

Directional Dynamics of the Non-Hermitian Skin Effect

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Supporting Information

Appendix A: Model Details and Light-Cone Velocity

1. Bulk Spectrum and Topological Phases

Under periodic boundary conditions, the Bloch Hamiltonian reads $\mathcal{H}(k) = \mathbf{d}(k) \cdot \boldsymbol{\sigma}$, where

$$d_x(k) = t_1 + t_2 \cos k, \quad d_y(k) = t_2 \sin k + i\gamma. \quad (\text{A1})$$

For $\gamma = 0$, the dispersion is

$$E(k) = \sqrt{t_1^2 + t_2^2 + 2t_1 t_2 \cos k}, \quad (\text{A2})$$

with gap $\Delta = 2|t_1 - t_2|$ and winding number $\nu = 1$ ($\nu = 0$) for $t_2 < t_1$ ($t_2 > t_1$) [1].

Table I summarizes NHSE parameters for representative γ values.

TABLE I. NHSE parameters for $t_1 = 1$, $L = 42$ sites.

γ	r	ξ	Direction	IPR
-0.4	1.53	2.4	Right	0.30
-0.2	1.22	4.9	Right	0.12
0	1.00	∞	—	0.024
+0.2	0.82	4.9	Left	0.12
+0.4	0.65	2.4	Left	0.30

2. Derivation of Light-Cone Velocity

Starting from the dispersion relation Eq. (A2), the quasiparticle group velocity is

$$v_g(k) = \frac{\partial E}{\partial k} = -\frac{t_1 t_2 \sin k}{E(k)}. \quad (\text{A3})$$

To find the maximum of $|v_g(k)|$, we analyze

$$|v_g(k)| = \frac{t_1 t_2 |\sin k|}{\sqrt{t_1^2 + t_2^2 + 2t_1 t_2 \cos k}}. \quad (\text{A4})$$

At the critical point $t_1 = t_2 \equiv t$, the dispersion simplifies to

$$E(k) = t\sqrt{2 + 2\cos k} = 2t \left| \cos \frac{k}{2} \right|, \quad (\text{A5})$$

and the group velocity becomes

$$|v_g(k)| = \frac{t^2 \cdot 2 |\sin(k/2) \cos(k/2)|}{2t |\cos(k/2)|} = t \left| \sin \frac{k}{2} \right|, \quad (\text{A6})$$

which is maximized at $k = \pi$, yielding $v_{\max} = t$.

For general parameters, the maximum group velocity is given by

$$v_{\max} = \min(t_1, t_2). \quad (\text{A7})$$

To prove this, assume $t_2 \leq t_1$ and set $v_g(k^*) = t_2$. This gives $t_1 \sin k^* = E(k^*)$, which after squaring and simplifying yields $(t_1 \cos k^* + t_2)^2 = 0$, hence $\cos k^* = -t_2/t_1$. This solution exists when $t_2 \leq t_1$, and the corresponding velocity is exactly $v_g(k^*) = t_2 = \min(t_1, t_2)$. This result is analogous to the transverse-field Ising model, where $v_{\max} = \min(J, h)$ with J the exchange coupling and h the transverse field [2].

Table II compares the exact numerical values with the approximate formula for $t_1 = 1$.

TABLE II. Comparison of exact maximum group velocity with the approximate formula $v_{\max} \approx \min(t_1, t_2)$ for $t_1 = 1$.

t_2	Phase	$v_{\max}$ (exact)	$\min(t_1, t_2)$
0.6	Topological	0.60	0.6
0.8	Topological	0.80	0.8
1.0	Critical	1.00	1.0
1.2	Trivial	1.00	1.0
1.4	Trivial	1.00	1.0

3. Similarity Transformation and GBZ Spectrum

The non-Hermitian SSH Hamiltonian H_{NH} can be mapped to an equivalent Hermitian model via a similarity transformation $H_{\text{NH}} = S^{-1} H_{\text{eff}} S$. Define the diagonal matrix S with

$$S_{\alpha,j} = r^{-j}, \quad S_{\beta,j} = r^{-j-1}, \quad (\text{A8})$$

where j labels the unit cell and $r = \sqrt{(t_1 - \gamma)/(t_1 + \gamma)}$ [Eq. (3) in the main text]. Under this transformation: *Intracell hopping* (j -independent):

$$\begin{aligned} (SH_{\text{NH}}S^{-1})_{\alpha\beta} &= S_{\alpha,j}(t_1 + \gamma)S_{\beta,j}^{-1} = r^{-j}(t_1 + \gamma)r^{j+1} = (t_1 + \gamma)r, \\ (SH_{\text{NH}}S^{-1})_{\beta\alpha} &= S_{\beta,j}(t_1 - \gamma)S_{\alpha,j}^{-1} = r^{-j-1}(t_1 - \gamma)r^j = (t_1 - \gamma)/r. \end{aligned} \quad (\text{A9})$$

Setting $(t_1 + \gamma)r = (t_1 - \gamma)/r$ gives $r^2 = (t_1 - \gamma)/(t_1 + \gamma)$, confirming r as defined. The common value is

$$\tilde{t}_1 = (t_1 + \gamma)r = \sqrt{(t_1 + \gamma)(t_1 - \gamma)} = \sqrt{t_1^2 - \gamma^2}. \quad (\text{A10})$$

Intercell hopping (cell $j \rightarrow j+1$):

$$\begin{aligned} (SH_{\text{NH}}S^{-1})_{\alpha,j+1;\beta,j} &= r^{-(j+1)} \cdot t_2 \cdot r^{j+1} = t_2, \\ (SH_{\text{NH}}S^{-1})_{\beta,j;\alpha,j+1} &= r^{-j-1} \cdot t_2 \cdot r^{j+1} = t_2. \end{aligned} \quad (\text{A11})$$

Thus H_{eff} is a Hermitian SSH model with intracell hopping $\tilde{t}_1 = \sqrt{t_1^2 - \gamma^2}$ and unchanged intercell hopping t_2 . Equivalently, on the generalized Brillouin zone [3] with $e^{ik} \rightarrow re^{i\theta}$, the OBC spectrum is

$$E(\theta) = \pm \sqrt{\tilde{t}_1^2 + t_2^2 + 2\tilde{t}_1 t_2 \cos \theta}, \quad (\text{A12})$$

and the maximum group velocity per unit cell is $\min(\tilde{t}_1, t_2)$, giving

$$v_{\text{max}}^{\text{GBZ}}(\gamma) = 2 \min(\sqrt{t_1^2 - \gamma^2}, t_2) \quad (\text{A13})$$

in units of sites per time. This velocity is *direction-independent*: the GBZ dispersion Eq. (A12) is symmetric in $\theta \rightarrow -\theta$, so the wavefront propagates at the same speed in both directions.

4. Lieb-Robinson Bound and Its Breakdown

For Hermitian systems, the Lieb-Robinson theorem [4, 5] provides a rigorous upper bound on information propagation:

$$\| [A_x(t), B_0] \| \leq C \|A\| \|B\| e^{-\alpha(|x| - v_{\text{LR}}|t|)}, \quad (\text{A14})$$

where $v_{\text{LR}} \sim 2J_{\text{max}}$. For the Hermitian SSH model ($\gamma = 0$): $v_{\text{LR}}^{\text{H}} = 2 \max(t_1, t_2) = 2.0$.

