FRONTIERS OF PHYSICS

RESEARCH ARTICLE
Volume 19 / Issue 4 / 41201 / 2024

Unravelling-based (auto)control of back-action in atomic Bose–Einstein condensate

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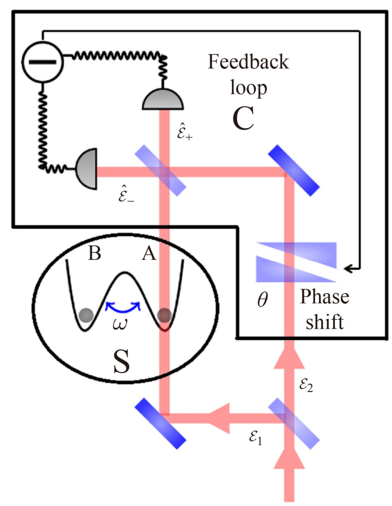
Received August 12, 2023; accepted December 10, 2023

ABSTRACT

We present a novel feedback control method for quantum systems. Feedback does not affect the controlled system itself. Instead, it controls the unravelling of the quantum channel of interaction between the system and its environment. This interaction can be represented as a history of events. If their informational content is changed, their back-action on the system is also modified. Feedback action is trigged by the events, thus granting the system the degree of control over its own state. The efficiency of the proposed scheme is demonstrated on the example of two-mode atomic Bose-Einstein condensate, with one of its modes subject to phase-contrast imaging in a Mach–Zehnder interferometer. The histories of photocounts in the output channels of the interferometer are used for feedback. Its capabilities of state engineering are studied for different settings of the feedback loop and different numbers of events in the recorded histories.



Keywords quantum feedback control, quantum measurements, two-mode Bose–Einstein condensates

1 Introduction

Modern problems in quantum computation and quantum metrology require efficient methods of controlling quantum systems. Most of these methods are feedback-based. Feedback gives the system ability to partially control its own evolution. The value of auto-controlling classical systems and the role of feedback as a tool for achieving this were already well recognized at the dawn of cybernetics [1]. Feedback in quantum systems is special because it exploits their inherent randomness that reveals itself in interaction with classical environment. When feedback is implemented, this interaction originates from measurement performed on the system. This results in classical information gain, which in its turn acts back on the system[1)]. The feedback loop is then closed *through* a quantum system. In recent work [6] it was noted that it is possible to close the loop *outside* a quantum system, in a kind of border zone between quantum and classical systems. Nevertheless, the system gains the autocontrol ability. The present work elaborates on this idea.

In many cases the interaction of monitored quantum system with environment can be represented as a sequence of time-resolved events of various types. Photoemissions into a set of channels is a good example; the channel defines the type of the event and is tied to a certain detector. The spontaneous nature of each event makes its type and time of occurrence random. The system thus becomes a generator of a random event sequence. Each event carries a certain information about

1) This is extremely brief and simplified description of the measurement-based feedback paradigm [2], the one that is important in the context of the present work. Because it involves classical environment, such a feedback can be called semiclassical or hybrid. A completely different paradigm of fully quantum (coherent) feedback was proposed by Lloyd [3–5].

Higher Education Press

https://doi.org/10.1007/s11467-023-1375-2

the system and therefore affects its state via back-action effect.

Because of this back-action, the measurement acts themselves can be used for control. Such control strategies were successfully used for control of simplest quantum systems (qubits) [7–12], as well as more complex quantum-optical systems – fields and atomic ensembles [13–17]. The works [18, 19] studied feedback operating in a pulse-like, stroboscopic manner and are particularly worth noting. But the time moments in which the measurements on the system were implemented were chosen arbitrarily, mainly in a periodic manner. A very interesting set-up arises if this choice is left to the system itself. Consider an open quantum system interacting with environment via a feedback loop. If this interaction is also time-local, it is possible to organize feedback that is initiated by the system-environment interaction events. The authors have previously reported the efficiency of such kind of feedback [20–22] with atomic Bose–Einstein condensate (BEC) as an example of controlled system.

Atomic BECs have been an extremely popular object of study since the development of experimental techniques for control of such systems. These techniques rely on making BECs interact with external optical fields of specific configurations. Tuning them in real-time manner, it is possible to implement measurement of various BEC observables with controlled strength. Sophisticated optimal-control schemes for manipulation of the states of atomic BECs were developed [23–25]. Feedback control schemes have been applied to atomic many-body systems to create highly non-classical states [26, 27], control phase transitions in an optical lattice [28], create and maintain strongly correlated many-body states [29]. However, implementing such complex control techniques even with modern technologies is a challenging task. Although the scheme we suggest in present paper does not aim to achieve the same spectacular results, it is relatively simple to realize experimentally and offers an alternative to familiar quantum feedback paradigms.

