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Evolution of decoherence in a two-mode entangled system under amplitude decay

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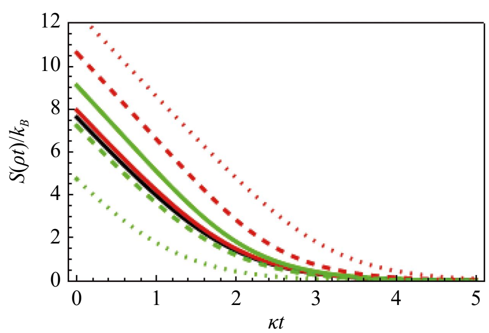
ABSTRACT

Based on our recently published work [*Front. Phys.* 21, 093201 (2026)], we extend the analysis to the density-operator evolution of a two-mode entangled quantum system subject to amplitude decay and show that the output state retains the same functional structure as the initial density operator throughout the decay. We further examine the evolutions of several quantum statistical distributions and the von Neumann entropy used to quantify entanglement during the process. This work offers a systematic approach to studying two-mode entangled open systems and yields results on the von Neumann entropy that may help facilitate entanglement measurement.



Keywords two-mode entangled quantum system, operator ordering method, amplitude decay, quantum statistical distribution, von Neumann entropy

1 Introduction

Quantum decoherence, first discussed in pioneering work by Zeh and later developed extensively [1, 2], describes the loss of quantum coherence that drives a quantum system toward effectively classical behavior, i.e., a quantum-to-classical transition. This occurs because an open quantum system inevitably becomes entangled with its surrounding environment, leading to a gradual suppression of phase coherence [3–5]. Numerous experiments have confirmed decoherence effects. For example, in the double-slit experiment, interference fringes can disappear when which-path information becomes available through environmental coupling, reflecting the loss of phase coherence.

Decoherence is a rapid degradation of quantum coherence and a central obstacle to the development of quantum technology. In particular, it plays a critical role in quantum computing, where coherence loss directly limits computational fidelity, efficiency, and stability. Since the concept was introduced, decoherence has attracted broad attention across quantum physics. Reference [6] shows that decoherence can lead to environment-induced superselection (einselection), a process associated with the selective suppression of information in certain bases. The decoherence dynamics of continuous-variable two-qubit systems have also been studied under the simultaneous action of two different noise channels (e.g., amplitude and phase damping), which better reflect realistic experimental environments [7]. In addition, Ref. [8] reviews decoherence mechanisms, the evolution of geometric quantum correlations and quantum coherence in various noisy channels, and the adverse impacts of decoherence

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on quantum-information tasks. Overall, decoherence arising from coupling to a noisy environment is unavoidable, and mitigating its effects remains essential for executing quantum information protocols reliably.

For any open quantum system, quantum noise inevitably arises in its fundamental dynamical processes. Notably, amplitude decay is one of the most common causes of nonclassicality deterioration in open systems. Here, we consider a two-mode bosonic system coupled to a bosonic amplitude-decay environment (a Markovian process). In the interaction picture, the time-dependent density operator ρ_t of the system satisfies the following master equation [9–11]:

$$\frac{\mathrm{d}\rho_t}{\mathrm{d}t} = \kappa \sum_{i=a,b} \left(2i\rho_t i^\dagger - i^\dagger i\rho_t - \rho_t i^\dagger i\right), \tag{1}$$

where the decay rate κ for the two modes is assumed to be the same, and $a^\dagger$ and $b^\dagger$ are the boson creation operators of the two-mode system, respectively. Unlike phase damping and thermal processes, amplitude decay represents energy transfer from the system to a zero-temperature environment. To solve Eq. (1), we review the thermal entangled state $|\zeta\rangle$ and its relevant properties, where $|\zeta\rangle = \mathrm{e}^{-|\zeta|^2/2+\zeta a^\dagger - \zeta^* \tilde{a}^\dagger + a^\dagger \tilde{a}^\dagger}|00\rangle$ describes the entanglement between the system and its surrounding environment, and $\tilde{a}^\dagger$ represents the creation operator of the environment. In the entangled state $|\zeta=0\rangle$, there exist operator correspondence relations between the environment mode $(\tilde{a}, \tilde{a}^\dagger)$ and the system mode $(a, a^\dagger)$, i.e., $a \Leftrightarrow \tilde{a}^\dagger$, $a^\dagger \Leftrightarrow \tilde{a}$ and $(a^\dagger a)^n \Leftrightarrow (\tilde{a}^\dagger \tilde{a})^n$ [12, 13]. Using these operator relations, Eq. (1) can be successfully solved, and its explicit solution can be written in the operator-sum form [12, 14, 15], i.e.,

$$\rho_t = \sum_{l,m=0}^{\infty} \boldsymbol{M}_{l,m} \rho_0 \boldsymbol{M}_{l,m}^\dagger, \tag{2}$$

where $\boldsymbol{M}_{l,m}$ is the Kraus operator, i.e.,

$$\boldsymbol{M}_{l,m} = \sqrt{\frac{\mathcal{T}^{l+m}}{l!m!}} \mathrm{e}^{-\kappa t\left(a^\dagger a + b^\dagger b\right)} a^l b^m, \ \ \mathcal{T} = 1 - \mathrm{e}^{-2\kappa t}. \tag{3}$$

Using the operator ordering method and the operator identities

$$\mathrm{e}^{\lambda i^\dagger i} i \mathrm{e}^{-\lambda i^\dagger i} = \mathrm{e}^{-\lambda} i, \ \mathrm{e}^{\lambda i^\dagger i} =: \exp\left[\left(\mathrm{e}^{\lambda} - 1\right) i^\dagger i\right]:, \tag{4}$$

where the symbol $: \ :$ means normal ordering, showing that all creation operators are on the left side of the annihilation operators [16–18], λ is an arbitrary parameter, we can check $\sum_{l,m=0}^{\infty} \boldsymbol{M}_{l,m}^\dagger \boldsymbol{M}_{l,m} = 1$, such that $\mathrm{tr}\rho_t = 1$.

Recently, solving various master equations that describe quantum decoherence processes and investigating the time evolutions of quantum states in these processes have received great attention. A comprehensive review is given in Ref. [12], which used the entangled-state approach to derive the evolutions of quantum states in phase diffusion, damping, amplification, and phase-sensitive processes, and revealed the decoherence mechanism in an intuitive way. Other studies typically involve at most two types of simple quantum states or decoherence processes [7, 19, 20]. In view of the current progress in theoretical research, investigations of the evolutions of two-mode entangled states governed by known Hamiltonians have not, to the best of our knowledge, been reported. So, different from the previous work [6–8, 12, 19–22], here we make full use of the operator ordering method to find the entangled density operator describing the two-mode entangled quantum system (e.g., a non-degenerate optical parametric amplifier system) with the Hamiltonian

$$H_0 = \varrho a^\dagger b^\dagger + \omega\left(a^\dagger a + b^\dagger b\right) + \varrho^* ab, \tag{5}$$

and investigate its decoherence evolution in the amplitude decay process, where ϱ is the entanglement coefficient of the system and ω refers to the frequency of input signal light [23]. The reason for choosing this topic is twofold. One is that the entangled density operator corresponding to the Hamiltonian H_0, especially its normal ordering product, and the amplitude decay model provide a more general and realistic framework for investigating the evolutions of two-mode open systems. The other is that the operator ordering method and the integration within ordered products offer a powerful tool for investigating the evolutions of two-mode entangled systems and the von Neumann entropy describing quantum entanglement, and this method can also be applied to other two-mode or multi-mode open quantum systems.