In non-Hermitian systems, LR bounds generically break down [6–8]. The non-unitary evolution operator e^{-iHt} can amplify signals along certain directions, enabling correlations to spread faster than the Hermitian limit. A naive upper bound using the maximum hopping amplitude $J_{\text{max}} = \max(t_1 + |\gamma|, t_2)$ gives $v_{\text{LR}}^{\text{NH}} = 2J_{\text{max}}$, which is looser than the Hermitian bound.

5. Direction-Dependent Onset Times

Using the propagator relation Eq. (9) in the main text, the direction-dependent effective velocities are listed in Table III. The symmetry $v_{\text{eff}}^{\text{right}}(\gamma) \approx v_{\text{eff}}^{\text{left}}(-\gamma)$ reflects the $\gamma \rightarrow -\gamma$ plus spatial inversion equivalence.

That all effective velocities exceed $v_{\text{LR}}^{\text{H}} = 2.0$ is consistent with the generic LR bound breakdown in non-Hermitian systems [6–8].

6. Numerical Verification of the Imaginary-Gauge Propagator Identity and Comparison with Wavepacket Drift

This appendix provides direct numerical evidence for the propagator identity Eq. (9) of the main text and a

TABLE III. Direction-dependent effective velocities for $t_1 = 1$, $t_2 = 0.5$, $L = 42$, extracted from linear fits to $t^*(d)$ for $d \leq 10$.

γ	$v_{\text{eff}}^{\text{right}}$	$v_{\text{eff}}^{\text{left}}$	ratio
−0.3	3.38	2.48	1.36
0	2.93	3.01	0.97
+0.3	2.32	3.40	0.68

side-by-side comparison of the QLIF asymmetry $\Delta_{\mathbb{T}}$ with the wavepacket center-of-mass drift $\Delta x(t) = \langle x(t) \rangle - j_0$, the latter serving as a conventional non-Hermitian transport diagnostic. All data are obtained for $L = 42$, $t_1 = 1$, $t_2 = 0.5$, observation time $t_{\text{obs}} = 10$, with a single-site initial state $|\psi_0\rangle = |j_0\rangle$ at the central α -site $j_0 = 20$, and observation sites placed at $j_0 \mp 2d_{\text{cells}}$ ($\alpha\alpha\alpha$ configuration). Figure 1 summarizes the four diagnostics.

a. *Panel (a): exact match between data and the similarity transformation.* For $\gamma = 0.1, 0.2, 0.3$ and all distances $d = 1\text{--}9$ sites, the measured propagator ratios agree with the staircase prediction $(S_{i,i}/S_{j,j})^2$ from $S_{\alpha,n} = r^{-n}$, $S_{\beta,n} = r^{-n-1}$ [Eq. (96) of this Supplemental Material] to better than 10^{-4} , spanning two decades of the bias ratio. The exact left-right product $|G_{\text{NH}}(j_0+d)/G_{\text{eff}}|^2 \cdot |G_{\text{NH}}(j_0-d)/G_{\text{eff}}|^2 = 1$ at all d confirms the diagonal structure of S . The bias is therefore not a smooth $r^{|d|}$ in site distance but a unit-cell staircase: each step of two sites multiplies the squared ratio by r^2 . This is the imaginary-gauge content of the directional NHSE that the QLIF asymmetry inherits.

b. *Panel (b): $\Delta_{\mathbb{T}}(\gamma)$ at multiple d .* The QLIF asymmetry is antisymmetric under $\gamma \rightarrow -\gamma$ at machine precision, consistent with the sign rule Eq. (8). It exhibits a peak at $\xi \sim d_{\text{site}}$ (e.g., $d = 3$: peak at $\gamma \approx 0.15$, $\xi \approx 6.6$; $d = 4$: peak at $\gamma \approx 0.20$, $\xi \approx 4.9$), then decays toward zero at both $|\gamma|$ extremes. Peak amplitudes drop by more than an order of magnitude from $d = 3$ to $d = 4$, faster than the r^{2d} bias alone: the Hermitian baseline $|G_{\text{eff}}(2d, t)|^2$ that multiplies the bias also depends on d through the SSH light-cone structure, contributing an additional d -dependent suppression.

c. *Panels (c)–(d): comparison with wavepacket drift.* The wavepacket drift $\Delta x(\gamma)$ is monotone in γ , takes the same sign as the dominant skin direction (negative for $\gamma > 0$, leftward), and is independent of the QLIF observation distance d . The Hermitian-limit residual $\Delta x(\gamma = 0) \approx +0.55$ originates from the sublattice asymmetry of the α -site initial state and is unrelated to non-reciprocity. Panel (d) plots $\Delta_{\mathbb{T}}$ and Δx on a common ξ axis for $d = 4$, $\gamma > 0$: both quantities point leftward (same sign) and share a common imaginary-gauge origin via the $r^{\pm d}$ bias, but their shapes are qualitatively different— $\Delta x(\xi)$ is monotone (saturating in the strong-skin limit $\xi \rightarrow 0$), while $\Delta_{\mathbb{T}}(\xi)$ exhibits a pronounced peak at $\xi \sim d_{\text{site}}$ before being suppressed. The peak