Many works on quantum control consider feedback that affects the internal parameters of the controlled system. However, in the case of quantum system the state engineering and the measurement process are effectively inseparable. This fact opens way for a different approach. It is possible to construct a feedback that controls the very way the system interacts with its environment (including the measurement procedure). Neither the state nor the form of the system's "openness" is manipulated directly, but rather the information content of events is modified – the unravelling [30] of communication channel between the system and its environment. This might be convenient when direct control of parameters that define the system state is difficult or undesirable. Such a scheme is also interesting from methodic viewpoint, because it proposes a novel control scheme, with feedback loop not touching the controlled system at all and only localized within the controller. In Ref. [6] an application of such a scheme to control of a steady state of two-mode Bose–Einstein condensates (BECs) was studied. One of the modes was subject to interferometric phase-contrast imaging. Feedback was controlling the unravelling of photodetection, switching between two pre-set unravellings based on the type of the last recorded photodetection. In present work we investigate a more complicated variant of this scheme, involving longer event sequences. Our consideration is still preliminary in its nature and in no way can serve as a detailed program for a proposed experiment.

2 General considerations

Types of events and their statistics are defined by the structure of environment of a quantum system. Its modification via event-triggered feedback gives the system a function of auto-control. Being able to specifically design this function gives an effective tool for quantum state engineering. This can be further explained by the following formal construction, its particular case being the subject of the present work. Let us represent the state of a monitored quantum system by a set of statistical operators

$$\hat{\varrho}^{(\sigma_1\ldots\sigma_k)}, \tag{1}$$

that are conditioned by histories – series of registered events of types $\sigma_1, \sigma_2, \ldots, \sigma_k$ (ordered from left to right, from the earliest to the latest).

$$\hat{\varrho} = \sum_{\sigma_1}\cdots\sum_{\sigma_k}\hat{\varrho}^{(\sigma_1\ldots\sigma_k)} \tag{2}$$

is a usual statistical operator of the system, and

$$p(\sigma_1,\ldots,\sigma_k) = \mathrm{Tr}\hat{\varrho}^{(\sigma_1\ldots\sigma_k)} \tag{3}$$

is the probability to register a given event sequence. We have explicitly assumed a finite memory by setting a fixed length of the history.

The master equation for (1) reads

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\hat{\varrho}^{(\sigma_1\ldots\sigma_k)} + \mathrm{i}[\hat{H}, \hat{\varrho}^{(\sigma_1\ldots\sigma_k)}]\\ &= \sum_{\sigma}\Big(2\hat{\mathcal{E}}_{\sigma_k}(\sigma\sigma_1\ldots\sigma_{k-1})\hat{\varrho}^{(\sigma\sigma_1\ldots\sigma_{k-1})}\hat{\mathcal{E}}^{\dagger}_{\sigma_k}(\sigma\sigma_1\ldots\sigma_{k-1})\\ &\quad - \{\hat{\mathcal{E}}^{\dagger}_{\sigma}(\sigma_1\ldots\sigma_k)\hat{\mathcal{E}}_{\sigma}(\sigma_1\ldots\sigma_k), \hat{\varrho}^{(\sigma_1\ldots\sigma_k)}\}\Big)\end{aligned} \tag{4}$$

and apart from the term with Hamiltonian $\hat{H}$ it also includes a right-hand side (curly brackets denote the anti-commutator) describing the system-environment interaction. The operator $\hat{\mathcal{E}}_{\sigma}(\sigma_1\ldots\sigma_k)$ describes the back-action from the σ-type event. It is exactly these operators that reflect the nature of how environment monitors the system. We assume that the feedback with a memory of

k-event history is capable of modifying this monitoring upon registration of each event.

Note that the right-hand side of Eq. (4) is a generalization of Lindblad structure. The main difference is including the dependence of the previous history of system-environment interactions. The first term in round brackets (the so-called sandwich term) reflects the back-action of the latest event – the σ_k event. The anti-commutator corresponds to the next event being of any type, thus the summation over σ. On the contrary, such summation in the sandwich term reflects the presence of event prior to σ_1. This is because we want to consider the system state in Eq. (1) as a function of k-event histories. The dependence of $\hat{\mathcal{E}}$-operators in Eq. (4) on the registered history may be completely general. In what follows a special type of this dependence will be important. It is given by a so-called unravelling of the sandwich term in Eq. (4). Assume that the system may generate events of N types (determined by a the type or number of the detector that registers it) described by a basis of operators $\hat{\mathcal{E}}_1$,...,$\hat{\mathcal{E}}_N$. The dependence on a history $\sigma_1 \ldots \sigma_k$ is given by a linear combination:

$$\hat{\mathcal{E}}_\sigma(\sigma_1 \ldots \sigma_k) = \sum_{\sigma'=1}^{N} U_{\sigma\sigma'}(\sigma_1 \ldots \sigma_k)\hat{\mathcal{E}}_{\sigma'}. \tag{5}$$

The right-hand side of Eq. (5) suggests the interpretation of a σ event for such unravelling as a kind of superposition of 'basic' events to which the operators $\hat{\mathcal{E}}_\sigma$ correspond. Aforementioned coefficients form a matrix $\boldsymbol{U}(\sigma_1 \ldots \sigma_k)$ from $SU(N)$. For any such matrix

$$\sum_{\sigma=1}^{N} \hat{\mathcal{E}}_\sigma^\dagger(\sigma_1 \ldots \sigma_k)\hat{\mathcal{E}}_\sigma(\sigma_1 \ldots \sigma_k) = \sum_{\sigma=1}^{N} \hat{\mathcal{E}}_\sigma^\dagger \hat{\mathcal{E}}_\sigma, \tag{6}$$

which leads to anticommutator in Eq. (4) being independent on $\sigma_1 \ldots \sigma_k$. The general event statistics (regardless of their type) also does not depend on the chosen unravelling – the matrix $\boldsymbol{U}$. At the same time it modifies the sandwich term. It is capable of radical modification of the informational content of the event of a given type, and hence of how it acts back on the system state. Note that reducing the dependence of $\hat{\mathcal{E}}$-operators on the history to the matrix elements of $\boldsymbol{U}$ still leaves much freedom of choice of that dependence.