The rest of the work is arranged as follows. In Section 2, we obtain the two-mode entangled state ρ_0 belonging to Hamiltonian H_0. In Section 3, we further obtain the evolution of the density operator ρ_0 in the amplitude decay process. Sections 4, 5 and 6 are, respectively, devoted to discussing the evolutions of the average photon number, photon number distribution and Wigner distribution function in this process. The evolved von Neumann entropy of the state ρ_0 in this process is investigated in Section 7. Finally, we summarize some main results of this work in Section 8.

2 Entangled density operator

To investigate the evolution of the entangled system with the Hamiltonian H_0 under amplitude decay, we first obtain the normalized density operator of the entangled system. Using the theory of statistical ensembles, the normalized density operator ρ_0 of the entangled system is represented as

$$\rho_0 = \mathrm{e}^{-\beta H_0}/\left(\mathtt{tr}\mathrm{e}^{-\beta H_0}\right), \tag{6}$$

where $\beta = 1/(kT)$, k is the Boltzmann constant and T is the temperature of the thermal field. Given that the operators $a^\dagger b^\dagger, a^\dagger a + b^\dagger b + 1$, and ab yield the standard su(1,1) Lie algebra, we obtain the following decomposition [24]

$$\begin{aligned}&\exp\left[ma^\dagger b^\dagger + g\left(a^\dagger a + b^\dagger b\right) + nab\right]\\&= \mathrm{e}^{-g}\exp\left(\frac{m}{E\coth E - g}a^\dagger b^\dagger\right)\\&\quad\times\exp\left[\left(a^\dagger a + b^\dagger b + 1\right)\ln\frac{E}{E\cosh E - g\sinh E}\right]\\&\quad\times\exp\left(\frac{n}{E\coth E - g}a^\dagger b^\dagger\right)\end{aligned} \tag{7}$$

with $E^2 = g^2 - mn$. Using Eqs. (5)–(7), we obtain the normal ordering product of $\mathrm{e}^{-\beta H_0}$ as

$$\mathrm{e}^{-\beta H_0} = \varkappa\colon\exp\left[\alpha a^\dagger b^\dagger + \sigma\left(a^\dagger a + b^\dagger b\right) + \alpha^* ab\right]\colon, \tag{8}$$

where the parameters are, respectively,

$$\begin{aligned}&\varkappa = \mathrm{e}^{-\omega'}(\sigma+1),\ \alpha = \frac{-\varrho'}{\Lambda\coth\Lambda - \omega'},\\&\sigma = \frac{\Lambda}{\Lambda\cosh\Lambda - \omega'\sinh\Lambda} - 1,\\&\Lambda = \left(\omega'^2 - |\varrho'|^2\right)^{1/2},\ \omega' = -\beta\omega,\ \varrho' = -\beta\varrho.\end{aligned} \tag{9}$$

On the other hand, using the completeness relation of two-mode coherent states $|z_1 z_2\rangle$, i.e., $\pi^{-2}\int \mathrm{d}^2 z_1 \mathrm{d}^2 z_2 |z_1 z_2\rangle\langle z_1 z_2| = 1$ and the integral formula

$$\int\frac{\mathrm{d}^2\gamma}{\pi}\mathrm{e}^{h|\gamma|^2 + s\gamma + \eta\gamma^*} = -\frac{1}{h}\mathrm{e}^{-\frac{s\eta}{h}},\ \ \mathrm{Re}h < 0, \tag{10}$$

we obtain

$$\mathtt{tr}\mathrm{e}^{-\beta H_0} = \frac{\varkappa}{\sigma^2 - |\alpha|^2}. \tag{11}$$

Therefore, the normalized entangled density operator ρ_0 is given by

$$\rho_0 = \mathrm{e}^{\Delta}\colon\exp\left[\alpha a^\dagger b^\dagger + \sigma\left(a^\dagger a + b^\dagger b\right) + \alpha^* ab\right]\colon, \tag{12}$$

where $\mathrm{e}^{\Delta} = \sigma^2 - |\alpha|^2$. Eqs. (10) and (12) imply that, in order to guarantee that the state ρ_0 is physically valid, the parameters α, σ must satisfy the conditions: $\sigma < 0$ and $\sigma^2 - |\alpha|^2 > 0$ or equivalently, $\sigma < 0$ and $\sigma + |\alpha| < 0$.

Obviously, the density operator ρ_0 is in general since it can involve squeezing, vacuum, mixedness, etc. For examples, for $\alpha = 0$, ρ_0 becomes the thermal state with the density operator $\sigma^2\mathrm{e}^{\left(a^\dagger a + b^\dagger b\right)\ln(\sigma+1)} \equiv \rho_{\mathrm{th}}$ in terms of Eq. (4), and for $\alpha = 0$ and $\sigma = -1$, ρ_0 reduces to vacuum $(|00\rangle\langle 00|)$ via the normal ordering product of two-mode vacuum projection operator, i.e.,

$$|00\rangle\langle 00| = \colon\exp\left[-\left(a^\dagger a + b^\dagger b\right)\right]\colon. \tag{13}$$

However, for $\alpha \neq 0$, ρ_0 refers to the two-mode squeezed vacuum or thermal state.

3 Evolution of ρ_0 under amplitude decay

In this section, we investigate how the two-mode entangled state ρ_0 evolves when it undergoes amplitude decay. By observing the bosonic operator-ordering structure in the Kraus operator-sum representation of ρ_t in Eq. (2), we find that the antinormally ordered form of ρ_0 is particularly convenient for discussing the evolution of ρ_0 under amplitude decay. Therefore, inserting Eq. (12) into the formula that rewrites any two-mode boson operator in antinormal ordering [25, 26], namely,

$$\begin{aligned}\rho = &\int\frac{\mathrm{d}^2 z_1 \mathrm{d}^2 z_2}{\pi^2}\langle -z_1, -z_2|\rho|z_1, z_2\rangle\\&\vdots\mathrm{e}^{|z_1|^2 + |z_2|^2 + z_1^* a - z_1 a^\dagger + z_2^* b - z_2 b^\dagger + a^\dagger a + b^\dagger b}\vdots,\end{aligned} \tag{14}$$

where $\vdots\ \vdots$ denotes antinormal ordering, whose ordering rules are opposite to those of normal ordering, and using the integration formula in Eq. (10) twice, we obtain the antinormal ordering of ρ_0 as