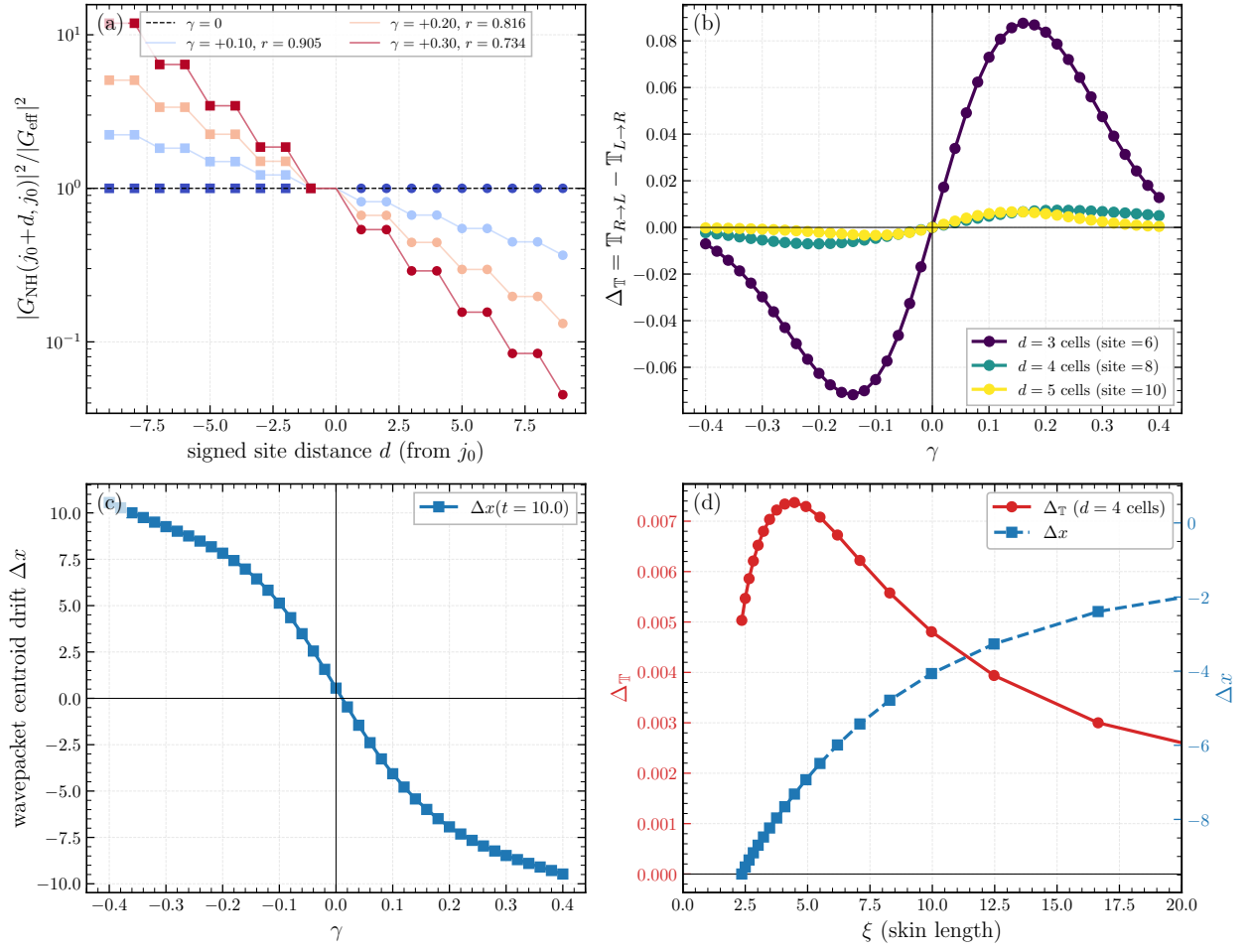


FIG. 1. Numerical verification of the imaginary-gauge picture and benchmarking against a conventional transport diagnostic. (a) Propagator amplitude ratio $|G_{\text{NH}}(j_0 \pm d, j_0; t)|^2 / |G_{\text{eff}}(j_0 \pm d, j_0; t)|^2$ versus signed site distance, on a logarithmic scale, for $\gamma = 0, +0.1, +0.2, +0.3$ (circles: right-going; squares: left-going; staircases: similarity-transform prediction $(S_{i,i}/S_{j,j})^2$). (b) QLIF asymmetry $\Delta_T(\gamma)$ for unit-cell observation distance $d = 3, 4, 5$. (c) Wavepacket center-of-mass drift $\Delta x(\gamma)$ at $t_{\text{obs}} = 10$. (d) Δ_T (red, left axis) and Δx (blue, right axis) versus the skin length ξ on the $\gamma > 0$ branch.

structure is a specific signature of the entropy-difference construction of QLIF: in the strong-skin limit the normalization $|\psi(t)|^2$ becomes concentrated at the boundary, suppressing bulk-site occupation and therefore the local conditional entropies that QLIF integrates. This feature is absent in Δx and identifies QLIF as a complementary, rather than redundant, diagnostic of non-Hermitian directional dynamics.

Appendix B: Numerical Methods

1. Exact Diagonalization

We numerically diagonalize the non-Hermitian SSH Hamiltonian using standard linear algebra routines. For non-Hermitian matrices, we compute both right and left eigenvectors, though for QLIF calculations only the right eigenvectors are needed.

2. Time Evolution and QLIF Implementation

For a single-particle state $|\psi\rangle = \sum_j \psi_j c_j^\dagger |0\rangle$, the site entropy reduces to binary entropy:

$$S_A = -|\psi_A|^2 \ln |\psi_A|^2 - (1 - |\psi_A|^2) \ln(1 - |\psi_A|^2). \quad (\text{B1})$$

The freezing operation sets all matrix elements connected to site B to zero:

$$(H_{\text{f}})_{ij} = \begin{cases} 0 & \text{if } i = B \text{ or } j = B, \\ H_{ij} & \text{otherwise.} \end{cases} \quad (\text{B2})$$

Time evolution is performed via exact diagonalization:

$$|\psi(t)\rangle = e^{-iHt} |\psi_0\rangle = \sum_n e^{-iE_n t} |n\rangle \langle n | \psi_0\rangle, \quad (\text{B3})$$

where $\{E_n, |n\rangle\}$ are eigenvalues and right eigenvectors. For non-Hermitian H , biorthogonal normalization is employed.

3. Inverse Participation Ratio (IPR)

The inverse participation ratio (IPR) is a standard measure of localization [9] and has been widely used to characterize the skin effect in non-Hermitian systems [10, 11]. It is defined as

$$\text{IPR} = \sum_j |\psi_j|^4 / \left(\sum_j |\psi_j|^2 \right)^2, \quad (\text{B4})$$

ranging from $1/L$ (extended) to 1 (fully localized).

Here we derive the IPR for eigenstates exhibiting the non-Hermitian skin effect.

For a skin-effect eigenstate localized at the left boundary, the probability amplitude decays exponentially:

$$|\psi_n|^2 = A r^{2(n-1)}, \quad n = 1, 2, \dots, N, \quad (\text{B5})$$

where n labels the unit cell, r is the skin parameter [Eq. (3) in the main text], and A is the normalization constant.

a. Normalization. From $\sum_{n=1}^N |\psi_n|^2 = 1$:

$$A \sum_{n=1}^N r^{2(n-1)} = A \frac{1 - r^{2N}}{1 - r^2} = 1, \quad (\text{B6})$$

giving

$$A = \frac{1 - r^2}{1 - r^{2N}}. \quad (\text{B7})$$

b. IPR calculation. The IPR is

$$\begin{aligned} \text{IPR} &= \sum_{n=1}^N |\psi_n|^4 = A^2 \sum_{n=1}^N r^{4(n-1)} \\ &= A^2 \frac{1 - r^{4N}}{1 - r^4} = \left(\frac{1 - r^2}{1 - r^{2N}} \right)^2 \frac{1 - r^{4N}}{(1 - r^2)(1 + r^2)} \\ &= \frac{(1 - r^2)(1 - r^{4N})}{(1 - r^{2N})^2(1 + r^2)}. \end{aligned} \quad (\text{B8})$$

Using the identity $1 - r^{4N} = (1 - r^{2N})(1 + r^{2N})$:

$$\text{IPR}_{\text{th}} = \frac{1 - r^2}{1 + r^2} \cdot \frac{1 + r^{2N}}{1 - r^{2N}}. \quad (\text{B9})$$

c. Limiting cases.

- **Strong localization** ($r \ll 1$, N large): $\text{IPR} \rightarrow (1 - r^2)/(1 + r^2) \rightarrow 1$.
- **Extended states** ($r = 1$): Using L'Hôpital's rule, $\text{IPR} \rightarrow 1/N$, the expected value for uniform distribution.
- **Finite-size correction**: For $r < 1$ and large N , $\text{IPR} \approx (1 - r^2)/(1 + r^2)$.

The IPR values listed in Table I are obtained from direct numerical diagonalization, which includes finite-size corrections and deviations from the idealized exponential decay near boundaries. For the Hermitian case ($r = 1$), the table correctly shows $\text{IPR} = 1/L \approx 0.024$ for $L = 42$ sites.