Earlier, an example of a system interacting with environment by emitting spontaneous quanta into channels numbered by $\sigma = 1, 2, \ldots, N$ was given. These channels may be related to different directions of the emission. Let the photodetection in a σ channel be described by the operator $\hat{\mathcal{E}}_\sigma$. The unravelling in Eq. (5) may then be organized by means of linear optics, by placing a certain interface between the system and a set of detectors. Numbers $U_{\sigma\sigma'}(\sigma_1 \ldots \sigma_k)$ are related to the parameters of the interface – transmissions of the beamsplitters, phase shifts, etc. These parameters are feedback-controlled,

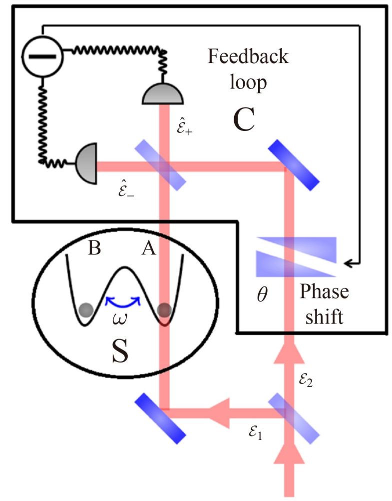


Fig. 1 Double-well BEC in the Mach–Zehnder interferometer. The control is implemented by phase switchings in the condensate-free arm. Note that the feedback loop lies outside of BEC.

and so the feedback loop appears to be closed within the interface, not directly involving the quantum system itself. The system, as a source of photoemissions, by the current unravelling defines the informational content of the event that follows, and the corresponding back-action. The parameters of the feedback loop that define a set of available unravellings and how the switchings between them occur, may be chosen depending on the task of preparation of a specific target quantum state. After observing a certain sequence of events $\sigma_1 \ldots \sigma_k$ the system state (1) becomes close to the target state, provided the proper settings of the feedback loop. This effect was shown in Ref. [6] on the example of interferometric probing of atomic BEC. The simplest variant of one-event histories was chosen, i.e., the states $\hat{\varrho}^{(\sigma_1)}$ were considered. As will be shown below, taking into account two events in a conditional state $\hat{\varrho}^{(\sigma_1\sigma_2)}$ allow for significantly better results.

3 The model

Just like in Ref. [6], we consider the simplest model of two-mode atomic BEC localized in a symmetric double-well potential. Only those properties of BEC which are taken into account which we can not dispense with. It suffices to assume that BEC consists of spinless particles without interaction between them. We also assume that the total density matrix is diagonal with respect to number of atoms in BEC. The latter lets us consider the fixed number of atoms N_{at} with possible averaging over it later. We assume that the number of atoms is conserved throughout the evolution of the system. The Mach–Zehnder interferometer with balanced input and output beamsplitters is used for probing the BEC state

(Fig. 1). Atom-atom interaction in the BEC is not essential for the scheme to work and will be excluded from the model. Symmetry of the potential allows reducing all the dynamics to tunnelling between the wells A and B. It is described by a following Hamiltonian:

$$\hat{H} = \omega\left(\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a}\right), \tag{7}$$

with operators of annihilation and creation of atoms in a certain well. Frequency-valued parameter ω is the tunnelling rate (Planck constant is set to unity).

One of the wells (let it be well A) is probed by an optical beam propagating through the arm of interferometer. The probing beam is interacting with the BEC in a dispersive (non-resonant) regime, performing phase-contrast imaging of the condensate [14]. As a result, the beam gains a phase shift proportional to the number of atoms in that well (we believe that all atoms end up in the beam's region). The radiation is then detected by two photodetectors placed in the exit channels of the interferometer. The controlled phase shift φ in the other arm of interferometer is important element of the scheme. The feedback action is initiated by photodetection events. It rapidly (compared to ω^{-1}) changes this phase shift into one of two values, depending on which phodetector clicked. The considered system can be divided into two parts – the atomic BEC, treated as a quantum system and designated by S in Fig. 1, and the feedback loop, designated by C. The latter part of the scheme is classical, and its time evolution consists of jump-like switchings between two possible values of the phase shift. The model thus includes two types of events. In what follows we will mark them with $\sigma = +$ and $\sigma = -$. These same symbols will also mark the states of the feedback loop (the controller) conditioned by the *latest* recorded event. The probability of switchings $(\sigma') \to (\sigma)$ depends on the BEC state existing before switching: $Tr(\hat{\mathcal{E}}^\dagger_\sigma(\varphi_{\sigma'})\hat{\mathcal{E}}_\sigma(\varphi_{\sigma'})\hat{\varrho}^{(\dots\sigma')})$, where