$$\begin{aligned}\rho = &\mathrm{e}^{\Delta}\int\frac{\mathrm{d}^2 z_1 \mathrm{d}^2 z_2}{\pi^2}\mathrm{e}^{\alpha z_1^* z_2^* + \alpha^* z_1 z_2}\\&\vdots\mathrm{e}^{-(1+\sigma)\left(|z_1|^2 + |z_2|^2\right) + z_1^* a - z_1 a^\dagger + z_2^* b - z_2 b^\dagger + a^\dagger a + b^\dagger b}\vdots\\= &\mathrm{e}^{\Delta}\mathcal{G}\vdots\exp\left\{\mathcal{G}\alpha a^\dagger b^\dagger + \left[1 - \mathcal{G}(1+\sigma)\right]\right.\\&\left.\times\left(a^\dagger a + b^\dagger b\right) + \mathcal{G}\alpha^* ab\right\}\vdots,\end{aligned} \tag{15}$$

where we have used the inner product $\langle -z_1, -z_2|z_1, z_2\rangle = \mathrm{e}^{-2\left(|z_1|^2 + |z_2|^2\right)}$, $\mathcal{G} = \left[(1+\sigma)^2 - |\alpha|^2\right]^{-1}$. Further, inserting Eq. (15) into Eq. (2) and using the completeness relation of two-mode coherent states $|z_1, z_2\rangle$, we have

$$\begin{aligned}\rho_t = &\mathrm{e}^{\Delta}\mathcal{G}\sum_{l,m=0}^{\infty}\frac{\mathcal{T}^{l+m}}{l!m!}\mathrm{e}^{-\kappa t\left(a^\dagger a + b^\dagger b\right)}\vdots a^l b^m\exp\left\{\mathcal{G}\alpha a^\dagger b^\dagger\right.\\&\left.+\left[1 - \mathcal{G}(1+\sigma)\right]\left(a^\dagger a + b^\dagger b\right) + \mathcal{G}\alpha^* ab\right\}\\&\cdot b^{\dagger m} a^{\dagger l}\vdots\mathrm{e}^{-\kappa t\left(a^\dagger a + b^\dagger b\right)}\\= &\mathrm{e}^{\Delta}\mathcal{G}\int\frac{\mathrm{d}^2 z_1 \mathrm{d}^2 z_2}{\pi^2}\mathrm{e}^{\mathcal{T}\left(|z_1|^2 + |z_2|^2\right)}\mathrm{e}^{-\kappa t\left(a^\dagger a + b^\dagger b\right)}|z_1, z_2\rangle\\&\cdot\langle z_1, z_2|\mathrm{e}^{-\kappa t\left(a^\dagger a + b^\dagger b\right)}\exp\left\{\mathcal{G}\alpha z_1^* z_2^*\right.\\&\left.+\left[1 - \mathcal{G}(\sigma+1)\right]\left(|z_1|^2 + |z_2|^2\right) + \mathcal{G}\alpha^* z_1 z_2\right\}.\end{aligned} \tag{16}$$

Using the identity $e^{-\lambda a^\dagger a}|\alpha\rangle = \exp\left[-\frac{1}{2}(1 - e^{-2\lambda})|\alpha|^2\right]|\alpha e^{-\lambda}\rangle$, we can rewrite ρ_t as

$$\rho_t = e^{\Delta}\mathcal{G}\int\frac{d^2z_1 d^2z_2}{\pi^2}\left|z_1e^{-\kappa t}, z_2e^{-\kappa t}\right\rangle \cdot\left\langle z_1e^{-\kappa t}, z_2e^{-\kappa t}\right|\exp\left\{\mathcal{G}\alpha z_1^* z_2^* + [1-\mathcal{G}(1+\sigma)]\left(|z_1|^2+|z_2|^2\right)+\mathcal{G}\alpha^* z_1 z_2\right\}. \quad (17)$$

Finally, using the integral formula in Eq. (10) twice, we obtain

$$\rho_t = e^{\Delta_t}\colon \exp\left[\alpha_t a^\dagger b^\dagger + \sigma_t\left(a^\dagger a + b^\dagger b\right) + \alpha_t^* ab\right]\colon, \quad (18)$$

where $\mathcal{H} = \left\{[(1+\sigma)\mathcal{G} - \mathcal{T}]^2 - \mathcal{G}^2|\alpha|^2\right\}^{-1}$, and the parameters α_t, σ_t and Δ_t are, respectively,

$$\begin{aligned}
\alpha_t &= \mathcal{H}\mathcal{G}\alpha e^{-2\kappa t},\\
\sigma_t &= \mathcal{H}[\mathcal{G}(1+\sigma)-\mathcal{T}]e^{-2\kappa t} - 1,\\
e^{\Delta_t} &= e^{\Delta}\mathcal{G}\mathcal{H} \equiv \sigma_t^2 - |\alpha_t|^2.
\end{aligned} \quad (19)$$

Obviously, Eq. (18) shows that the time-dependent density operator ρ_t always has the same form as the initial density operator ρ_0. However, under the influence of amplitude-decay noise, ρ_t is always a mixed quantum state regardless of whether the initial state ρ_0 is pure or mixed, which results from the loss of coherence in the initial state ρ_0.

Further, we need to verify whether the output state ρ_t satisfies the condition $\sigma_t + |\alpha_t| < 0$ in order to show that the state ρ_t is always physically valid in the amplitude decay process. Since $\mathcal{T} = 1 - e^{-2\kappa t}$ increases monotonically with time t, and $\lim_{t\to 0}\mathcal{T}\to 0$, $\lim_{t\to+\infty}\mathcal{T}\to 1$, we have $0<\mathcal{T}<1$. Further, using $\sigma + |\alpha| < 0$, we obtain the inequality $(\sigma+|\alpha|+1)\mathcal{T} < \mathcal{T} < 1$ or equivalently, $(\sigma - |\alpha| + 1)\mathcal{G} - \mathcal{T} > 0$. Hence, we have

$$\begin{aligned}
\mathcal{H}^{-1} &= [(\sigma+1)\mathcal{G}-\mathcal{T}]^2 - \mathcal{G}^2|\alpha|^2\\
&= [(\sigma+|\alpha|+1)\mathcal{G}-\mathcal{T}]\\
&\quad\times[(\sigma-|\alpha|+1)\mathcal{G}-\mathcal{T}]\\
&> 0,
\end{aligned} \quad (20)$$

which leads to $\sigma_t + |\alpha_t| < 0$, that is, ρ_t is a physically valid state.