4. Oscillation Analysis

Late-time QLIF oscillations have period $T_{\text{osc}} \approx 2.2$ independent of system size L , ruling out finite-size boundary reflections. The period is determined by intrinsic energy-level spacing: $T_{\text{osc}} \sim 2\pi/\Delta E$ where $\Delta E \approx 2.9$ matches the bulk bandwidth.

The NHSE protects oscillation amplitude: comparing $L = 20$ to $L = 120$, the Hermitian amplitude decays by $\sim 3\times$ while the non-Hermitian ($\gamma = 0.3$) amplitude decays by only $\sim 1.3\times$, because skin-localized eigenstates maintain high local probability density.

5. Sublattice Configuration for QLIF Measurements

As discussed in the main text (Fig. 4), mixed-sublattice configurations introduce a structural asymmetry $|\Delta_{\text{T}}| \sim 0.07$ at $t = 10$ even for $\gamma = 0$, while same-sublattice configurations ($\alpha\alpha\alpha$, $\beta\beta\beta$) preserve inversion symmetry to machine precision ($\sim 10^{-15}$). To isolate genuine non-Hermitian effects, the observation distance d is chosen as an even integer to maintain the same sublattice, with $d = 6$ (3 unit cells) as the standard configuration.

6. Numerical Convergence and Error Analysis

The QLIF evaluation involves two potential sources of numerical error: (i) finite system size L , and (ii) the practical implementation of site freezing. The time step dt does not contribute, because the propagator e^{-iHt} is constructed via exact matrix exponentiation (`scipy.linalg.expm`) rather than an iterative integrator: across $dt \in \{0.005, 0.01, 0.02, 0.05, 0.1\}$ the value $\Delta_{\text{T}}(t=10) = 4.7479844544 \times 10^{-2}$ is reproduced bit-for-bit, so dt enters only through the spacing of the cumulative-QLIF curve and carries no truncation error. We quantify the two remaining contributions at the working point $t_1 = 1$, $t_2 = 0.5$, $\gamma = 0.3$, $d = 6$, $t = 10$; the results are summarized in Fig. 2.

a. System size. For $L \in \{20, 30, 42, 60, 80\}$ with $d = 6$ held fixed in lattice-site units, Δ_{T} stabilizes beyond $L \approx 30$: the value at $L = 42$ agrees with $L = 80$ to better than 2×10^{-9} [Fig. 2(a)]. The small $L = 20$ deviation reflects edge proximity of the observation sites and is already removed at $L = 30$, well below our working $L = 42$. At $L = 60$ the residual approaches the floating-point cancellation floor $|\Delta_{\text{T}}| \epsilon \approx 10^{-17}$ set by subtracting two near-identical $\mathcal{O}(10^{-2})$ values in double precision; the underlying physical correction may be smaller still but is no longer resolvable.

b. Freezing operation. We compare the hard-freezing scheme used in the main text (zeroing out the row and column of the frozen site) with a soft-freezing scheme that adds a large on-site chemical potential $\mu|\text{site}\rangle\langle\text{site}|$ to H . The two schemes coincide in

the limit $\mu \rightarrow \infty$. Across $\mu \in \{10, 10^2, \dots, 10^8\}$ the soft-freeze value converges to the hard-freeze value as $1/\mu^2$ [Fig. 2(b)], reaching agreement to 1.45×10^{-9} at $\mu = 10^4$ and saturating at floating-point precision for $\mu \geq 10^6$. The hard-freezing scheme is therefore the exact $\mu \rightarrow \infty$ limit of soft freezing, and the systematic error of this implementation is negligible.

c. Hermitian baseline. An independent consistency check is the $\gamma = 0$ limit, where time-reversal symmetry enforces $\mathbb{T}_{R \rightarrow L}(t) = \mathbb{T}_{L \rightarrow R}(t)$ identically. This is verified on the Hatano–Nelson chain in Appendix E: the residual asymmetry is bounded by $\sim 10^{-15}$ across the entire evolution window, confirming that any asymmetry reported in the main text is unambiguously attributable to $\gamma \neq 0$ rather than to numerical artifacts.

d. Error estimate for main-text figures. Combining the time-step and system-size residuals in quadrature, $\sigma_{\Delta_{\mathbb{T}}} = \sqrt{\sigma_{dt}^2 + \sigma_L^2} \approx 2 \times 10^{-9}$, which lies seven orders of magnitude below the signal $\Delta_{\mathbb{T}} \sim 5 \times 10^{-2}$ and far below the marker size in Figs. 2 and 3. The uncertainty on all displayed asymmetry values is therefore bounded by this estimate.

Appendix C: Scaling Analysis: Role of Observation Distance and System Size

The directional asymmetry $\Delta_{\mathbb{T}}$ depends on the observation distance d and the skin length ξ . Since our QLIF is computed from the normalized density matrix [Eq. (7) in the main text], the absolute eigenstate weight r^{j_0} at the initial position j_0 cancels between numerator and denominator. The physically relevant quantity is the *ratio* of eigenstate amplitudes at the two observation sites:

$$\frac{|\psi(j_0 + d)|}{|\psi(j_0 - d)|} = r^{2d} = e^{-2d/\xi}, \quad (\text{C1})$$

which is independent of j_0 . This justifies the use of d/ξ as the controlling scale for the directional asymmetry.

To verify this, we perform two numerical experiments.

1. System-Size Independence

We compute $\Delta_{\mathbb{T}}(\gamma)$ at fixed observation distance $d = 6$ for four system sizes $L = 20, 42, 60, 80$ (corresponding to $j_0 = 10, 20, 30, 40$). As shown in Fig. 3, the curves converge rapidly with increasing L . In particular:

- The suppression at large $|\gamma|$ is insensitive to L , confirming that it is not governed by j_0/ξ .
- The residual L -dependence at moderate $|\gamma|$ reflects finite-size effects (boundary reflections in short chains), which diminish as L increases.

2. Observation-Distance Dependence

We compute $\Delta_{\mathbb{T}}(\gamma)$ at fixed $L = 42$ for four observation distances $d = 3, 6, 9, 12$. As shown in Fig. 4, d has a dramatic effect:

- The peak amplitude at $d = 3$ is roughly four times that at $d = 6$, while $d = 12$ yields a nearly vanishing signal.
- The peak position in $|\gamma|$ shifts with d : smaller d allows the asymmetry to persist to larger $|\gamma|$ values, consistent with the d/ξ scaling.

Comparing Figs. 3 and 4, the d -dependence is far stronger than the L -dependence, confirming that d/ξ —not j_0/ξ —is the relevant scale for the directional asymmetry measured by QLIF.

3. Robustness of Three Temporal Regimes

To demonstrate that the three temporal regimes identified in the main text persist in the thermodynamic limit, we compute the full QLIF time evolution for system sizes $L = 20, 42, 60, 80$ at fixed $\gamma = 0.3$ and $d = 6$. Figure 5 shows that all three regimes—onset (I), stabilization (II), and oscillation (III)—are clearly visible for every L .