$$\hat{\mathcal{E}}_\pm(\varphi) \doteq \frac{1}{\sqrt{2}}\left(\mathrm{e}^{\mathrm{i}\chi\hat{a}^\dagger\hat{a}} \pm \mathrm{e}^{\mathrm{i}\varphi}\right), \tag{8}$$

and χ is a phase shift incurred by a single atom. In the model considered the feedback loop state (the current value of the phase shift in the BEC-free arm of interferometer) depends only on the type of the latest recorded photodetection. The memory of the controller is thus as short as possible. For operators from Eq. (5) one gets: $\hat{\mathcal{E}}_\sigma(\sigma_1\dots\sigma_k) = \hat{\mathcal{E}}_\sigma(\sigma_k)$. The same approximation was used in Ref. [6], where it was noted that Eq. (8) is indeed a single-parameter unravelling. In Ref. [6] a model of one-event histories was also expanded to BEC states: $\hat{\varrho}(\sigma_1\dots\sigma_k) = \hat{\varrho}(\sigma_k)$. We will further show some interesting consequences of lifting this restriction.

Consider operators $\hat{\varrho}^{(\sigma_1\sigma_2)}$ conditioned by two-event histories. Master equation written in a way similar to Eq. (4) reads

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\hat{\varrho}^{(\sigma_1\sigma_2)} = &-\mathrm{i}[\hat{H}, \hat{\varrho}^{(\sigma_1\sigma_2)}] + \sum_\sigma\bigg(2\hat{\mathcal{E}}_{\sigma_2}(\varphi_{\sigma_1})\hat{\varrho}^{(\sigma\sigma_1)}\\ &\times\hat{\mathcal{E}}^\dagger_{\sigma_2}(\varphi_{\sigma_1}) - \{\hat{\mathcal{E}}^\dagger_\sigma(\varphi_{\sigma_2})\hat{\mathcal{E}}_\sigma(\varphi_{\sigma_2}), \hat{\varrho}^{(\sigma_1\sigma_2)}\}\bigg).\end{aligned} \tag{9}$$

Obviously, in case of "forgetting" the information about the type of the earliest of two events one should obtain the operators conditioned by one-event histories:

$$\sum_{\sigma_1}\hat{\varrho}^{(\sigma_1\sigma_2)} = \hat{\varrho}^{(\sigma_2)}. \tag{10}$$

It can be easily shown that after performing such summation in Eq. (9) one gets the same master equations for one-event operators first obtained in Ref. [6]:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\hat{\varrho}^{(\sigma_2)} = &-\mathrm{i}[\hat{H}, \hat{\varrho}^{(\sigma_2)}] + \sum_\sigma\bigg(2\hat{\mathcal{E}}_{\sigma_2}(\varphi_\sigma)\hat{\varrho}^{(\sigma)}\hat{\mathcal{E}}^\dagger_{\sigma_2}(\varphi_\sigma)\\ &- \{\hat{\mathcal{E}}^\dagger_\sigma(\varphi_{\sigma_2})\hat{\mathcal{E}}_\sigma(\varphi_{\sigma_2}), \hat{\varrho}^{(\sigma_2)}\}\bigg).\end{aligned} \tag{11}$$

This equation describes irreversible evolution, and due to that the indices σ_1 and σ_2 are not symmetric. Summation over σ_2 does not shorten the history, but instead makes a gap in it – because information about the type of the earlier event is lost.

The solution of Eq. (9) relies on the approach similar to the one used in Ref. [6] for solution of Eq. (11). We will limit the study to the case in which the feedback should be most efficient – namely, the case of rapid decoherence. In terms of parameters this corresponds to the region of $|\omega| \ll 1$. It is then possible to use the following ansatz for two-event operators:

$$\begin{aligned}\hat{\varrho}^{(\sigma_1\sigma_2)} = \sum_{n=0}^{N_{at}}\bigg(&p_n^{(\sigma_1\sigma_2)}|n\rangle\langle n| + q_n^{(\sigma_1\sigma_2)}|n+1\rangle\langle n|\\ &+ \bar{q}_n^{(\sigma_1\sigma_2)}|n\rangle\langle n+1|\bigg),\end{aligned} \tag{12}$$

where $|n\rangle = |n\rangle_A \otimes |N_{at}-n\rangle_B$ are the states of Fock basis with respect to the number of atoms in the A well. The solution of the resulting system on the matrix elements $p_n^{(\sigma_1\sigma_2)}, q_n^{(\sigma_1\sigma_2)}$ is straightforward. This system can be presented in the following form:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}p_n^{(\sigma_1\sigma_2)} = &\frac{\omega^2}{2}\cdot(n+1)(N_{at}-n)\big(p_{n+1}^{(\sigma_1\sigma_2)} - p_n^{(\sigma_1\sigma_2)}\big)\\ &+ \frac{\omega^2}{2}\cdot n(N_{at}-n+1)\big(p_{n-1}^{(\sigma_1\sigma_2)} - p_n^{(\sigma_1\sigma_2)}\big)\\ &- 4p_n^{(\sigma_1\sigma_2)} + 2\big(1 + \sigma_2\cdot\cos(\chi n - \varphi_{\sigma_1})\big)p_n^{(\sigma_1)}\\ &+ \frac{\omega}{2}\sqrt{(n+1)(N_{at}-n)}Q_n^{(\sigma_1\sigma_2)}\\ &- \frac{\omega}{2}\sqrt{n(N_{at}-n+1)}Q_{n-1}^{(\sigma_1\sigma_2)},\end{aligned} \tag{13}$$