We now consider several special cases. For the case of $\alpha = 0$, $e^{\Delta} = \sigma^2$, $\mathcal{G} = (\sigma+1)^{-2}$, $\mathcal{H} = \left[(\sigma+1)^{-1} - \mathcal{T}\right]^{-2}$, we therefore have

$$\begin{aligned}
\alpha_t &= 0,\\
\sigma_t &= \frac{e^{-2\kappa t}}{(\sigma+1)^{-1}-\mathcal{T}} - 1,\\
e^{\Delta_t} &= \frac{\sigma^2}{[1-(\sigma+1)\mathcal{T}]^2},
\end{aligned} \quad (21)$$

thus ρ_t reduces to the evolution of the two-mode thermal state ρ_{th} under amplitude decay, where we have used the identity in Eq. (4). For $\sigma = -1$, owing to $e^{\Delta} = 1 - |\alpha|^2$, $\mathcal{G} = -|\alpha|^{-2}$, $\mathcal{H} = \left(\mathcal{T}^2 - |\alpha|^{-2}\right)^{-1}$, thus

$$\begin{aligned}
\alpha_t &= \frac{\alpha e^{-2\kappa t}}{1-\mathcal{T}^2|\alpha|^2},\\
\sigma_t &= \frac{\mathcal{T}|\alpha|^2 e^{-2\kappa t}}{1-\mathcal{T}^2|\alpha|^2} - 1,\\
e^{\Delta_t} &= \frac{1-|\alpha|^2}{1-\mathcal{T}^2|\alpha|^2},
\end{aligned} \quad (22)$$

so ρ_t becomes the evolution of the two-mode squeezed vacuum with the density operator $\rho_{\text{squ}} = \left(1-|\alpha|^2\right)e^{\alpha a^\dagger b^\dagger}|00\rangle\langle 00|e^{\alpha^* ab}$ in the amplitude decay process. In the limiting case, $\kappa t = 0$, $\mathcal{T} = 0$, $\mathcal{H} = G^{-1}$, $\mathcal{G}\mathcal{H} = 1$, $\alpha_t \to \alpha$, $\sigma_t \to \sigma$, $e^{\Delta_t} \to e^{\Delta}$, thus Eq. (18) naturally becomes the initial density operator ρ_0. In the long-time limit, $\kappa t \to \infty$, $\mathcal{T} \to 1$, $\mathcal{G}\mathcal{H} \to \left(\sigma^2 - |\alpha|^2\right)^{-1}$, $\alpha_t \to 0$, $\sigma_t \to -1$, $e^{\Delta_t} \to 1$, so $\rho_{t\to\infty} \to |00\rangle\langle 00|$.

4 Average photon number

The average photon number is one of the fundamental parameters that characterizes the amount of light radiation, and its measurement is of great significance in the continuous-variable quantum key distribution. For the state ρ_t, the average photon number in the a mode reads

$$\langle n_a\rangle = e^{\Delta_t}\mathrm{tr}\left\{\colon a^\dagger \exp\left[\alpha_t a^\dagger b^\dagger + \sigma_t\left(a^\dagger a + b^\dagger b\right) + \alpha_t^* ab\right] a\colon\right\} - 1. \quad (23)$$

Inserting the completeness relation of two-mode coherent states $|z_1, z_2\rangle$ and using the mathematical integral formula

$$\int\frac{d^2\gamma}{\pi}\gamma^n\gamma^{*m}e^{h|\gamma|^2+s\gamma+\eta\gamma^*} = e^{-\frac{s\eta}{h}}\sum_{l=0}^{\min(n,m)}\binom{m}{l}\binom{n}{l}\frac{l!s^{m-l}\eta^{n-l}}{(-h)^{m+n+1-l}} \quad (24)$$

and

$$\int\frac{d^2\gamma}{\pi}\gamma^n\gamma^{*m}e^{h|\gamma|^2} = \delta_{m,n}n!\left(-\frac{1}{h}\right)^{n+1}, \quad (25)$$

where $\mathrm{Re}h < 0$, we thus obtain the evolved average photon number of a-mode under amplitude decay, i.e.,

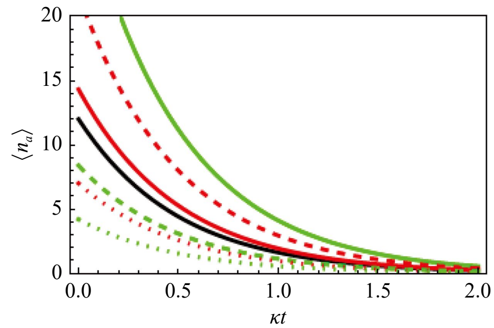

Fig. 1 The evolution of average photon number of a-mode in the state ρ_0 with the decay time κt for different values of the parameters β, ω, and ϱ under amplitude decay. The values of (β, ω, ϱ) are, respectively, $(0.3, 0.3, 0.1)$ (thick black), $(0.3, 0.3, 0.15)$ (thick red), $(0.3, 0.3, 0.24)$ (thick green), $(0, 3, 0.2, 0.1)$ (dashed thick red), $(0.3, 0.4, 0.1)$ (dashed thick green), $(0, 5, 0.3, 0.1)$ (dotted thick red), and $(0.8, 0.3, 0.1)$ (dotted thick green).

$$\begin{aligned}\langle n_a \rangle &= \mathrm{e}^{\Delta_t} \int \frac{\mathrm{d}^2 z_1 \mathrm{d}^2 z_2}{\pi^2} |z_1|^2 \exp\left[\alpha_t z_1^* z_2^* \right. \\ &\quad \left. + \sigma_t \left(|z_1|^2 + |z_2|^2\right) + \alpha_t^* z_1 z_2\right] - 1 \\ &= -\frac{\sigma_t}{\sigma_t^2 - |\alpha_t|^2} - 1. \end{aligned} \tag{26}$$

Clearly, $\langle n_a \rangle = \langle n_b \rangle$. In particular, for $\alpha = 0$ or $\sigma = -1$, Eq. (26) reduces to the evolved average photon number of the a mode in the state ρ_{th} or ρ_{squ} in the amplitude decay process, i.e., $\langle n_a \rangle_{\text{th}} = -\left(1 + \sigma^{-1}\right) \mathrm{e}^{-2\kappa t}$ or $\langle n_a \rangle_{\text{squ}} = |\alpha_t|^2 / \left(1 - |\alpha_t|^2\right)$, where α_t is given in Eq. (22). In the limit, $\kappa t \to \infty$, $\langle n_a \rangle \to 0$.

Figure 1 shows the evolved average photon number $\langle n_a \rangle$ as a function of the decay time κt for the parameters β, ω and ϱ. For any β, ω and ϱ, $\langle n_a \rangle$ decreases gradually with increasing κt. Especially, $\langle n_a \rangle$ has different initial values as $\kappa t \to 0$ and approaches zero when κt is sufficiently large. This is consistent with the analysis in Sec. 3, as a result of the decoherence effect caused by amplitude-decay noise, which confirms that amplitude decay leads to a decrease in the average photon number. For a fixed κt, $\langle n_a \rangle$ increases with increasing ϱ but decreases with increasing β and ω. For small κt, the changes of $\langle n_a \rangle$ with the parameters β, ω and ϱ are more pronounced.

5 Photon number distribution

The photon number distribution describes the statistical probability features of the photon quantity in a quantum light field and often used to characterize the non-classicality of the light field [27, 28].