Table IV lists the quantitative regime boundaries extracted from these data. The onset time ($t_I^* = 3.90$), the first oscillation time ($t_{II}^* = 6.55$), and the oscillation period ($T_{\text{osc}} = 2.0$) are all independent of L to within numerical precision. The onset time is determined by the observation distance and the Lieb-Robinson velocity ($t_I^* \approx d/v_{\text{LR}}$), while the oscillation period reflects intrinsic energy-level spacing ($T_{\text{osc}} \sim 2\pi/\Delta E$). Only the oscillation *amplitude* depends on L , decreasing as $L^{-\alpha}$ with $\alpha \approx 0.2$ for $\gamma = 0.3$, because the NHSE concentrates eigenstate weight near boundaries and protects against uniform dilution.

TABLE IV. Regime boundaries for different system sizes at $\gamma = 0.3$, $d = 6$.

L	j_0	t_I^* (onset)	t_{II}^* (1st osc.)	T_{osc}
20	10	3.90	6.55	2.10
42	20	3.90	6.55	2.00
60	30	3.90	6.55	2.00
80	40	3.90	6.55	2.00

4. Robustness of Velocity Ordering

The velocity ordering $v_{\text{eff}}(\gamma < 0) > v_{\text{eff}}(0) > v_{\text{eff}}(\gamma > 0)$ is a key prediction of NHSE blocking. To test its robustness, we measure the onset time t^* versus distance d

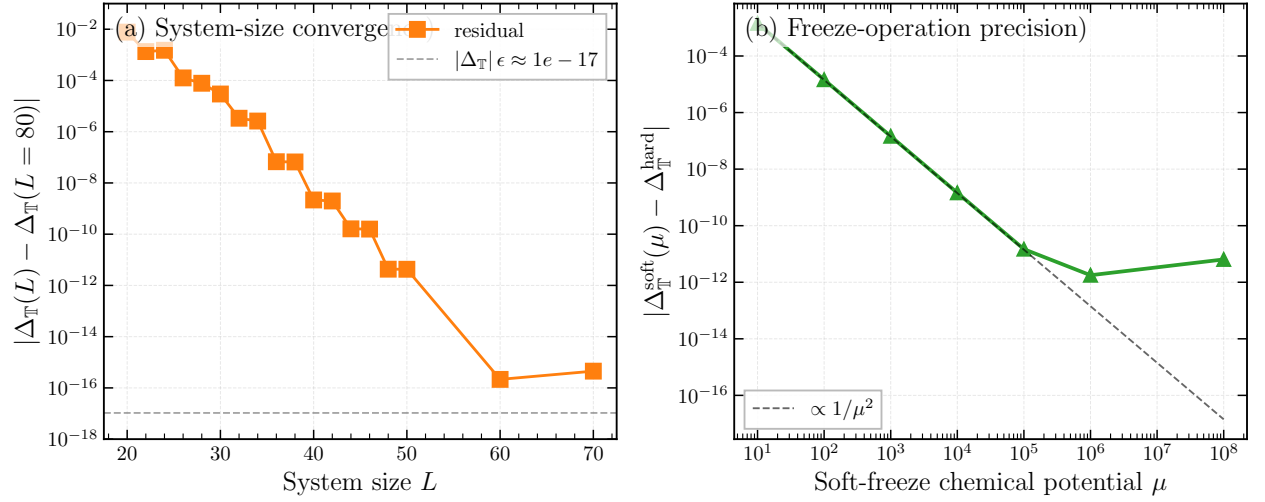


FIG. 2. Numerical convergence of the QLIF asymmetry $\Delta_{\mathbb{T}}(t=10)$ at the working point $d = 6$, $\gamma = 0.3$. (a) System-size convergence: $|\Delta_{\mathbb{T}}(L) - \Delta_{\mathbb{T}}(L=80)|$ decays exponentially in L ; agreement with the $L = 80$ reference is better than 2×10^{-9} at the working size $L = 42$. The grey dashed line marks the floating-point cancellation floor $|\Delta_{\mathbb{T}}| \epsilon \approx 10^{-17}$. (b) Freezing-operation precision at $L = 42$: soft freeze converges to hard freeze as $1/\mu^2$, reaching $\sim 10^{-9}$ at $\mu = 10^4$ and saturating near floating-point precision for $\mu \geq 10^6$; the hard-freezing scheme used in the main text is the exact $\mu \rightarrow \infty$ limit. The time step dt is omitted because exact matrix exponentiation makes $\Delta_{\mathbb{T}}$ bit-identical across $dt \in [0.005, 0.1]$.

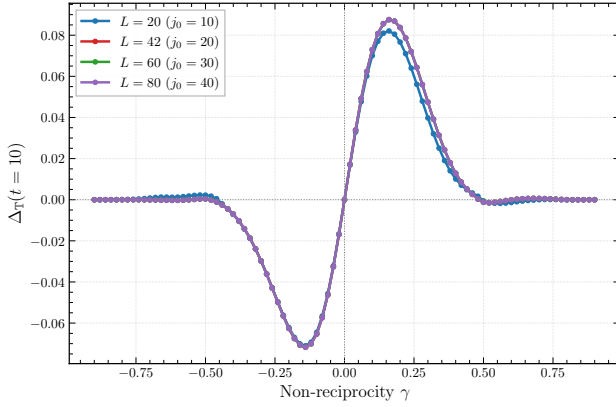


FIG. 3. Directional asymmetry $\Delta_{\mathbb{T}}(t = 10)$ versus γ at fixed observation distance $d = 6$ for system sizes $L = 20, 42, 60, 80$. The curves converge with increasing L , demonstrating that $j_0 = L/2$ is not the controlling scale. Initial state: $|\psi_0\rangle = |j_0, \alpha\rangle$ at the central unit cell; $\alpha\alpha\alpha$ configuration; $t_1 = 1$, $t_2 = 0.5$.

for three system sizes $L = 42, 60, 80$ at $\gamma = -0.3, 0, +0.3$. Figure 6 shows that the $t^*(d)$ curves are identical across all L values, with effective velocities:

$$v_{\text{eff}}(-0.3) = 3.19, \quad v_{\text{eff}}(0) = 2.72, \quad v_{\text{eff}}(+0.3) = 2.13, \quad (\text{C2})$$

independent of system size. This confirms that the velocity ordering is a bulk property determined by the local hopping structure, not a finite-size artifact.

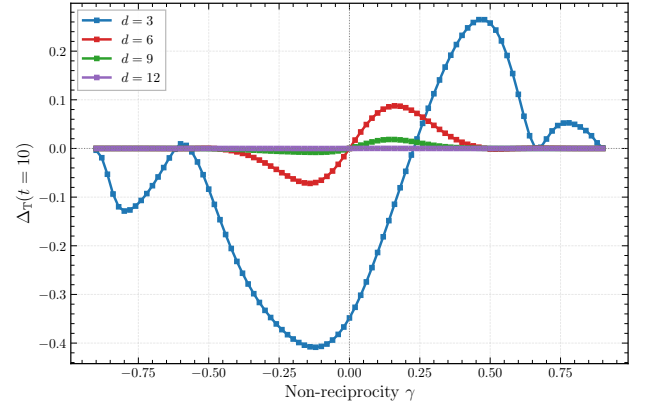


FIG. 4. Directional asymmetry $\Delta_{\mathbb{T}}(t = 10)$ versus γ at fixed $L = 42$ for observation distances $d = 3, 6, 9, 12$. The observation distance d has a dramatic effect on both the amplitude and peak position, in sharp contrast to the weak L -dependence in Fig. 3. Initial state: $|\psi_0\rangle = |j_0, \alpha\rangle$ at the central unit cell ($j_0 = 20$). Parameters: $t_1 = 1$, $t_2 = 0.5$. Note: $d = 6, 12$ yield $\alpha\alpha\alpha$ configurations (all sites on α sublattice), while $d = 3, 9$ yield $\alpha\beta\beta$ configurations (observation sites on β sublattice), which introduces a structural offset at $\gamma = 0$ as discussed in the main text (Fig. 4).