with

$$Q_n^{(\sigma_1\sigma_2)} = Im\left[q_n^{(\sigma_1)}\left(e^{i\chi}(1+\sigma_2\cdot e^{i\chi n - i\varphi_{\sigma_1}}) + 1 + \sigma_2\cdot e^{-i\chi n + i\varphi_{\sigma_1}}\right)\right]. \tag{14}$$

As one can see, the solution can be expressed in terms of matrix elements of one-event operators, that immediately follow from Eq. (11) by using the same ansatz as in Eq. (12) (see Appendix). It is also evident that the exactly same procedure can be used for step-by-step construction of master equations conditioned by arbitrarily long histories from Eq. (4), taking into account Eqs. (7) and (8).

4 Results and discussion

Before proceeding with the detailed analysis of the results, let us have a closer look on the equations (13) and (14). The three bottom lines of (13) depend on the matrix elements of one-event operators, $p_n^{(\sigma)}$ and $q_n^{(\sigma)}$. Due to their symmetry relations (20) and (23) derived in the Appendix, one can immediately conclude that similar relations also hold for two-event operators: probability distribution $p_n^{(++)}$ turns into $p_n^{(--)}$ upon a transformation $\varphi_\pm \to \varphi_\mp + \pi$. These relations originate from the fact that the interferometer beamsplitters are balanced and the Lindblad operators are of the form (8), and so they are rigorous and will hold for arbitrarily long *same-event* histories. It is then sufficient to consider only one of these distributions. However, for distributions conditioned by *different-event* histories, $p_n^{(+-)}$ and $p_n^{(-+)}$, no such strict relations seem to exist. Numerical evaluations show that they are identical up to $O(\omega)$ precision, i.e. the difference is of the order of coherence between adjacent Fock states. From Eq. (13) it can be seen that the term with $Q^{(\sigma_1\sigma_2)}$ generated by this coherence is linear with ω, and hence its contribution to the evolution of diagonal matrix elements $p^{(\sigma_1\sigma_2)}$ is of the same order. It is exactly this term that provides physical inequivalence of histories $(+-)$ and $(-+)$, because the rest of the terms in Eq. (14) make for identical evolution of $p_n^{(+-)}$ and $p_n^{(-+)}$.

The main goal of the present work is to study the effect of feedback. To this end, it is useful to compare the steady-state solutions of Eq. (13) with some "reference" distributions arising in absence of feedback, or in case of ignoring the history of photocounts, which effectively negates the feedback action. As a quantitative characteristic of this difference, the Kullback–Leibler divergence (KL-divergence) will be used [31]. The most striking results appear for distributions generated by same-event histories. In Fig. 2 KL-divergences for normalized distributions (designated by capital letters) generated by one, two- and three-event histories are presented. The values of $K^{(\sigma_1\ldots\sigma_k)}$ in the left column are KL-divergences from uniform distribution[2)] $p_n = 1/(N_{at}+1)$ as a function of phase shifts (φ_+, φ_-) that determine unravelling:

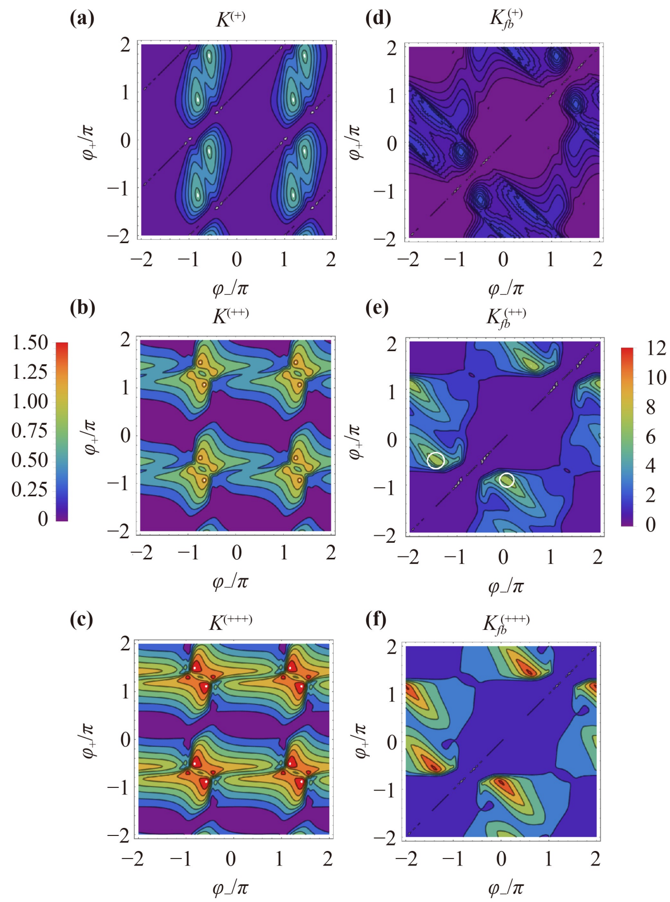


Fig. 2 KL-divergences between normalized distributions $P_n^{(+)}, P_n^{(++)}, P_n^{(+++)}$ and: **(a–c)** uniform distribution $p_n = 1/(N_{at}+1)$; **(d–f)** distributions $R_n^{(+)}, R_n^{(++)}, R_n^{(+++)}$, evaluated for $\chi = \pi/20$, $\omega = 0.07$, $N_{at} = 10$. Color scales to the left and to the right correspond to the respective column. A pair of adjacent maxima are marked on the (e) plot with coordinates $(\varphi_+, \varphi_-) = (-0.4\pi, -1.4\pi)$ and $(\varphi_+, \varphi_-) = (-0.9\pi, 0.1\pi)$.

$$K^{(\sigma_1\ldots\sigma_k)} = \sum_{n=0}^{N_{at}} P_n^{(\sigma_1\ldots\sigma_k)} \cdot \ln\left(\frac{P_n^{(\sigma_1\ldots\sigma_k)}}{p_n}\right). \tag{15}$$

Just like in Ref. [6], the maxima of this quantity correspond to non-trivial values of (φ_+, φ_-).

The right column in Fig. 2 represents KL-divergences $K_{fb}^{(\sigma_1\ldots\sigma_k)}$ from reference distributions $R_n^{(\sigma_1\ldots\sigma_k)}(\varphi) \doteq P_n^{(\sigma_1\ldots\sigma_k)}(\varphi,\varphi)$ for $\varphi = (\varphi_+ + \varphi_-)/2$. They are evaluated at the point on the main diagonal of the plain that is closest to (φ_+, φ_-):

[2)] This distribution is stationary in absence of feedback [32], or if one ignores the history of events in case of rapid decoherence.

$$K_{fb}^{(\sigma_1\ldots\sigma_k)} = \sum_{n=0}^{N_{at}} P_n^{(\sigma_1\ldots\sigma_k)} \cdot \ln\left(\frac{P_n^{(\sigma_1\ldots\sigma_k)}}{R_n^{(\sigma_1\ldots\sigma_k)}}\right). \tag{16}$$

Points of that diagonal correspond to the case of no feedback, because there are no phase switchings upon registration of events ($\varphi_+ = \varphi_-$). Naturally, the values of the said divergences on this diagonal are exactly zero.

Both series of plots presented in Fig. 2 have some common patterns. One could note that divergences for two- and three-event histories are in good qualitative agreement. The maxima on the plots are arranged in symmetrically placed pairs [one of them is marked in Fig. 2(e)]. When lengthening the history from two to three events, the value of maxima increase, but their positions do not change significantly; the typical order of the displacement is again $O(\omega)$. That said, despite the used approximation of strong decoherence, the non-zero value ω is crucial for the observed effects. For one-event histories [Figs. 2(a) and (d)] the shapes of maxima are somewhat different, but the patterns of their placement are more or less the same. The values of the symmetrically placed maxima are the same, but the corresponding distributions are different (Fig. 3). For each of them, only one of the wells is dominantly populated. In a sense, the state becomes "anti-squeezed" as an opposite to the number-squeezed state, in which the zero population imbalance between the wells becomes most probable [33]. These maxima are rather close to each other, which gives potential ability to transfer atomic population from one well to another without significant change of unravelling.

From Fig. 3 it can be seen that the "population imbalance" in the wells in the $\hat{\varrho}^{(++)}$ conditional state for fixed φ can be greater than in absence of feedback. It may seem that, if the goal is to prepare the BEC in a state with maximally non-uniform distribution, then there is no need to actually implement feedback that switches between φ_+ and φ_-. However, this statement would be incorrect. Consider again the left column of Fig. 2. It can be seen that the maximal KL-divergences between the feedback-generated distributions and the uniform one are obtained for non-diagonal values of (φ_+, φ_-), i.e., the feedback is necessary.

In absence of optical probing (and feedback) the BEC would evolve according to the Hamiltonian (7). This case is opposite to the one investigated above and corresponds to slow (close to zero) decoherence. However, it is still worth investigating the difference between the population distribution for this case and the feedback-generated one. Tunnelling between symmetric wells leads to equiprobable appearance of any atom in any well. The probability to find exactly n atoms in the A well is then given by a binomial distribution: $p_n^{(binom)} = C_{N_{at}}^n / 2^{N_{at}}$. The KL-divergences of feedback-generated distributions with this one are presented in Fig. 4 (the parameters are the same as in Fig. 2). Again,