Using the relation between the number state $|n_a, n_b\rangle$ and the unnormalized coherent state $\|z_1, z_2\rangle$, i.e., $|n_a, n_b\rangle = (n_a! n_b!)^{-1/2} \partial_{z_1}^{n_a} \partial_{z_2}^{n_b} \|z_1, z_2\rangle|_{z_1, z_2 = 0}$ and the inner product $\langle z' | z \rangle = \mathrm{e}^{-|z'|^2/2 - |z|^2/2 + z'^* z}$, we obtain the photon number distribution of ρ_t as

$$\begin{aligned} &P(n_a, n_b, t) \\ &= \mathrm{e}^{\Delta_t} \langle n_a, n_b| : \mathrm{e}^{\alpha_t a^\dagger b^\dagger + \sigma_t \left(a^\dagger a + b^\dagger b\right) + \alpha_t^* ab} : |n_a, n_b\rangle \\ &= \mathrm{e}^{\Delta_t} \sum_{l=0}^{\min(n_a, n_b)} \binom{n_a}{l} \binom{n_b}{l} \frac{|\alpha_t|^{2l} (\sigma_t + 1)^{n_a + n_b}}{(\sigma_t + 1)^{2l}}. \end{aligned} \tag{27}$$

In general, letting $n_a \leqslant n_b$, and using the standard definition of Jocobi polynomials $\mathrm{P}_m^{(x,y)}(\cdot)$, we can rewrite $P(n_a, n_b, t)$ as

$$\begin{aligned} P(n_a, n_b, t) &= \mathrm{e}^{\Delta_t} (\sigma_t + 1)^{n_b - n_a} \left((\sigma_t + 1)^2 - |\alpha_t|^2\right)^{n_a} \\ &\quad \times \mathrm{P}_{n_a}^{(0, n_b - n_a)} \left(\frac{(\sigma_t + 1)^2 + |\alpha_t|^2}{(\sigma_t + 1)^2 - |\alpha_t|^2}\right). \end{aligned} \tag{28}$$

Especially, when $\alpha = 0$, $P(n_a, n_b, t)$ represents the photon number distribution evolution of the state ρ_{th} under amplitude decay, i.e.,

$$P_{\text{th}}(n_a, n_b, t) = \frac{\sigma^2}{[1 - \mathcal{T}(\sigma + 1)]^2} \left[\frac{(\sigma + 1)\,\mathrm{e}^{-2\kappa t}}{1 - \mathcal{T}(\sigma + 1)}\right]^{n_a + n_b} \tag{29}$$

and when $\sigma = -1$, Eq. (27) reduces to the evolved photon number distribution of the state ρ_{squ} under amplitude decay, i.e., $P_{\text{squ}}(n_a, n_b, t) = (1 - |\alpha_t|^2) n_a! n_b! \, |\alpha_t|^{n_a + n_b}$, where α_t can be found in Eq. (22).

In Fig. 2, the evolved photon number distribution $P(n_a, n_b, t)$ of the state ρ_0 under amplitude decay is shown for different parameters β, ω, ϱ, and κt. With the increase of β and ω, the distribution $P(n_a, n_b, t)$ is obvious only for a few small photon number states, especially for the number states with $n_a = n_b$, and is almost zero for others, more and more photons occur in vacuum. Also, when β and ω are sufficiently large, almost all photons will appear in vacuum. So, we say that the parameters β and ω have the same role in the distribution $P(n_a, n_b, t)$, however the impacts of the parameters ϱ and κt on the distribution $P(n_a, n_b, t)$ are exactly the opposite.

6 Wigner distribution function

The Wigner distribution function is a phase-space quasi-probability distribution, which is often used to assess the fidelity of logical qubits in quantum error correction experiments and is also applied to the study of non-commutative space quantum mechanics.

Using the two-mode Wigner operator $\Delta(z_1, z_2) = \Delta(z_1) \otimes \Delta(z_2)$ in the coherent state representation, where

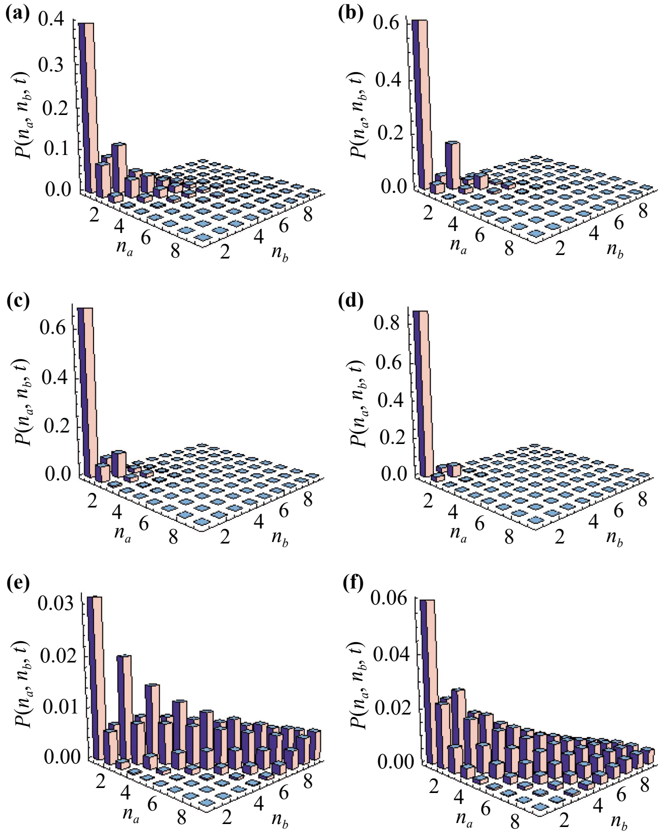

Fig. 2 The evolved photon number distribution of the state ρ_0 for different values of the parameters β, ω, ϱ, and κt under amplitude decay. The parameter sets for **(a)**, **(b)**, **(c)**, **(d)**, **(e)** and **(f)** are, respectively, $(\beta, \omega, \varrho, \kappa t) = (2, 1.2, 1, 0.01)$, $(4, 1.2, 1, 0.01)$, $(2, 1.5, 1, 0.01)$, $(2, 2, 1, 0.01)$, $(2, 2, 1.99, 0.01)$, $(2, 2, 1.99, 0.5)$.

$\Delta(z_j) = \pi^{-2} e^{|z_j|^2} \int d^2 z_j' \left|z_j'\right\rangle \left\langle -z_j'\right| e^{2(z_j z_j'^* - z_j^* z_j')}$, $(j = 1, 2)$ [29–34], together with Eq. (18) and the integral formula in Eq. (10), we obtain the Wigner distribution function for the state ρ_t as

$$\begin{aligned} w(z_1, z_2, t) = {} & \frac{e^{|z_1|^2 + |z_2|^2} e^{\Delta_t}}{\pi^2} \int \frac{d^2 z_1' d^2 z_2'}{\pi^2} \\ & e^{-(\sigma_t + 2)\left(|z_1'|^2 + |z_2'|^2\right) + \alpha_t^* z_1' z_2' + \alpha_t z_1'^* z_2'^*} \\ & \times e^{2(z_1 z_1'^* - z_1^* z_1') + 2(z_2 z_2'^* - z_2^* z_2')} \\ = {} & \frac{\Gamma e^{\Delta_t}}{\pi^2} \exp\left\{[1 - 4\Gamma(\sigma_t + 2)]\left(|z_1|^2 + |z_2|^2\right) \right. \\ & \left. + 4\Gamma \alpha_t^* z_1 z_2 + 4\Gamma \alpha_t z_1^* z_2^*\right\}, \end{aligned} \tag{30}$$

where $\Gamma = \left[(\sigma_t + 2)^2 - |\alpha_t|^2\right]^{-1}$. Equation (30) shows that the evolved Wigner distribution function $w(z_1, z_2, t)$ remains Gaussian under amplitude decay.