Appendix D: Robustness across the Topological Phase Boundary

To assess whether the main conclusions of this work depend on the topological character of the underlying Hermitian band structure, we repeat the scissors-effect analysis in the trivial phase by raising t_2 from 0.5 to 1.5, keeping all other parameters as in the main-text simula-

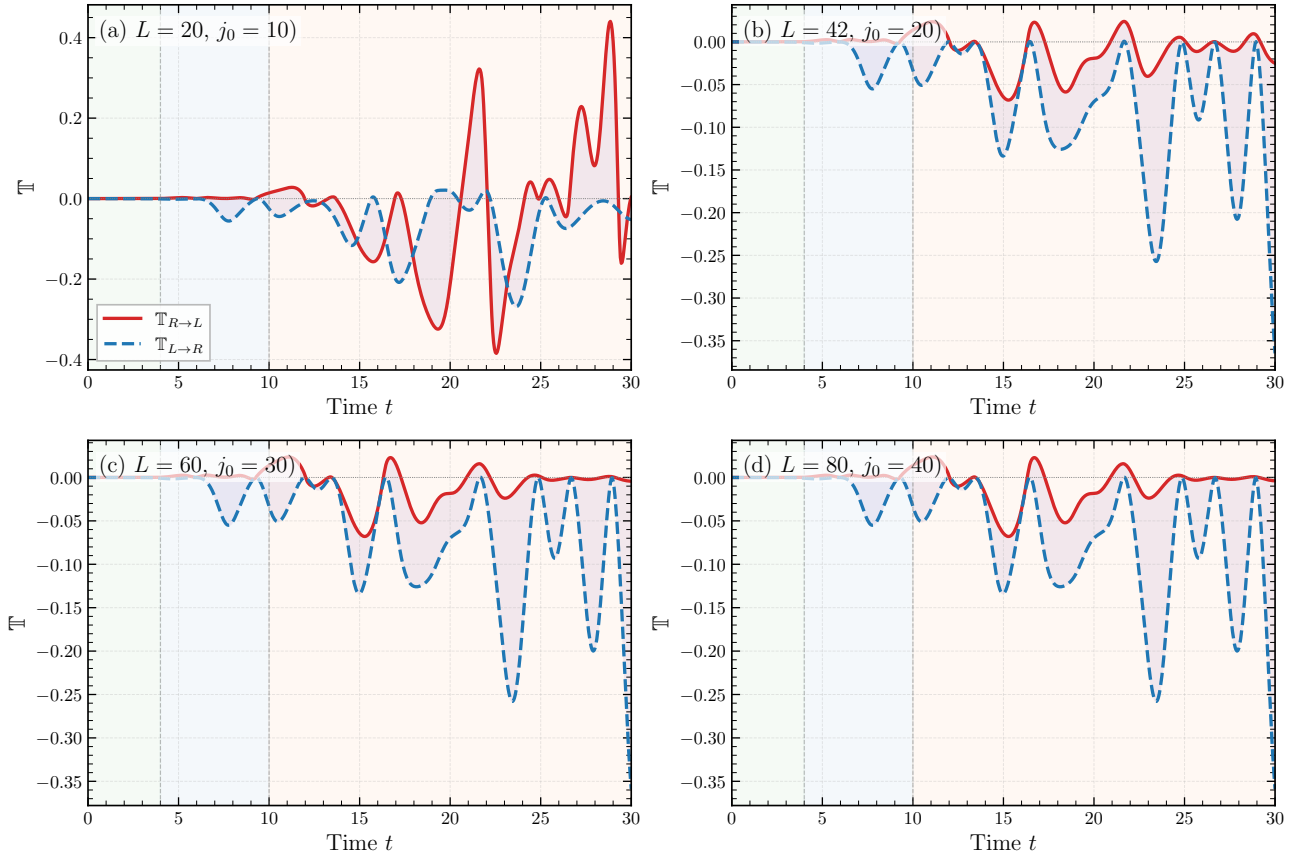


FIG. 5. Time evolution of $\mathbb{T}_{R \rightarrow L}$ (solid red) and $\mathbb{T}_{L \rightarrow R}$ (dashed blue) for (a) $L = 20$, (b) $L = 42$, (c) $L = 60$, (d) $L = 80$, at fixed $\gamma = +0.3$ and $d = 6$. Shaded regions mark regimes I (onset), II (stabilization), and III (oscillation). The three-regime structure is robust across all system sizes; only the oscillation amplitude decreases with L . Initial state: $|\psi_0\rangle = |j_0, \alpha\rangle$ at the central unit cell; $\alpha\alpha\alpha$ configuration; $t_1 = 1$, $t_2 = 0.5$.

tions ($L = 42$, $t_1 = 1$, $d = 6$, $T_{\max} = 20$).

Figure 7 compares the two phases. Panels (a)–(c) show the time evolution of the cumulative QLIF at three representative values of the non-reciprocity parameter; panel (d) shows $\Delta_{\mathbb{T}}(\gamma)$ at $t = 10$. Three features established in the topological phase remain valid in the trivial phase: (i) the Hermitian limit $\gamma = 0$ enforces a vanishing directional asymmetry; (ii) the sign rule $\text{sgn}(\Delta_{\mathbb{T}}) = \text{sgn}(\gamma)$ holds throughout the scanned range; and (iii) the slope of $\Delta_{\mathbb{T}}(\gamma)$ near $\gamma = 0$ is essentially unchanged across the phase boundary.

The only notable phase-dependent feature is a vertical offset of the cumulative QLIF at $\gamma = 0$, which we attribute to the change in the group-velocity scale $v_{\max} = \min(t_1, t_2)$ that controls how rapidly the initially localized state spreads beyond the observation window. This offset affects the overall magnitude but not the γ -induced directional asymmetry that constitutes the central phenomenon of this paper.

We emphasize that this parallel verification does not resolve the behavior of QLIF at the topological transition point $t_2 = t_1$, where the gap closes and the generalized Brillouin zone becomes non-analytic. A systematic inves-

tigation of the interplay between non-Hermitian topology and the skin effect on directional information flow is left to future work, as discussed in Section V.

Appendix E: QLIF on the Hatano–Nelson Chain: Model Independence of the Directional Signatures

To test whether the directional QLIF phenomena reported in the main text are specific to the SSH sublattice structure or are intrinsic to the non-Hermitian skin effect itself, we repeat the central analysis on the Hatano–Nelson (HN) chain [12], the simplest one-dimensional non-reciprocal model. The HN chain has a single band, no sublattice, and no line-gap topology; any feature that survives on this model is therefore a property of non-reciprocity per se, not of SSH topology.