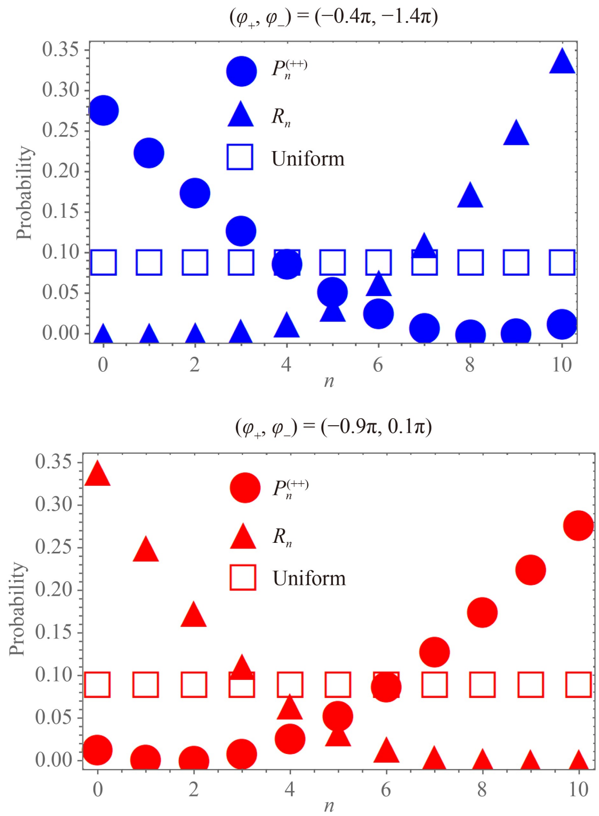


Fig. 3 Unity-normalized steady-state distributions for values of unravellings (φ_+, φ_-), corresponding to the KL-divergence' maxima highlighted in Fig. 2(e). Filled circles represent actual feedback-induced distributions, while triangles represent no-feedback distributions for unravelling set at $((\varphi_+ + \varphi_-)/2, (\varphi_+ + \varphi_-)/2)$. Empty squares represent uniform distributions.

there is a common trend for longer histories. Also, the feedback still demonstrates its efficiency because the maximal values of the divergences are achieved for non-diagonal values of unravelling angles.

The main effect of taking into account longer histories is increasing the maximum values of KL-divergences. For example, for three events it is ~12 for $(\varphi_+, \varphi_-) = (-0.6\pi, -1.5\pi)$, while for one event it is only ~2 for $(\varphi_+, \varphi_-) = (-0.2\pi, -0.8\pi)$. This becomes most pronounced for same-event histories. Therefore, the considered feedback scheme uses the information gained from the system efficiently. With more events in the controller memory, it is possible to engineer more non-uniform states of the BEC.

Real process of loading the optical traps that correspond to the wells discussed in the text is random. The results above can then be readily generalized. Assume that the total number of atoms N_{at} obeys Poissonian distribution P_{Pois}, which is a reasonable assumption for a gaussian state. Then the actual distributions P' of atoms inside the wells should be calculated as follows:

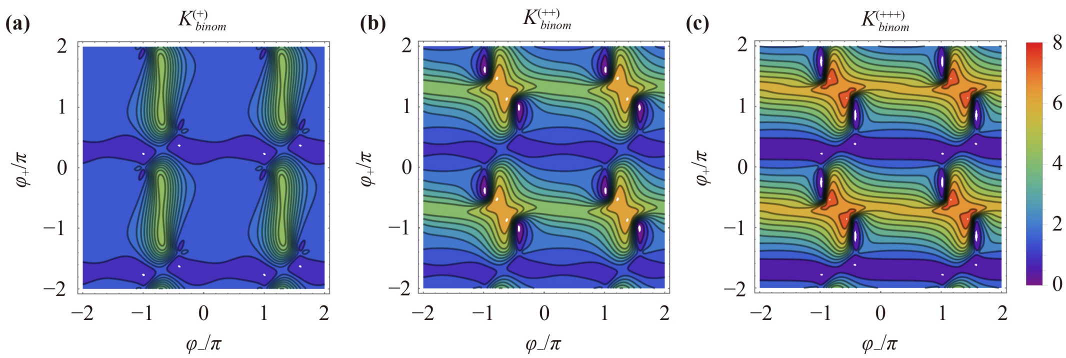


Fig. 4 KL-divergences between $P_n^{(+)}, P_n^{(++)}, P_n^{(+++)}$ and the binomial distribution $p_n^{(binom)}$. The color scale is common for all three plots.

$$P_n^{\prime(\sigma_1\ldots\sigma_k)} = \sum_{N=n}^{+\infty} P_{n,N-n}^{(\sigma_1\ldots\sigma_k)} \cdot P_{Pois}(N). \tag{17}$$

Here $N \equiv N_{at}$ and $P_{n,N-n}^{(\sigma_1\ldots\sigma_k)}$ is the history-conditioned probability distribution of N atoms between two spatial modes. Averaging over N_{at} should be applied to KL divergencies as well.

5 Conclusion

From classical viewpoint, the results obtained in the present work are quite counter-intuitive. Note that modified unravelling deals with a portion of probing radiation which either escaped interaction with BEC, or has already passed it (depending on one or another internal arm of MZI). This modification takes place at the stage when probing has already finished but the information on its result has not yet got fixed in the classical form. Despite the fact that feedback does not directly influence the state of the quantum system, but only controls the way it interacts with environment, it is nevertheless capable of altering its evolution in the form of states conditioned by histories of photoregistrations. In the studied model of atomic BEC it became possible due to entangling its state with the state of the probing field. The phase of the latter acquires operator nature with respect to atoms as a result of interaction with the BEC. Feedback appears in the conditional states of the quantum system that are bound to the history of its interactions with environment. Considering longer histories leads to more efficient control, which is reflected in greater differences of the obtained conditional states from the no-feedback ones. The procedure for constructing the master equations for operators of conditional states is quite straightforward and can be done in an iterative manner, much like the way equation (13) was derived. Moreover, these equations remain Markovian, making the solution significantly easier.