In particular, for $\alpha = 0$, using Eq. (21), $w(z_1, z_2, t)$ reduces to the evolved Wigner distribution function for the state $\rho_{\rm th}$ in the amplitude decay process, that is

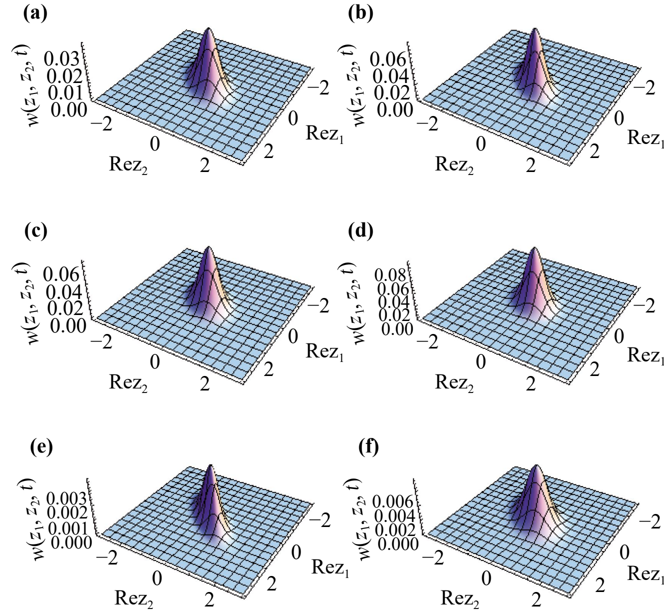

Fig. 3 The evolved Wigner distribution function for the state ρ_0 in the amplitude decay process is shown for different values of the parameters β, ω, ϱ, and κt, where the values of $(\beta, \omega, \varrho, \kappa t)$ are the same as those of Fig. 2.

$$w_{\rm th}(z_1, z_2, t) = \frac{\sigma^2 \mathcal{A}_1^2}{\pi^2} \exp\left[\mathcal{A}_1 \mathcal{A}_2 \left(|z_1|^2 + |z_2|^2\right)\right], \tag{31}$$

where $\mathcal{A}_1 = [(\sigma + 1)(1 - 2\mathcal{T}) + 1]^{-1}$, and $\mathcal{A}_2 = (\sigma + 1)(1 + 2\mathcal{T}) - 3$. However, for $\sigma = -1$, using Eq. (22), we obtain the evolution of the Wigner distribution function for the state $\rho_{\rm squ}$ under amplitude decay. In the long-time limit $(\kappa t \to \infty)$, $w(z_1, z_2, t) \to \pi^{-2} e^{-2\left(|z_1|^2 + |z_2|^2\right)}$, which corresponds to the Wigner distribution function for vacuum.

Figure 3 clearly shows that, for arbitrary values of β, ω, ϱ, and κt, the function $w(z_1, z_2, t)$ remains Gaussian, consistent with the analytical result above. With increasing β and ϱ, the function $w(z_1, z_2, t)$ exhibits stronger squeezing in phase space, whereas the frequency ω and the decay time κt evidently weaken the squeezing of the state ρ_0. Especially, when κt is sufficiently large, the function $w(z_1, z_2, t)$ evolves into the Gaussian wave packet corresponding to vacuum in phase space, which indicates that the initial state ρ_0 eventually loses nonclassicality and decays to vacuum under the long-term influence of amplitude-decay noise.

Besides, the peak value of the function $w(z_1, z_2, t)$ increases with increasing β and ω, but decreases with increasing entanglement coefficient ϱ and decay time κt.

7 Von Neumann entropy

Von Neumann entropy unifies the concepts of thermodynamic entropy and information entropy through density matrices, and it is a core tool in quantum information theory because it can be used to quantify the degree of entanglement in a system [35–37]. Its main applications include quantum computing, black hole physics, and many-body physics. To obtain the time-evolved von Neumann entropy of the state ρ_t in the amplitude decay process, we first use Eq. (12) to derive the following operator identity:

$$\begin{aligned}&:\ \exp\left[\alpha a^\dagger b^\dagger+\sigma\left(a^\dagger a+b^\dagger b\right)+\alpha^* ab\right]: \\ &=\frac{\mathrm{e}^{\omega'}}{\sigma+1}\exp\left[\varrho' a^\dagger b^\dagger+\omega'\left(a^\dagger a+b^\dagger b\right)+\varrho'^* ab\right],\end{aligned}\tag{32}$$

where

$$\begin{aligned}\omega'&=\frac{\Lambda\left[(\sigma+1)\cosh\Lambda-1\right]}{(\sigma+1)\sinh\Lambda},\\ \varrho'&=-\frac{\alpha\Lambda}{(\sigma+1)\sinh\Lambda},\end{aligned}\tag{33}$$

and

$$\cosh\Lambda=\frac{\sigma+1}{2}+\frac{1-|\alpha|^2}{2(\sigma+1)}.\tag{34}$$

The proof of Eq. (32) is as follows. Using Eq. (9), we have

$$\begin{aligned}\omega'\Lambda^{-1}\sinh\Lambda&=\cosh\Lambda-\frac{1}{\sigma+1},\\ \varrho'\Lambda^{-1}\sinh\Lambda&=-\frac{\alpha}{\sigma+1},\end{aligned}\tag{35}$$

which leads to the identity

$$\begin{aligned}&\left(\Lambda^{-1}\sinh\Lambda\right)^2\left(\omega'^2-|\varrho'|^2\right)\\ &=\left(\cosh\Lambda-\frac{1}{\sigma+1}\right)^2-\frac{|\alpha|^2}{(\sigma+1)^2},\\ &\sinh^2\Lambda=\cosh^2\Lambda-\frac{2\cosh\Lambda}{\sigma+1}+\frac{1-|\alpha|^2}{(\sigma+1)^2},\\ &\cosh\Lambda=\frac{\sigma+1}{2}+\frac{1-|\alpha|^2}{2(\sigma+1)}.\end{aligned}\tag{36}$$

Thus, Eq. (32) follows from Eqs. (8), (9), (35) and (36).

Therefore, using Eq. (32), we can rewrite Eq. (18) as

$$\begin{aligned}\rho_t&=\mathrm{e}^{\Delta_t}:\ \exp\left[\alpha_t a^\dagger b^\dagger+\sigma_t\left(a^\dagger a+b^\dagger b\right)+\alpha_t^* ab\right]:\\ &=\frac{\mathrm{e}^{\Delta_t}\mathrm{e}^{\omega_t'}}{\sigma_t+1}\exp\left[\varrho_t' a^\dagger b^\dagger+\omega_t'\left(a^\dagger a+b^\dagger b\right)+\varrho_t'^* ab\right],\end{aligned}\tag{37}$$

where ω_t', ϱ_t' and Λ_t are obtained by making the substitutions $\alpha\to\alpha_t$, $\sigma\to\sigma_t$ in Eqs. (33) and (34), and α_t, σ_t can be found in Eq. (19).