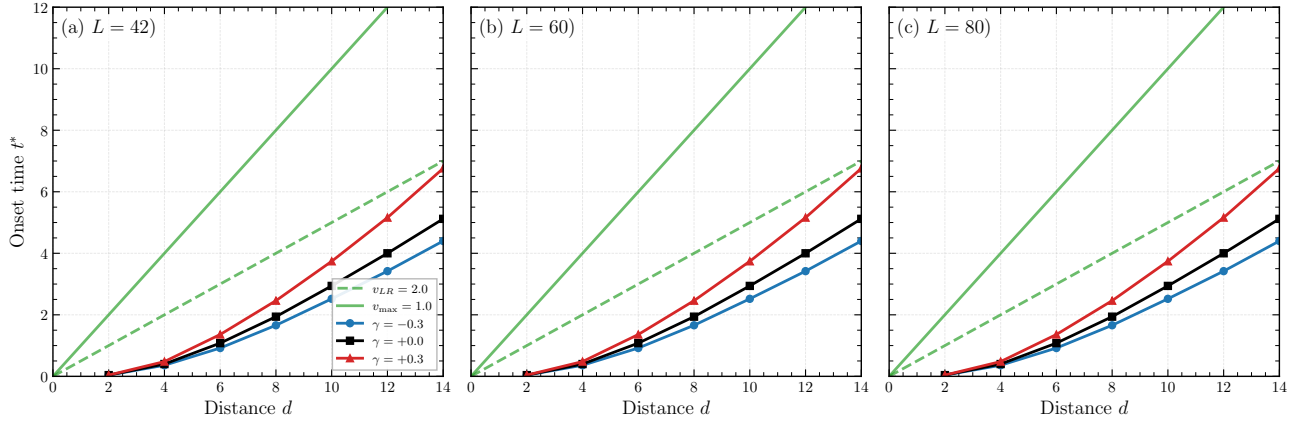


FIG. 6. Onset time t^* (threshold $\epsilon = 10^{-6}$) versus distance d for (a) $L = 42$, (b) $L = 60$, (c) $L = 80$, at $\gamma = -0.3$ (blue circles), 0 (black squares), $+0.3$ (red triangles). Solid green: $v_{\max} = 1.0$; dashed green: $v_{LR} = 2.0$. The velocity ordering is identical across all system sizes, confirming its robustness in the thermodynamic limit. Initial state: $|\psi_0\rangle = |j_0, \beta\rangle$ at the central unit cell; $t_1 = 1$, $t_2 = 0.5$.

1. Model and protocol

The Hatano–Nelson Hamiltonian, in single-particle form with open boundary conditions, reads

$$H = \sum_{j=1}^{L-1} \left[(t + \gamma) c_{j+1}^\dagger c_j + (t - \gamma) c_j^\dagger c_{j+1} \right], \quad (\text{E1})$$

with $t = 1$ and non-reciprocity parameter γ . We use $L = 43$ sites (odd, so that the central site is unambiguous) and initialize $|\psi_0\rangle = c_{j_0}^\dagger |0\rangle$ at $j_0 = 21$. Observation sites are placed symmetrically at $j_0 \pm d$ with $d = 6$. The cumulative QLIF and the freezing prescription follow Sec. II of the main text and Appendix B without modification.

The imaginary-gauge argument used in Sec. IV of the main text applies verbatim: the similarity transformation $S = \text{diag}(r^{-j})$ with $r = \sqrt{(t - \gamma)/(t + \gamma)}$ maps Eq. (E1) to a Hermitian reference Hamiltonian. The ratio of eigenstate amplitudes at the two observation sites is again $r^{\pm 2d}$, so the same mechanism that drives the SSH scissors effect should drive the HN scissors effect.

2. Numerical results

Figure 8 shows the cumulative QLIF on the HN chain for $\gamma = 0, +0.3, -0.3$ together with the asymmetry scan

$\Delta_{\mathbb{T}}(\gamma)$. Three features mirror the SSH results of Sec. III:

- Hermitian-limit symmetry.** At $\gamma = 0$ the two directional QLIFs agree to machine precision across the entire evolution window: $\max_t |\mathbb{T}_{R \rightarrow L}(t) - \mathbb{T}_{L \rightarrow R}(t)|_{\gamma=0} \lesssim 10^{-15}$.
- Sign rule and mirror antisymmetry.** For $\gamma = \pm 0.3$ the asymmetry satisfies $\Delta_{\mathbb{T}}(+0.3) = -\Delta_{\mathbb{T}}(-0.3)$ to all displayed digits; the sign of $\Delta_{\mathbb{T}}$ is opposite to the skin-localization direction, reproducing the rule $\text{sgn}(\Delta_{\mathbb{T}}) = -\text{sgn}(\text{skin direction})$ established in the SSH analysis.
- Optimal regime.** $\Delta_{\mathbb{T}}(\gamma)$ is non-monotonic, with peaks near $|\gamma| \approx 0.20$. The slope at the origin, $d\Delta_{\mathbb{T}}/d\gamma \approx -0.11$, has magnitude comparable to (within $\sim 20\%$ of) the SSH value of Fig. 1; the opposite sign is a convention artifact, with $\gamma > 0$ favoring $+x$ in the HN convention and $-x$ in the SSH/Yao–Wang convention.

The reproduction of all three features—on a model without any of the SSH structural ingredients—confirms that the directional QLIF signatures originate in non-reciprocity itself. SSH topology only modulates quantitative details (notably through the t_2 -dependent effective skin length), not the existence of the phenomena.