While the BEC model we used in the paper is very simple, it allowed to deduce potentially interesting results. A more realistic model would include many-body effects, and most of all interatomic interaction. This would result in the kinetics of atoms between condensed and uncondensed fraction, as well as modified fluctuations of the number of atoms in the wells.

The wide range of possible unravellings that can be implemented with this kind of feedback is also worth noting. For example, the unitary matrix in Eq. (5) can be made dependent not only on the history of system-environment interactions, but also on the current estimated state of the system. This would make the master equation non-linear, but would provide more possibilities for quantum state engineering.

Declarations The authors declare that they have no competing interests and there are no conflicts.

Acknowledgements This work was supported by the State order (Project AAAA-A21-121021800168-4) at the Institute of Automation and Electrometry SB RAS.

Appendix

Consider a system (11), with $\sigma_2 = \pm$. Assume that the coherence decays rapidly, i.e. that in the Fock basis of the atomic states the matrices $\hat{\varrho}^{(\pm)}$ are three-diagonal:

$$\hat{\varrho}^{(\sigma)} = \sum_{n=0}^{N_{at}} \Big(p_n^{(\sigma)} |n\rangle\langle n| + q_n^{(\sigma)} |n+1\rangle\langle n| + \overline{q}_n^{(\sigma)} |n\rangle\langle n+1| \Big). \tag{18}$$

Equations on $q_n^{(\sigma)}$ immediately follow

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} q_n^{(\sigma)} &= \mathrm{i}\omega\sqrt{(n+1)(N_{at}-n)} \cdot (p_n^{(\sigma)} - p_{n+1}^{(\sigma)}) \\ &\quad + (\mathrm{e}^{\mathrm{i}\chi}+1) \cdot \sum_{\sigma'} q_n^{(\sigma')} - 4 q_n^{(\sigma)} \\ &\quad + \sigma \cdot \mathrm{e}^{\mathrm{i}\chi(n+1)} \sum_{\sigma'} \mathrm{e}^{-\mathrm{i}\varphi_{\sigma'}} q_n^{(\sigma')} \\ &\quad + \sigma \cdot \mathrm{e}^{-\mathrm{i}\chi n} \sum_{\sigma'} \mathrm{e}^{\mathrm{i}\varphi_{\sigma'}} q_n^{(\sigma')}. \end{aligned} \tag{19}$$

Coefficients $p_n^{(+)}$ and $p_n^{(-)}$ sum up to unconditional distribution $p_n^{(+)} + p_n^{(-)} = p_n = \langle n|\hat{\varrho}|n\rangle$, which is unform in a steady state: $p_n = 1/(N_{at}+1)$. In this case, $p_n^{(-)} - p_{n+1}^{(-)} = -(p_n^{(+)} - p_{n+1}^{(+)})$. It is now clear that equations (19) for different values of σ transform one into another upon changing signs of $q_n^{(\sigma)}$, i.e.,

$$q_n^{(-)} = -q_n^{(+)}. \tag{20}$$

This leads to a major simplification of equations on $p_n^{(\sigma)}$. The equation on $p_n^{(+)}$ reads

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}p_n^{(+)} + W_n(\varphi_+,\varphi_-)\cdot n(N_{at}-n+1)\big(p_{n-1}^{(+)} - p_n^{(+)}\big)\\
&+ W_{n-1}(\varphi_+,\varphi_-)\cdot (n+1)(N_{at}-n)\big(p_{n+1}^{(+)} - p_n^{(+)}\big)\\
&= 2\Big(V_n(\varphi_+,\varphi_-) - 2\Big)p_n^{(+)} + 2\Big(1+\cos(\chi n - \varphi_-)\Big)p_n.
\end{aligned} \tag{21}$$

with

$$\begin{aligned}
&V_n(\varphi_+,\varphi_-) \approx \cos(\chi n - \varphi_+) - \cos(\chi n - \varphi_-),\\
&W_n(\varphi_+,\varphi_-) \approx\\
&\omega^2\cdot\frac{\cos(\chi/2)V_{n+1/2}(\varphi_+,\varphi_-) - 2}{V_{n+1/2}^2(\varphi_+,\varphi_-) - 4\cos(\chi/2)V_{n+1/2}(\varphi_+,\varphi_-) + 4}.
\end{aligned} \tag{22}$$

Remembering that $p_n^{(+)} + p_n^{(-)} = p_n$, it is straightforward to show that the equation on $p_n^{(-)}$ differs from (21) by a substitution $\varphi_\pm \to \varphi_\mp + \pi$, i.e.,

$$p_n^{(-)}\Big|_{\varphi_\pm\to\varphi_\mp+\pi} = p_n^{(+)}. \tag{23}$$

Only the last line of the equation should be considered, because the rest of it is clearly invariant under such a transformation.

These relations can be used for obtaining solutions of Eqs. (13, 14).

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