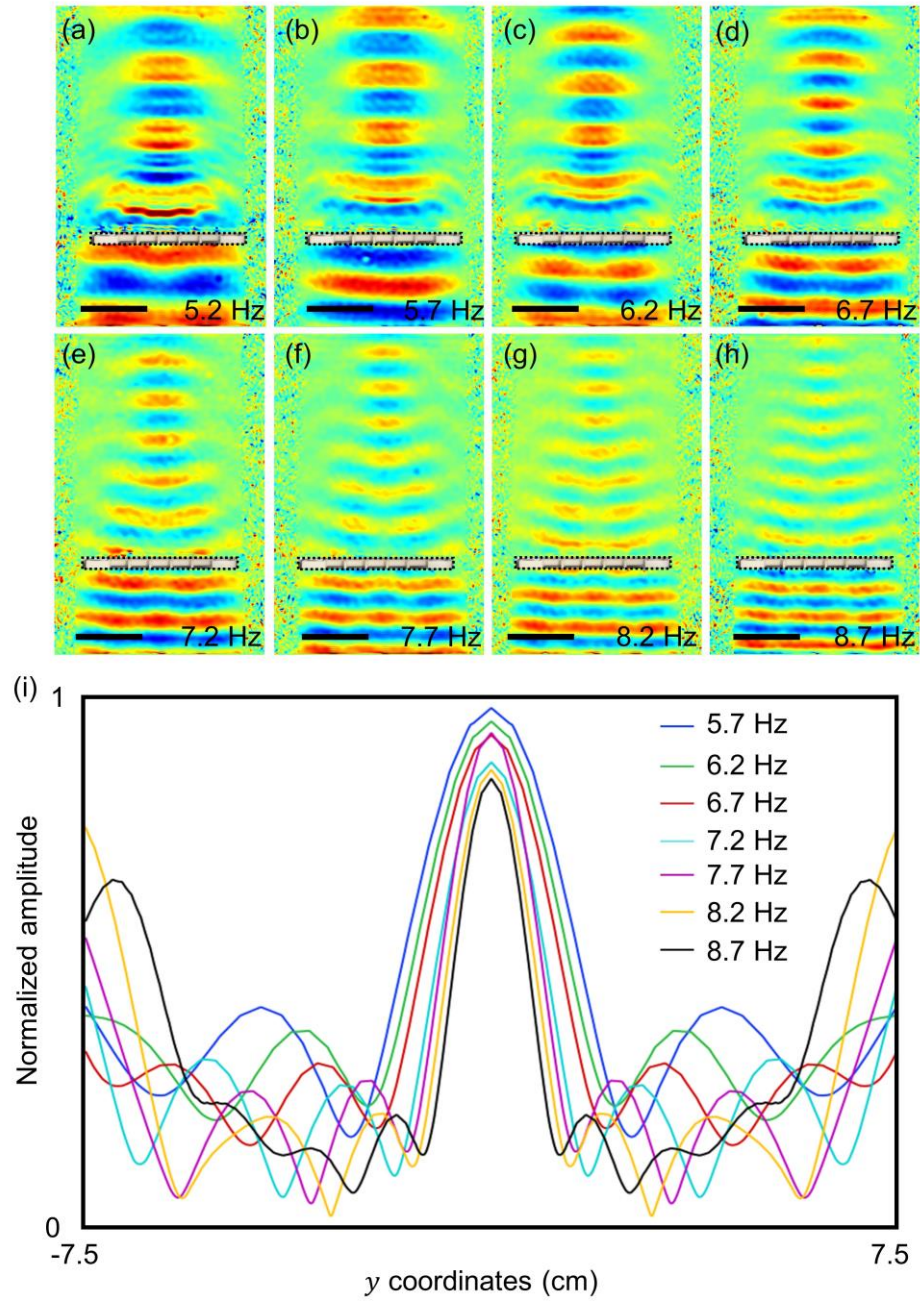


Fig. S6 (a-h) Measured results of water waves passing through the focal WMGs at different frequencies. The scale bar is 6 cm. **(i)** normalized amplitude along the y -direction where the focal point is located.