Using the von Neumann entropy, $S(\rho_t)=-k_B\mathrm{tr}(\rho_t\ln\rho_t)$ [19], together with Eqs. (18) and (37), we have

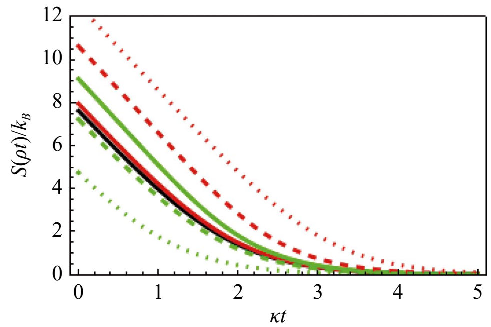

Fig. 4 The von Neumann entropy evolution of the state ρ_0 with the decay time κt for different values of the parameters ω, ϱ, and β under amplitude decay, where the values of (ω,ϱ,β) are, respectively, $(0.6, 0.01, 0.1)$ (thick black), $(0.6, 0.15, 0.1)$ (thick red), $(0.6, 0.29, 0.1)$ (thick green), $(0.3, 0.15, 0.1)$ (dashed thick red), $(0.8, 0.15, 0.1)$ (dashed thick green), $(0.6, 0.15, 0.01)$ (dotted thick red), and $(0.6, 0.15, 0.5)$ (dotted thick green).

$$\begin{aligned}&S(\rho_t)/k_B\\ &=-\mathrm{tr}\left[\rho_t\ln\left(\frac{\mathrm{e}^{\Delta_t}\mathrm{e}^{\omega_t'}}{\sigma_t+1}\mathrm{e}^{\varrho_t' a^\dagger b^\dagger+\omega_t'\left(a^\dagger a+b^\dagger b\right)+\varrho_t'^* ab}\right)\right]\\ &=-\ln\frac{\mathrm{e}^{\Delta_t}\mathrm{e}^{\omega_t'}}{\sigma_t+1}-\mathrm{e}^{\Delta_t}\mathrm{tr}(:\ \mathrm{e}^{\alpha_t a^\dagger b^\dagger+\sigma_t\left(a^\dagger a+b^\dagger b\right)+\alpha_t^* ab}:\\ &\quad\times\left[\varrho_t' a^\dagger b^\dagger+\omega_t'\left(a^\dagger a+b^\dagger b\right)+\varrho_t'^* ab\right]).\end{aligned}\tag{38}$$

Similar to the derivation of Eq. (26), we have

$$\begin{aligned}\langle ab\rangle&=\mathrm{e}^{\Delta_t}\mathtt{tr}\left[:\ \mathrm{e}^{\alpha_t a^\dagger b^\dagger+\sigma_t\left(a^\dagger a+b^\dagger b\right)+\alpha_t^* ab}ab:\ \right]\\ &=\mathrm{e}^{\Delta_t}\int\frac{\mathrm{d}^2z_1\mathrm{d}^2z_2}{\pi^2}z_1z_2\mathrm{e}^{\alpha_t z_1^* z_2^*++\sigma_t\left(|z_1|^2+|z_2|^2\right)+\alpha_t^* z_1z_2}\\ &=\frac{\alpha_t}{\sigma_t^2-|\alpha_t|^2}=\left\langle a^\dagger b^\dagger\right\rangle^*,\end{aligned}\tag{39}$$

and using

$$\langle n_a\rangle=\langle n_b\rangle=-\frac{\sigma_t}{\sigma_t^2-|\alpha_t|^2}-1,\tag{40}$$

we finally obtain the von Neumann entropy of the state ρ_t, i.e.,

$$\begin{aligned}S(\rho_t)/k_B=&-\ln\frac{\left(\sigma_t^2-|\alpha_t|^2\right)\mathrm{e}^{\omega_t'}}{\sigma_t+1}-\frac{\varrho_t'\alpha_t^*+\varrho_t'^*\alpha_t}{\sigma_t^2-|\alpha_t|^2}\\ &+2\omega_t'\left(\frac{\sigma_t}{\sigma_t^2-|\alpha_t|^2}+1\right).\end{aligned}\tag{41}$$

As a special case, setting $\alpha=0$, we obtain the evolved von Neumann entropy of the state ρ_{th} under amplitude decay, that is

$$S(\rho_t)/k_B = -\ln\frac{\sigma_t^2 e^{\omega_t''}}{\sigma_t+1} + \frac{2\omega_t''(\sigma_t+1)}{\sigma_t}, \tag{42}$$

where σ_t can be found in Eq. (21), and

$$\omega_t'' = \frac{[(\sigma_t+1)\cosh\ln(\sigma_t+1)-1]\ln(\sigma_t+1)}{(\sigma_t+1)\sinh\ln(\sigma_t+1)}. \tag{43}$$

However, for $\sigma = -1$, using Eq. (22), Eq. (41) becomes the entropy evolution of the state ρ_{squ} in the amplitude decay process.

To illustrate the effects of the parameters ω, ϱ, and β on the von Neumann entropy $S(\rho_t)$ in the amplitude decay process, we plot $S(\rho_t)/k_B$ as a function of the decay time κt for different values of ω, ϱ, and β. Clearly, the initial value of the entropy $S(\rho_t)$ is determined by ω, ϱ, and β. As the decay time κt increases, the entropy $S(\rho_t)$ gradually decreases and eventually approaches zero, regardless of the values of ω, ϱ, and β. Moreover, when ϱ is larger or when ω and β are smaller, the initial value of $S(\rho_t)$ is larger and the entropy approaches zero more slowly as κt increases.

In addition, for a fixed decay time κt, the entropy $S(\rho_t)$ increases with increasing ϱ, but decreases with increasing ω and β. Since the von Neumann entropy can quantify the degree of entanglement of the system, a larger ϱ or smaller ω and β enhances the entanglement of the initial state ρ_0 and leads to a slower reduction of entanglement over the decay time κt.

8 Conclusions

In summary, using the operator ordering method, we obtained the normalized entangled density operator ρ_0 corresponding to the Hamiltonian H_0 and found that the conditions $\sigma < 0$ and $\sigma + |\alpha| < 0$ ensure that ρ_0 is physically valid. Starting from the initial state ρ_0, we derived the time evolution of the density operator ρ_0 in the amplitude decay process using the Kraus-operator representation of the master equation for amplitude decay, and found that the density operator ρ_t retains the same functional form as the initial density operator ρ_0. We further showed $\sigma_t + |\alpha_t| < 0$, which guarantees that the evolved density operator ρ_t is physically valid. Clearly, the decoherence evolution is fully determined by the decay rate κ and the system parameters ω and ϱ. In the long-time limit $\kappa t \to \infty$, the initial state ρ_0 decays to vacuum due to amplitude decay.

In addition, the evolution of ρ_0 under amplitude decay was characterized by examining its average photon number, photon number distribution, and Wigner distribution function. We found that larger β and ϱ enhance the nonclassicality of the initial state ρ_0, whereas larger κ and ω lead to stronger decoherence. Moreover, it is worth emphasizing that larger ϱ or smaller ω and β yields stronger entanglement of the state ρ_0. These results suggest that examining the decoherence evolution of a two-mode entangled quantum system under amplitude decay is broadly relevant and may have applications in studies of decoherence models in quantum optics and quantum information.