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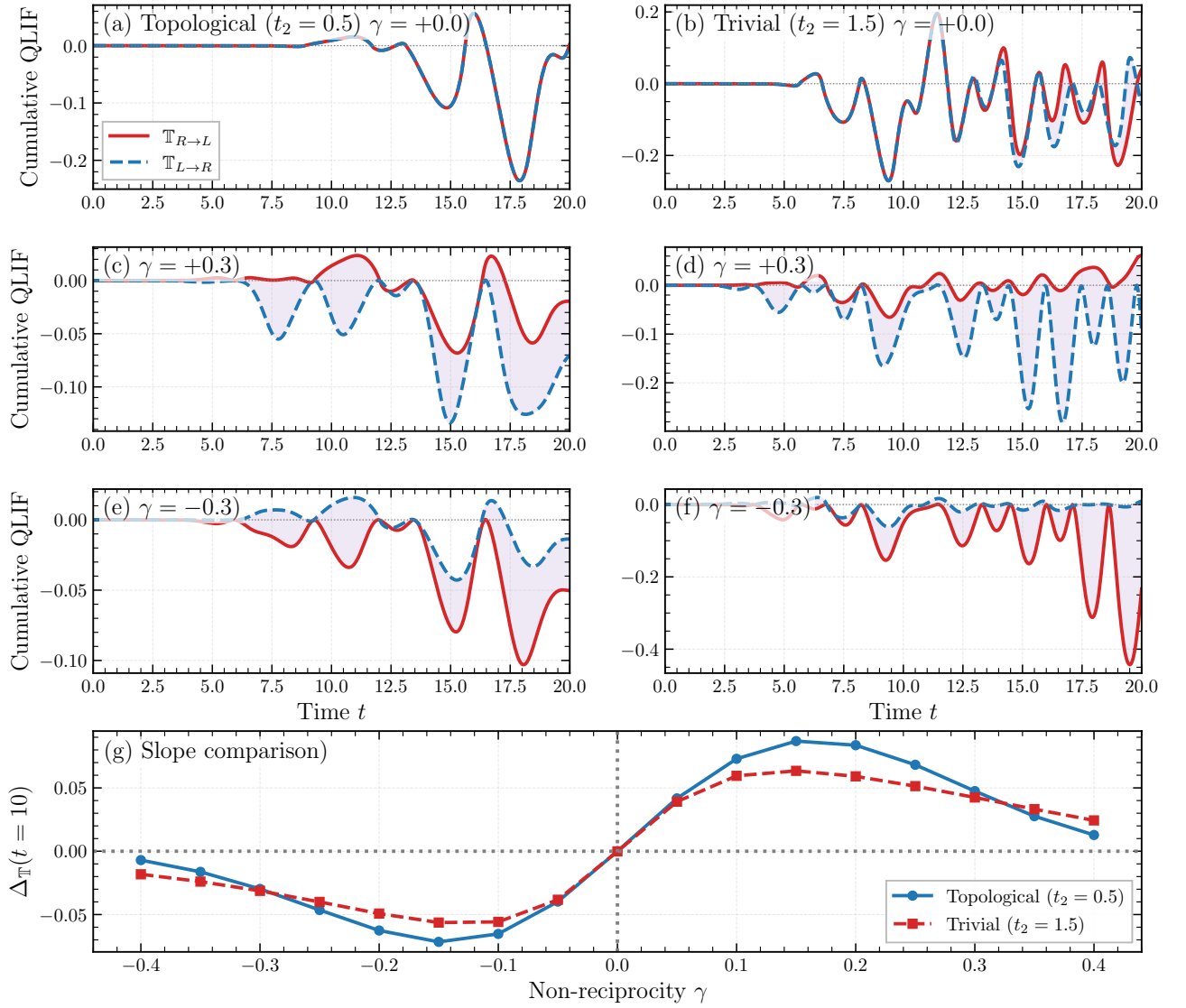


FIG. 7. Robustness of the scissors effect across the topological phase boundary. Left column: topological phase ($t_2 = 0.5$). Right column: trivial phase ($t_2 = 1.5$). Rows (a)–(c): cumulative QLIF $\mathbb{T}_{R \rightarrow L}$ (solid) and $\mathbb{T}_{L \rightarrow R}$ (dashed) at $\gamma = 0, +0.3, -0.3$; the shaded band indicates the directional asymmetry $\Delta_{\mathbb{T}}$. Row (d): $\Delta_{\mathbb{T}}(t = 10)$ versus γ for both phases; the linear slopes near $\gamma = 0$ are $+0.14$ (topological) and $+0.13$ (trivial). Other parameters as in Fig. 2.

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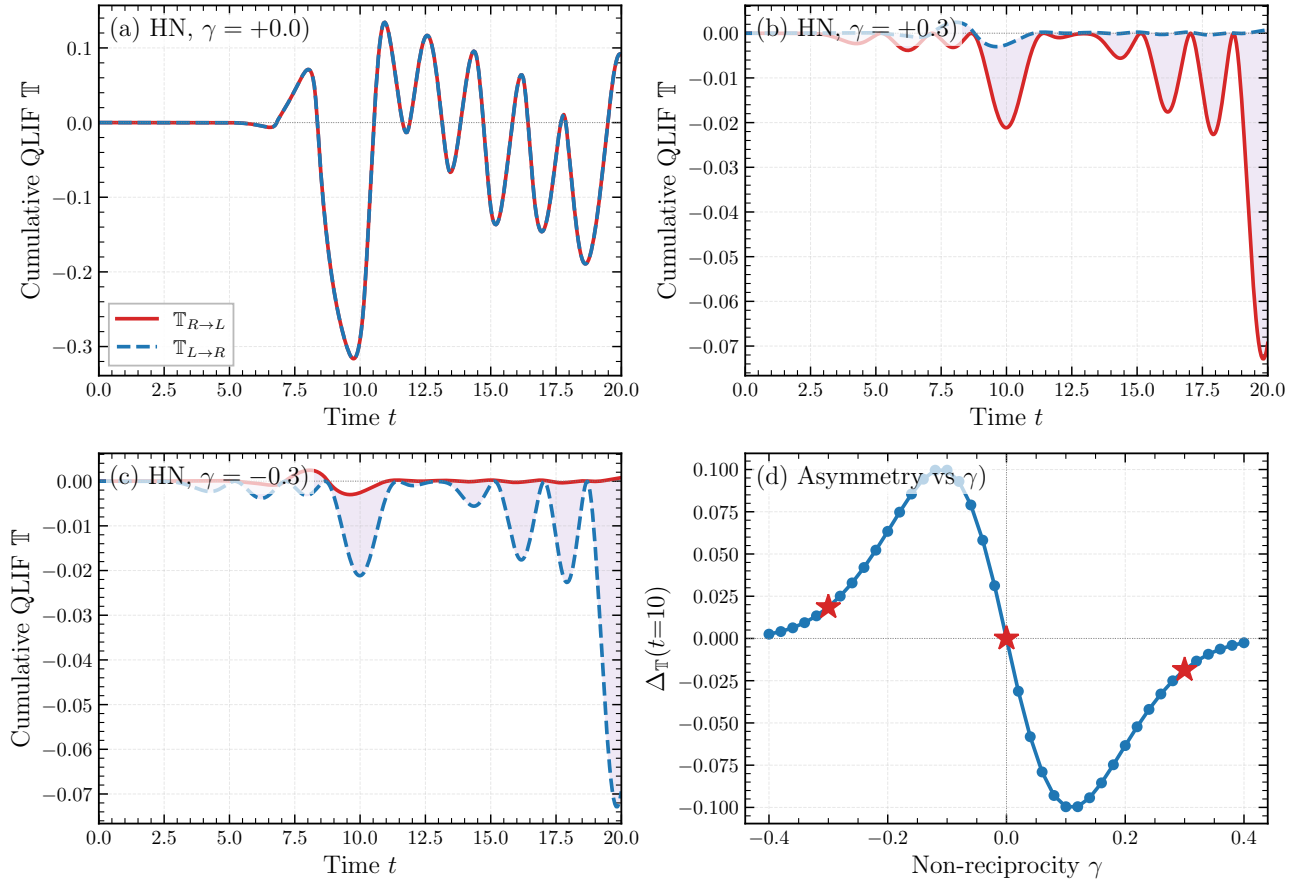


FIG. 8. Directional QLIF on the Hatano–Nelson chain ($L = 43$, $j_0 = 21$, $d = 6$, $t = 1$). (a)–(c) Cumulative $\mathbb{T}_{R \rightarrow L}$ and $\mathbb{T}_{L \rightarrow R}$ versus time for $\gamma = 0, +0.3, -0.3$. The Hermitian case (a) shows exact symmetry; the non-reciprocal cases (b),(c) are perfect mirror images of each other. (d) Asymmetry $\Delta_{\mathbb{T}}(t = 10)$ versus γ ; red stars mark the three main-panel values. The scissors effect, sign rule, and non-monotonic optimal-regime peak all reproduce the SSH results of Fig. 1, despite the absence of any sublattice or topological structure.