Declarations The authors declare that they have no competing interests and there are no conflicts.

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References

1. H. D. Zeh, On the interpretation of measurement in quantum theory, *Found. Phys.* 1(1), 69 (1970)
2. E. Joos and H. D. Zeh, The emergence of classical properties through interaction with the environment, *Z. Phys. B* 59(2), 223 (1985)
3. S. J. Xiong, Z. Sun, and X. G. Wang, Quantum interferometric power and non-Markovianity in the decoherence channels, *Chin. Phys. B* 32(8), 080302 (2023)
4. W. H. Zurek, Decoherence and the transition from quantum to classical, *Phys. Today* 44(10), 36 (1991)
5. C. J. Zu and X. Y. Yu, Memory effect in time fractional Schrödinger equation, *Chin. Phys. B* 33(2), 020501 (2024)
6. W. H. Zurek, Decoherence, einselection, and the quantum origins of the classical, *Rev. Mod. Phys.* 75(3), 715 (2003)
7. X. Y. Chen, Simultaneous amplitude and phase damping of x–p symmetric Gaussian states and their separability, *Phys. Rev. A* 73(2), 022307 (2006)
8. M. L. Hu, X. Y. Hu, J. C. Wang, Y. Peng, Y. R. Zhang, and H. Fan, Quantum coherence and geometric quantum discord, *Phys. Rep.* 762–764, 1 (2018)
9. C. H. Henry and R. F. Kazarinov, Quantum noise in photonics, *Rev. Mod. Phys.* 68(3), 801 (1996)
10. H. J. Carmichael, Statistical Methods in Quantum Optics, Springer Press, 1999
11. C. W. Gardner and P. Zoller, Quantum Noise, Springer Press, 2000
12. H. Y. Fan and L. Y. Hu, Operator-sum representation of density operators as solutions to master equations obtained via the entangled state approach, *Mod. Phys. Lett. B* 22(25), 2435 (2008)
13. K. Zhang, L. L. Li, and H. Y. Fan, Wigner function of optical cumulant operator and its dissipation in thermo-entangled state representation, *Chin. Phys. B* 33(6), 060307 (2024)
14. X. G. Meng, B. L. Liang, J. M. Liu, and X. C. Zhou,

Entropy evolution law of general quadratic state in a laser channel, *Front. Phys. (Beijing)* 20(2), 022205 (2025)

15. X. G. Meng, K. C. Li, J. S. Wang, Z. S. Yang, X. Y. Zhang, Z. T. Zhang, and B. L. Liang, Multi-variable special polynomials using an operator ordering method, *Front. Phys. (Beijing)* 15(5), 52501 (2020)
16. H. Y. Fan, Operator ordering in quantum optics theory and the development of Dirac's symbolic method, *J. Opt. B* 5(4), R147 (2003)
17. X. G. Meng, X. S. Goan, J. S. Wang, and R. Zhang, Nonclassical thermal-state superpositions: Analytical evolution law and decoherence behavior, *Opt. Commun.* 411(7), 15 (2018)
18. K. Zhang, L. L. Li, D. W. Guo, and H. Y. Fan, New light fields based on integration theory within the Weyl ordering product of operators, *Chin. Phys. B* 32(4), 040301 (2023)
19. Y. Yang, L. L. Xu, Z. Z. Gao, and X. G. Meng, The von Neumann entropy and quantum statistical features of a degenerate parametric amplifier system, *Physica A* 678, 130954 (2025)
20. X. G. Meng, J. S. Wang, and B. L. Liang, Entangled State Representations in Quantum Optics, Springer Press, 2023
21. Z. L. Wan, The entropy variation and quantum statistical behaviors of a general linear state in a laser channel, *Chin. J. Phys.* 94, 215 (2025)
22. Z. L. Wan, H. C. Yuan, X. L. Yin, C. Y. Wang, and Y. X. Wang, Decoherence dynamics of a general quadratic state in a thermal environment, *Chin. J. Phys.* 98, 14 (2025)
23. L. Mandel and E. Wolf, Optical Coherence and Quantum Optics, Cambridge University Press, 1995
24. H. Y. Fan and L. Y. Hu, New convenient way for deriving exponential operators' disentangling formulas and their applications (II), *Commum. Theor. Phys.* 51(3), 506 (2009)
25. H. Y. Fan, New antinormal ordering expansion for density operators, *Phys. Lett. A* 161(1), 1 (1991)
26. H. Y. Fan, Entangled states, squeezed states gained via the route of developing Dirac's symbolic method and their applications, *Int. J. Mod. Phys. B* 18(10–11), 1387 (2004)
27. S. Kono, Y. Masuyama, T. Ishikawa, Y. Tabuchi, R. Yamazaki, K. Usami, K. Koshino, and Y. Nakamura, Nonclassical photon number distribution in a superconducting cavity under a squeezed drive, *Phys. Rev. Lett.* 119(2), 023602 (2017)
28. E. Waks, E. Diamanti, and Y. Yamamoto, Generation of photon number states, *New J. Phys.* 8, 4 (2006)
29. Y. Guo, X. Zhang, Q. P. Du, Z. S. Yang, and J. P. Li, Study on statistical properties of mode coupling and dispersion in mode-division multiplexing systems, *J. Liaocheng Univ. (Nat. Sci. Ed.)* 36(3), 39 (2023)
30. J. D. Zhang and S. Wang, Noise filtering of optical communication systems based on two modulated quantum numbers, *J. Liaocheng Univ. (Nat. Sci. Ed.)* 37(1), 23 (2024)
31. X. G. Meng, J. S. Wang, X. Y. Zhang, Z. S. Yang, B. L. Liang, and Z. T. Zhang, Non-classicality via the super-positions of photon addition and subtraction and quantum decoherence for thermal noise, *Ann. Phys.* 532(12), 2000219 (2020)
32. X. G. Meng, K. C. Li, J. S. Wang, X. Y. Zhang, Z. T. Zhang, Z. S. Yang, and B. L. Liang, Continuous-variable entanglement and Wigner-function negativity via adding or subtracting photons, *Ann. Phys.* 532(5), 1900585 (2020)
33. X. G. Meng, J. S. Wang, B. L. Liang, and C. X. Han, Evolution of a two-mode squeezed vacuum for amplitude decay via continuous-variable entangled state approach, *Front. Phys. (Beijing)* 13(5), 130322 (2018)
34. X. Y. Zhang, J. S. Wang, L. Wang, X. G. Meng, and B. L. Liang, Non-classicality of photon-modulated spin coherent states in the Holstein–Primakoff realization, *Chin. Phys. B* 31(5), 054205 (2022)
35. J. Chen, Q. Chen, and X. Q. Wang, Eigenstate properties of the disordered Bose-Hubbard chain, *Front. Phys. (Beijing)* 19(4), 43207 (2024)
36. X. Li and Z. Q. Yin, Improve variational quantum eigensolver by many-body localization, *Front. Phys. (Beijing)* 20(2), 023202 (2025)
37. Y. J. Chu, F. Huang, M. X. Li, and Z. J. Zheng, An entropy function of a quantum channel, *Quantum Inform. Process.* 22(1), 27 (2022)